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Entropy of thin shells in a $(2+1)$-dimensional asymptotically AdS spacetime and the BTZ black hole limit

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The thermodynamic equilibrium states of a static thin ring shell in a $(2+1)$-dimensional spacetime with a negative cosmological constant are analyzed. Inside the ring, the spacetime is pure anti–de Sitter, whereas outside it is a Bañados-Teitelbom-Zanelli spacetime and thus asymptotically anti–de Sitter. The first law of thermodynamics applied to the thin shell, plus one equation of state for the shell's pressure and another for its temperature, leads to a shell's entropy, which is a function of its gravitational radius alone. A simple example for this gravitational entropy, namely, a power law in the gravitational radius, is given. The equations of thermodynamic stability are analyzed, resulting in certain allowed regions for the parameters entering the problem. When the Hawking temperature is set on the shell and the shell is pushed up to its own gravitational radius, there is a finite quantum backreaction that does not destroy the shell. One then finds that the entropy of the shell at the shell's gravitational radius is given by the Bekenstein-Hawking entropy.

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I. INTRODUCTION

Due to the long-range interaction of the gravitational field, gravitating systems have important and interesting thermodynamic properties, such as negative specific heat, making the systems unstable with consequent gravitational collapse or energy loss through evaporation. This happens both in Newtonian gravitation and in general relativity. A well-known instance of this fact is given by the black hole system, whose thermodynamic properties were understood by Bekenstein [1] and put on a firm basis by Hawking, by discovering that through quantum effects it radiates at a definite temperature [2]. Refinement of the study of black hole thermodynamics appeared in many guises, in particular by the introduction of a formalism useful for studying general relativistic systems in a canonical ensemble [3,4].

Another gravitating system in general relativity prone to a thermodynamic study is a thin shell and the spacetime it generates. Spurred by the interest in black hole thermodynamics, some studies have analyzed the thermodynamics of thin shells in black hole spacetimes [5], or of pure thin shells in $3+1$ spacetimes, notably in [6], where several thermodynamic quantities of thin shells are discussed and a stability analysis of them is performed. Other studies on the thermodynamics of thin shells are [7,8]. For related studies of thermodynamics of gravitating matter, especially quasi-black holes, i.e., stars on the verge of becoming a black hole, see [9,10]. All of these works are in the usual $3+1$ dimensions.

Now, in many senses, it is interesting to reduce the spatial dimension by 1 and study general relativity in $2+1$ dimensions. This plays an important role in the understanding of systems in curved spacetime, as the decrease in dimensionality with respect to the usual $3+1$ spacetime reduces the degrees of freedom to a few. This leaves possible complications aside and keeps the essential physical features. The interest in $(2+1)$-dimensional general relativity underwent a boost after a black hole solution was found in spacetimes with negative cosmological constant Λ, i.e., spacetimes with an anti–de Sitter (AdS) background [11,12]. This $(2+1)$-dimensional black hole, the Bañados-Teitelbom-Zanelli (BTZ) black hole, belongs to a family of solutions, which, depending on the parameters of the solution, includes the BTZ black holes themselves, positive mass naked singularities, the AdS spacetime, and negative mass naked singularities. The BTZ black hole, the most important solution in the family, is a black hole solution in its simplest form. The singularity it hides is not a curvature singularity, but rather is a much milder topological singularity, akin to the conical singularities [11,12].

In the realm of thermodynamics and its connection to the quantum world, the BTZ black hole has a Bekenstein-Hawking entropy $S_{\rm BH} = \frac{1}{4}\frac{A_{\rm h}}{l_{\rm p}}$, where $A_{\rm h}$ is the horizon area, or in $2+1$ dimensions a circumference with $A_{\rm h} = 2\pi r_+$, where r_+ is the horizon radius, and $l_{\rm p}$ is the Planck length given by $l_{\rm p} = G_3\hbar$, G_3 being the three-dimensional gravitational constant and $\hbar$ Planck's constant, and a Hawking temperature given by $T_{\rm H} = \frac{l_{\rm p}}{2\pi G_3 l^2} r_+$ [11], were l is the AdS length defined through $-\Lambda = \frac{l}{l^2}$. (We put $k_{\rm B} = 1$ and

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$c = 1$.) The thermodynamic and entropy properties of the BTZ black hole have been further explored in, e.g., [13–16].

In $2+1$ dimensions, as in the $3+1$ case, it is also interesting to study the thermodynamics of self-gravitating thin shells, since the BTZ black hole can form from the gravitational collapse of such thin shells [17–19] which, when static, can be stable or unstable according to their intrinsic parameters [20]. In $2+1$ dimensions, the thermodynamics of thin shells in spacetimes with zero cosmological constant has been studied [21]. Motivated in part by this study [21] and also from the fact that in $2+1$ dimensions in general relativity with a cosmological constant there are BTZ black holes with interesting thermodynamic properties, we want to study in this article the thermodynamics of static thin matter shells in $2+1$ dimensions. In particular, we intend to find the shell's entropy and analyze their thermodynamic stability, as well as scrutinize the limit when the radius of the shell R goes into its own gravitational radius. To find the spacetime solution we employ the junction conditions formalism for a thin shell [22], where in this $2+1$ case the shell is simply a ring. One can then determine the pressure and the mass density of the shell in order for it to be static in a spacetime with a negative cosmological constant, in which the interior to the shell is pure AdS and the exterior is asymptotically AdS. Employing the first law of thermodynamics and using the formalism for the usual thermodynamic systems [23], which was developed by Martinez to apply to thin matter shell systems in $3+1$ general relativity [6] (see, also, [3,4]), one finds then the entropy for these gravitating systems, the desired thermodynamic properties, and the quasiblack hole limit.

The paper is organized as follows. In Sec. II, we compute the components of the extrinsic curvature of the ring shell that leads to the shell's linear density and pressure. We also discuss the no-trapped-surface condition and the dominant energy condition. In Sec. III we review the thermodynamics of the shell. We use the entropy representation and assume that the state variables are the proper mass of the shell and its perimeter, or radius. We then use the first law of thermodynamics for a one-dimensional system to display the integrability and the stability conditions of the thermodynamic system. In Sec. IV we present the equation of state for the pressure in terms of the proper mass and radius of the shell, and we derive the equation of state for the temperature of the shell as a function of the state variables. In Sec. V we use the previously obtained integrability conditions for the first law of thermodynamics in order to simplify the entropy differential of the shell. This allows us to obtain an expression for the entropy up to an arbitrary function of the gravitational radius. In Sec. VI we consider a phenomenological expression for the arbitrary function, consisting in a power law of the gravitational radius, which allows us to obtain an explicit expression for the entropy. We then analyze the thermodynamic stability of the system by calculating the permitted intervals of the free parameters in order for the shell to remain thermodynamically stable. In Sec. VII, the arbitrary function will be equated to the inverse Hawking temperature, and it will be found that the Bekenstein-Hawking entropy of the BTZ black hole naturally arises when the shell is pushed up to its gravitational radius. Finally, in Sec. VIII we draw some conclusions.

II. THE THIN SHELL SPACETIME

Einstein's equation in $2+1$ dimensions is written as

$$G_{\alpha\beta} - \Lambda g_{\alpha\beta} = 8\pi G_3 T_{\alpha\beta}, \tag{1}$$

where $G_{\alpha\beta}$ is the Einstein tensor, Λ is the cosmological constant, $g_{\alpha\beta}$ is the spacetime metric, $8\pi G_3$ is the coupling with G_3 being the gravitational constant in $2+1$ dimensions, and $T_{\alpha\beta}$ is the energy-momentum tensor. We keep units where the velocity of light is $c = 1$, and thus G_3 has units of the inverse of mass. Greek indices are spacetime indices and run as $\alpha, \beta = 0, 1, 2$, with 0 being the time index. Since we want to work in an AdS background where the cosmological constant is negative, we define the AdS length l through the equation

$$-\Lambda = \frac{1}{l^2}. \tag{2}$$

We now consider a one-dimensional timelike shell, i.e., a ring, with radius R in a $(2+1)$-dimensional spacetime. The ring divides spacetime into two parts, an inner region $\mathcal{V}_-$ and an outer region $\mathcal{V}_+$. To find the corresponding spacetime solution, we follow [22].

In the inner region $\mathcal{V}_-$ ($r \le R$), inside the ring, we consider a spherically symmetric AdS metric, with cosmological length l, given by

$$\begin{aligned} ds_-^2 &= g^-_{\alpha\beta} dx^\alpha dx^\beta \\ &= -\frac{r^2}{l^2} dt^2 + \frac{dr^2}{\frac{r^2}{l^2}} + r^2 d\phi^2, \qquad r \le R, \end{aligned} \tag{3}$$

where polar coordinates $x^{\alpha -} = (t, r, \phi)$ are used. In the outer region $\mathcal{V}_+$ ($r \ge R$), outside the shell, the spacetime is described by the BTZ line element

$$\begin{aligned} ds_+^2 = g^+_{\alpha\beta} dx^\alpha dx^\beta = -\left(\frac{r^2}{l^2} - 8G_3 m\right) dt^2 \\ + \frac{dr^2}{(\frac{r^2}{l^2} - 8G_3 m)} + r^2 d\phi^2, \qquad r \ge R, \end{aligned} \tag{4}$$

written also in polar $x^{\alpha +} = (t, r, \phi)$ coordinates. Here, m is a constant which is interpreted as the Arnowitt-Deser-Misner (ADM) mass, or energy. At $r \to \infty$ the spacetime is asymptotically AdS. On the hypersurface itself, the induced metric h_{ab} yields the line element

$$ds^2_\Sigma = h_{ab}dy^a dy^b = -d\tau^2 + R^2(\tau)d\phi^2, \tag{5}$$

where we have chosen $y^a = (\tau, \phi)$ as coordinates on the shell and where we have adopted the convention to use latin indexes for the components on the hypersurface. The shell ring is at radius $R = R(\tau)$, and the parametric equations of the ring hypersurface for both the $\mathcal{V}_-$ and $\mathcal{V}_+$ are $r = R(\tau)$ and $t = T(\tau)$. The induced metric h_{ab} is written in terms of the metrics $g^\pm_{\alpha\beta}$ as

$$h^\pm_{ab} = g^\pm_{\alpha\beta} e^\alpha_{\pm a} e^\beta_{\pm b}, \tag{6}$$

where $e^\alpha_{\pm a}$ are tangent vectors to the hypersurface viewed from each side of it.

The formalism employed in [22] uses two conditions in order to assure the smoothness of the metric across the hypersurface. These are the junction conditions. The first junction condition states that

$$[h_{ab}] = 0, \tag{7}$$

where the parentheses symbolize the jump in the quantity across the hypersurface, here the induced metric. This condition leads to the relation

$$\begin{aligned}&\left(\frac{r^2}{l^2} - 8G_3 m\right)\dot{T}^2 - \frac{\dot{R}^2}{(\frac{r^2}{l^2} - 8G_3 m)}\\ &= \frac{r^2}{l^2}\dot{T}^2 - \left(\frac{r^2}{l^2}\right)^{-1}\dot{R}^2 = 1,\end{aligned} \tag{8}$$

where a dot denotes differentiation with respect to τ. The second junction condition makes use of the extrinsic curvature $K^a{}_b$ defined as

$$K^a_{\pm b} = \nabla_\beta n_\alpha e^\alpha_{\pm c} e^\beta_{\pm b} h^{ca}_\pm, \tag{9}$$

where ∇_β denotes the covariant derivative and n_α is the normal to the shell. When the jump in this quantity is non-null, there exists a thin matter shell with stress-energy tensor $S^a{}_b$ given by

$$S^a{}_b = -\frac{1}{8\pi G_3}([K^a{}_b] - [K]h^a{}_b), \tag{10}$$

where $K = h^b{}_a K^a{}_b$. For the line elements (3) and (4), and using Eq. (8), one can compute the nonzero components of $K^a{}_b$. They are

$$K^\tau_{+\tau} = \frac{\frac{R}{l^2} + \ddot{R}}{\sqrt{-8G_3 m + \frac{R^2}{l^2} + \dot{R}^2}}, \tag{11}$$

$$K^\tau_{-\tau} = \frac{\frac{R}{l^2} + \ddot{R}}{\sqrt{\frac{R^2}{l^2} + \dot{R}^2}}, \tag{12}$$

$$K^\phi_{+\phi} = \frac{1}{R}\sqrt{-8G_3 m + \frac{R^2}{l^2} + \dot{R}^2}, \tag{13}$$

$$K^\phi_{-\phi} = \frac{1}{R}\sqrt{\frac{R^2}{l^2} + \dot{R}^2}. \tag{14}$$

Imposing that the shell is static, i.e., $\dot{R} = 0$ and $\ddot{R} = 0$, one finds the non-null components of the stress-energy tensor for a static shell,

$$S^\tau{}_\tau = \frac{\sqrt{-8G_3 m + \frac{R^2}{l^2}} - \frac{R}{l}}{8\pi G_3 R} \tag{15}$$

$$S^\phi{}_\phi = \frac{1}{8\pi G_3}\frac{R}{l^2}\left(\frac{1}{\sqrt{-8G_3 m + \frac{R^2}{l^2}}} - \frac{1}{\frac{R}{l}}\right). \tag{16}$$

If, in addition, we consider the shell to be made of a fluid with linear energy density λ and pressure p, the stress-energy tensor will have the form

$$S^a{}_b = (\lambda + p)u^a u_b + p h^a{}_b, \tag{17}$$

where u^a is the 3-velocity of a shell element. Thus, for such a fluid, we find

$$S^\tau{}_\tau = -\lambda, \tag{18}$$

$$S^\phi{}_\phi = p. \tag{19}$$

Note that for a one-dimensional fluid with linear energy density and pressure in a $(2 + 1)$-dimensional spacetime, there are only two possible degrees of freedom to characterize the system. Therefore, the stress-energy tensor (17) is the most general one that one can consider in this setting, and it is thus seen to be the stress-energy tensor for a perfect fluid. Equations (15) and (16) together with Eqs. (18) and (19) yield

$$\lambda = \frac{1}{8\pi G_3 R}\left(\frac{R}{l} - \sqrt{-8G_3 m + \frac{R^2}{l^2}}\right), \tag{20}$$

$$p = \frac{1}{8\pi G_3}\frac{R}{l^2}\left(\frac{1}{\sqrt{-8G_3 m + \frac{R^2}{l^2}}} - \frac{1}{\frac{R}{l}}\right). \tag{21}$$

Now, from Eq. (4), one finds that the gravitational radius r_+ of the shell is given by

$$r_+ = \sqrt{8G_3 m l}. \tag{22}$$

It is useful to define a variable k as

$$k \equiv \sqrt{1 - \frac{r_+^2}{R^2}}. \tag{23}$$

Then Eqs. (20) and (21) can be rewritten as

$$\lambda = \frac{1}{8\pi G_3 l}(1-k), \tag{24}$$

$$p = \frac{1}{8\pi G_3 l}\left(\frac{1}{k} - 1\right). \tag{25}$$

Note that when $\lambda = 0$ and $p = 0$, Eqs. (24) and (25) [or, if one prefers, Eqs. (20) and (21)] give $m = 0$, which from Eq. (4) implies that the outside spacetime is pure AdS. Since the inside is also pure AdS, there is no shell in this case, only AdS spacetime.

Having treated the static problem and having found λ and p, there are mechanical constraints on the shell that should be imposed. One constraint is that the shell must be outside any trapped surface, so that the spacetime defined by Eqs. (3) and (4) makes sense. Imposing that there are no trapped surfaces gives

$$R \geq r_+; \tag{26}$$

i.e., the shell is outside its own gravitational radius. One can also see where the energy conditions lead to. The weak energy condition is automatically satisfied as we impose λ and p non-negative. On the other hand, the dominant energy condition $p \leq \lambda$ is equivalent to the relation $k^2 - \frac{2+\frac{l^2}{R^2}}{\sqrt{1+\frac{l^2}{R^2}}}k + 1 \leq 0$. This is satisfied for $\frac{1}{\sqrt{1+\frac{l^2}{R^2}}} \leq k \leq \sqrt{1+\frac{l^2}{R^2}}$. The right inequality is trivially obeyed; the left inequality leads to

$$R \geq \frac{1}{\sqrt{1 - \frac{r_+^2}{l^2}}} r_+, \tag{27}$$

which is the equation the shell must obey in order that the dominant energy condition holds. It is new. It is more stringent than the no-trapped-surface condition Eq. (26). The condition (27) is plausible on physical grounds. Indeed, for large l, the spacetime is weakly AdS, and so it is an almost flat spacetime, which in three dimensions means no, or negligible, gravity. The condition Eq. (27) gives then that $R \gtrsim r_+$; i.e., any shell that makes sense, in the sense of Eq. (26), is possible. As l decreases, the spacetime becomes strongly AdS, there is strong gravitational attraction, and the shell can satisfy the dominant energy condition only for sufficient large R. When $l = r_+$, the shell has to have infinite radius in order to obey the dominant energy condition and for even smaller l there is no shell that obeys the dominant energy condition. There are also stability conditions as Eiroa and Simeone have shown [20]. Although not explicitly shown in [20], presumably the radius $R = R(r_+, l)$ at which the shell becomes unstable is slightly larger than the R given in Eq. (27).

III. THERMODYNAMICS AND STABILITY CONDITIONS FOR THE THIN SHELL: GENERICS

Now, we assume that the shell is a hot shell; i.e., it possesses a temperature T as measured locally and has an entropy S.

In the entropy representation, as stated in [23], the entropy S of a system is given in terms of the state independent variables. Following [23], when S is known, the thermodynamical system is known. We consider as the natural state independent variables the proper local mass M of the shell, and its size, here denoted by the perimeter of the ring shell A. Thus, for the shell,

$$S = S(M, A). \tag{28}$$

Thus, when S in Eq. (28) is known, the thermodynamical properties of the system follow.

Using these variables, the first law of thermodynamics can be written as

$$TdS = dM + pdA, \tag{29}$$

where T and p are the temperature and the pressure conjugate to A. In order to find S, one has to know the equations of state for these quantities, i.e.,

$$p = p(M, A), \tag{30}$$

and

$$\beta = \beta(M, A), \tag{31}$$

where $\beta = 1/T$ is the inverse temperature.

Given Eq. (29), one can find its integrability condition for the differential of the entropy. It is given by

$$\left(\frac{\partial \beta}{\partial A}\right)_M = \left(\frac{\partial \beta p}{\partial M}\right)_A. \tag{32}$$

There is then the possibility of studying the local intrinsic stability of the shell at a thermodynamical level, which is guaranteed as long as the inequalities

$$\left(\frac{\partial^2 S}{\partial M^2}\right)_A \leq 0, \tag{33}$$

$$\left(\frac{\partial^2 S}{\partial A^2}\right)_M \leq 0, \tag{34}$$

$$\left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial A^2}\right) - \left(\frac{\partial^2 S}{\partial M \partial A}\right)^2 \geq 0, \tag{35}$$

are satisfied. For a derivation of this type of equations, see [23].

IV. THE TWO EQUATIONS OF STATE: EQUATION FOR THE PRESSURE AND EQUATION FOR THE TEMPERATURE

A. The two independent thermodynamic variables

In order to find the entropy, one needs to have the equations of state $p = p(M, A)$ and $\beta = \beta(M, A)$ [see Eqs. (30) and (31)], with M and A being the independent variables.

Note, however, that

$$A = 2\pi R, \tag{36}$$

so that the perimeter A and the radius R can be swapped at will as the independent variables. The definition of the shell's rest mass M is

$$M = 2\pi R \lambda, \tag{37}$$

where λ is given above in Eq. (24), and so

$$M = \frac{R}{4G_3 l}(1 - k). \tag{38}$$

We should now put some of the basic quantities, m, r_+, and k, in terms of M and R. Equation (38) together with Eqs. (22) and (23) implies that the ADM mass m is given in terms of the shell's proper mass M and radius R by

$$m(M, R) = \frac{-2G_3^2 M^2 + G_3 M \frac{R}{l}}{G_3}, \tag{39}$$

so that when there is no shell, i.e., $M = 0$, one has that the ADM mass of the spacetime is zero, $m = 0$. Also now Eq. (22) should be written as

$$r_+(M, R) = \sqrt{8G_3 m(M, R)}\, l, \tag{40}$$

where $m(M, R)$ is given in Eq. (39), and k should be also seen as $k = k(M, R)$, i.e.,

$$k(M, R) \equiv \sqrt{1 - \frac{r_+^2(M, R)}{R^2}}, \tag{41}$$

where $r_+(M, R)$ is given in Eq. (40).

B. The pressure equation of state

With this rationale in mind, and following the notation of Eq. (30), we write the equation for the pressure (21) in the form

$$p(M, R) = \frac{1}{8\pi G_3 l}\left(\frac{1}{k(M, R)} - 1\right). \tag{42}$$

It is clear that Eq. (25) yields the equation we aimed for as an equation of state for the shell. This equation is an exclusive effect of the gravitational equations, and the junction conditions on the ring, and it does not depend on the essence of the fields of matter composing the shell. In brief, it is compulsory that the matter fields obey this equation of state so that mechanical equilibrium is maintained.

C. The temperature equation of state

Now we turn to the other equation of state, Eq. (31), the equation for $p(M, R)$. Inserting Eq. (42) and the differential of Eq. (38) into the first law (29), using Eq. (36), and changing variables from (M, A) to (r_+, R) to simplify the calculations, where r_+ is the gravitational radius of the ring given in Eq. (40), we obtain

$$dS = \beta(r_+, R)\frac{r_+}{4G_3 l R k} dr_+, \tag{43}$$

where now S can be seen as $S = S(r_+, R)$, and the same for β, $\beta(r_+, R) \equiv 1/T(r_+, R)$. Equation (43) is integrable as long as an appropriate form for β is given. To find this appropriate form for β we use the integrability condition Eq. (32), which upon changing to the (r_+, R) variables reads

$$\left(\frac{\partial \beta}{\partial R}\right)_{r_+} = \frac{\beta}{R k^2}, \tag{44}$$

where k is envisaged now as $k = k(r_+, R)$; see Eq. (41). It can be shown that Eq. (44) has the following analytic solution:

$$\beta(r_+, R) = \frac{R}{l} k(r_+, R) b(r_+), \tag{45}$$

where $b(r_+)$ is an arbitrary function of the gravitational radius r_+. Note that $b(r_+)$ has units of inverse temperature and can be interpreted as the inverse of the temperature the shell would possess if located at $R = \sqrt{l^2 + r_+^2}$, as can be seen from Eq. (45). Equation (45) follows from the integrability condition for the entropy and is directly related to the equivalence principle for systems at a temperature different from zero. It is the Tolman relation for the temperature in a gravitational system.

Note that b is forced to depend on the state variables (M, R) through the specific function $r_+(M, R)$. However, the integrability condition does not yield a precise form for b. As stated in [6] (see also [5]), this is expected on physical grounds as the Euclideanized AdS geometry inside the ring can be identified with any period in the partition function stemming from a path integral approach, and thus the hot AdS space inside the shell can have any temperature, not fixing b *a priori*. Any specific function $b(r_+(M, R))$ must resort to the specificity of the matter itself contained in the shell. This state of affairs is common in thermodynamics. In order to find the equation of state of a gas, one can resort to generic and consistency considerations, and find for instance that the temperature T is $T = T(\rho, p, V)$ quite generally, where ρ, p, and V are the density, pressure, and volume of the gas, respectively. To have then a concrete form for the function $T = T(\rho, p, V)$, one has to know the specificities of the gas, whether it is an ideal gas or a Van der Waals gas with its two specifying constants, or any other gas; see, e.g., [23].

V. ENTROPY OF THE THIN SHELL

We are now in a position to find the entropy S of the thin shell spacetime. Inserting Eq. (45) into Eq. (43), one is led to the specific form for the differential of the entropy

$$dS(r_+) = b(r_+)\frac{r_+}{4G_3 l^2}dr_+. \tag{46}$$

Integrating Eq. (46) yields the following entropy: $S(r_+) = \frac{1}{4G_3 l^2}\int b(r_+)r_+ dr_+ + S_0$, where S_0 is an integration constant. By noting that a zero ADM mass shell, i.e., $m = 0$ or equivalently $r_+ = 0$, should naturally have zero entropy, for any regular integrand in the entropy formula just given we must have $S(r_+ \to 0) \to 0$, i.e., $S_0 = 0$. So,

$$S(r_+) = \frac{1}{4G_3 l^2}\int b(r_+)r_+ dr_+. \tag{47}$$

In the same way as $b(r_+)$, S is also forced to depend on the state variables (M, R) through the specific function $b(r_+(M, R))$. This dependence of S on $r_+(M, R)$ seen in the formula (47) comes directly from the self-gravitating nature of the setup. It is the result of the matching conditions which determine the mass m and the pressure p as in Eqs. (39) and (42), respectively, and of the equivalence principle in the form of the redshift factor of the Tolman temperature given in (45). As explained above, a precise shape for the function $b(r_+(M, R))$ has to emanate from definite, thermodynamic or otherwise, configurations for the matter fields.

Equation (47) opens the possibility of studying the local intrinsic stability of the shell at the thermodynamic level, which is guaranteed as long as the inequalities (33)–(35) are satisfied. It also permits us to study the spacetime thermodynamics in the limit the shell approaches its own gravitational radius.

VI. A SPECIFIC EQUATION OF STATE FOR THE TEMPERATURE OF THE THIN MATTER SHELL: ENTROPY AND STABILITY

A. The temperature equation and the entropy

In order to implement the calculation for the entropy, one must resort to a specific fluid or a specific gas and give exactly the function $b(r_+(M, R))$. One could think of many, and not wanting to treat here the specificities of the matter, we resort to the most simple suggestion for $b(r_+)$ as given in [6], i.e., a power law equation of the form

$$b(r_+) = 4\alpha G_3 l^2 \frac{r_+^a}{l_{\rm p}^{(2+a)}}, \tag{48}$$

where a is a free parameter, essentially a number, the factors $4G_3$, l^2, and $l_{\rm p} = G_3\hbar$ appear for dimensional and useful reasons, with $l_{\rm p}$ being the Planck's length in a three-dimensional spacetime, and $\hbar$ Planck's constant, and α is another free parameter without units that can be some function of $l/l_{\rm p}$. For instance, one can choose $\alpha = \bar{\alpha}\frac{l_{\rm p}^{(2+a)}}{l^{(2+a)}}$, with $\bar{\alpha}$ a number, but many other choices are possible. Boltzmann's constant is taken as equal to 1. In order to further justify the choice of the form of $b(r_+)$ in Eq. (48), we recall that in many thermodynamic instances one recurs to power law functions, most notably near or at a phase transition point, where the temperature goes as the power of the density (or the mass) of the fluid and a power of its specific volume (or the volume itself), for instance. These thermodynamic treatments do not even need to know the details of the fine grain constituency of the fluid. Such power laws are assumed and indeed represent well the fluid behavior. Here, since the temperature, or what is the same, b, cannot be any function of M nor any function of R, and so not a power of M times a power of R as one could be led to think from the usual thermodynamic treatments, but has to be a function such that M and R appear through $r_+(M, R)$, a natural and simple choice for b is that $b(r_+(M, R))$ is a power of $r_+(M, R)$, as we have written in Eq. (48).

Inserting Eq. (48) into (47) and integrating, we get

$$S(r_+) = \frac{\alpha}{a+2}\left(\frac{r_+}{l_{\rm p}}\right)^{(a+2)}, \tag{49}$$

valid for any a as long as we consider the case $a = -2$ as yielding a logarithmic function, $S(r_+) = \alpha \ln r_+/l$, as it should. Using this formula for the entropy, one is able to analyze the stability conditions imposed on the free parameters, and despite the fact that the values of the parameters α and a do not have specific values as long as some type of nature of the matter fields is not prescribed, it is possible to constrain a nonetheless, such that thermodynamic equilibrium states of the shell are possible.

B. The stability conditions for the specific temperature ansatz

As for the stability equations, we leave the detailed analysis for the Appendix and present here the main result. It is then possible to show that Eqs. (33)–(35) altogether applied to the temperature and entropy formulas, Eqs. (48) and (49), respectively, imply that the shell is stable when

$$a = -1 \text{ and } \frac{R}{r_+} \to \infty, \tag{50}$$

with $r_+ > 0$.

Thus, the thermodynamic stability conditions for the shell given in Eq. (50) are more restrictive than the dominant energy condition Eq. (27) and the no-trapped-surface condition Eq. (26) for the shell radius R.

When comparing the stability conditions given in Eq. (50) with those for the Schwarzschild shell in $3+1$ dimensions given in [6], we see that the stability conditions for the $2+1$ BTZ shell are much more confining, even though the temperature and the entropy were considered to have a power law dependence in r_+ both for the Schwarzschild and the BTZ shells. Different dependences on r_+ will give different values and ranges for the relevant parameters.

VII. THE ENTROPY OF THE THIN SHELL IN THE BTZ BLACK HOLE LIMIT

A case of particular interest is the stable case $a = -1$. In this case, the inverse temperature $b(r_+)$ of the shell, taken from Eq. (48), is

$$b(r_+) = \frac{4\alpha G_3 l^2}{l_\mathrm{p}} \frac{1}{r_+}, \tag{51}$$

and the entropy of the shell, taken from Eq. (49), has the explicit form

$$S(r_+) = \alpha \frac{r_+}{l_\mathrm{p}}. \tag{52}$$

This is valid for any radius R of the shell ($R \geq r_+$), since from the integrability condition the entropy does not depend on the radius of the shell R.

In particular, if we take the limit $R \to r_+$, then the shell hovers at its own gravitational radius. One then expects that quantum fields are present [2], and the backreaction will diverge unless one chooses the matter to be at the Hawking temperature [11]

$$T_\mathrm{H} = \frac{l_\mathrm{p}}{2\pi G_3 l^2} r_+. \tag{53}$$

This fixes the function $b = 1/T_\mathrm{H}$ to be

$$b(r_+) = \frac{2\pi G_3 l^2}{l_\mathrm{p}} \frac{1}{r_+}, \tag{54}$$

which means that $\alpha = \pi/2$ in Eq. (51). Then the entropy (52) of the shell at its own gravitational radius is

$$S_\mathrm{BH} = \frac{\pi}{2} \frac{r_+}{l_\mathrm{p}}. \tag{55}$$

The area A_h of the horizon, which is actually a perimeter in $2+1$ dimensions, is $A_\mathrm{h} = 2\pi r_+$. Therefore

$$S_\mathrm{BH} = \frac{1}{4} \frac{A_\mathrm{h}}{l_\mathrm{p}}. \tag{56}$$

This is precisely the Bekenstein-Hawking entropy of the $(2+1)$-dimensional BTZ black hole [11], now derived from the properties of the spacetime of the shell of matter and from the fact that the shell is at its own gravitational radius. At $R = r_+$ one has from Eq. (23) that $k = 0$, and so from Eqs. (24) and (25) one finds that $\lambda = \frac{1}{8\pi G_3 l}$ (i.e., $M = \frac{1}{4G_3} \frac{r_+}{l}$) and $p = \frac{1}{8\pi G_3 l} \frac{1}{k} \to \infty$, characteristic of certain quasiblack holes, objects which also yield the Bekenstein-Hawking entropy [9,10]. Indeed, the shell at its own gravitational radius is a quasiblack hole.

VIII. CONCLUSIONS

In this paper we have considered the thermodynamics and entropy of a $(2+1)$-dimensional shell, a ring, in an AdS spacetime. Inside the ring, the spacetime is given by the AdS metric, characterized by a negative cosmological constant $-\Lambda = \frac{1}{l^2}$, and outside it is given by the BTZ metric, characterized by a mass m, by the same negative cosmological constant $-\Lambda = \frac{1}{l^2}$, and by being asymptotically AdS. The ring shell at radius R has a mass density λ (or equivalently a mass $M = 2\pi R\lambda$) and a pressure p associated with it required to achieve a static equilibrium.

The first law of thermodynamics implies that we need two equations of state: one for the pressure p of the shell and another for its temperature T. The pressure p is given in terms of the state variables M and R through the junction conditions. The temperature T, or its inverse, is found to be a function of the gravitational radius r_+ of the system alone. This r_+ is itself a particular known function of the state variables M and R. The entropy of the shell can then be found as a function of r_+ alone. To give an example of a hot shell, inspired in several usual thermodynamics systems which have the temperature as given in power laws of the state variables, we have chosen the inverse temperature b to be proportional to a power law in r_+, r_+^a, for some number a. The computation of the specific form of the entropy led to an analysis of the parameter regions for which the ring is thermodynamically stable. We have found that $a = -1$ and the shell must be located at infinity, $R \to \infty$. This means that the BTZ shell is much more restrictive than the Schwarzschild shell in $3+1$ dimensions for the same power law of the entropy as a function of the shell's gravitational radius r_+.

In the case $a = -1$, the shell can be chosen to have a Hawking type temperature at the outset. One can then tune the temperature to be precisely the Hawking temperature, including all numerical factors, and push the shell up to its gravitational radius, since at this temperature there is a finite backreaction at the horizon that does not destroy the solution. The entropy found is then the Bekenstein-Hawking entropy as it is appropriate for a quasiblack hole.

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APPENDIX: ANALYSIS OF THE STABILITY EQUATIONS

We use here the stability conditions Eqs. (33)–(35) in the temperature and entropy formulas, Eqs. (48) and (49), respectively, to show the results presented in Sec. VIB.

It is possible to show that Eq. (33) implies the inequality

$$\frac{a}{a+1} R^2 \leq r_+^2, \tag{A1}$$

which, together with the condition that the shell is above or at its own gravitational radius, i.e., $r_+^2 \leq R^2$, sets up the restricted values for R relative to r_+, namely,

$$\frac{a}{a+1} R^2 \leq r_+^2 \leq R^2. \tag{A2}$$

For $0 \leq a < \infty$ this inequality always holds. If $-1 \leq a < 0$, the lower limit will assume negative values, but the inequality is satisfied nonetheless. For $a < -1$, the left half of the inequality will exceed the right half and thus $a < -1$ is excluded. Thus, Eq. (A2) restricts the interval of a to

$$-1 \leq a < \infty. \tag{A3}$$

Turning now to Eq. (34), it leads to the relation

$$r_+^2 \geq 2R^2 \left(1 - \sqrt{1 - \frac{r_+^2}{R^2}}\right), \tag{A4}$$

which after some manipulation yields

$$\frac{r_+}{R} = 0. \tag{A5}$$

Since, we want a system with a horizon, i.e., $r_+ > 0$, Eq. (A5) means that R obeys

$$\frac{R}{r_+} \to \infty. \tag{A6}$$

In addition, from Eq. (35) one obtains the inequality

$$a + 1 \leq 0. \tag{A7}$$

Eqs. (A3) and (A7) together mean that

$$a = -1. \tag{A8}$$

In brief, the chosen thermodynamic system with a temperature given in Eq. (48) and the entropy given in Eq. (49) is stable when

$$a = -1 \text{ and } \frac{R}{r_+} \to \infty \tag{A9}$$

with $r_+ > 0$.

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