Do we have unitary and (super)renormalizable quantum gravity below the Planck scale?

Filipe de O. Salles[1,*] and Ilya L. Shapiro[1,2,†]

[1]*Departamento de Física, ICE, Universidade Federal de Juiz de Fora, 36036-330 Minas Gerais, Brazil*
[2]*Tomsk State Pedagogical University and Tomsk State University, Tomsk 634061, Russia*
(Received 25 January 2014; published 11 April 2014)

We explore how the stability of metric perturbations in higher-derivative theories of gravity depends on the energy scale of the initial seeds of such perturbations and on a typical energy scale of the gravitational vacuum background. It is shown that, at least in the cases of specific cosmological backgrounds, the unphysical massive ghost which is present in the spectrum of such theories is not growing up as a physical excitation and remains in the vacuum state until the initial frequency of the perturbation is close to the Planck order of magnitude. In this situation, the existing versions of renormalizable and super-renormalizable theories can be seen as very satisfactory effective theories of quantum gravity below the Planck scale.

DOI: 10.1103/PhysRevD.89.084054 PACS numbers: 04.60.-m, 04.60.Bc, 04.62.+v, 11.10.Jj

I. INTRODUCTION

The situation in quantum gravity (QG) has always been shadowed by the conflict between renormalizability and unitarity. From one side, general relativity, which seems to be *the theory* of classical gravity, leads to a nonrenormalizable quantum theory [1–3]. One can achieve renormalizability by including general four-derivative covariant terms in the action [4], but such terms lead to unphysical ghost excitations in the particle spectrum of the theory. Trying to remove these ghosts from the physical spectrum violates the unitarity of the gravitational S matrix. So, the renormalizable QG is nonunitary without ghosts, while the unitary version of QG is nonrenormalizable. As a result of this conflict, the idea of quantum gravity went far beyond the conventional approach of perturbatively quantizing the gravitational field. However, there is an important remaining question: to which extent one should be afraid of higher-derivative ghosts, which are the source of the difficulty?

The problem of higher derivatives and related instabilities has attracted a lot of attention for a long time. Already in 1850, Ostrogradski described these exponential-type instabilities [5]. Later on, in 1963, Veltman discussed a process of quantum scattering of a large-mass, negative-energy particle and a much lighter, positive-energy particle [6]. In a simplified qualitative form, the net result of this study is that typically, the negative-energy particle (massive ghost) gains even more negative kinetic energy, and consequently, the positive-energy particle gains more positive kinetic energy. In the case of higher-derivative gravity, even if we do not observe the ghost due to its huge mass, there should be intensive graviton emission, which can destroy the "pacific" classical solution. More recently, the subject was treated both in the framework of QG [7–11,14], in classical gravity [15–22], and for the simplified model theories, mainly based on higher-derivative oscillators [23–27].

One can note that the mentioned approaches are in fact very different. The QG-based approaches [7,9,14] are related to the assumption that the ghost pole gains a gauge-dependent imaginary contribution at the quantum level, leading to the unitary S matrix. Unfortunately, the one-loop result [28–31] is not sufficient for checking whether this desirable quantum effect really takes place.[1] Another "quantum" proposal [11] can be described as an idea to modify quantum field theory formalism such that the ghost will always be treated together with the graviton and is not regarded as an independent particle. For a while, it was not clear how to put this idea into practice.

The classical approaches [16,17,19,20] are related to the exploration of stability for a given (cosmological or black hole) solution. In the cosmological case, it is reduced to the stability with respect to the perturbations of the conformal factor of the metric (see also Refs. [33,34]) and also to the stability for the gravitational-wave-type perturbations [35–39]. It is remarkable that the perturbations in higher-derivative theories do not show, actually, such strong instabilities as one would expect in the theory with unphysical ghosts. It is important to notice that the mentioned works do not deal just with the linear perturbations, because the latter propagate on a nontrivial metric background.

The purpose of the present contribution is to consider the relation between the presence of ghosts and gravitational instabilities in a spirit of effective quantum field theory. Our

[*]fsalles@fisica.ufjf.br
[†]shapiro@fisica.ufjf.br

[1]In our opinion, the situation is qualitatively similar also for the existing nonperturbative methods (see, e.g., Ref. [32]), but the existence of such methods looks very promising.

consideration will be simple, purely classical, and to some extent close to that of Refs. [20] and [26]. We are going to present some arguments in favor of the idea that the behavior of the gravitational perturbations is closely related to the presence of ghosts, but only if the energy scale is sufficient to generate such a ghost. Our consideration will be based on the linear perturbations on a nontrivial gravitational background. Our results will not be conclusive and should be seen, hopefully, as a contribution for further investigation of the problem.

The paper is organized as follows: In the next section, we shall briefly review the reasons to introduce higher-derivative terms, and consequently massive unphysical ghosts, in the quantum theory. Section III includes the derivation of the equation for a low-energy gravitational wave on an arbitrary low-energy gravitational background and qualitative discussion about the possible effect of such a background on the time evolution of the gravitational wave modes. In Sec. IV, the analysis of the metric perturbations is performed in the relatively simple cases of the cosmological background for renormalizable and super-renormalizable versions of the higher-derivative theory of gravity. We show that the explosive nature of ghosts really takes place, but only for the initial frequencies of the Planck order of magnitude. At the same time, nothing like this can be observed for smaller energies of gravitational perturbations. Finally, in Sec. V, we draw our conclusions and discuss possible continuations of this work.

II. GENERAL SITUATION WITH MASSIVE UNPHYSICAL GHOSTS

One can start by formulating a few general questions concerning higher-derivative ghosts, e.g., as follows: (i) Can we survive without them? (ii) What is really bad about these ghosts? (iii) Can we somehow get rid of them? Let us start from the beginning and show that the answer to the first question (i) is negative.

A. Can quantum theory survive without gravitational higher derivatives?

In order to understand why we need higher derivatives in the gravitational action, one has to start with the relatively simple situation in which only matter fields are quantized and gravity is a classical background. In this semiclassical theory, one has to introduce the action of vacuum, which is a functional of the external classical metric. It has been well known for a long time [40] (for general proofs see, e.g., Refs. [41–43]) that such a theory may be renormalizable, but only if one introduces the following terms into the classical action of vacuum:

$$S_{\rm vac} = S_{\rm EH} + S_{\rm HD}, \tag{1}$$

where

$$S_{\rm EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\{R + 2\Lambda\} \tag{2}$$

is the Einstein-Hilbert term with a cosmological constant, and

$$S_{\rm HD} = \int d^4x\sqrt{-g}\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\} \tag{3}$$

includes higher-derivative terms. Here we use the notations

$$C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + \frac{1}{3}R^2,$$
$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 \tag{4}$$

for the square of the Weyl tensor and for the Lagrange density of the Gauss-Bonnet topological term (Euler density) in $d = 4$, respectively.

The sufficiency of the higher-derivative terms [Eq. (3)] for renormalizability has been consequently proved in a formal way (see, e.g., Ref. [44] for an introduction and further references). The most difficult part is to prove that the diffeomorphism invariance is preserved at the quantum level, and this can be done, including the case in which noncovariant gauges are used for the background metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ [43]. Furthermore, one has to remember that all possible UV counterterms are local expressions. After that, the problem reduces to the evaluation of the superficial degree of divergence in the diagrams with internal lines of matter fields and external lines of $h_{\mu\nu}$. The theory which is renormalizable in flat space-time has only mass dimension-4 logarithmic divergences. An important observation is that adding external lines of $h_{\mu\nu}$ does not increase the degree of divergence (see, e.g., Ref. [45] for more detailed consideration). Therefore, only dimension-4 divergences will emerge in the same theory, even in curved space. This means one has to introduce all such terms at the classical level; that is why we need all terms of Eq. (1) in the vacuum action.

One has to note the great importance of higher-derivative terms [Eq. (3)] for the most important applications of semiclassical theory. For example, the Hawking radiation [46] and the general version of Starobinsky inflation [33] can be derived from the conformal anomaly [47], and the latter results from the renormalization of the terms in Eq. (3).

In quantum gravity, the higher-derivative term with C^2 in Eq. (3) means a massive ghost, a spin-2 particle with negative kinetic energy. This leads to the problem with unitarity, at least at the tree level. But, in the semiclassical theory, gravity is external, and the unitarity of the gravitational S matrix may be not considered really important. The consistency conditions in this case can include the existence of physically reasonable solutions and their stability under small metric perturbations. We shall discuss the relation of such a stability to the presence of massive ghosts in what follows.

Let us now consider the situation in the quantum theory of gravitational fields. Once again, one can prove that the diffeomorphism invariance is preserved at the quantum level (see, for example, the consideration in Ref. [4], which can be generalized to a wide class of the theories of gravity). The evaluation of the superficial degree of divergence D for a Feynman diagram of the field $h_{\mu\nu}$ can be performed by means of the general formula

$$D + d = \sum_{l_{\rm int}} (4 - r_l) - 4n + 4 + \sum_{\nu} K_\nu, \tag{5}$$

with an additional topological relation

$$l_{\rm int} = p + n - 1. \tag{6}$$

Here, d is the number of derivatives on external lines of the diagram, r_l is the power of momenta in the inverse propagator of internal lines, n is the number of vertices, and K_ν is the power of momenta in a given vertex. In Eq. (6), $l_{\rm int}$ and p are the number of internal lines and loops, correspondingly. The $D = 0$ case corresponds to the logarithmic divergences, and then d indicates the number of derivatives in the requested counterterms.

For the quantum version of general relativity, we have $r_l = 2$ and $K_\nu = (2, 0)$. It is easy to see from Eqs. (5) and (6) that the final expression for the logarithmic divergences is $d = 2 + 2p$, and this means the theory is not renormalizable.

If we start from the action in Eq. (1), which includes fourth-derivative terms [Eq. (3)], then $r_l = 4$ and $K_\nu = (4, 2, 0)$. It is easy to see that the maximal power of derivatives in the logarithmic counterterms is $d = 4$, so this theory is renormalizable at all loop orders.

One can introduce more derivatives by considering the action [49]

$$\begin{aligned} S = S_{\rm EH} + \int d^4x\sqrt{-g}\{ & a_1 R^2_{\mu\nu\alpha\beta} + a_2 R^2_{\mu\nu} + a_3 R^2 + \cdots \\ & + b_1 R_{\mu\nu\alpha\beta}\Box R^{\mu\nu\alpha\beta} + b_2 R_{\mu\nu}\Box R^{\mu\nu} + b_3 R\Box R + \mathcal{O}(R^3_{\dots}) \\ & + c_1 R_{\mu\nu\alpha\beta}\Box^k R^{\mu\nu\alpha\beta} + c_2 R_{\mu\nu}\Box^k R^{\mu\nu} + c_3 R\Box^k R + \cdots \\ & + \mathcal{O}(R^{k+2}_{\dots})\}. \end{aligned} \tag{7}$$

For this theory, in the general case, we have $r_l = 4 + 2k$ and $K_\nu = (0, 2, \ldots, 4 + 2k)$. By means of Eqs. (5) and (6), for the logarithmic divergences one has $d = 4 + k(1 - p)$. This formula has three important consequences. First, the theory is super-renormalizable for $k \geq 1$, and only one-loop divergences are present for $k \geq 3$. Second, all divergences are fourth-derivative ones or less. This means that most of the terms in Eq. (7) are not renormalized. Third, the zeroth-derivative, second-derivative, and fourth-derivative counterterms depend on the choice of coefficients in the highest-derivative terms. Let us note, incidentally, that the power counting in the popular Hořava-Lifshits gravity [50] is exactly the same as that described above. This means that the fourth-derivative-in-time logarithmic divergences in this theory are very likely to show up, but they can perhaps be canceled by a special fine-tuning of highest-derivative terms. Only explicit calculation can demonstrate whether this really happens or not, and one-loop calculation would be sufficient for $k \geq 3$. Anyway, as far as a target is *pure* QG, the Hořava-Lifshits gravity has a good chance. At the same time, there is a more serious difficulty related to the contribution of matter fields. If the Lorentz violation in the matter sector is not assumed, these fields will always produce covariant $R^2_{\mu\nu}$-type divergences at all loop orders, and hence it is unclear how one can construct a theory without fourth-order time derivatives. Some support for this consideration comes, also, from the direct calculations for a scalar field in Ref. [51].

The massive ghosts are still present in the theory [Eq. (7)]. For the case of real poles, we have [49]

$$G_2(k) = \frac{A_0}{k^2} + \frac{A_1}{k^2 + m_1^2} + \frac{A_2}{k^2 + m_2^2} + \cdots + \frac{A_{N+1}}{k^2 + m_{N+1}^2}, \tag{8}$$

where the signs alternate,

$$A_j \cdot A_{j+1} < 0, \tag{9}$$

for any sequence with growing real masses:

$$0 < m_1^2 < m_2^2 < m_3^2 < \cdots < m_{N+1}^2. \tag{10}$$

In principle, it would be interesting to explore the cases of imaginary and negative poles (e.g., looking for some kind of a seesaw mechanism for the ghost poles), but we shall leave such a consideration for future work. In the present paper, our attention will be restricted by the case in Eq. (8), and we shall discuss the relation between the presence of ghosts and gravitational instabilities of the vacuum state of the theory.

Looking at the expression in Eq. (8), one can see that this theory has one (in the case $k = 1$) or more (for $k \geq 2$) ghost degrees of freedom in the tensor sector. We conclude that, in general, the price of (super)renormalizability is the presence of ghosts (see also additional discussion of this issue and further references in Ref. [52]). However, from the general perspective, the most important argument in favor of higher derivatives comes from the quantization of matter fields. Taking into account the importance of $S_{\rm HD}$ in Eq. (3) for constructing a renormalizable action of vacuum for quantum matter fields, it is really difficult to see how one can achieve a consistent theory without covariant higher derivatives, so it is worthwhile to take the presence of ghosts seriously and see how we can deal with them.

B. Can one get rid of massive ghosts?

Massive ghosts are tensor (spin-2) massive states with negative kinetic energy. The corresponding components of the propagator do not depend on the gauge fixing and can be seen as physical degrees of freedom. Creation of the particle with negative kinetic energy from the vacuum state is not protected by energy conservation; this means that in the theory with ghosts, one should expect the continuous creation of ghosts and a lot of high-energy gravitons (remember that "our" ghost has Planck-order mass). Even if we do not see the ghost itself, we are going to observe a huge destructing outflux of gravitational energy, which is supposed to explode any classical gravitational solution (see, e.g., Ref. [25] for a recent review).

There were, as we have already mentioned in the Introduction, several interesting attempts to get rid of the massive ghosts. The most obvious idea is to assume that the initial $|\mathrm{in}\rangle$ state in the classical scattering of gravitational perturbations does not contain ghosts. The problem is that, due to the nonpolynomial nature of gravity, ghosts have infinitely complicated interactions with gravitons, and as a result of this interaction, there should be ghosts in the $|\mathrm{out}\rangle$ state. Then the theory will be nonunitary. Let us mention a recent work (Ref. [53]), where it was shown that the theory *without* ghosts is unitary. In our opinion, this is not a real solution, because the problem is exactly of how one can remove the ghost from the spectrum.

One of the interesting ideas is related to the possible role of quantum corrections on the unphysical massive pole [7–9]. As we have already discussed in the Introduction, the existing methods do not enable one to perform nonperturbative analysis, which is needed to make a final conclusion about this possibility [10]. Let us note that the situation can be somewhat better in the super-renormalizable version of the theory [Eq. (7)], where it is technically possible to calculate exact β functions and thus arrive at the leading approximation to the full quantum-corrected propagator. We shall leave this possibility for future work and will concentrate, instead, on a much simpler, direct approach to the problem of ghosts and instabilities.

An interesting possibility has been suggested in Ref. [54] and developed further recently in Refs. [55,56] (see further references on classical applications therein). The idea is to continue an expansion in Eq. (7) to the infinite order in derivatives. The expectation is that one can achieve the following form of the bilinear expansion of the classical action (for simplicity, we take a flat background here, and assume that an appropriate gauge-fixing term is included):

$$S^{(2)} = \frac{1}{2} h^{\alpha\beta}\{cM_P^2\Box + f(\Box)\}h_{\alpha\beta}, \tag{11}$$

such that $c = const$ and $f(\Box)$ is chosen such that the sum $cM_P^2\Box + f(\Box)$ is a specially designed entire function of the argument $\Box$. It is assumed that the resulting theory is (super)renormalizable and that the propagator of the gravitational perturbation $h^{\alpha\beta}$ has a unique pole at $k^2 = 0$. The idea looks very nice and beautiful, but there are certain doubts about whether this scheme will work for QG. First, in order to claim that the theory is (super) renormalizable, one has to arrive at the Feynman rules for $h^{\alpha\beta}$, and to this end, one needs to perform quantization of the theory. It is not clear how this can be done in a nonpolynomial-in-derivatives theory like Eq. (11). Second, in the theory in Eq. (11), one has both r_l and K_ν infinite. Therefore, the evaluation of the superficial degree of divergence [Eq. (5)] in this theory will produce an indefinite $(\infty - \infty)$-type result, so it is unclear what one can say about this theory being super-renormalizable, renormalizable, or nonrenormalizable. In the present case, the possibility of nonrenormalizable theory means, in particular, that the form of the function $f(\Box)$ may eventually change under quantum corrections, such that the massive pole will come back to the theory. Starting from the expression of the actions like

$$S_{i\mathrm{HD}} = \int d^4x\sqrt{-g}\{cR + R^{\mu\nu}h(\Box)R_{\mu\nu} + Rh_1(\Box)R\}, \tag{12}$$

we arrive at the inverse propagator (in momentum space, for the spin-2 sector) of the form

$$G^{-1}(k) = c_1k^2 + k^4\psi(k^2), \tag{13}$$

where $\psi(k^2)$ is an analytic function and $c_1 \neq 0$. One can provide an absence of extra poles with $k \neq 0$ (real or complex) in such a case by setting, e.g.,

$$c_1 + k^2\psi(k^2) = c_1e^{-k^2/M^2}, \tag{14}$$

or in some other similar way [54], but it is not obvious that this form of the function will hold after quantum corrections are taken into account. Finally, the proposal of Ref. [54] is very interesting, but the statement that the theory based on Eq. (11) really solves the conflict between renormalizability and unitarity looks a little bit premature and has not been clarified until now.

A qualitatively distinct approach has been suggested in Ref. [11]. It is based on the observation that the ghost is not an independent particle, but rather a companion of the graviton in the linearized gravity. The separation of different degrees of freedom in higher-derivative theories is a nontrivial issue even in the case of linearized theories (see, e.g., Ref. [57]). Needless to say, the situation should be more complicated in gravity, which has a nonpolynomial interaction structure. However, up to now it has not been clear how one can put the proposal of Ref. [11] into practice and how the new quantum theory of gravity should look. Anyway, these two proposals show that the situation with ghosts is not completely hopeless and should be explored in more detail.

Finally, let us mention the literature of avoiding the ghosts in the models of massive gravity [12]. An alternative approach here is to admit that the unphysical ghost may exist, but it is harmless, because its interaction with the rest of the particles is nonlocal and is suppressed by some large parameter [13]. It looks tentative to find some mechanism with similar final output for the much more relevant (at least, in our opinion) case of higher derivatives. In what follows, we consider the possibility that the corresponding ghost exists only as a vacuum excitation but never shows up as a physical particle, and therefore, may be harmless at energies below the Planck scale.

III. GRAVITATIONAL WAVES ON AN ARBITRARY BACKGROUND

Let us remember the assumptions which were made to deal with the ghost problem in higher-derivative theory:

(i) One can draw conclusions about the gravity theory by using linearized approximation. The S matrix of gravitons should be the main object of our interest.

(ii) Ostrogradsky instabilities [5] or Veltman scattering [6] are relevant independent of the energy scale; in all cases, they produce runaway solutions and the Universe explodes.

There is a simple way to directly check most of these assumptions at once. Let us take a higher-derivative theory of gravity and verify the stability with respect to the linear perturbations on some physically interesting, classical solution. If the mentioned assumptions are correct, we will observe rapidly growing modes even for the low-energy (i.e., low-curvature) background. However, if there are no growing modes at the linear level, there will not be such modes even at higher orders. Let us remember that the ghost problem is a tree-level one, and therefore we do not need to worry about loop effects. Moreover, according to the known mathematical theorem [58], if the system is stable with respect to linear fluctuations, it will be stable at the nonlinear level too, at least for the sufficiently small amplitudes of perturbations.

Finally, our general purpose is to explore the time dynamics of the gravitational waves on an arbitrary "low-energy" background, in a higher-derivative theory of gravity. In what follows, we shall start from the theory in Eq. (1) on a general background and show that there are some arguments in favor of its irrelevance for the sufficiently low-energy fluctuations. In the consequent section, we shall deal with the reduced problem and identify the relations between the presence of growing modes and existence of massive ghosts on the cosmological background.

A. Riemann normal coordinate expansions

Let us consider the fourth-derivative theory [Eq. (1)] and set to zero the cosmological constant. This is justified when we are interested in the behavior of the gravitational waves, because the cosmological constant is irrelevant at distances much smaller than the size of the Universe. The action which we will deal with can be cast into the form

$$S_{4dQG} = \int d^4x\sqrt{-g}\left\{-\frac{M_P^2}{16\pi}R + a_1C^2 + a_2E + a_3\Box R + a_4R^2\right\}. \quad (15)$$

The unique dimensional parameter in this theory is the Planck mass M_P, because all other coefficients are dimensionless. Of course, the a_k's are arbitrary parameters, and we can choose them to be as great as we like, but let us make a moderate choice, assuming that the values of the a_k's are close, in the orders of magnitude, to unity. Then the Planck mass M_P defines the unique scale of the theory. This means that all those quantities which are much smaller than M_P are very small in this theory. One can note that this feature has been extensively used in establishing the effective approach to QG [59].

The low-energy approach to the dynamics of gravitational perturbations on an arbitrary metric background means that the following inequalities are satisfied:

$$|R_{\mu\nu\alpha\beta}| \ll M_P^2 \quad \text{and} \quad \mathbf{k}^2 \ll M_P^2, \quad (16)$$

where $R_{\mu\nu\alpha\beta}$ are components of the Riemann tensor of a background and $\mathbf{k}$ is a wave vector for the perturbation.

The equation of our interest is

$$H^{\mu\nu,\alpha\beta}\bar{h}^{\perp}_{\alpha\beta} = 0, \text{where} \quad H^{\mu\nu,\alpha\beta} = \frac{\delta^2 S_{4dQG}}{\delta g_{\mu\nu}\delta g_{\alpha\beta}}. \quad (17)$$

The gauge-fixing term is irrelevant, since we are interested only in the traceless and completely transverse components of the gravitational perturbation $\bar{h}^{\perp}_{\alpha\beta}(x)$, which will be denoted $h_{\alpha\beta}$ in what follows. We will assume that $h_{\alpha\beta}$ satisfies the constraints

$$h_{\alpha\beta}g^{\alpha\beta} = 0 \quad \text{and} \quad \nabla^\alpha h_{\alpha\beta} = 0. \quad (18)$$

As an illustration, let us write separately the zeroth-order-in-curvature terms in Eq. (17) as

$$a_1\left(\Box^2 - \frac{M_P^2}{32\pi a_1}\Box\right)h^{\alpha\beta} = 0, \quad (19)$$

which corresponds to the mass of the ghost, $m_2 = M_P/\sqrt{32\pi a_1}$.

The full equation includes the terms in Eq. (19) and also terms linear and quadratic in curvature. One can easily obtain this equation from the works on higher derivative quantum gravity (HDQG), e.g., Ref. [44] or Ref. [31]. In the first order in curvature and taking into account Eq. (18), this equation has the form of Eq. (17) with[2]

$$H_{\mu\nu,\alpha\beta} = -\frac{a_1}{2}\delta_{\mu\nu,\alpha\beta}\Box^2 + D^{\rho\lambda}_{\mu\nu,\alpha\beta}\nabla_\rho\nabla_\lambda + W_{\mu\nu,\alpha\beta}, \quad (20)$$

where

$$D^{\rho\lambda}{}_{\mu\nu,\alpha\beta} = 2a_1 g_{\nu\beta}R_{\alpha\cdot\cdot\mu}{}^{\rho\lambda} + a_1 g^{\rho\lambda}(2g_{\nu\beta}R_{\alpha\mu} - R_{\mu\alpha\nu\beta}) + \left(\frac{M_P^2}{64\pi} - \frac{a_1}{6}R - \frac{a_4}{2}R\right)g^{\rho\lambda}\delta_{\mu\nu,\alpha\beta},$$

$$W_{\mu\nu,\alpha\beta} = \frac{M_P^2}{64\pi}(R_{\mu\alpha\nu\beta} + 3R_{\mu\alpha}g_{\nu\beta} - R\delta_{\mu\nu,\alpha\beta}). \quad (21)$$

The reason to keep only linear terms in curvature is due to our interest in the behavior of metric perturbations in Eq. (17) when both the background and the perturbations have typical energies much smaller that the Planck scale. This means, in particular, that we can ignore all $\mathcal{O}(R^2_{\dots})$ terms. Of course, it would be interesting to explore higher orders at some point, but in the present work we will try to make calculations as simple as possible.

It is natural to use some technique which enables one to treat curvature tensor components as small perturbations. The covariant formalism of this kind is based on the Riemann normal coordinates [60]. This approach is traditionally used for describing the propagator [61], in our case for gravitons. The method is also useful in other situations, mainly related to the evaluation of loop effects [62,63], but now we intend to discuss only the tree-level approximation.

The normal coordinates method assumes an expansion around a chosen point in the space-time—let us call it $P(x'^\mu)$. The quantities corresponding to this point will be labeled by small zeros; for instance, the metric is $\overset{o}{g}_{\alpha\beta}$. Also, we shall need the curvature tensor and its covariant derivatives at this point. The nice feature of normal coordinates is that the coordinate lines are specially designed geodesic lines, and an expansion can be done covariantly with respect to the point P. The deviation from the point P is parameterized by the quantities $y^\mu = x^\mu - x'^\mu$, which are zero at P. As far as we consider the components of the curvatures to be small, the consideration can be restricted by the first-order terms. For the sake of generality, we shall also perform part of the expansion until the second order; the corresponding results are settled in Appendix A.

The expansion for the metric has the form

$$g_{\alpha\beta}(y) = \overset{o}{g}_{\alpha\beta} - \frac{1}{3}\overset{o}{R}_{\mu\alpha\nu\beta}y^\mu y^\nu + \cdots. \quad (22)$$

[2]In these expressions, the symmetrization over the pairs of indices $(\mu\nu)$ and $(\alpha\beta)$ is assumed. The complete forms including second-order terms can be found in Ref. [31].

One can always choose the metric in the expansion point to be the Minkowski one, $\overset{o}{g}_{\alpha\beta} = \eta_{\alpha\beta}$. For the Christoffel symbol, one has

$$\Gamma^\lambda_{\alpha\beta} = \frac{2}{3}\overset{o}{R}{}^{\lambda}_{\cdot(\alpha\beta)\nu}y^\nu + \cdots.$$

Let us start from the normal coordinate expansion for $\Box h^{\alpha\beta}$. The expansion represents a power series in both curvature components $\overset{o}{R}{}^{\lambda}_{\cdot\alpha\beta\nu}$ and y^μ. In what follows, we label by $A^{(n)}$ the order n of the expansion in y^μ for the quantity A, for instance

$$\Box h^{\alpha\beta} = (\Box h^{\alpha\beta})^{(0)} + (\Box h^{\alpha\beta})^{(1)} + (\Box h^{\alpha\beta})^{(2)} + \cdots, \quad (23)$$

where the dots indicate the omitted terms of higher orders in y^μ and of higher orders in the curvature tensor and its covariant derivatives at the point P. Direct calculation yields the following results up to the second order in y^μ:

$$(\Box h^{\alpha\beta})^{(0)} = \eta^{\mu\nu}\left[\partial_\mu\partial_\nu h^{\alpha\beta} - \frac{1}{3}\overset{o}{R}{}^{\alpha}_{\cdot\nu\lambda\mu}h^{\lambda\beta} - \frac{1}{3}\overset{o}{R}{}^{\beta}_{\cdot\nu\lambda\mu}h^{\alpha\lambda}\right], \quad (24)$$

$$(\Box h^{\alpha\beta})^{(1)} = -\frac{4}{3}\eta^{\mu\nu}[\overset{o}{R}{}^{\alpha}_{\cdot(\nu\lambda)\tau}\partial_\mu h^{\lambda\beta} + \overset{o}{R}{}^{\beta}_{\cdot(\nu\lambda)\tau}\partial_\mu h^{\alpha\lambda}]y^\tau, \quad (25)$$

$$(\Box h^{\alpha\beta})^{(2)} = \frac{1}{3}\overset{o}{R}{}^{\mu\ \nu}_{\cdot\tau\cdot\rho}(\partial_\mu\partial_\nu h^{\alpha\beta})y^\tau y^\rho. \quad (26)$$

B. Zeroth-order approximation

The next step would be to make a Fourier transformation in the spatial sector,

$$h_{\mu\nu}(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3}h_{\mu\nu}(\mathbf{k}, t)e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (27)$$

As a useful approximation, we can treat the wave vector $\mathbf{k}$ as constant and will be therefore interested only in the time evolution of the perturbation $h_{\mu\nu}$. The validity of such a treatment is restricted to the long-wave perturbations, where we assume that the modes $h_{\mu\nu}(\mathbf{k}, t)$ have independent time dynamics. This treatment enables one to trade the complicated partial differential equation [Eq. (17)] for the much simpler ordinary differential equations for individual modes. Since in the theory under discussion the unique scale parameter is given by the Planck mass, a long wavelength is just one which is larger than the Planck length.

Let us now see what the approximation of independent modes $h_{\mu\nu}(\mathbf{k}, t)$ means, from the practical side. Looking at Eqs. (24), (25), and (26), it is clear that the Eq. (17) has two complications: those related to the derivatives like $\partial h^{\alpha\beta}(\mathbf{r}, t)/\partial y^\mu$, and those related to the factors of y^μ. Obviously, $\partial h^{\alpha\beta}(\mathbf{r}, t)/\partial y^\mu$ reduces, after using Eq. (27), to $ik_\mu h^{\alpha\beta}(\mathbf{k},t)$. The treatment of the factors of y^μ is a bit more complicated and goes as follows:

$$\int \frac{d^3k}{(2\pi)^3} y^\mu h^{\alpha\beta}(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{r}} = \int \frac{d^3k}{(2\pi)^3} h^{\alpha\beta}(\mathbf{k},t) \frac{\partial}{i\partial k_\mu} e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{28}$$

One can integrate by parts in the last expression. The surface term at infinity can be neglected, because we can assume $h^{\alpha\beta}(|\mathbf{k}| \to \infty) \to 0$, since all perturbations are suppressed beyond the Planck scale. In this way, we arrive at the relation

$$\int \frac{d^3k}{(2\pi)^3} y^\mu h^{\alpha\beta}(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{r}} = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\partial}{i\partial k_\mu} h^{\alpha\beta}(\mathbf{k},t). \tag{29}$$

At that point, we conclude that the expansion in normal coordinates y^μ means an expansion of modes $h^{\alpha\beta}(\mathbf{k},t)$ in the series in k^μ. In the simplest possible approximation, we assume that the modes do not depend on k^μ—that is, $h^{\alpha\beta}(\mathbf{k},t) = h^{\alpha\beta}(t)$. This means we can restrict our consideration to the zeroth-order approximation in y^μ in Eq. (17).

For the $\Box^2$ term, one can write

$$(\Box^2 h^{\alpha\beta})^{(0)} = \eta^{\mu\nu}\left\{[\partial_\mu\partial_\nu(\Box h^{\alpha\beta})]^{(0)} - \frac{1}{3}\overset{\circ}{R}{}^{\alpha}{}_{\nu\tau\mu}(\Box h^{\tau\beta})^{(0)} - \frac{1}{3}\overset{\circ}{R}{}^{\beta}{}_{\nu\tau\mu}(\Box h^{\alpha\tau})^{(0)}\right\}. \tag{30}$$

Let us introduce the following notations for the expansions in Eqs. (25) and (26)[3]:

$$\begin{aligned}(\Box h^{\alpha\beta})^{(1)} &= \Delta^{\alpha\beta}{}_{\chi} y^\chi,\\ (\Box h^{\alpha\beta})^{(2)} &= \Lambda^{\alpha\beta}{}_{\chi\omega} y^\chi y^\omega.\end{aligned} \tag{31}$$

After a very small amount of algebra, we obtain

$$\begin{aligned}(\Box^2 h^{\alpha\beta})^{(0)} = {}& \eta^{\mu\nu}\partial_\mu\partial_\nu(\Box h^{\alpha\beta})^{(0)} + \eta^{\mu\nu}\Big[2\partial_\nu\Delta^{\alpha\beta}{}_{\mu} + 2\Lambda^{\alpha\beta}{}_{\nu\mu}\\ & - \frac{1}{3}\overset{o}{R}{}^{\alpha}{}_{\nu\tau\mu}(\Box h^{\tau\beta})^{(0)} - \frac{1}{3}\overset{o}{R}{}^{\beta}{}_{\nu\tau\mu}(\Box h^{\alpha\tau})^{(0)}\Big],\end{aligned} \tag{32}$$

where $(\Box h^{\alpha\beta})^{(0)}$ has been defined in Eq. (24). Taking this together with Eq. (21), we arrive at the expression

$$\begin{aligned}H_{\mu\nu,\alpha\beta} = {}& -\frac{a_1}{2}\delta_{\mu\nu,\alpha\beta}(\Box^2 h^{\alpha\beta})^{(0)} + 2a_1\eta_{\nu\beta}\overset{\circ}{R}{}_{\alpha\cdot\cdot\mu}^{\;\;\rho\lambda}\partial_\rho\partial_\lambda\\ & + \left[\left(\frac{a_1}{6}\overset{\circ}{R} + \frac{a_4}{2}\overset{\circ}{R} + \frac{M_P^2}{64\pi}\right)\delta_{\mu\nu,\alpha\beta}\right.\\ & \left. + 2a_1\eta_{\nu\beta}\overset{\circ}{R}_{\alpha\mu} - a_1\overset{\circ}{R}_{\mu\alpha\nu\beta}\right]\Box\\ & + \frac{M_P^2}{64\pi}(\overset{\circ}{R}_{\mu\alpha\nu\beta} + 3\eta_{\mu\alpha}\overset{\circ}{R}_{\nu\beta} - \overset{\circ}{R}\delta_{\mu\nu,\alpha\beta}).\end{aligned} \tag{33}$$

By incorporating Eq. (32) into the last formula, we obtain the equation for the metric perturbation in the zeroth-order approximation in y^μ:

$$\begin{aligned}\Box^2 h_{\mu\nu} & - \frac{1}{3}(\overset{\circ}{R}_{\lambda\mu}\Box h^\lambda_\nu + \overset{\circ}{R}_{\lambda\nu}\Box h^\lambda_\mu) + \frac{4}{3}(\overset{\circ}{R}{}^{\lambda\rho\tau}_{\cdots\mu}\partial_\rho\partial_\tau h_{\nu\lambda}\\ & + \overset{\circ}{R}{}^{\lambda\rho\tau}_{\cdots\nu}\partial_\rho\partial_\tau h_{\mu\lambda}) - 2\overset{\circ}{R}{}_{\lambda\cdot\cdot\mu}^{\;\;\rho\tau}\partial_\rho\partial_\tau h^\lambda_\nu - 2\overset{\circ}{R}{}_{\lambda\cdot\cdot\nu}^{\;\;\rho\tau}\partial_\rho\partial_\tau h^\lambda_\mu\\ & - 2\overset{\circ}{R}_{\tau\mu}\Box h^\tau_\nu - 2\overset{\circ}{R}_{\tau\nu}\Box h^\tau_\mu + 2\overset{\circ}{R}_{\mu\rho\nu\tau}\Box h^{\rho\tau}\\ & + \frac{2}{3}\overset{\circ}{R}{}^{\rho\tau}\partial_\rho\partial_\tau h_{\mu\lambda} + \frac{a_1 + 3a_4}{a_1}\overset{\circ}{R}\Box h_{\mu\nu}\\ & - \frac{M_P^2}{32\pi a_1}[(\Box - \overset{\circ}{R})h_{\mu\nu} + (\overset{\circ}{R}_{\mu\lambda\nu\tau} + 3\eta_{\mu\lambda}\overset{\circ}{R}_{\nu\tau})h^{\lambda\tau}] = 0.\end{aligned} \tag{34}$$

We note that Eq. (34) is a flat-space differential equation, which depends on the curvature tensor components in a given point P, $\overset{\circ}{R}{}^{\alpha}_{\cdot\beta\mu\nu}$. In particular, here we assume a flat d'Alembertian operator, $\Box = \eta^{\rho\tau}\partial_\rho\partial_\tau$. Of course, the complete expression is an infinite-series expansion in both k^μ and $\overset{\circ}{R}_{\mu\nu\alpha\beta}$, so Eq. (34) is just the lowest-order nontrivial approximation to it. Equation (34) is a generalization of the basic equation (19), and the difference between the two is represented by the terms linear in curvature which are partially hidden and partially omitted in Eq. (19). The investigation of the time dynamics of $h_{\mu\nu}$ with a constant $\mathbf{k}$ can be performed on the basis of Eq. (34). One can expect that the nonlinearities, presented by a nontrivial background, will be responsible for relatively small corrections to the dynamics of Eq. (19) in flat space. This statement can be correct or not, and at the moment we are unable to give a definite answer on the basis of Eq. (34). Instead, we shall perform partial verification of this statement for the case of the cosmological background, in the next section.

Equation (34) contains relevant information about the evolution of the traceless and transverse mode of the metric perturbation in the regime $|\overset{\circ}{R}_{\alpha\beta\tau\lambda}| \ll M_P^2$. We postpone the analysis of this complicated equation for future work. In the next section, we shall consider another approximation, which is not related to the expansion around the flat space. To some extent, the results of this consideration will justify

[3]Higher-order expressions for $\Delta^{\alpha\beta}{}_{\chi}$ and $\Lambda^{\alpha\beta}{}_{\chi\omega}$ can be found in Eqs. (A5) and (A6) in Appendix A.

the system of approximations which were used in deriving Eq. (34).

IV. PERTURBATIONS ON THE COSMOLOGICAL BACKGROUND

Let us now turn to a very different approach and consider the cosmological background metric. In this case, the consideration is not related to the weak curvature approximation, but the background is of course a very special one. Anyway, this consideration can be useful in collecting evidence in favor of (in)stability of the background in higher-derivative gravity theory. One has to note that classical cosmological solutions can be very different, and hence the problem is technically not completely trivial.

The consideration of metric perturbations in higher-derivative theories on a cosmological background has been previously studied in Refs. [35–37] for the particular case of an inflationary (dS) background, and recently in Ref. [39] for more general cosmological metrics, namely radiation- and dust-dominated cases. In all these works, the equations were derived on the basis of higher-derivative theory with semiclassical corrections, and in all cases no instabilities were detected. Here we restrict our attention to the purely classical theory [Eq. (15)]. Compared to the previous publications, we shall extend the set of initial conditions and finally discover the unstable case, in exactly the situation which will confirm the main assumptions formulated in the previous sections.

We consider the perturbations

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu} \tag{35}$$

over an isotropic and homogeneous cosmological background,

$$g^0_{\mu\nu} = \mathrm{diag}\{1, -\delta_{ij}a^2(t)\}. \tag{36}$$

One pertinent observation is in order here. The action in Eq. (15) without a cosmological constant has only one term which can affect the solution for the scale function $a(t)$ of the cosmological background. Remember that the Weyl tensor is zero for the metric in Eq. (36), and the Gauss-Bonnet term does not contribute to the equations of motion in $d = 4$. At the end of the day, the only relevant higher-derivative term for the background is[4] a_4R^2. Then, as long as we consider the low-energy situation with $|R| \ll M_P^2$, the classical solutions of GR can be seen as precise approximations for the theory in Eq. (15). For this reason, we shall consider the metric perturbations over the background [Eq. (36)], with $a(t)$ corresponding to the standard cosmological solutions of GR, such as a matter-dominated or radiation-dominated Universe, and to the exponential case. In the last case, the accelerated expansion is due to the cosmological constant only.

The initial conditions for the perturbations will be chosen to originate from the fluctuations of free quantum fields. The spectrum is identical to a scalar quantum field in Minkowski space (see, e.g., Ref. [64]),

$$h(x,\eta) = h(\eta)e^{\pm i\mathbf{k}.\mathbf{r}}, \qquad h(\eta) \propto \frac{e^{\pm ik\eta}}{\sqrt{2k}}, \tag{37}$$

where we employ the conformal time η, $a(\eta)d\eta = dt$, $\mathbf{k}$ is the wave-number vector, and $k = |\mathbf{k}|$. A normalization constant is not necessary for the case of linear perturbations. Initial amplitudes are supposed to have a quantum origin and depend on $\mathbf{k}$ according to

$$h_0 \propto \frac{1}{\sqrt{2k}}, \qquad \dot{h}_0 \propto \sqrt{\frac{k}{2}}, \qquad \ddot{h}_0 \propto \frac{k^{3/2}}{\sqrt{2}}, \qquad \dddot{h}_0 \propto \frac{k^{5/2}}{\sqrt{2}}, \tag{38}$$

where the derivatives are taken with respect to the cosmic time. Let us stress that the vacuum stability is related to the asymptotic behavior of perturbations at $t \to \infty$, and therefore the choice of initial conditions is, to a great extent, irrelevant. However, all plots presented below correspond to Eq. (38).

In order to study the time dynamics of $h(t,\mathbf{r})$, one can perform a Fourier transform,

$$h_{\mathbf{k}}(t) = \frac{1}{(2\pi)^{3/2}} \int h(t,\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3x. \tag{39}$$

Now we are ready to analyze the presence (or not) of growing modes for the particular cases. We shall present only the final form of the equations; more details can be found in the previous works [38,39]. In Ref. [38], similar equations were obtained for the pre-big-bang scenario.

To derive the wave equations, we will use the conditions (where $\mu = 0, i = 0,\ 1,\ 2,\ 3$)

$$\partial_i h^{ij} = 0 \quad \text{and} \quad h_{kk} = 0, \tag{40}$$

together with the synchronous coordinate condition $h_{\mu 0} = 0$.

A. Stability analysis

In this section, we will begin to analyze if there is (or is not) stability for the cosmological solutions in the theory according to Eq. (15). The consideration will be based on the combination of semianalytical and numerical methods, where the latter is mainly used for control and illustration purposes.

The basis of the semianalytical method is as follows: After applying Eq. (39), we obtain a fourth-order ordinary

[4]It is interesting that the a_1C^2 term is much more relevant for the metric perturbations than the a_4R^2 term, so the situations for the background and for metric perturbations are just opposite.

differential equation for the tensor part of metric perturbations. One can easily transform it into the system of four first-order equations, and then the problem is reduced to the analysis of eigenvalues of the corresponding characteristic equation. The details are briefly described in Appendix B; the reader can also consult Ref. [39]. It is easy to see that one always has to calculate the quantity Δ, given in Eq. (B7) based on the ancient Cardano approach [67]. This quantity contains all relevant information about the asymptotic behavior of the solution.

One can distinguish the following cases:

(1) $\Delta < 0$. The three roots are real and distinct. Then we have one of the following situations:
  (a) All roots are negative: Stable solution.
  (b) Some root is positive: Unstable, and instability generally increases with an increasing number of positive roots; in a sense, one needs more severe initial conditions to avoid instability.

(2) $\Delta = 0$. The roots are real, and two or three are equal. Then
  (a) All roots are either negative or have negative real parts: Stable.
  (b) Some root has a positive real part: Unstable, and this instability increases with an increasing number of such positive roots.

(3) $\Delta > 0$. One real root and two complex roots.
  (a) All roots are negative or have negative real parts: Stable.
  (b) Some root has a positive real part: Unstable.

In what follows, we shall perform the analysis separately for each case, namely for flat space-time, exponential expansion, radiation, and matter in fourth-derivative theory, and we shall also consider the flat case for a super-renormalizable theory. In each case, we shall consider many different values of k and will try to see in which range of frequencies the growing modes will show up.

B. Flat case

In order to fix the method, consider first the flat case, when $g^0_{\mu\nu} = (1, -\delta_{ij})$. The action of our interest [Eq. (15)] can be presented as

$$S_{HDQG} = S_0 + S_1 + S_3, \tag{41}$$

where

$$S_0 = f_0 \int d^4x\sqrt{-g}R, \qquad S_1 = f_1 \int d^4x\sqrt{-g}C^2,$$
$$S_3 = f_3 \int d^4x\sqrt{-g}R^2. \tag{42}$$

The metric perturbations are defined as

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}, \tag{43}$$

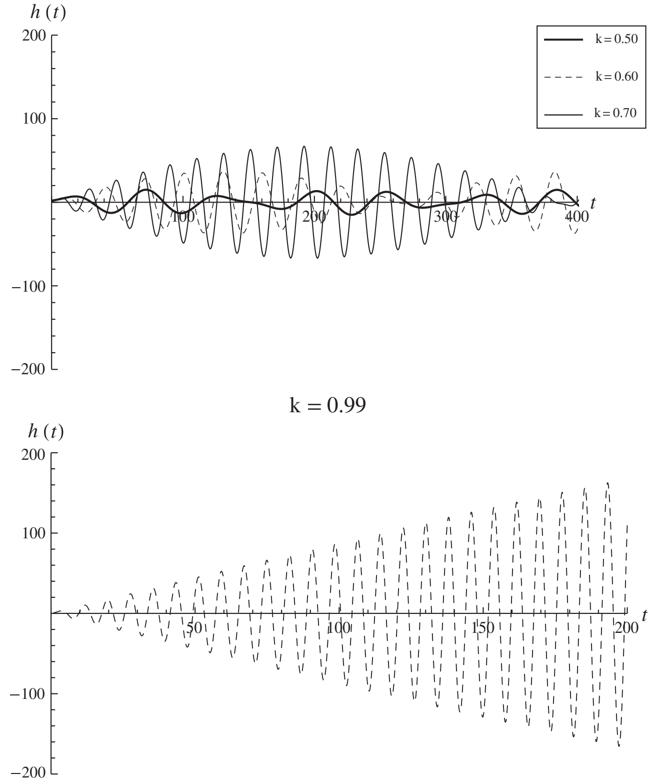

FIG. 1. The plots for the flat-space case. k is measured in units of M_P, and the growing modes appear for k close to 1. Oscillations here mean that the eigenvalues with both positive and negative real parts have imaginary components. For smaller values of k, the amplitude greatly increases, but asymptotically it goes to zero (out of the plot).

and the synchronous and harmonic gauge-fixing conditions [Eq. (40)] are imposed. Then the second variations of the actions yield the following results (here $h \equiv \bar{h}^{\perp}_{ij}$):

$$S_0^{(2)} = f_0\left[h\ddot{h} + \frac{3}{4}\dot{h}\,\dot{h} - \frac{1}{4}h\nabla^2 h\right],$$
$$S_1^{(2)} = f_1\left[\frac{1}{2}\ddot{h}^2 + \frac{1}{2}(\nabla^2 h)(\nabla^2 h) + \ddot{h}\nabla^2 h + 2\dot{h}\nabla^2\dot{h}\right],$$
$$S_3^{(2)} = 0, \tag{44}$$

where also $\nabla = \nabla_k, k = 1, 2, 3$. As always, the R^2 term does not contribute to the tensor part of the gravitational perturbation in the flat case. Taking the sum of the three terms in Eq. (44), we arrive at the equation for perturbations,

$$f_1\ddddot{h} - 2f_1\nabla^2\ddot{h} + f_1\nabla^4 h + \frac{1}{2}f_0\ddot{h} - \frac{1}{2}f_0\nabla^2 h = 0, \tag{45}$$

which is nothing else but the equation equivalent to Eq. (19),[5]

[5]We adopt the notations $h^l_k h^k_l = h^2$, $h^l_k \dot{h}^k_l = h\dot{h}$ and use $\Box h^l_k = \ddot{h}^l_k - \nabla^2 h^l_k$.

$$(f_1\Box^2 + f_0\Box)h = 0. \tag{46}$$

Let us present the results for the growing modes:
Semianalytical analysis: For $a_1 > 0$, we find runaway solutions for all values of k.
For $a_1 < 0$ and $k < 0.90$, we have $\Delta < 0$, and all eigenvalues are real and negative. So, we have stability in this case. For $k > 0.90$, we find two positive eigenvalues. Therefore, we can observe instability, i.e., runaway solutions.
Numerical analysis: Using MATHEMATICA software [65], we find that the growing modes show up from $k \geq 0.99$. The illustrating plots for the initial period of time are shown in Fig. 1.

One can see that for $a_1 < 0$, growing modes exist for the magnitude of the wave vector being equal to or greater than the Planck mass. For much smaller frequencies k, we do not observe the effect of a ghost, probably because its mass is too large. It is important for our general understanding that for $a_1 > 0$ there are exponentially growing modes for all values of k. In this case, the massless mode (graviton) is actually a ghost, so there is no energy gap for generating the runaway solutions. Obviously, a huge energy gap exists for the $a_1 < 0$ case.

Let us make one more observation concerning the marginal value of k, starting from which the growing modes are observed. According to Eq. (19), this value depends on the ratio $M_P/\sqrt{-a_1}$. In the consideration presented above, we have used $a_1 = -1$ and consequently found that the marginal value of k is close to M_P.

C. Cosmological solutions

Let us now consider the dynamics of the gravitational waves on the cosmological background. It proves useful to present the action of Eq. (15) using different notations. After performing some integrations by parts, we arrive at

$$S = \int d^4x\sqrt{-g}L, \tag{47}$$

where

$$L = \sum_{s=0}^{5} f_s L_s = (f_0 R + f_1 R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} + f_2 R^{\alpha\beta}R_{\alpha\beta} + f_3 R^2), \tag{48}$$

and the coefficients $f_{0,\ldots,3}$ are defined according to

$$\begin{aligned} f_0 &= -\frac{M_P{}^2}{16\pi}, \\ f_1 &= a_1 + a_2, \\ f_2 &= -2a_1 - 4a_2, \\ f_3 &= \frac{a_1}{3} + a_2. \end{aligned} \tag{49}$$

As one should expect, the coefficient a_2 of the Gauss-Bonnet topological term does not affect the equations.

Let us consider the background cosmological solution $g^0_{\mu\nu} = \{1, -\delta_{ij}a^2(t)\}$. Then, one can arrive at the following expressions for the bilinear parts of the partial Lagrangians from Eq. (48):

$$\begin{aligned} L_0 &= a^3 f_0\left[h^2\left(\frac{3}{2}\dot{H} + 3H^2\right) + h\ddot{h} + 4Hh\dot{h} + \frac{3}{4}\dot{h}^2 - \frac{h}{4}\frac{\nabla^2 h}{a^2}\right] + \mathcal{O}(h^3), \\ L_1 &= a^3 f_1\Big[\dot{h}^2(2H^2 - 2\dot{H}) - h\ddot{h}(4H^2 + 4\dot{H}) - h^2(3\dot{H}^2 + 6\dot{H}H^2 + 6H^4) - h\dot{h}(8H\dot{H} + 16H^3) + \ddot{h}^2 \\ &\quad + 4H\dot{h}\,\ddot{h} + \left(\frac{\nabla^2 h}{a^2}\right)^2 + 2\dot{h}\frac{\nabla^2\dot{h}}{a^2} + (H^2 h - 2H\dot{h})\frac{\nabla^2 h}{a^2}\Big] + \mathcal{O}(h^3), \\ L_2 &= a^3 f_2\Big[-h\dot{h}(12\dot{H}H + 24H^3) - \frac{\dot{h}^2}{2}\left(5\dot{H} + \frac{18}{4}H^2\right) - h^2(3\dot{H}^2 + 9\dot{H}H^2 + 9H^4) - h\ddot{h}(4\dot{H} + 6H^2) + \frac{\ddot{h}^2}{4} + \frac{3}{2}H\dot{h}\,\ddot{h} \\ &\quad + \frac{1}{4}\left(\frac{\nabla^2 h}{a^2}\right)^2 - \frac{1}{2}(\ddot{h} + 3H\dot{h} - \dot{H}h - 3H^2 h)\frac{\nabla^2 h}{a^2}\Big] + \mathcal{O}(h^3), \\ L_3 &= -6a^3 f_3(\dot{H} + 2H^2)\left[h^2\left(\frac{3}{2}\dot{H} + 3H^2\right) + 2h\ddot{h} + 8Hh\dot{h} + \frac{3}{2}\dot{h}^2 - \frac{h}{2}\frac{\nabla^2 h}{a^2}\right] + \mathcal{O}(h^3), \end{aligned} \tag{50}$$

Omitting higher-order terms $\mathcal{O}(h^3)$ in the expressions of Eq. (50) and taking a variational derivative with respect to $h_{\mu\nu}$, we arrive at the equation for tensor mode[6]:

[6]Which is, in fact, a part of the more complicated equation with quantum corrections that was explored in Ref. [39].

$$
\begin{aligned}
&\left(2f_1+\frac{f_2}{2}\right)\overset{....}{h}+[3H(4f_1+f_2)]\dddot{h}+\left[3H^2\left(6f_1+\frac{f_2}{2}-4f_3\right)+6\dot{H}(f_1-f_3)+\frac{1}{2}f_0\right]\ddot{h}-(4f_1+f_2)\frac{\nabla^2\ddot{h}}{a^2}\\
&+\left[-21H\dot{H}\left(\frac{1}{2}f_2+2f_3\right)-\ddot{H}\left(\frac{3}{2}f_2+6f_3\right)-9H^3(f_2+4f_3)+\frac{3}{2}Hf_0\right]\dot{h}-H(4f_1+f_2)\frac{\nabla^2\dot{h}}{a^2}\\
&-[(36\dot{H}H^2+18\dot{H}^2+24H\ddot{H}+4\dddot{H})(f_1+f_2+3f_3)]h+f_0[2\dot{H}+3H^2]h+\Big[H^2(4f_1+4f_2+12f_3)\\
&+2\dot{H}(f_1+f_2+3f_3)-\frac{1}{2}f_0\Big]\frac{\nabla^2 h}{a^2}+\left(2f_1+\frac{1}{2}f_2\right)\frac{\nabla^4 h}{a^4}=0. \qquad (51)
\end{aligned}
$$

This equation can be used for different cosmological solutions. In what follows, we consider three examples, namely the exponential expansion, and the radiation- and matter-dominated epochs.

1. Exponential expansion

Semianalytical analysis: For $a_1 > 0$, there are runaway solutions for all k values.
In the case $a_1 < 0$, for $k < 0.036$ we have $\Delta < 0$ and all eigenvalues are real and negative; hence there are no instabilities in this case. For $k > 0.036$, there is one positive eigenvalue. So, starting from this frequency, one can observe instability (i.e., runaway solutions) for the exponential expansion of the Universe.

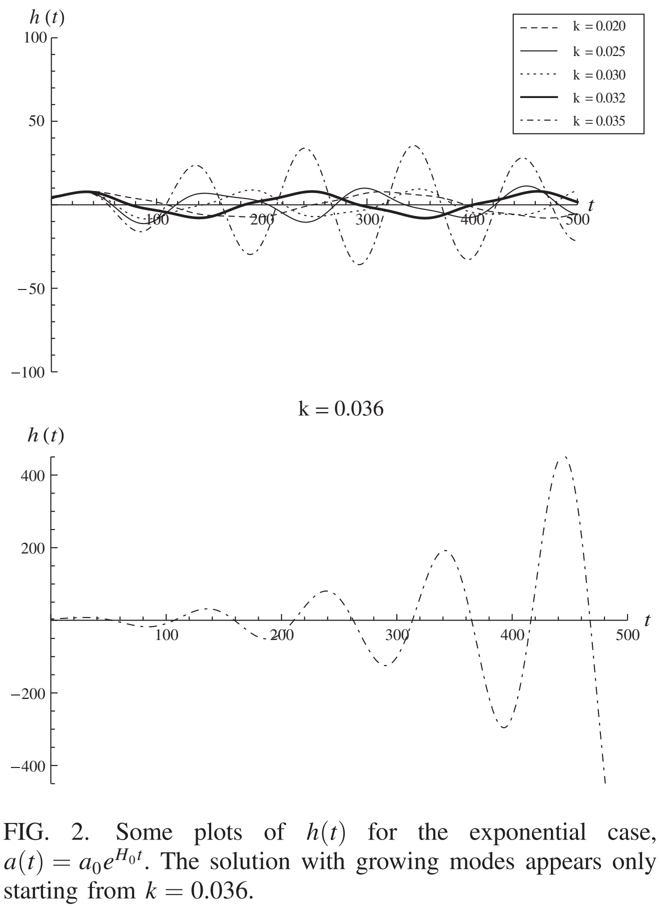

FIG. 2. Some plots of $h(t)$ for the exponential case, $a(t) = a_0 e^{H_0 t}$. The solution with growing modes appears only starting from $k = 0.036$.

Numerical analysis: The result described above is perfectly well confirmed by numerical analysis by using MATHEMATICA software. For the exponential expansion, the growing modes emerge only when $k \geq 0.036$, as is illustrated in Fig. 2. For smaller frequencies, there are no runaway solutions.

We can see that the effect of the nontrivial background manifests itself mainly in the small modification of the marginal value of k, after which we observe growing modes. In view of the consideration in Sec. III, this is an expected result, because we saw that the weak (compared to the Planck scale) background will produce only small corrections to Δ, and hence to the growing modes. Let us see whether the situation is the same for other cosmological solutions.

2. Radiation-dominated epoch

Semianalytical analysis: We find runaway solutions for all k values for radiation when $a_1 > 0$, exactly as in the exponential expansion case.
If we choose $a_1 < 0$, we have $\Delta < 0$ for $k < 0.50$, and all eigenvalues are real and negative. Thus, we have stability for this frequency range. But for $k > 0.50$, we find extremely large values of $h(t)$ and two positive eigenvalues, so we have growing modes.
Numerical analysis: Again, the results found in the semianalytical method agree perfectly with the analysis done by the MATHEMATICA software. For the case of radiation, as we can see in Fig. 3, we have runaway solutions only when $k \geq 0.44$. For smaller frequencies, we do not have this kind of solution.

3. Matter-dominated epoch

Semianalytical analysis: Once again, for $a_1 > 0$, we have runaway solutions for all values of k. For $a_1 < 0$, we have $\Delta < 0$ for the k values up to $k = 0.80$, and all eigenvalues are real and negative; therefore, we have stability. But for

$k > 0.80$, we find two positive eigenvalues, indicating the presence of growing modes.

Numerical analysis: Using the MATHEMATICA software, one can see that runaway solutions appear starting from the frequencies $k \geq 1$, in good agreement with the semianalytical analysis. The illustrative plots are shown in Fig. 4.

One can see that the runaway solutions take place for smaller values of k in the case of exponential expansion than for radiation or, finally, for the dust (matter). The marginal values satisfy the inequalities

$$k_{\text{runaway}}^{\text{Inflation}} < k_{\text{runaway}}^{\text{Radiation}} < k_{\text{runaway}}^{\text{Matter}} \tag{52}$$

However, in all cases, the growing modes appear only when we have k close to the Planck scale, for a negative a_1.

D. Super-renormalizable theory

In order to check our understaning of the relation between the energy gap for the runaway solution and the presence of massive unphysical ghosts with the Planck-order mass, let us consider the simplest possible example of the super-renormalizable theory of gravity [Eq. (7)] by including two next-order terms compared to the fourth-derivative theory:

$$S = S_{\text{EH}} + \int d^4x\sqrt{-g}\{a_1 R_{\mu\nu}^2 + a_2 R^2 + \cdots + b_1 R_{\mu\nu}\Box R^{\mu\nu} + b_2 R\Box R + b_{4,5,\ldots}\mathcal{O}(R_{\ldots}^3) + \cdots + b_{3,4,\ldots}\mathcal{O}(R_{\ldots}^3)\}. \tag{53}$$

As we have already mentioned in Sec. II, this theory has exactly the same amount of ghosts as the fourth-derivative theory [Eq. (15)], because an extra spin-2 degree of freedom has positive kinetic energy, and also Planck-order mass. Then one should expect that the conditions of stability in the two theories [Eqs. (53) and (15)] should be very similar.

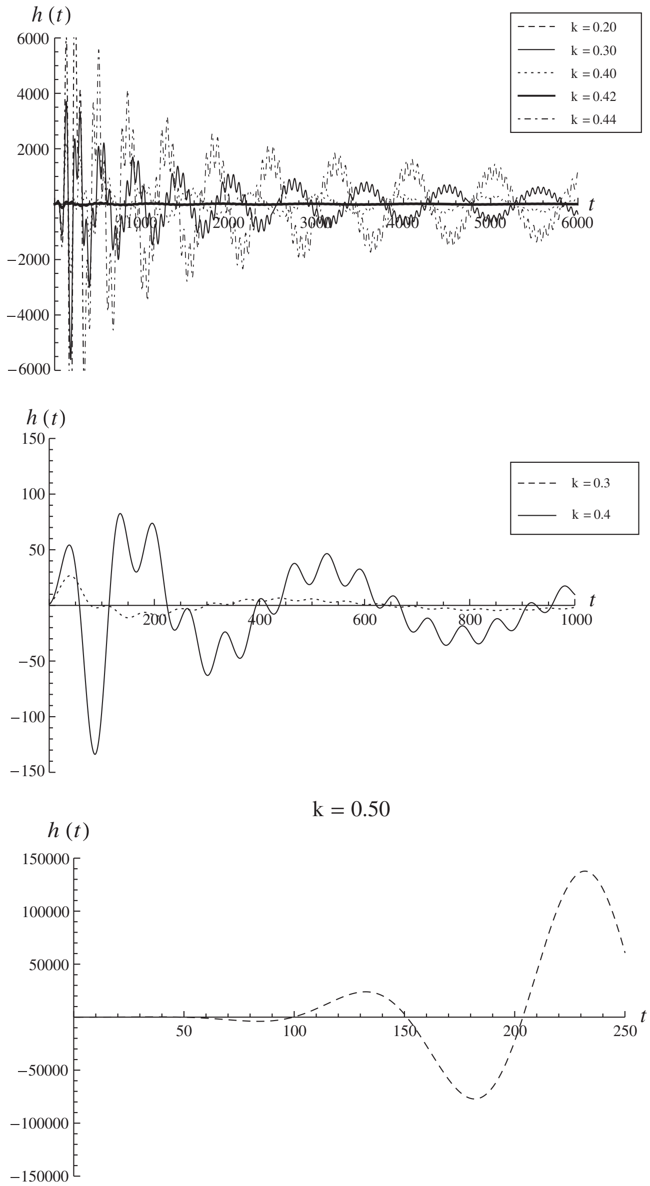

FIG. 3. Graph for $h(t)$ perturbation as a function of time analyzed for radiation, when $a(t) = a_0 t^{1/2}$. Starting from $k \sim 0.50$, the solutions become "violent", as one can see on the last plot. However, below this value, there are no growing modes.

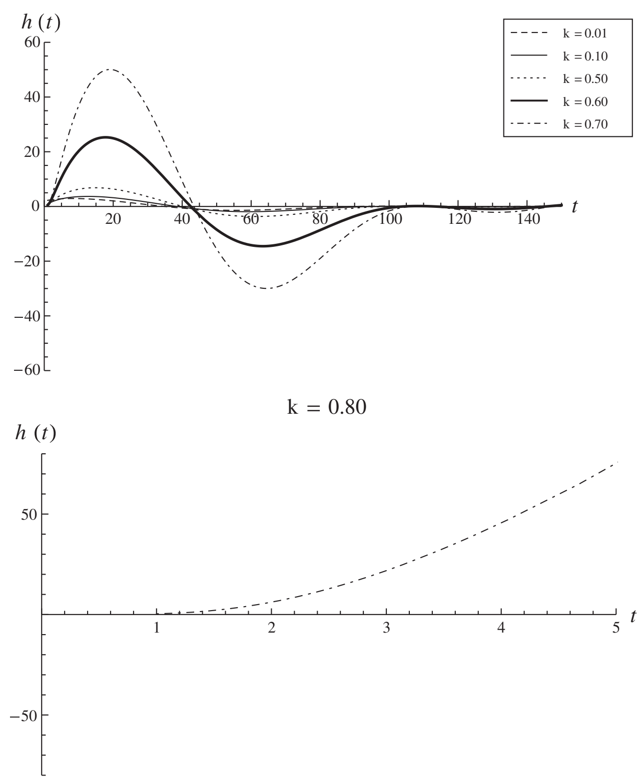

FIG. 4. Graph for $h(t)$ perturbation as a function of time analyzed for matter, when $a(t) = a_0 t^{3/2}$.

The consideration presented above is valid for the structure of poles in the spin-2 sector, according to

$$G_2(k) = \frac{A_0}{k^2} + \frac{A_1}{k^2 + m_1^2} + \frac{A_2}{k^2 + m_2^2}, \qquad (54)$$

with growing real masses of poles,

$$0 < m_1^2 < m_2^2. \qquad (55)$$

In this case, we have $A_0 > 0$ and $A_2 > 0$, while $A_1 < 0$, according to Eq. (9). This feature indicates that the first massive particle, with a negative sign of A_1, is a ghost, while the second massive particle, with a positive sign of A_2, is just a positively defined spin-2 particle with a huge mass. From the physical side, the presence of such an extra particle cannot lead to any extra instability, and this is what we intend to check here.

The first question is how to provide this structure of poles. Let us first establish the necessary conditions for the coefficients a_1 and b_1 in the action. Making the expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we can easily derive the bilinear terms of this action (the spin-2 part only, of course) in the form

$$S_2^{(2)} = \int d^4x h_{\mu\nu} \left(\frac{M_P^2}{64\pi} \Box - \frac{a_1}{2} \Box^2 - \frac{b_1}{2} \Box^3 \right) h_{\mu\nu}. \qquad (56)$$

For the inverse propagator, we meet the expression

$$G_6^{-1}(k) = \frac{b_1}{2} k^2 \left(k^4 - \frac{a_1}{b_1} k^2 - \frac{M_P^2}{32\pi b_1} \right). \qquad (57)$$

The two relevant observations can be done at this moment. First, if we want to have a positive-energy graviton, the sign of b_1 should be positive. This is clear already from Eq. (56). Second, if we want the Planck mass to be the unique scale-defining parameter of the theory, then the coefficient b_1 should be taken as $b_1 = B_1/M_P^2$, with B_1 being a dimensionless parameter of the order 1.

With these choices, we arrive at the following representation:

$$G_6^{-1}(k) = \frac{b_1}{2} k^2 (k^2 - m_1^2)(k^2 - m_2^2), \qquad (58)$$

where

$$m_{1/2}^2 = M_P^2 \left[\frac{a_1}{2B_1} \mp \sqrt{\frac{1}{32\pi B_1} + \frac{a_1^2}{4B_1^2}} \right]. \qquad (59)$$

Obviously, one has to choose, in order to achieve the structure of poles of Eq. (54), the positive sign of a_1, which is the opposite to the four-derivative case. Furthermore, the inequality

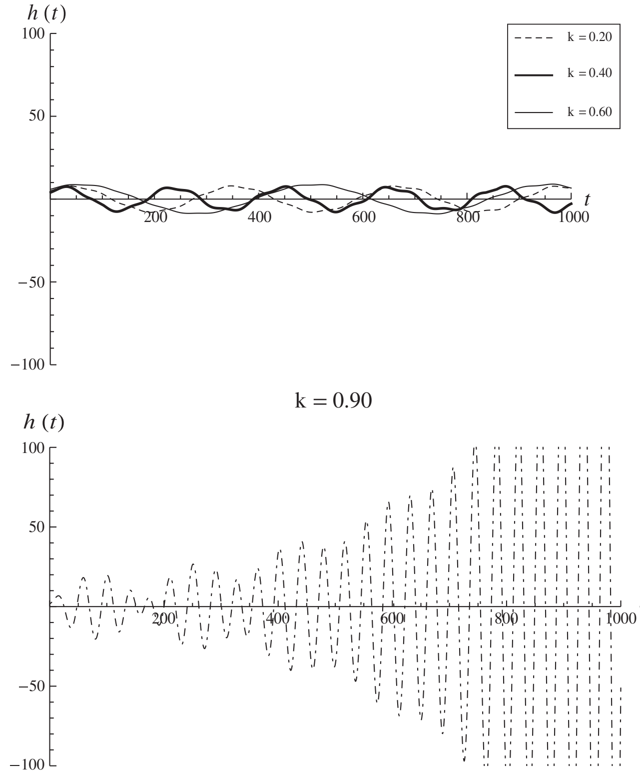

FIG. 5. Once again, growing modes appear only close to the Planck scale.

$$a_1^2 > \frac{B_1}{8\pi} \qquad (60)$$

is requested to provide positive real poles for the propagator. It is obvious that all these conditions can be satisfied if we choose, for example, $a_1 = B_1 = 1$. This will be our choice for the given theory; with it we shall explore the time dynamics of the gravitational perturbations in the flat case. The choice of the flat background is natural, since it is the simplest one, and hence we avoid complications in the comparison of the stability limits for the theories in Eqs. (53) and (15).

The analysis of stability performs exactly as in the fourth-derivative case, so we can directly go to the results. *Semianalytical analysis*. If we choose $b_1 < 0$, we find, for $k < 0.90$, that $\Delta < 0$ and all eigenvalues are real and negative. So, we have stability in this case. For $k > 0.90$, we find two complex eigenvalues with positive real parts, indicating instability. For $a_1 > 0$, we find runaway solutions for all values of k.
Numerical analysis: Again, as in the cases which were considered before, the semianalytical method agrees with the numerical results. In both cases, there are growing modes when $k \geq 0.90$. For Eq. (58), the plot is shown in Fig. 5. The conditions and the behavior of the perturbations look very much like those in the case of the theory in Eq. (15).

V. CONCLUSIONS

We considered the stability of higher-derivative gravity theories under the gauge-independent part of metric perturbations. It was shown that at least cosmological solutions are stable. Due to the similarity with the general situation as it is described in Sec. III, it might happen that this is true for any classical solution. The perturbations which we have dealt with were taken at the linear level, but over the nontrivial metric background, so according to the known theorems [58], the linear stability should be a sufficient condition of the stability even beyond the linear approximation, if the amplitude of initial perturbations is sufficiently small.

One can ask two natural questions concerning this situation:

First, as we have already mentioned in the Introduction, any kind of classical solution is obviously not protected by energy conservation from a process in which one massive ghost and a large amount of gravitons are created at the same time. So, the first question is how one can reconcile this with the stability properties. Let us confess that we have no definite answer to this question. At the same time, physical intuition tells us that the situation in which we need to accumulate a Planck-order energy density of gravitons in the vicinity of a certain space-time point, where the ghost should be created, means that we should go to the physics at the Planck anergy scale. As long as we intend to have a consistent QG theory at the energy scale a few orders beyond M_P, there is a hope to achieve a consistent solution to this discrepancy. For example, in recent papers [66], one can find a discussion of the possible limits on the occupation number of gravitons in a gravitational field. It might happen that such limits can be very useful for understanding the situation with the creation of ghosts from vacuum in higher-derivative QG. Furthermore, we cannot rule out that the solution of the problem can be achieved even for the Planck scale of energy, if we better understand the physical principles behind such limits.

Second, are the cosmological solutions sufficiently general to draw general conclusions? In our opinion, the answer is negative. We mainly dealt with these solutions because they are the simplest ones and the technique of corresponding perturbations is better developed. At the same time, it would be very interesting to explore, using effective framework, the stability of the static black hole metric, where we have contradicting results (Refs. [16] and [17]). It would be very important to have certain results on the stability of this and other relevant solutions, e.g., for the Kerr metric.

Finally, let us note that one single definite example of an unstable physically relevant solution in the theory with higher derivatives would mean that the situation with the (in)stability of vacuum in this theory becomes definitely negative. In view of the great relevance of higher derivatives, especially for the quantization of matter fields on a curved background, this would mean the necessity of some dramatic changes in our understanding, starting from the semiclassical approach to gravity. However, after considerations presented in this work, we have an expectation that the situation with higher derivatives in a theory based on a unique Planck scale can be resolved.

ACKNOWLEDGMENTS

I. S. is grateful to R. Balbinot and P. Creminelli for useful (and critical) discussions. The authors acknowledge the contribution of Grigori Chapiro, who gave us useful hints concerning semianalytical methods for exploring stability. We are grateful to Sebastiao Mauro for the discussion of the black hole case. Furthermore, the authors are grateful to CAPES, CNPq, and FAPEMIG for partial support. I. S. was also supported by an ICTP Associate Membership program.

APPENDIX A: SECOND-ORDER EXPANSIONS

The expansion for the metric has the form

$$g_{\alpha\beta}(y) = \overset{\circ}{g}_{\alpha\beta} - \frac{1}{3}\overset{\circ}{R}_{\mu\alpha\nu\beta}y^\mu y^\nu - \frac{1}{3!}\overset{\circ}{R}_{\mu\alpha\nu\beta;\sigma}y^\mu y^\nu y^\sigma + \frac{1}{5!}\left(\frac{16}{3}\overset{\circ}{R}{}^{\lambda}_{\cdot\mu\alpha\nu}\overset{\circ}{R}_{\lambda\rho\beta\sigma} - 6\overset{\circ}{R}_{\alpha\mu\beta\nu;\rho\sigma}\right)y^\mu y^\nu y^\rho y^\sigma + \cdots. \tag{A1}$$

One can always choose the metric in the expansion point to be the Minkowski one, $\overset{\circ}{g}_{\alpha\beta} = \eta_{\alpha\beta}$. For the Christoffel symbol, one has

$$\Gamma^\lambda_{\alpha\beta} \approx \frac{2}{3}\overset{\circ}{R}{}^{\lambda}{}_{(\alpha\beta)\nu}y^\nu + \frac{1}{8}\left(\overset{\circ}{R}{}^{\lambda}{}_{\mu\nu\beta;\alpha} + \overset{\circ}{R}{}^{\lambda}{}_{\alpha\nu\beta;\mu} + 2\overset{\circ}{R}{}^{\lambda}{}_{\beta\mu\alpha;\nu}\right)y^\mu y^\nu. \tag{A2}$$

Let us present the results of the expansions for $\Box h^{\alpha\beta}$. We will label by $A^{(0)}$ the order of expansion in y^μ for the quantity A, such that

$$\Box h^{\alpha\beta} = (\Box h^{\alpha\beta})^{(0)} + (\Box h^{\alpha\beta})^{(1)} + (\Box h^{\alpha\beta})^{(2)} + \cdots. \tag{A3}$$

The direct calculation yields the following results in zeroth and first order in the deviation y^μ:

$$(\Box h^{\alpha\beta})^{(0)} = \eta^{\mu\nu}\left(\partial_\mu\partial_\nu h^{\alpha\beta} - \frac{1}{3}\overset{\circ}{R}{}^{\alpha}{}_{\nu\lambda\mu}h^{\lambda\beta} - \frac{1}{3}\overset{\circ}{R}{}^{\beta}{}_{\nu\lambda\mu}h^{\alpha\lambda}\right) \tag{A4}$$

and

$$(\Box h^{\alpha\beta})^{(1)} = \eta^{\mu\nu}\Big[-\frac{2}{3}\Big(\overset{\circ}{R}{}^{\alpha}{}_{\nu\lambda\chi} + \overset{\circ}{R}{}^{\alpha}{}_{\lambda\nu\chi}\Big)\partial_\mu h^{\lambda\beta} - \frac{2}{3}\Big(\overset{\circ}{R}{}^{\beta}{}_{\nu\lambda\chi} + \overset{\circ}{R}{}^{\beta}{}_{\lambda\nu\chi}\Big)\partial_\mu h^{\alpha\lambda} + \frac{1}{4}\Big(\overset{\circ}{R}{}^{\alpha}{}_{\chi\mu\lambda;\nu} + \overset{\circ}{R}{}^{\alpha}{}_{\nu\mu\lambda;\chi} + 2\overset{\circ}{R}{}^{\alpha}{}_{\lambda\chi\nu;\mu}\Big)h^{\lambda\beta}$$
$$+ \frac{2}{3}\overset{\circ}{R}{}^{\lambda}_{\mu\nu\chi}\partial_\lambda h^{\alpha\beta} + \frac{1}{4}\Big(\overset{\circ}{R}{}^{\beta}{}_{\chi\mu\lambda;\nu} + \overset{\circ}{R}{}^{\beta}{}_{\nu\mu\lambda;\chi} + 2\overset{\circ}{R}{}^{\beta}{}_{\lambda\chi\nu;\mu}\Big)h^{\alpha\lambda}\Big]y^\chi, \tag{A5}$$

where a semicolon indicates a covariant derivative taken at the point P. Furthermore, in the second order in y^μ, we meet

$$(\Box h^{\alpha\beta})^{(2)} = \frac{1}{3}\overset{\circ}{R}{}^{\mu}{}_{\chi}{}^{\nu}{}_{\omega}\Big\{\partial_\mu\partial_\nu h^{\alpha\beta} - \frac{1}{3}\Big(\overset{\circ}{R}{}^{\alpha}{}_{\nu\lambda\mu} - \overset{\circ}{R}{}^{\alpha}{}_{\lambda\nu\mu}\Big)h^{\lambda\beta} - \frac{1}{3}\Big(\overset{\circ}{R}{}^{\beta}{}_{\nu\lambda\mu} - \overset{\circ}{R}{}^{\beta}{}_{\lambda\nu\mu}\Big)h^{\alpha\lambda}\Big\}y^\chi y^\omega$$
$$+ \eta^{\mu\nu}\Big\{\frac{1}{8}\Big(\overset{\circ}{R}{}^{\alpha}{}_{\chi\omega\lambda;\nu} + \overset{\circ}{R}{}^{\alpha}{}_{\nu\omega\lambda;\chi} + 2\overset{\circ}{R}{}^{\alpha}{}_{\lambda\chi\nu;\omega}\Big)\partial_\mu h^{\lambda\beta} + \frac{1}{8}\Big(\overset{\circ}{R}{}^{\beta}{}_{\chi\omega\lambda;\nu} + \overset{\circ}{R}{}^{\beta}{}_{\nu\omega\lambda;\chi} + 2\overset{\circ}{R}{}^{\beta}{}_{\lambda\chi\nu;\omega}\Big)\partial_\mu h^{\alpha\lambda}$$
$$- \frac{1}{8}\Big(\overset{\circ}{R}{}^{\lambda}{}_{\chi\omega\lambda;\mu} + \overset{\circ}{R}{}^{\lambda}{}_{\mu\omega\nu;\chi} + 2\overset{\circ}{R}{}^{\lambda}{}_{\nu\chi\mu;\omega}\Big)\partial_\lambda h^{\alpha\beta} - \frac{2}{9}\overset{\circ}{R}{}^{\lambda}{}_{\mu\nu\chi}\Big[\Big(\overset{\circ}{R}{}^{\alpha}{}_{\lambda\tau\omega} + \overset{\circ}{R}{}^{\alpha}{}_{\tau\lambda\omega}\Big)h^{\tau\beta} + \Big(\overset{\circ}{R}{}^{\beta}{}_{\lambda\tau\omega} + \overset{\circ}{R}{}^{\beta}{}_{\tau\lambda\omega}\Big)h^{\alpha\tau}\Big]$$
$$+ \frac{1}{8}\Big(\overset{\circ}{R}{}^{\alpha}{}_{\chi\omega\lambda;\mu} + \overset{\circ}{R}{}^{\alpha}{}_{\mu\omega\lambda;\chi} + 2\overset{\circ}{R}{}^{\alpha}{}_{\lambda\chi\mu;\omega}\Big)\partial_\nu h^{\lambda\beta} + \frac{1}{9}\Big(\overset{\circ}{R}{}^{\alpha}{}_{\mu\lambda\chi} + \overset{\circ}{R}{}^{\alpha}{}_{\lambda\mu\chi}\Big)\Big[\Big(\overset{\circ}{R}{}^{\lambda}{}_{\nu\tau\omega} + \overset{\circ}{R}{}^{\lambda}{}_{\tau\nu\omega}\Big)h^{\tau\beta} + \Big(\overset{\circ}{R}{}^{\beta}{}_{\nu\tau\omega} + \overset{\circ}{R}{}^{\beta}{}_{\tau\nu\omega}\Big)h^{\lambda\tau}\Big]$$
$$+ \frac{1}{9}\Big(\overset{\circ}{R}{}^{\beta}{}_{\mu\lambda\chi} + \overset{\circ}{R}{}^{\beta}{}_{\lambda\mu\chi}\Big)\Big[\Big(\overset{\circ}{R}{}^{\lambda}{}_{\nu\tau\omega} + \overset{\circ}{R}{}^{\lambda}{}_{\tau\nu\omega}\Big)h^{\alpha\tau} + \Big(\overset{\circ}{R}{}^{\alpha}{}_{\nu\tau\omega} + \overset{\circ}{R}{}^{\alpha}{}_{\tau\nu\omega}\Big)h^{\lambda\tau}\Big] + \frac{1}{8}\Big(\overset{\circ}{R}{}^{\beta}{}_{\chi\omega\lambda;\mu} + \overset{\circ}{R}{}^{\beta}{}_{\mu\omega\lambda;\chi} + 2\overset{\circ}{R}{}^{\beta}{}_{\lambda\chi\mu;\omega}\Big)\partial_\nu h^{\alpha\lambda}\Big\}y^\chi y^\omega. \tag{A6}$$

Altogether, we find

$$\Box h^{\alpha\beta} = (\Box h^{\alpha\beta})^{(0)} + (\Box h^{\alpha\beta})^{(1)} + (\Box h^{\alpha\beta})^{(2)} + \cdots, \tag{A7}$$

where the dots indicate the terms of higher orders in y^μ and terms of higher orders in the curvature tensor and its covariant derivatives at the point P.

APPENDIX B: BACKGROUNDS OF OUR SEMIANALYTICAL METHOD

We found the following type of fourth-order differential equation for tensor perturbations:

$$b_4\ddddot{h} + b_3\dddot{h} + b_2\ddot{h} + b_1\dot{h} + b_0 h = 0, \tag{B1}$$

where b_0, b_1, b_2, b_3, and b_4 are the coefficients of this equation. One can reduce this fourth-order equation to a system of four first-order equations. Changing the variables, we have

$$h_0 = h, \qquad h_1 = \dot{h}_0 = \dot{h},$$
$$h_2 = \dot{h}_2 = \ddot{h}, \qquad h_3 = \dot{h}_2 = \dddot{\mathrm{h}}. \tag{B2}$$

Now, we can rewrite these as

$$\dot{h}_3 = -\frac{1}{b_4}(b_3h_3 + b_2h_2 + b_1h_1 + b_0h_0),$$
$$\dot{h}_2 = h_3,$$
$$\dot{h}_1 = h_2,$$
$$\dot{h}_0 = h_1.$$

Rewriting the differential equation, we arrive at

$$\dot{h}_3 = -\frac{1}{b_4}(b_3h_3 + b_2h_2 + b_1h_1 + b_0h_0),$$
$$\dot{h}_2 = h_3,$$
$$\dot{h}_1 = h_2,$$
$$\dot{h}_0 = h_1.$$

We can rewrite this linear system of four equations in a matrix form and easily compute the eigenvalues and eigenvectors. Thus, we can write in simplified form

$$\dot{h}_k = A_k^l h_l, \tag{B3}$$

where $k = 0, 1, 2, 3$, and the matrix $A = A_k^l$ has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ d_0 & d_1 & d_2 & d_3 \end{pmatrix},$$

where we have called $d_k = -b_k/b_4$. We need to find the eigenvalues of A, and for this end we consider

$$\det \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ d_0 & d_1 & d_2 & (d_3-\lambda) \end{pmatrix} = 0. \qquad \text{(B4)}$$

The algebraic equation is

$$\lambda^4 - d_3\lambda^3 - d_2\lambda^2 - d_1\lambda^1 - d_0 = 0. \qquad \text{(B5)}$$

After some algebraic operations due to Cardano [67], one can reduce this fourth-order equation to the second-order one,

$$z^2 + \xi_1 z + \xi_2 = 0, \qquad \text{(B6)}$$

where the most important quantity is given by

$$\Delta = \xi_1 + \frac{4}{27}\xi_2^3 = 4\left[\left(\frac{\xi_1}{2}\right)^2 + \left(\frac{\xi_2}{3}\right)^3\right]. \qquad \text{(B7)}$$

The value of Δ will tell us the nature of these roots, as explained in the text. To find Eq. (B7), we use the fact that

$$\xi_1 = \frac{-\alpha}{3} + \beta \quad \text{and} \quad \xi_2 = \left(\frac{2\alpha^3}{27} + \frac{3\gamma - \beta\gamma}{3}\right),$$
$$\alpha = \frac{5}{2}p, \qquad \gamma = \frac{1}{8}(q^2 - 4p^2 + 4pr), \quad \text{and}$$
$$\beta = 2p^2 - r, \qquad p = -\frac{39}{8}d_3^2 + d_2,$$
$$q = \frac{d_3^2}{8} - \frac{d_3 d_2}{2} + d_1, \quad \text{and}$$
$$r = -\frac{3d_3^4}{256} + \frac{d_2 d_3^2}{16} - \frac{d_2 d_1}{4} + d_0, \qquad \text{(B8)}$$

where $b_k/b_4 = -d_k$, and the b_k's are the coefficients of the fourth-order differential equation.

[1] G. 't Hooft and M. J. G. Veltman, Ann. Inst. Henri Poincaré, A **20**, 69 (1974).
[2] S. Deser and P. van Nieuwenhuizen, Phys. Rev. D **10**, 401 (1974); **10**411 (1974); S. Deser, H. Sh. Tsao, and P. van Nieuwenhuizen, Phys. Rev. D **10**, 3337 (1974).
[3] M. H. Goroff and A. Sagnotti, Nucl. Phys. **B266**, 709 (1986).
[4] K. S. Stelle, Phys. Rev. D **16**, 953 (1977).
[5] M. Ostrogradski, Mem. Ac. St. Petersbourg, **6**, 385 (1850).
[6] M. J. G. Veltman, Physica (Utrecht) **29**, 186 (1963).
[7] E. T. Tomboulis, Phys. Lett. **70B**, 361 (1977); **97B**, 77 (1980); Phys. Rev. Lett. **52**, 1173 (1984).
[8] A. Salam and J. Strathdee, Phys. Rev. D **18**, 4480 (1978).
[9] I. Antoniadis and E. T. Tomboulis, Phys. Rev. D **33**, 2756 (1986).
[10] D. A. Johnston, Nucl. Phys. **B297**, 721 (1988).
[11] S. W. Hawking and T. Hertog, Phys. Rev. D **65**, 103515 (2002).
[12] K. Hinterbichler, Rev. Mod. Phys. **84**, 671 (2012).
[13] M. Jaccard, M. Maggiore, and E. Mitsou, Phys. Rev. D **88**, 044033 (2013).
[14] I. Antoniadis, E. Dudas, and D. M. Ghilencea, Nucl. Phys. **B767**, 29 (2007).
[15] K. S. Stelle, Gen. Relativ. Gravit. **9**, 353 (1978).
[16] B. Whitt, Phys. Rev. D **32**, 379 (1985).
[17] Y. S. Myung, Phys. Rev. D **88**, 024039 (2013).
[18] A. L. Maroto and I. L. Shapiro, Phys. Lett. B **414**, 34 (1997).
[19] A. V. Toporensky and P. V. Tretyakov, Int. J. Mod. Phys. D **16**, 1075 (2007).
[20] N. Deruelle, M. Sasaki, Y. Sendouda, and A. Youssef, J. Cosmol. Astropart. Phys. 03 (2011) 040; J. High Energy Phys. 09 (2012) 009.
[21] P. D. Mannheim, Found. Phys. **37**, 532 (2007).
[22] V. I. Tkach, Mod. Phys. Lett. A **27**, 1250131 (2012).
[23] A. V. Smilga, Nucl. Phys. **B706**, 598 (2005); SIGMA **5**, 017 (2009).
[24] V. V. Nesterenko, Phys. Rev. D **75**, 087703 (2007).
[25] R. Woodard, Lect. Notes Phys. **720**, 403 (2007).
[26] J. B. Jimenez, E. Dio, and R. Durrer, J. High Energy Phys. 04 (2013) 030.
[27] S. I. Kruglov, Ann. Fond. Louis de Broglie **31**, 343 (2006).
[28] J. Julve and M. Tonin, Nuovo Cimento B **46**, 137 (1978).
[29] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B201**, 469 (1982).
[30] I. G. Avramidi and A. O. Barvinsky, Phys. Lett. **159B**, 269 (1985); I. G. Avramidi, Sov. J. Nucl. Phys. **44**, 255 (1986); Ph.D. thesis, Moscow University, 1986 [arXiv:hep-th/9510140].
[31] G. de Berredo-Peixoto and I. L. Shapiro, Phys. Rev. D **71**, 064005 (2005).
[32] A. Codello and R. Percacci, Phys. Rev. Lett. **97**, 221301 (2006); Ann. Phys. (Amsterdam) **324**, 414 (2009).
[33] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980); in *Quantum Theory of Gravity II, Moscow 1981*, edited by (INR Press,

Moscow, 1982), pp. 58–72; JETP Lett. **30**, 719 (1979); **34**, 460 (1981); Let. Astr. J. **9**, 579 (1983).
[34] A. M. Pelinson, I. L. Shapiro, and F. I. Takakura, Nucl. Phys. **B648**, 417 (2003).
[35] A. A. Starobinsky, Pisma Zh. Eksp. Teor. Fiz. **30**, 719 (1979) [JETP Lett. **30**, 682 (1979)]; Zh. Eksp. Teor. Fiz. **34**, 460 (1981) [Sov. Astron. Lett. **9**, 302 (1983)].
[36] S. W. Hawking, T. Hertog, and H. S. Reall, Phys. Rev. D **63**, 083504 (2001).
[37] J. C. Fabris, A. M. Pelinson, and I. L. Shapiro, Nucl. Phys. **B597**, 539 (2001); **B602**644(E) (2001).
[38] M. Gasperini, Phys. Rev. D **56**, 4815 (1997).
[39] J. C. Fabris, A. M. Pelinson, F. O. Salles and I. L. Shapiro, J. Cosmol. Astropart. Phys. 02 (2012) 019.
[40] R. Utiyama and B. S. DeWitt, J. Math. Phys. (N.Y.) **3**, 608 (1962).
[41] I. L. Buchbinder, Theor. Math. Phys. **61**, 393 (1984).
[42] D. J. Toms, Phys. Lett. **126B**, 37 (1983).
[43] P. M. Lavrov and I. L. Shapiro, Phys. Rev. D **81**, 044026 (2010).
[44] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (IOP Publishing, Bristol, 1992).
[45] I. L. Shapiro, Classical Quantum Gravity **25**, 103001 (2008).
[46] S. W. Hawking, Nature (London) **248**, 30 (1974); Commun. Math. Phys. **43**, 199 (1975).
[47] S. M. Christensen and S. A. Fulling, Phys. Rev. D **15**, 2088 (1977); see also more detailed analysis, based on anomaly-induced effective action of gravity in Ref. [48].
[48] R. Balbinot, A. Fabbri, and I. L. Shapiro, Phys. Rev. Lett. **83**, 1494 (1999); Nucl. Phys. **B559**, 301 (1999).
[49] M. Asorey, J. L. López, and I. L. Shapiro, Int. J. Mod. Phys. A **12**, 5711 (1997).
[50] P. Hořava, Phys. Rev. D **79**, 084008 (2009).
[51] G. Giribet, D. L. Nacir, and F. D. Mazzitelli, J. High Energy Phys. 09 (2010) 009.
[52] I. L. Shapiro, Phys. Rep. **357**, 113 (2002).
[53] G. Narain and R. Anishetty, J. Phys. Conf. Ser. **405**, 012024 (2012).
[54] E. T. Tomboulis, arXiv:hep-th/9702146.
[55] L. Modesto, Phys. Rev. D **86**, 044005 (2012); Astronomicheskii Vestnik / Astronomy Review **8**, 4 (2013); F. Briscese, L. Modesto, and S. Tsujikawa, Phys. Rev. D **89** 024029 (2014); L. Modesto and S. Tsujikawa, Phys. Lett. B **727**, 48 (2013).
[56] T. Biswas, E. Gerwick, T. Koivisto, and A. Mazumdar, Phys. Rev. Lett. **108**, 031101 (2012).
[57] F. J. de Urries, J. Julve, and E. J. Sánchez, J. Phys. A **34**, 8919 (2001); F. J. de Urries and J. Julve, J. Phys. A **31**, 6949 (1998); A. Bartoli, J. Julve, and E. J. Sánchez, Classical Quantum Gravity **16**, 2283 (1999).
[58] A. M. Lyapunov, Ph.D. thesis, Moscow University, 1892; [English translations: *Stability of Motion* (Academic Press, New York, 1966); and *The General Problem of the Stability of Motion* (Taylor and Francis, London, 1992)]; For the modern treatment, see, e.g., L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations,* (Springer, New York, 2012), 3rd ed.
[59] J. F. Donoghue, Phys. Rev. Lett. **72**, 2996 (1994); Phys. Rev. D **50**, 3874 (1994).
[60] A. Z. Petrov, *Einstein Spaces* (Pergamon, Oxford, 1969).
[61] T. S. Bunch and L. Parker, Phys. Rev. D **20**, 2499 (1979).
[62] L. Parker and D. J. Toms, Phys. Rev. D **29**, 1584 (1984).
[63] F. Sobreira, B. J. Ribeiro, and I. L. Shapiro, Phys. Lett. B **705**, 273 (2011).
[64] N. D. Birell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
[65] Wolfram Research, Inc., Mathematica, Version 9.0, Champaign, IL (2013).
[66] G. Dvali, S. Folkerts, and C. Germani, Phys. Rev. D **84**, 024039 (2011); G. Dvali and C. Gomez, Fortschr. Phys. **61**, 742 (2013).
[67] G. Cardano, *Artis Magnae, Sive de Regulis Algebraicis Liber Unus*. (1545); [*Ars magna or The Rules of Algebra* (Dover, New York, 1993)]; For a modern reference, see M. Artin, *Algebra* (Prentice Hall, Englewood Cliffs, NJ, 1991).