

Gravitational collapse of Bose-Einstein condensate dark matter halos

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We study the mechanisms of the gravitational collapse of the Bose-Einstein condensate dark matter halos, described by the zero temperature time-dependent nonlinear Schrödinger equation (the Gross-Pitaevskii equation), with repulsive interparticle interactions. By using a variational approach, and by choosing an appropriate trial wave function, we reformulate the Gross-Pitaevskii equation with spherical symmetry as Newton's equation of motion for a particle in an effective potential, which is determined by the zero-point kinetic energy, the gravitational energy, and the particles interaction energy, respectively. The velocity of the condensate is proportional to the radial distance, with a time-dependent proportionality function. The equation of motion of the collapsing dark matter condensate is studied by using both analytical and numerical methods. The collapse of the condensate ends with the formation of a stable configuration, corresponding to the minimum of the effective potential. The radius and the mass of the resulting dark matter object are obtained, as well as the collapse time of the condensate. The numerical values of these global astrophysical quantities, characterizing condensed dark matter systems, strongly depend on the two parameters describing the condensate, the mass of the dark matter particle, and of the scattering length, respectively. The stability of the condensate under small perturbations is also studied, and the oscillations frequency of the halo is obtained. Hence these results show that the gravitational collapse of the condensed dark matter halos can lead to the formation of stable astrophysical systems with both galactic and stellar sizes.

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I. INTRODUCTION

The recently published Planck satellite data [1] have generally confirmed the predictions of the standard Λ CDM cold dark matter (Λ CDM) cosmological model, as well as the matter composition of the Universe. The Λ CDM model successfully describes the accelerated expansion of the Universe, the observed temperature fluctuations in the cosmic microwave background radiation, the large scale matter distribution, and the main aspects of the formation and the evolution of virialized cosmological objects. On the other hand the latest cosmic microwave background data, as well as the observations of the distant Type IA supernovae, baryon acoustic oscillations, weak gravitational lensing, and the abundance of galaxy clusters, provide compelling evidence that about 95% of the content of the Universe resides in two unknown forms of matter/energy, called dark matter and dark energy, respectively: the first residing in bound objects as nonluminous matter at the galactic and extragalactic scale [2], while the latter is in the form of a zero-point energy that pervades the whole Universe [3,4]. The dark matter is assumed to be composed of cold neutral weakly interacting massive particles, beyond those existing in the Standard Model of particle physics, and not yet detected in accelerators or in dedicated direct and indirect searches. There are many possible

candidates for dark matter, the most popular ones being the axions and the weakly interacting massive particles (for a review of the particle physics aspects of dark matter see [5]). The interaction cross sections of dark matter particles with normal baryonic matter, while extremely small, are expected to be nonzero, and we may expect to detect them directly [6]. Scalar fields or other long range coherent fields coupled to gravity have also intensively been used to model galactic dark matter [7]. Alternative theoretical models to explain the galactic rotation curves have also been elaborated recently [8].

Despite its important achievements, at galactic scales of the order of ~ 10 kpc, the Λ CDM model faces major challenges in explaining the observed distribution of the dark matter around the luminous one. In fact, N -body simulations, performed in the Λ CDM scenario, have shown that bound halos surrounding galaxies must have very characteristic density profiles that feature a well pronounced central cusp, $\rho_{\text{NFW}}(r) = \rho_s / (r/r_s)(1 + r/r_s)^2$ [9], where r_s is a scale radius and ρ_s is a characteristic density. On the observational side, high-resolution rotation curves show, instead, that the actual distribution of dark matter is much shallower than the simulated one, and it presents a nearly constant density core: $\rho_B(r) = \rho_0 r_0^3 / (r + r_0)(r^2 + r_0^2)$ [10], where r_0 is the core radius and ρ_0 is the central density. Therefore, to solve this contradiction between observation and theory, new models and a new understanding of dark matter and its properties are required.

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The observation of the Bose-Einstein condensation in 1995 in dilute alkali gases, such as vapors of rubidium and sodium, confined in a magnetic trap, and cooled to very low temperatures [11] did represent a major breakthrough in condensed matter and statistical physics. At very low temperatures, all particles in a dilute Bose gas condense to the same quantum ground state, forming a Bose-Einstein condensate (BEC), i.e., a sharp peak over a broader distribution in both coordinates and momentum space. Particles become correlated with each other when their wavelengths overlap; that is, the thermal wavelength λ_T is greater than the mean interparticle distance l . This happens at a temperature $T < 2\pi\hbar^2 n^{2/3}/mk_B$, where m is the mass of the particle in the condensate, n is the number density, and k_B is Boltzmann's constant [12–16]. A coherent state develops when the particle density is high enough, or the temperature is sufficiently low. From experimental point of view, the occurrence of the condensation is indicated by a sharp peak in the velocity distribution, observed below a critical temperature. This shows that all the atoms have condensed in the same ground state, with a narrow peak in the momentum and coordinate space [11]. Quantum degenerate gases have been created by a combination of laser and evaporative cooling techniques, opening several new lines of research, at the border of atomic, statistical and condensed matter physics [12–16].

Since the Bose-Einstein condensation is a phenomenon observed and well studied in the laboratory, the possibility that it may occur on astrophysical or cosmic scales cannot be rejected *a priori*. Thus, dark matter, which is required to explain the dynamics of the neutral hydrogen clouds at large distances from the Galactic center, and which is a cold, bosonic system, could also be in the form of a Bose-Einstein condensate [17]. In these early studies either a phenomenological approach was used, or the nonrelativistic Gross-Pitaevskii equation describing the condensate was investigated numerically. A systematic study of the condensed galactic dark matter halos and of their properties was performed in [18]. By introducing the Madelung representation of the wave function, the dynamics of the dark matter halo can be formulated in terms of the continuity equation and of the hydrodynamic Euler equations. Hence condensed dark matter can be described as a nonrelativistic, Newtonian gas, whose density and pressure are related by a barotropic equation of state. In the case of a condensate with quartic nonlinearity, the equation of state is polytropic with index $n = 1$ [18].

To test the validity of the condensed dark matter model the Newtonian tangential velocity equation was fitted with a sample of rotation curves of low surface brightness and dwarf galaxies, respectively. A very good agreement was found between the theoretical rotation curves and the observational data. Therefore dark matter halos can be described as an assembly of light individual bosons that acquire a repulsive interaction by occupying the same

ground energy state. The repulsive interaction prevents gravity from forming the central density cusps. The condensate particle is light enough to naturally form condensates of very small masses that later may coalesce, forming the structures of the Universe in a similar way than the hierarchical clustering of the bottom-up CDM picture. Then, at large scales, BEC perfectly mimic an ensemble of cold particles, while at small scales quantum mechanics drives the mass distribution.

The properties of the Bose-Einstein condensed dark matter halos, as well as their cosmological implications, have been investigated recently. The recently observed size evolution of very massive compact galaxies in the early universe can be explained, if dark matter is in a Bose-Einstein condensate state [19]. The size of the dark matter halos and galaxies depends on the correlation length of dark matter and, hence, on the expansion of the Universe. The BEC predicts that the size of the galaxies increases as the Hubble radius of the Universe even without merging, which agrees well with the recent observational data. In [20] it was shown that the finite length scale of the condensate dark matter can explain the recently observed common central mass of the Milky Way satellites ($\sim 10^7 M_\odot$) independent of their luminosity, if the mass of the dark matter particle is about 10^{-22} eV.

The validity of the BEC model on the galactic scale by using observed rotation curves was tested in [21] by comparing the tangential velocity equation of the model with a sample of eight rotation curves of dwarf galaxies. A good agreement was found between the theoretically predicted rotation curves (without any baryonic component) and the observational data. The mean value of the logarithmic inner slope of the mass density profile of dwarf galaxies was also obtained, and it was shown that the observed value of this parameter is in agreement with the theoretical results. The study of the galactic rotation curves in the BEC model was considered in [22] and [23]. The BEC model predicts that all galaxies must be very similar and exist for bigger redshifts than in the Λ CDM model. In [22] the fits of high-resolution rotation curves of a sample of thirteen low surface brightness galaxies were compared with fits obtained using a Navarro-Frenk-White and pseudoisothermal (PI) profiles. A better agreement with the BEC model and PI profiles was found. The mean value of the logarithmic inner density slopes is -0.27 ± 0.18 . A natural way to define the core radius, with the advantage of being model independent, was also introduced. Using this new definition in the BEC density profile it was found that the recent observation of the constant dark matter central surface density can be reproduced. The BEC model gives a similar fit to the Navarro-Frenk-White (NFW) dark matter model for all high surface brightness and low surface brightness galaxies in a sample of 9 galaxies [23]. For dark matter dominated dwarf galaxies the addition of the BEC component improved more upon the purely baryonic fit

than the NFW component. Thus despite the sharp cutoff of the halo density, the BEC dark matter candidate is consistent with the rotation curve data of all types of galaxies [23]. The dynamics of rotating Bose condensate galactic dark matter halos, made of ultralight spinless bosons, and the impact of the halo rotation on the galactic rotation curves was analyzed in [24].

Finite temperature effects on dark matter halos were analyzed in [25], where the condensed dark matter and thermal cloud density and mass profiles at finite temperatures were explicitly obtained. The obtained results show that when the temperature of the condensate and of the thermal cloud are much smaller than the critical Bose-Einstein transition temperature, the zero temperature density and mass profiles give an excellent description of the dark matter halos. The angular momentum and vortices in BEC galactic dark matter halos were studied in [26–29], respectively.

In [30] it was proposed that the dark matter content of galaxies consist of a cold bosonic fluid, composed of weakly interacting slim particles, represented by spin-0 axionlike particles and spin-1 hidden bosons, thermalized in the Bose-Einstein condensation state and bounded by their self-gravitational potential. By comparing this model with data obtained from 42 spiral galaxies and 19 low surface brightness galaxies, the dark matter particle mass was constrained to the range $10^{-6} - 10^{-4}$ eV, and the lower bound for the scattering length was found to be of the order of 10^{-14} fm.

The possibility that due to their superfluid properties some compact astrophysical objects may contain a significant part of their matter in the form of a Bose-Einstein condensate was investigated in [31]. To study the condensate the Gross-Pitaevskii equation was used, with arbitrary nonlinearity. In this way a large class of stable astrophysical objects was obtained, whose basic astrophysical parameters (mass and radius) sensitively depend on the mass of the condensed particle, and on the scattering length.

The Bose-Einstein condensation process in a cosmological context, by assuming that this process can be described (at least approximately) as a first order phase transition was studied in [32]. It was shown that the presence of the condensate dark matter and of the Bose-Einstein phase transition could have modified drastically the cosmological evolution of the early universe, as well as the large scale structure formation process. The effects of the finite dark matter temperature on the properties of the Bose-Einstein condensed dark matter halos were analyzed, in a cosmological context, in [33]. The basic equations describing the finite temperature condensate, representing a generalized Gross-Pitaevskii equation that takes into account the presence of the thermal cloud were formulated. The static condensate and thermal cloud in thermodynamic equilibrium was analyzed in detail, by using the Hartree-Fock-Bogoliubov

and Thomas-Fermi approximations. It was also shown that finite temperature effects may play an important role in the early stages of the cosmological evolution of the dark matter condensates. The cosmological perturbations in the cosmological models with condensed dark matter were studied in [34–36]. The large scale perturbative dynamics of the BEC dark matter in a model where this component coexists with baryonic matter and cosmological constant was investigated in [37]. The perturbative dynamics was studied using neo-Newtonian cosmology (where the pressure is dynamically relevant for the homogeneous and isotropic background) which is assumed to be correct for small values of the sound speed. BEC dark matter effects can be seen in the matter power spectrum if the mass of the condensate particle lies in the range $15 \text{ meV} < m < 700 \text{ meV}$, leading to a small, but perceptible, excess of power at large scales.

Simulation codes that are designed to study the behavior of the dark matter galactic halos in the form of a Bose-Einstein condensate were developed in [38] and [39]. In [40] it was shown that once appropriate choices for the dark matter particle mass and scattering length are made, the galactic dark matter halos composed by axionlike Bose-Einstein condensed particles, trapped by a self-gravitating potential, may be stable in the Thomas-Fermi approximation. The validity of the Thomas-Fermi approximation for the halo system was also discussed, and it was shown that the kinetic energy contribution is indeed negligible. The Thomas-Fermi approximation for the study of the condensed dark matter halos was also discussed in [41]. The Thomas-Fermi approximation is based on the assumption that in the presence of a large number of particles, the kinetic term in the Gross-Pitaevskii energy functional can be neglected. However, this assumption is violated near the condensate surface. It was also shown that the total energy of the self-gravitating condensate in the Thomas-Fermi approximation is positive.

A major recent experimental advance in the study of the Bose-Einstein condensation processes was the observation of the collapse and subsequent explosion of the condensates [42]. A dynamical study of an attractive ^{85}Rb BEC in an axially symmetric trap was done, where the interatomic interaction was manipulated by changing the external magnetic field, thus exploiting a nearby Feshbach resonance. In the vicinity of a Feshbach resonance the atomic scattering length a can be varied over a huge range, by adjusting an external magnetic field. Consequently, the sign of the scattering length is changed, thus transforming a repulsive condensate of ^{85}Rb atoms into an attractive one, which naturally evolves into a collapsing and exploding condensate.

From a simple physical point of view the collapse of the Bose-Einstein condensates can be described as follows. When the number of particles becomes sufficiently large, so that $N > N_c$, where N_c is a critical number, the attractive interparticle energy overcomes the quantum pressure, and

the condensate implodes. In the course of the implosion stage, the density of particles increases in the small vicinity of the trap center. When it approaches a certain critical value, a fraction of the particles gets expelled. In a time period of an order of few milliseconds, the condensate again stabilizes. There are two observable components at the final stage of the collapse: remnant and burst particles. The remnant particles are those which remain in the condensate. The burst particles have an energy much larger than that of the condensed particles. There is also a fraction of particles, which is not observable. This fraction is referred to as the missing particles [43].

The study of the Bose-Einstein collapse within a model of a gas of free bosons described by a semi-classical Fokker-Planck equation was performed in [44]. A striking similarity between the Bose-Einstein condensation in the canonical ensemble, and the gravitational collapse of a gas of classical self-gravitating Brownian particles was found. It was also shown that at the Bose-Einstein condensation temperature T_c , the chemical potential $\mu(t)$ vanishes exponentially with a universal rate. After t_{coll} , the finite time interval in which $\sqrt{\mu(t)}$ vanishes, the mass of the condensate grows linearly with time, and saturates exponentially to its equilibrium value for large times.

It is the purpose of the present paper to study the dynamics of gravitationally self-bound Bose-Einstein dark matter condensates of collisionless particles, without exterior trapping potentials. In particular, we focus on the description, mechanism and properties of the condensate collapse. In order to study the gravitational collapse, and to solve the Gross-Pitaevskii equation describing the dynamics of the condensate, we employ a variational method. By appropriately choosing a trial wave function, the dynamical evolution of the condensate can be described by an effective time-dependent action, with the equation of motion of the condensate being given by the equation of motion of a single particle in an effective potential. The effective potential contains the contributions of the zero-point kinetic energy, of the gravitational energy, and of the interaction energy, respectively. The effective equation of motion of the collapsing dark matter condensate is studied by using both analytical and numerical methods. The collapse of the condensate ends with the formation of a stable astrophysical configuration, corresponding to the minimum of the effective potential. The radius and the mass of the resulting dark matter object are obtained, as well as the collapse time of the condensate by numerically solving the effective equation of motion. Approximate expressions for the radius of the stable configuration and of the collapse time are also obtained. The numerical values of these global astrophysical quantities, characterizing condensed dark matter systems, strongly depend on the two parameters describing the condensate, the mass of the dark matter particle, and of the scattering length, respectively. The stability of the condensate under small perturbations is also

studied, and the oscillations frequency of the halo is obtained. Hence the results obtained in the present paper show that the gravitational collapse of the condensed dark matter halos can lead to the formation of stable astrophysical systems on both galactic and stellar scales.

The present paper is organized as follows. The basic physical properties of the static Bose-Einstein condensed dark matter halos are reviewed briefly in Sec. II. The variational formulation of the Gross-Pitaevskii equation, the choice of the trial wave function, and the formulation of the effective dynamics of the condensate as a motion of a single particle in an effective potential are presented in Sec. III. The physical parameters of the time-dependent dark matter halos (density, gravitational potential, and the physical parameters of the effective potential) are determined, within the framework of the variational approach in Sec. IV. The gravitational collapse of the dark matter halos is analyzed in Sec. V. The stability properties of the dark matter halos formed after the gravitational collapse are investigated in Sec. VI. We discuss and conclude our results in Sec. VII.

II. PHYSICAL PARAMETERS OF THE STATIC CONDENSED DARK MATTER HALOS

In the following we assume that the dark matter halos are composed of a strongly coupled, dilute Bose-Einstein condensate at absolute zero temperature. Therefore almost all the dark matter particles are in the condensate. In a dilute and cold gas, only binary collisions at low energy are relevant, and, hence, independently of the details of the two-body potential, the collisions are characterized by a single physical parameter, the s -wave scattering length a . Consequently, one can replace the interaction potential with an effective interaction $V_I(\vec{r}' - \vec{r}) = u_0 \delta(\vec{r}' - \vec{r})$, where the coupling constant u_0 is related to the scattering length a through $u_0 = 4\pi\hbar^2 a/m_\chi$, where m_χ is the mass of the dark matter particle [12]. The ground state properties of the dark matter are described by the mean-field Gross-Pitaevskii (GP) equation.

The density distribution ρ_{BE} of the static gravitationally bounded single component dark matter Bose-Einstein condensate is given by [18],

$$\rho_{\text{BE}}(r) = \rho_{\text{BE}}^{(c)} \frac{\sin kr}{kr}, \quad (1)$$

where $k = \sqrt{Gm_\chi^3/\hbar^2 a}$ and $\rho_{\text{BE}}^{(c)}$ is the central density of the condensate, $\rho_{\text{BE}}^{(c)} = \rho_{\text{BE}}(0)$.

The mass profile $m_{\text{BE}}(r) = 4\pi \int_0^r \rho_{\text{BE}}(r) r^2 dr$ of the Bose-Einstein condensate galactic halo is

$$m_{\text{BE}}(r) = \frac{4\pi\rho_{\text{BE}}^{(c)}}{k^2} r \left(\frac{\sin kr}{kr} - \cos kr \right), \quad (2)$$

with the boundary radius R_{BE} . At the boundary of the dark matter distribution $\rho_{\text{BE}}(R_{\text{BE}}) = 0$, giving the condition $kR_{\text{BE}} = \pi$, which fixes the radius of the condensate dark matter halo as

$$R_{\text{BE}} = \pi \sqrt{\frac{\hbar^2 a}{Gm_\chi^3}}. \quad (3)$$

The total mass M_{BE} of the condensate is

$$M_{\text{BE}}(R_{\text{BE}}) = 4\pi \int_0^{R_{\text{BE}}} \rho_{\text{BE}}(r) r^2 dr = \frac{4}{\pi} R_{\text{BE}}^3 \rho_{\text{BE}}^{(c)}, \quad (4)$$

which represent a simple cubic proportionality between mass and radius. The mass of the condensate is around three times smaller than the mass of a constant density sphere.

For $r > R_{\text{BE}}$ the density is smaller than zero, so that the $n = 1$ polytropic density profile cannot be extended to infinity, that is, beyond its sharp boundary. For small values of r the condensate dark matter profile can be written as

$$\rho_{\text{BE}}(r) \approx \rho_{\text{BE}}^{(c)} \left(1 - \frac{\pi^2}{6} \frac{r^2}{R_{\text{BE}}^2} + \frac{\pi^4}{120} \frac{r^4}{R_{\text{BE}}^4} + \dots \right), \quad (5)$$

$r \leq R_{\text{BE}}$.

The gravitational potential $V_{\text{BE}}^{(\text{grav})}(r)$ of the condensed dark matter distribution is determined by the equation

$$\begin{aligned} V_{\text{grav}}^{(\text{BE})}(r) &= -G \int_r^{R_{\text{BE}}} \frac{m_{\text{BE}}(r') dr'}{r'^2} \\ &= -\frac{4G\rho_c R_{\text{BE}}^3}{\pi^2 r} \sin\left(\frac{\pi r}{R_{\text{BE}}}\right), \\ &r \leq R_{\text{BE}}. \end{aligned} \quad (6)$$

At small radii the potential behaves as

$$\begin{aligned} V_{\text{grav}}^{(\text{BE})}(r) &\approx -G\rho_c R_{\text{BE}}^2 \left(\frac{4}{\pi} - \frac{2\pi}{3} \frac{r^2}{R_{\text{BE}}^2} + \frac{\pi^3}{30} \frac{r^4}{R_{\text{BE}}^4} \right) \\ &+ O(r)^6, \quad r \leq R_{\text{BE}}. \end{aligned} \quad (7)$$

The tangential velocity of a test particle moving in the condensed dark halo can be represented as [18]

$$V_{\text{BE}}^2(r) = \frac{Gm_{\text{BE}}(r)}{r} = \frac{4\pi G\rho_{\text{BE}}^{(c)}}{k^2} \left(\frac{\sin kr}{kr} - \cos kr \right). \quad (8)$$

With the use of Eq. (3) the mass of the particle in the condensate can be obtained from the radius of the dark matter halo in the form [18]

$$\begin{aligned} m_\chi &= \left(\frac{\pi^2 \hbar^2 a}{GR_{\text{BE}}^2} \right)^{1/3} \\ &\approx 2.58 \times 10^{-30} [a(\text{cm})]^{1/3} [R_{\text{BE}}(\text{kpc})]^{-2/3} \text{g} \\ &\approx 6.73 \times 10^{-2} [a(\text{fm})]^{1/3} [R_{\text{BE}}(\text{kpc})]^{-2/3} \text{eV}, \end{aligned} \quad (9)$$

where $1\text{g} = 5.598 \times 10^{32} \text{eV}$.

From this equation it follows that m_χ is of the order of eV. For $a \approx 1 \text{fm}$ and $R_{\text{BE}} \approx 10 \text{kpc}$, the mass is of the order of $m_\chi \approx 14 \text{meV}$. For values of a of the order of $a \approx 10^6 \text{fm}$, corresponding to the values of a observed in terrestrial laboratory experiments, $m_\chi \approx 1.44 \text{eV}$. These numerical values of the mass of the dark matter particle are perfectly consistent with the limit $m_\chi < 1.87 \text{eV}$, obtained for the mass of the condensate particle from cosmological considerations [45].

The properties of dark matter can be obtained observationally through the study of the collisions between clusters of galaxies, like the Bullet Cluster (1E 0657-56) and the baby Bullet Cluster (MACSJ0025-12) [46,47]. From these observations one can obtain constraints on the physical properties of the dark matter, such as its interaction cross section with baryonic matter, and the dark matter self-interaction cross section. If the ratio $\sigma_m = \sigma/m_\chi$ of the self-interaction cross section $\sigma = 4\pi a^2$ and of the dark matter particle mass m_χ is known from observations, with the use of Eq. (9) the mass of the dark matter particle in the Bose-Einstein condensate can be obtained as [33]

$$m_\chi = \left(\frac{\pi^{3/2} \hbar^2 \sqrt{\sigma_m}}{2G R_{\text{BE}}^2} \right)^{2/5}. \quad (10)$$

The comparison of the results obtained from x-ray, strong lensing, weak lensing, and optical observations with numerical simulations of the merging galaxy cluster 1E 0657-56, gives an upper limit (68% confidence) for σ_m of the order of $\sigma_m < 1.2 \text{cm}^2/\text{g}$ [46]. By choosing for σ_m a value of $\sigma_m = 1.25 \text{cm}^2/\text{g}$, we obtain for the mass of the dark matter particle an upper limit of the order

$$\begin{aligned} m_\chi &< 3.1933 \times 10^{-37} \times \left(\frac{R_{\text{BE}}}{10 \text{kpc}} \right)^{-4/5} \times \left(\frac{\sigma_m}{1.25 \text{cm}^2/\text{g}} \right)^{1/5} \text{g} \\ &= 0.1791 \times \left(\frac{R_{\text{BE}}}{10 \text{kpc}} \right)^{-4/5} \times \left(\frac{\sigma_m}{1.25 \text{cm}^2/\text{g}} \right)^{1/5} \text{meV}. \end{aligned} \quad (11)$$

By using this value of the particle mass we can estimate the scattering length l_a as

$$a < \sqrt{\frac{\sigma_m \times m_\chi}{4\pi}} = 1.7827 \times 10^{-19} \text{cm} = 1.7827 \times 10^{-6} \text{fm}. \quad (12)$$

A stronger constraint for σ_m was proposed in [47], so that $\sigma_m \in (0.00335 \text{cm}^2/\text{g}, 0.0559 \text{cm}^2/\text{g})$, giving a dark matter particle mass of the order [33]

$$m_\chi \approx (9.516 \times 10^{-38} - 1.670 \times 10^{-37}) \times \left(\frac{R_{\text{BE}}}{10 \text{ kpc}} \right)^{-4/5} \text{ g}$$

$$= (0.053 - 0.093) \times \left(\frac{R_{\text{BE}}}{10 \text{ kpc}} \right)^{-4/5} \text{ meV}, \quad (13)$$

and a scattering length of the order of

$$a \approx (5.038 - 27.255) \times 10^{-21} \text{ cm}$$

$$= (5.038 - 27.255) \times 10^{-8} \text{ fm}. \quad (14)$$

Therefore the galactic radii data and the Bullet Cluster constraints predict a condensate dark particle mass of the order of $m_\chi \approx 0.1 \text{ meV}$.

III. TIME-DEPENDENT EVOLUTION OF ZERO TEMPERATURE BOSE-EINSTEIN DARK MATTER HALOS

In the present section we introduce the variational formulation of the Gross-Pitaevskii equation, describing the time dynamics of the cold ($T = 0$) Bose-Einstein condensed dark matter halos, and we obtain the basic equation describing the evolution of the gravitationally bounded dark matter condensates. We begin our analysis with the study of the general relativistic Bose-Einstein condensates trapped in a gravitational field in the semi-classical approximation, and with the use of the Newtonian limit we obtain the basic equations describing the gravitational collapse of the Bose-Einstein condensates.

A. The Newtonian limit for general relativistic Bose-Einstein condensates

In formulating a general relativistic model of the Bose-Einstein condensates we consider that bosonic matter at temperatures below the critical temperature T_c in a gravitational field represents a hybrid system, in which the gravitational field remains classical, while the bosonic condensate is described by quantum fields. In the standard approach used for coupling quantum fields to a classical gravitational field (i.e., semiclassical gravity), the energy-momentum tensor that serves as the source in the Einstein equations is replaced by the expectation value of the energy-momentum operator $\hat{T}_{\mu\nu}$ with respect to some quantum state Ψ [48],

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4} \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle, \quad (15)$$

where $R_{\mu\nu}$ is the curvature tensor, R is the curvature scalar, and $g_{\mu\nu}$ is the metric tensor of the space-time. In the nonrelativistic limit the state function Ψ evolves according to the Gross-Pitaevskii equation for the condensate wave function ψ , with a quartic nonlinear term [12–16,18]

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \left[-\frac{\hbar^2}{2m_\chi} \nabla^2 + m_\chi V_{\text{ext}}(\vec{r}, t) + u_0 |\psi(\vec{r}, t)|^2 \right] \times \psi(\vec{r}, t), \quad (16)$$

where $u_0 = 4\pi\hbar^2 a/m_\chi$, a is the coherent scattering length (defined as the zero-energy limit of the scattering amplitude f_{scat}), m_χ is the mass of the condensate particle, and V_{ext} is the external potential. For the energy-momentum tensor we obtain $\langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle = \langle \psi | \hat{T}_{\mu\nu} | \psi \rangle = T_{\mu\nu}$, where $T_{\mu\nu}$ is the classical energy-momentum tensor of the condensed system. In a comoving frame the energy-momentum tensor is diagonal, with components $T_{\mu\nu} = (\rho c^2, -P, -P, -P)$, with ρc^2 and P denoting the effective energy density and the thermodynamic pressure of the condensed system. As for $V_{\text{ext}}(\vec{r}, t)$, we assume that it is the gravitational potential V_{grav} , $V_{\text{ext}}(\vec{r}, t) = V_{\text{grav}}(\vec{r}, t)$, and it is given by the Newtonian limit of Eq. (15). In this limit $g_{00} = 1 + 2V_{\text{grav}}(\vec{r}, t)/c^2$, and Eq. (15) can be written as $R_{00}^0 = (4\pi G/c^2)\rho$. By taking into account that $R_{00}^0 = (1/c^2)\Delta V_{\text{grav}}(\vec{r}, t)$, it follows that for a single component condensate the gravitational potential $V_{\text{grav}}(\vec{r}, t)$ satisfies the Poisson equation

$$\nabla^2 V_{\text{grav}}(\vec{r}, t) = 4\pi G \rho(\vec{r}, t), \quad (17)$$

where

$$\rho = m_\chi n(\vec{r}, t) = m_\chi |\psi(\vec{r}, t)|^2, \quad (18)$$

is the mass density inside the Bose-Einstein condensate, $n(\vec{r}, t) = |\psi(\vec{r}, t)|^2$ is the particle number density, and G is the gravitational constant, respectively. The probability density $|\psi(\vec{r}, t)|^2$ is normalized according to $\int n(\vec{r}, t) d^3\vec{r} = \int |\psi(\vec{r}, t)|^2 d^3\vec{r} = N$, where N is the total particle number in the condensate.

The Newtonian approximation is valid only if the condition $P \ll \rho c^2$ is satisfied. The equation of state of the Bose-Einstein condensate is $P(\rho) = U_0 \rho^2$, with $U_0 = 2\pi\hbar^2 a/m_\chi^3$ [18], which is a polytrope with index $n = 1$. Therefore the general relativistic corrections can be ignored for condensate dark matter densities satisfying the constraint $U_0 \rho \ll c^2$, or, equivalently,

$$\rho \ll \frac{c^2 m_\chi^3}{2\pi\hbar^2 a} = 1.289 \times 10^{-15} \times \left(\frac{m_\chi}{10^{-32} \text{ g}} \right)^3 \left(\frac{a}{10^{-7} \text{ cm}} \right) \text{ g/cm}^3. \quad (19)$$

The central densities of the galactic dark matter halos are of the order of $\rho_c \approx 10^{-27} \text{ g/cm}^3 - 10^{-24} \text{ g/cm}^3$ [46,47], and hence the Newtonian approximation can be applied for the study of the astrophysical properties of the condensate dark matter systems.

B. Variational formulation of the Gross-Pitaevskii equation

The basic equations describing the gravitationally trapped Bose-Einstein condensates in the Newtonian limit are given by Eqs. (16) and (17), respectively. The time-dependent Gross-Pitaevskii equation (16) can be derived from the action principle $\delta \int_{t_1}^{t_2} L_\psi dt = 0$, where the Lagrangian L_ψ is given by [15,49,50]

$$L_\psi = \int_V \frac{i\hbar}{2} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) d^3\vec{r} - E, \quad (20)$$

where $E = \int_V \varepsilon d^3\vec{r}$ is the total energy, with the energy density ε given by

$$\varepsilon = \frac{\hbar^2}{2m_\chi} |\nabla\psi|^2 + m_\chi V_{\text{grav}} |\psi|^2 + \frac{u_0}{2} |\psi|^4. \quad (21)$$

If one multiplies Eq. (16) by ψ^* and subtracts the complex conjugate of the resulting equation, one arrives at the equation

$$\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \left[\frac{\hbar}{2im_\chi} (\psi^* \nabla\psi - \psi \nabla\psi^*) \right] = 0. \quad (22)$$

Equation (22) has the form of a continuity equation for the particle density, and can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad (23)$$

where the velocity of the condensate is defined by

$$\vec{v} = \frac{\hbar}{2im_\chi} \frac{\psi^* \nabla\psi - \psi \nabla\psi^*}{|\psi|^2}. \quad (24)$$

Simple expressions for the density and velocity can be obtained if we write the wave function ψ in terms of its amplitude f and of a phase φ , $\psi = f \exp(i\varphi)$ (the Madelung representation). In this representation $\rho = f^2$, and the velocity \vec{v} is $\vec{v} = (\hbar/m_\chi) \nabla\varphi$. In the Madelung representation of the wave function, the dynamics of the Bose-Einstein condensate can be formulated in terms of the continuity equation and of the hydrodynamic Euler equations [18].

C. Dynamical time evolution of the Bose-Einstein condensed dark matter halos

In order to study the time evolution of the condensate we apply the variational approach, by assuming a trial form for the wave function ψ of the condensate. We assume that during the motion of the dark matter cloud, the density profile maintains its shape, but that its spatial extent $R = R(t)$ depends on time. The density distribution is determined

by the amplitude of the wave function, while the phase determines its velocity field.

In the following we assume that the velocity is in the radial direction, and it is proportional to r . Translated into the behavior of the wave function, this implies that the phase φ varies as r^2 , since the radial velocity of the condensate is given by $(\hbar/m_\chi) \partial\varphi/\partial r$. We therefore write the phase of the wave function as $\varphi = H(t) m_\chi r^2 / 2\hbar$, where $H(t)$ is a second parameter in the wave function, which determines the velocity of the condensate as

$$\vec{v} = H(t) \vec{r}. \quad (25)$$

Thus, the complete trial wave function is [15,49,50]

$$\psi(r, t) = \frac{A\sqrt{N}}{R^{3/2}(t)} f \left[\frac{r}{R(t)} \right] \exp \left[iH(t) \frac{m_\chi r^2}{2\hbar} \right], \quad (26)$$

where A is a normalization constant, and $f[r/R(t)]$ is an arbitrary function to be determined. The mass density of the condensate is thus given by

$$\begin{aligned} \rho(r, t) &= m_\chi |\psi(r, t)|^2 \\ &= \frac{A^2 N m_\chi}{R^3(t)} f^2 \left[\frac{r}{R(t)} \right] \\ &= \frac{A^2 M}{R^3(t)} f^2 \left[\frac{r}{R(t)} \right], \end{aligned} \quad (27)$$

where $M = Nm_\chi$ is the total mass of the condensate dark matter halo.

We now carry out the integration over r in Eq. (20) for this choice of the wave function, and obtain the Lagrangian of the condensate as a function of the two independent variables $H(t)$ and $R(t)$, as well as of the time derivative of $H(t)$. For the first term in Eq. (20) we obtain

$$\begin{aligned} &\int \frac{i\hbar}{2} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) dV \\ &= -\frac{1}{2} \dot{H}(t) \int m_\chi n(\vec{r}, t) r^2 dV \\ &= -\frac{1}{2} m_{\text{eff}} R^2(t) \dot{H}(t), \end{aligned} \quad (28)$$

where

$$m_{\text{eff}}(R) = Nm_\chi \frac{\langle r^2 \rangle}{R^2(t)}, \quad (29)$$

and

$$\langle r^2 \rangle = \frac{\int n(\vec{r}, t) r^2 d^3\vec{r}}{\int n(\vec{r}, t) d^3\vec{r}} \quad (30)$$

is the mean square radius of the dark matter halo. The total energy of the condensate is obtained as

$$\begin{aligned} E &= \frac{1}{2} m_{\text{eff}}(R) R^2(t) H^2(t) + E_{zp}(R) + E_{\text{grav}}(R) + E_{\text{int}}(R) \\ &= \frac{1}{2} m_{\text{eff}}(R) R^2(t) H^2(t) + U(R), \end{aligned} \quad (31)$$

where

$$U(R) = E_{zp}(R) + E_{\text{grav}}(R) + E_{\text{int}}(R), \quad (32)$$

and

$$E_{zp}(R) = \frac{\hbar^2}{2m_\chi} \int_V \left(\frac{d|\psi|}{dr} \right)^2 d^3\vec{r} = C_{zp} R^{-2}, \quad (33)$$

$$E_{\text{grav}}(R) = \int_V m_\chi V_{\text{grav}} |\psi|^2 d^3\vec{r} = -C_{\text{grav}} R^{-1}, \quad (34)$$

and

$$E_{\text{int}}(R) = \frac{u_0}{2} \int_V |\psi|^4 d^3\vec{r} = C_{\text{int}} R^{-3}, \quad (35)$$

respectively. The coefficients C_{zp} , C_{grav} , and C_{int} , respectively, are defined by Eqs. (33–35), and are given by

$$C_{zp} = \frac{2\pi\hbar^2 A^2 N}{m_\chi} \int_0^1 f'^2(\xi) \xi^2 d\xi, \quad (36)$$

$$C_{\text{grav}} = 4\pi m_\chi A^2 N \int_0^1 f^2(\xi) V_{\text{grav}}(\xi) \xi^2 d\xi, \quad (37)$$

and

$$C_{\text{int}} = 2\pi u_0 A^4 N^2 \int_0^1 f^4(\xi) \xi^2 d\xi, \quad (38)$$

respectively. The coefficient C_{zp} describes the so-called zero-point kinetic energy of the system. Therefore the Lagrangian of the Bose-Einstein condensate dark matter halo takes the form of an effective Lagrangian, which can be written as

$$L_{\text{eff}}(R, H, \dot{H}) = - \left[\frac{1}{2} m_{\text{eff}}(R) R^2 (\dot{H} + H^2) + U(R) \right]. \quad (39)$$

The Lagrange equation for \dot{H} ,

$$\frac{d}{dt} \left[\frac{\partial L_{\text{eff}}(R, H, \dot{H})}{\partial \dot{H}} \right] = \frac{\partial L_{\text{eff}}(R, H, \dot{H})}{\partial H}, \quad (40)$$

gives

$$H = \frac{\dot{R}}{R}, \quad (41)$$

while the Lagrange equation for R , $\partial L_{\text{eff}}(R, H, \dot{H}) / \partial R = 0$, gives the equation of motion of the condensate as

$$\left[\frac{1}{2} \frac{\partial m_{\text{eff}}(R)}{\partial R} R + m_{\text{eff}} \right] R (\dot{H} + H^2) = - \frac{\partial U(R)}{\partial R}. \quad (42)$$

By integrating Eq. (23) over the volume of the condensate, with the use of the Gauss theorem we obtain immediately the following particle conservation number equation:

$$\frac{\partial}{\partial t} N + \int_S \frac{\rho}{m_\chi} \vec{v} \cdot \vec{n} dS = 0, \quad (43)$$

where the total particle number $N = \int_V |\psi(\vec{r}, t)|^2 d^3\vec{r}$ is obtained by integrating the norm of the wave function over the entire spherical volume V of the condensate, with radius R_{BE} . The time variation of N can be due only to the gain or loss of the particles moving in or out of the condensate through the surface S encompassing the volume V . In the case of a static condensate, $\vec{v} \equiv 0$, and $\rho(R_{\text{BE}}) \equiv 0$, and then obviously the particle flux $\vec{j}(R_{\text{BE}}) = \rho(R_{\text{BE}}) \vec{v} \equiv 0$, leading, via Eq. (43), to the conservation of the total particle number in the system, $N = \text{constant}$.

Based on the static case, for a time evolving condensate, we would expect that on its boundary the condition $\rho[R_{\text{BE}}(t)] \equiv 0$ to be satisfied for all times, leading to a zero particle flux through the surface encompassing the time-dependent volume V of the condensate, $\vec{j}[R_{\text{BE}}(t)] = \rho[R_{\text{BE}}(t)] \vec{v}[R_{\text{BE}}(t)] \equiv 0$, $\forall t \geq 0$. Since the Hamiltonian of the Gross-Pitaevskii equation (16) is real, it is obviously Hermitian, and therefore the norm of the wave function, as well as the total particle number, are conserved. Since N is a constant, from Eq. (43) it follows that in the time-dependent case the particle flux through the boundary S of the condensate must be zero, that is, $\vec{j}[R_{\text{BE}}(t)] = \rho[R_{\text{BE}}(t)] \vec{v}[R_{\text{BE}}(t)] \equiv 0$, $\forall t \geq 0$, and there is no particle loss from the system.

IV. DARK MATTER DENSITY, GRAVITATIONAL POTENTIAL AND PHYSICAL PARAMETERS OF TIME-DEPENDENT BOSE-EINSTEIN CONDENSATE DARK MATTER HALOS

As a first step in the study of the time dynamics of the gravitationally bounded Bose-Einstein condensates we have to chose a variational trial wave function. Instead of fixing it in an arbitrary way (by assuming, for example, that the initial density profile of the condensate has a Gaussian form), we require that $|\psi|^2$ satisfies the continuity equation (23). For the density of the Bose-Einstein condensate we assume a general form

$$\rho(r, t) = \rho_0(t) + \rho_1(t)\rho_2(r), \quad (44)$$

where $\rho_0(t)$, $\rho_1(t)$ and $\rho_2(r)$ are arbitrary functions of t and r to be determined. From a physical point of view, the trial density profile $\rho(t)$ is the sum of two terms, the first representing a ‘‘cosmological’’ type homogenous term $\rho_{\text{hom}}(t)$, while the second term $\rho_{\text{inhom}}(t, r)$ represents the effect of the time-dependent inhomogeneities in the dark matter halo. The inhomogeneous term is assumed to be separable in the variables t and r , so that $\rho_{\text{inhom}}(t, r) = \rho_1(t)\rho_2(r)$.

A. The density profile and the gravitational potential of the dark matter halo

Substitution of Eqs. (25) and (44) into the continuity equation (23) gives

$$\begin{aligned} \dot{\rho}_0(t) + 3\rho_0(t)\frac{\dot{R}(t)}{R(t)} + \rho_2(r) \\ \times \left[\dot{\rho}_1(t) + 3\rho_1(t)\frac{\dot{R}(t)}{R(t)} + \frac{\dot{R}(t)}{R(t)}\rho_1(t)r\frac{\rho_2'(r)}{\rho_2(r)} \right] = 0. \end{aligned} \quad (45)$$

We determine the function $\rho_2(r)$ by imposing the condition $r\rho_2'(r)/\rho_2(r) = \text{constant} = \alpha > 0$, which leads first to

$$\rho_2(r) = C_1 r^\alpha, \quad (46)$$

where C_1 is an arbitrary constant of integration. Next we require that the term in the square bracket of Eq. (45) vanishes. Therefore, Eq. (45) gives the following two independent differential equations for the determination of the functions $\rho_0(t)$ and $\rho_1(t)$:

$$\dot{\rho}_0(t) + 3\rho_0(t)\frac{\dot{R}(t)}{R(t)} = 0, \quad (47)$$

and

$$\dot{\rho}_1(t) + (3 + \alpha)\rho_1(t)\frac{\dot{R}(t)}{R(t)} = 0, \quad (48)$$

respectively. Hence, the general solution of Eq. (45) can be obtained as

$$\rho(r, t) = \frac{1}{R^3(t)} \left[a_0 + b_0 \frac{r^\alpha}{R^\alpha(t)} \right], \quad (49)$$

where a_0 and b_0 are arbitrary constants of integration. Since at the vacuum boundary of the condensate, where $r = R(t)$, the density must satisfy the condition $\rho[R(t), t] = 0$, $\forall t \geq 0$, we obtain for the two integration constants the condition $a_0 + b_0 = 0$. Therefore the density profile of the condensate can be represented in the form

$$\rho(r, t) = \frac{a_0}{R^3(t)} \left[1 - \frac{r^\alpha}{R^\alpha(t)} \right]. \quad (50)$$

With this density profile the Poisson equation can be written as

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dV_{\text{grav}}}{dr} \right) = 4\pi G a_0 \frac{1}{R^3(t)} \left[1 - \frac{r^\alpha}{R^\alpha(t)} \right], \quad (51)$$

and can be integrated to give

$$\frac{dV_{\text{grav}}}{dr} = 4\pi G a_0 \frac{1}{R^3(t)} \left[\frac{r}{3} - \frac{r^{\alpha+1}}{(\alpha+3)R^\alpha(t)} \right] + \frac{C_2(t)}{r^2}, \quad (52)$$

where $C_2(t)$ is an arbitrary integration function. In order to avoid any singularities for $r \rightarrow 0$ we take $C_2(t) \equiv 0$. A new integration gives

$$\begin{aligned} V_{\text{grav}}(r, t) = 4\pi G a_0 \frac{1}{R^3(t)} \\ \times \left[\frac{r^2}{6} - \frac{r^{\alpha+2}}{(\alpha+2)(\alpha+3)R^\alpha(t)} \right] + V_{\text{grav}}^{(0)}(t). \end{aligned} \quad (53)$$

At the surface of the dark matter halo $r = R(t)$ the gravitational potential becomes $V_{\text{grav}}[R(t), t] = -GM/R(t)$, where $M = Nm_\chi$ is the total mass of the condensate, which gives

$$V_{\text{grav}}^{(0)}(t) = -\frac{GM + 2\pi G a_0 [\alpha(\alpha+5)/3(\alpha+2)(\alpha+3)]}{R(t)}, \quad (54)$$

leading to

$$\begin{aligned} V_{\text{grav}}(r, t) = \frac{\pi G a_0}{R^3(t)} \left[\frac{2r^2}{3} - \frac{4r^{\alpha+2}}{(\alpha+2)(\alpha+3)R^\alpha(t)} \right] \\ - \frac{GM + 2\pi G a_0 [\alpha(\alpha+5)/3(\alpha+2)(\alpha+3)]}{R(t)}. \end{aligned} \quad (55)$$

Since the integration constant a_0 is arbitrary, we determine it by imposing the condition

$$\pi G a_0 = GM + 4\pi G a_0 [\alpha(\alpha+5)/6(\alpha+2)(\alpha+3)], \quad (56)$$

giving

$$a_0 = \frac{3(\alpha+2)(\alpha+3)}{\pi(\alpha^2 + 5\alpha + 18)} M. \quad (57)$$

From the normalization condition $M = 4\pi \int_0^R \rho r^2 dr$ we obtain

$$a_0 = \frac{3}{4\pi} \frac{\alpha + 3}{\alpha} M. \quad (58)$$

By equating Eqs. (57) and (58) we find for α the value $\alpha = 2$, giving $a_0 = 15M/8\pi$. Therefore the density and the gravitational potential in the Bose-Einstein condensed dark matter halo become

$$\rho(r, t) = \frac{15}{8\pi} \frac{M}{R^3(t)} \left[1 - \frac{r^2}{R^2(t)} \right], \quad (59)$$

and

$$V_{\text{grav}}(r, t) = -\frac{15}{8} \frac{GM}{R} \left[1 - \frac{2}{3} \frac{r^2}{R^2(t)} + \frac{1}{5} \frac{r^4}{R^4(t)} \right], \quad (60)$$

respectively. For the wave function of the condensate we obtain

$$|\psi(r, t)|^2 = \frac{15}{8\pi} \frac{N}{R^3} \left[1 - \frac{r^2}{R^2(t)} \right]. \quad (61)$$

By comparing Eqs. (27) and (59) we obtain the expressions of the trial function $f[r/R(t)]$ and of the normalization constant A as

$$A = \sqrt{\frac{15}{8\pi}}, \quad f\left[\frac{r}{R(t)}\right] = \sqrt{1 - \frac{r^2}{R^2(t)}}. \quad (62)$$

B. The coefficients C_{zp} , C_{grav} and C_{int}

Once the wave function of the system is known, the effective mass m_{eff} and the constant coefficients in Eqs. (36–38) can be obtained immediately. For the effective mass m_{eff} we find

$$m_{\text{eff}} = \frac{3}{7} M = \text{constant}. \quad (63)$$

The coefficient C_{zp} , giving the zero-point kinetic energy of the condensate is given by

$$C_{zp} = \frac{15\hbar^2}{4m_\chi} \frac{M}{m_\chi} F(\xi), \quad (64)$$

where

$$F(\xi) = \int_0^\xi \frac{x^4}{(1-x^2)} dx. \quad (65)$$

In obtaining the zero-point kinetic energy E_{zp} by using the trial function given by Eq. (62), it turns out that the coefficient C_{zp} diverges at the surface of the condensate, where $r \rightarrow R(t)$. The divergence of the kinetic energy near the edge of the condensate cloud is a general feature of the

Bose-Einstein condensates, and it can be understood from simple physical considerations [15]. In the Thomas-Fermi approximation the particle number profile is given by $n(\vec{r}) = [\mu - V_{\text{grav}}(\vec{r})]/u_0$, where μ is the chemical potential of the condensate. If \vec{r}_0 is a point on the surface, then $\mu = V_{\text{grav}}(\vec{r}_0)$ defines the surface of the condensate. By expanding the external potential about \vec{r}_0 we obtain $n(\vec{r}) = \vec{F}_{\text{grav}} \cdot (\vec{r} - \vec{r}_0)/u_0$, where $\vec{F}_{\text{grav}} = -\nabla V_{\text{grav}}(\vec{r}_0)$, while the wave function is given by $\psi(r) = [F_{\text{grav}} \cdot (\vec{r} - \vec{r}_0)/u_0]^{1/2}$ [15,51]. If we denote the coordinate in the direction of $\nabla V(\vec{r}_0)$ by x , and denote the position of the surface as x_0 , the interior of the condensed dark matter halo corresponds to $x \leq x_0$. The wave function near the surface varies as $(x_0 - x)^{1/2}$, and hence the kinetic energy term in the energy functional behaves as $\hbar^2 |d\psi/dx|^2 / 2m_\chi |\psi|^2 \sim \hbar^2 / 2m_\chi (x_0 - x)^2$, and it diverges as x approaches x_0 from below. In fact the kinetic energy term dominates over the potential energy term for $x_0 - x \leq \delta$, where

$$\delta = \left(\frac{\hbar^2}{2m_\chi F_{\text{grav}}} \right)^{1/3}. \quad (66)$$

In order to obtain a better description of the surface properties of the Bose-Einstein condensates one must study the Gross-Pitaevskii equation [15,51]

$$-\left(\frac{\hbar^2}{2m_\chi} \right) \psi''(x) + F_{\text{grav}} x \psi + u_0 |\psi|^2 \psi = 0, \quad (67)$$

describing the surface properties of the condensate. An approximate solution of this equation is $\Psi = \sqrt{-y}$, $y \leq 0$, where $y = x/\delta$, and $\Psi = \psi / \sqrt{F_{\text{grav}} \delta / u_0}$. By using this solution one can evaluate the kinetic energy per unit area perpendicular to the x axis, $\langle p^2 / 2m_\chi \rangle = (\hbar^2 / 2m_\chi) \int |\nabla \psi|^2 dx$. Since this integral diverges for $x \rightarrow 0$, the integral must be evaluated for x less than some cutoff value $-l$, with the lower limit of integration taken as $-L$, where $L \gg \delta$. By choosing the cutoff distance as $-\delta$, with the use of the approximate solution for ψ given above we obtain $\langle p^2 / 2m_\chi \rangle \approx (\hbar^2 / 8m_\chi) (F_{\text{grav}} / u_0) \ln(L/l)$. If one uses the true wave function obtained numerically one finds $\langle p^2 / 2m_\chi \rangle \approx (\hbar^2 / 8m_\chi) (F_{\text{grav}} / u_0) \ln(4.160L/l)$ [15]. Hence in order to calculate the kinetic energy in more general situations an effective cutoff must be used, whose numerical value is of the order of $l = 0.240\delta$ [15,51].

In order to make the integral $F(\xi)$ convergent, ξ must be smaller than 1, $\xi < 1$, and therefore we must introduce a cutoff length ϵ , so that the upper integration limit becomes $\xi = 1 - \epsilon$. Since the force acting on a particle on the surface of the condensate is the gravitational force $F = GMm_\chi / R^2$, by taking into account the previous results we obtain the cutoff length $\epsilon = l/R$ as

$$\begin{aligned} \epsilon &= 0.240 \frac{\delta}{R} = 0.190488 \times \left(\frac{\hbar^2}{Gm_\chi^2 MR} \right)^{1/3} \\ &= 2.653 \times 10^{-16} \times \left(\frac{m_\chi}{10^{-32} \text{g}} \right)^{-2/3} \left(\frac{M}{10^6 M_\odot} \right)^{-1/3} \\ &\quad \times \left(\frac{R}{10 \text{ kpc}} \right)^{-1/3}. \end{aligned} \quad (68)$$

For $\epsilon = 2.653 \times 10^{-16}$ we obtain $F(1-\epsilon) = 17.0350669$, while for $\epsilon = 10^{-10}$ we have $F(1-\epsilon) = 10.5261657$. By assuming that the surface region of the dark matter halo represents around 1% of its spatial extension, we have $\epsilon = 0.01$, and $F(1-\epsilon) = 1.333$. In the following we consider two limiting cases for ϵ , $\epsilon = 0.01$ and $\epsilon = 2.653 \times 10^{-16}$, respectively.

Therefore the coefficients C_{zp} , C_{grav} and C_{int} can be obtained as

$$\begin{aligned} C_{zp} &= \frac{15\hbar^2 M}{4m_\chi m_\chi} F(\xi) \\ &= (1.110 \times 10^{50} - 1.419 \times 10^{51}) \\ &\quad \times \left(\frac{m_\chi}{10^{-32} \text{g}} \right)^{-2} \left(\frac{M}{10^6 M_\odot} \right) \text{g cm}^4/\text{s}^2, \end{aligned} \quad (69)$$

$$C_{\text{grav}} = \frac{10}{7} GM^2 = 3.81 \times 10^{71} \times \left(\frac{M}{10^6 M_\odot} \right)^2 \text{g cm}^3/\text{s}^2, \quad (70)$$

and

$$\begin{aligned} C_{\text{int}} &= \frac{15}{4\pi} u_0 \frac{M^2}{m_\chi^2} = 6.665 \times 10^{114} \\ &\quad \times \left(\frac{a}{10^{-7} \text{cm}} \right) \left(\frac{m_\chi}{10^{-32} \text{g}} \right)^{-3} \left(\frac{M}{10^6 M_\odot} \right)^2 \text{g cm}^5/\text{s}^2, \end{aligned} \quad (71)$$

respectively.

V. GRAVITATIONAL COLLAPSE OF BOSE-EINSTEIN CONDENSATE DARK MATTER HALOS

By taking into account the explicit form of H , given by Eq. (41), and the constancy of the effective mass m_{eff} , the equation of motion of the gravitationally bounded Bose-Einstein condensate is given by

$$m_{\text{eff}} \ddot{R} = \frac{2C_{zp}}{R^3} - \frac{C_{\text{grav}}}{R^2} + \frac{3C_{\text{int}}}{R^4}, \quad (72)$$

Eq. (72) can be rewritten in the form of an energy conservation equation as

$$\frac{1}{2} m_{\text{eff}} \dot{R}^2(t) + U(R) = E_0 = \text{constant}, \quad (73)$$

where

$$U(R) = \frac{C_{zp}}{R^2} - \frac{C_{\text{grav}}}{R} + \frac{C_{\text{int}}}{R^3}. \quad (74)$$

The variation of the potential $U(R)$ as a function of R is represented in Fig. 1.

Let us consider the case of the contraction, that is, the case of a Bose-Einstein dark matter halo beginning to contract at $t = 0$ from an initial radius $R = R_0 = R_{\text{max}}$. Then the initial velocity $\dot{R}(0)$ of the dark matter halo can be obtained from Eq. (73) as

$$\dot{R}(0) = \sqrt{\frac{2}{m_{\text{eff}}} [E_0 - U(R_0)]}. \quad (75)$$

After a time t_{coll} the condensate will eventually reach a stable radius R_{st} , where its velocity is zero, and the potential energy $U(R)$ is minimum, $\partial U/\partial R|_{R=R_{st}} = 0$, so that

$$\left. \frac{2C_{zp}}{R^3} - \frac{C_{\text{grav}}}{R^2} + \frac{3C_{\text{int}}}{R^4} \right|_{R=R_{st}} = 0. \quad (76)$$

From Eq. (76) we obtain the radius of the stable configuration as

$$\begin{aligned} R_{st} &= \frac{C_{zp} + \sqrt{C_{zp}^2 + 3C_{\text{grav}}C_{\text{int}}}}{C_{\text{grav}}} \\ &= \frac{C_{zp}}{C_{\text{grav}}} \left[1 + \sqrt{1 + \frac{3C_{\text{grav}}C_{\text{int}}}{C_{zp}^2}} \right]. \end{aligned} \quad (77)$$

Since $3C_{\text{grav}}C_{\text{int}}/C_{zp}^2 \gg 1$, the equilibrium radius can be obtained, in a very good approximation, as

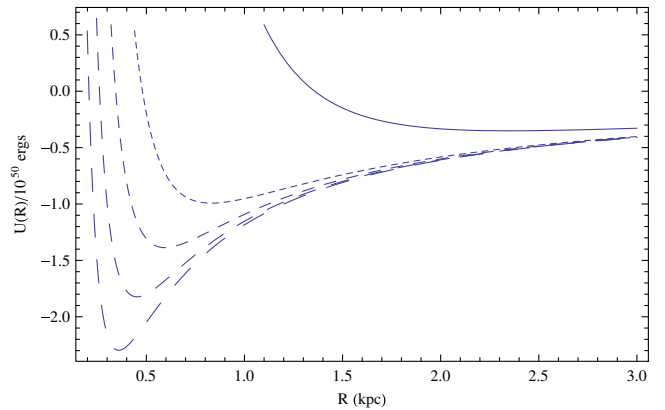


FIG. 1 (color online). Variation of the potential $U(R)$ as a function of R for a Bose-Einstein condensate dark matter halo with mass $M = 10^6 M_\odot$ and scattering length $a = 10^{-7} \text{cm}$. The mass of the condensate dark matter particle is $m = 10^{-32} \text{g}$ (solid curve), $m = 2 \times 10^{-32} \text{g}$ (dotted curve), $m = 2.5 \times 10^{-32} \text{g}$ (short dashed curve), $m = 3 \times 10^{-32} \text{g}$ (long dashed curve), and $m = 3.5 \times 10^{-32} \text{g}$ (ultra long dashed curve), respectively.

$$\begin{aligned}
R_{st} &\approx \sqrt{\frac{3C_{\text{int}}}{C_{\text{grav}}}} = \sqrt{\frac{63 u_0}{8\pi G m_\chi^2}} = \sqrt{\frac{63 \hbar^2 a}{2 G m_\chi^3}} \\
&= 1.106 \times 10^{21} \left(\frac{a}{10^{-7} \text{ cm}} \right) \left(\frac{m_\chi}{10^{-32} \text{ g}} \right)^{-3} \text{ cm}.
\end{aligned} \tag{78}$$

The condition of the zero final velocity of the halo, $\dot{R}(t_{\text{coll}}) = 0$ determines the total energy E_0 as given by

$$E_0 = U(R_{st}) = \frac{C_{zp}}{R_{st}^2} - \frac{C_{\text{grav}}}{R_{st}} + \frac{C_{\text{int}}}{R_{st}^3}. \tag{79}$$

Therefore the initial velocity of the halo $\dot{R}(0)$ is determined as a function of the final radius R_{st} and of the initial radius R_0 by the consistency condition (the conservation of energy),

$$\frac{m_{\text{eff}}}{2} \dot{R}^2(0) = \frac{C_{zp}}{R_{st}^2} - \frac{C_{\text{grav}}}{R_{st}} + \frac{C_{\text{int}}}{R_{st}^3} - \frac{C_{zp}}{R_0^2} + \frac{C_{\text{grav}}}{R_0} - \frac{C_{\text{int}}}{R_0^3}. \tag{80}$$

The general solution of Eq. (73) is given by

$$t - t_0 = \pm \sqrt{\frac{m_{\text{eff}}}{2}} \int_{R_0}^R \frac{dR}{\sqrt{E_0 - U(R)}}, \tag{81}$$

where + and – corresponds to the case of expansion and contraction, respectively. The variation of the collapse time as a function of radius is presented, for $R_0 = 20$ kpc and $t_0 = 0$, in Fig. 2.

Since $E_0 > U(R)$, $\forall R < R_0$, the function $1/\sqrt{E_0 - U(R)} = 1/\sqrt{E_0[1 - U(R)/E_0]}$ can be approximated in the first order as

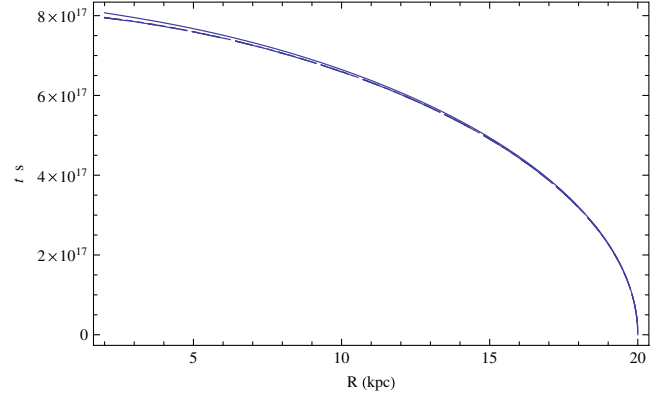


FIG. 2 (color online). Collapse time t as a function of R for a Bose-Einstein condensate dark matter halo with mass $M = 10^6 M_\odot$. The mass of the condensate dark matter particle is in the range $m_\chi \in (10^{-32} \text{ g}, 3.5 \times 10^{-32} \text{ g})$, and the scattering length is taken as $a = 10^{-7} \text{ cm}$.

$$\begin{aligned}
\frac{1}{\sqrt{E_0 - U(R)}} &\approx \frac{1}{\sqrt{E_0}} \left[1 + \frac{U(R)}{2E_0} \right] \\
&= \frac{1}{\sqrt{E_0}} \left(1 + \frac{C_{zp}}{2E_0} \frac{1}{R^2} - \frac{C_{\text{grav}}}{2E_0} \frac{1}{R} + \frac{C_{\text{int}}}{2E_0} \frac{1}{R^3} \right).
\end{aligned} \tag{82}$$

Therefore, within the framework of this approximation, the total collapse time,

$$t_{\text{coll}} = t(R_{st}) - t_0 = \sqrt{\frac{m_{\text{eff}}}{2}} \int_{R_0}^{R_{st}} \frac{dR}{\sqrt{E_0 - U(R)}}, \tag{83}$$

of the condensed dark matter halo between an initial radius R_0 and the radius of the stable configuration R_{st} is given by

$$t_{\text{coll}} \approx \frac{\sqrt{m_{\text{eff}}}}{2\sqrt{2}\sqrt{E_0}R_0^2R_{st}^2} \left\{ 2C_{\text{grav}}^{(0)}R_0^2R_{st}^2 \ln(R_{st}/R_0) + (R_0 - R_{st})[C_{\text{int}}^{(0)}(R_0 + R_{st}) + 2R_0R_{st}(C_{zp}^{(0)} + R_0R_{st})] \right\}, \tag{84}$$

where we have denoted $C_{zp}^{(0)} = C_{zp}/2E_0$, $C_{\text{grav}}^{(0)} = C_{\text{grav}}/2E_0$, and $C_{\text{int}}^{(0)} = C_{\text{int}}/2E_0$, respectively. By using Eq. (78) to express the radius of the final stable configuration, we obtain for the collapse time the expression

$$\begin{aligned}
t_{\text{coll}} &\approx \frac{\sqrt{m_{\text{eff}}}}{6\sqrt{2}C_{\text{int}}^{(0)}\sqrt{E_0}R_0^2} \times \left\{ 2\sqrt{3}C_{zp}^{(0)}R_0^2\sqrt{C_{\text{grav}}^{(0)}C_{\text{int}}^{(0)}} + C_{\text{int}}^{(0)}R_0 \left[R_0 \left(C_{\text{grav}}^{(0)} + 6R_0 - 6\sqrt{3}\sqrt{\frac{C_{\text{int}}^{(0)}}{C_{\text{grav}}^{(0)}}} \right) - 6C_{zp}^{(0)} \right] \right. \\
&\quad \left. + 3C_{\text{grav}}^{(0)}C_{\text{int}}^{(0)}R_0^2 \ln\left(\frac{3C_{\text{int}}^{(0)}}{C_{\text{grav}}^{(0)}R_0^2}\right) - 3(C_{\text{int}}^{(0)})^2 \right\}.
\end{aligned} \tag{85}$$

If the initial size of the condensed dark matter cloud R_0 is very high, so that the condition $E_0 = C_{zp}/R_0^2 - C_{\text{grav}}/R_0 + C_{\text{int}}/R_0^3 \approx 0$ is satisfied, and by assuming $C_{zp} = 0$, the collapse time of the condensate can be expressed in an exact form as

$$t_{\text{coll}} \approx \frac{\sqrt{2m_{\text{eff}}}}{3C_{\text{grav}}^3} \times \left\{ \frac{\sqrt{R_{st}} \left[(C_{\text{grav}}^3 C_{\text{int}})^{3/4} \sqrt{R_{st} - \frac{C_{\text{int}}}{C_{\text{grav}} R_{st}}} F \left(\sin^{-1} \left(\frac{\sqrt{\frac{C_{\text{int}}}{C_{\text{grav}}}}}{\sqrt{R_{st}}} \right) \middle| -1 \right) + C_{\text{grav}}^2 (C_{\text{int}} - C_{\text{grav}}^2 R_{st}) \right]}{\sqrt{C_{\text{grav}}^2 R_{st}^2 - C_{\text{int}}}} \right. \\ \left. - \frac{\sqrt{R_0} \left[(C_{\text{grav}}^3 C_{\text{int}})^{3/4} \sqrt{R_0 - \frac{C_{\text{int}}}{C_{\text{grav}} R_0}} F \left(\sin^{-1} \left(\frac{\sqrt{\frac{C_{\text{int}}}{C_{\text{grav}}}}}{\sqrt{R_0}} \right) \middle| -1 \right) + C_{\text{grav}}^2 (C_{\text{int}} - C_{\text{grav}}^2 R_0) \right]}{\sqrt{C_{\text{grav}}^2 R_0^2 - C_{\text{int}}}} \right\}, \quad (86)$$

where $F(\phi|m)$ denotes the elliptic integral of the first kind, $F(\phi|m) = \int_0^\phi (1 - m \sin^2 \theta)^{-1/2} d\theta$. By using Eq. (78) for the equilibrium radius, the collapse time can be written as

$$t_{\text{coll}} \approx \frac{\sqrt{2m_{\text{eff}} C_{\text{int}}^{3/4}}}{3C_{\text{grav}}^{5/4}} \left\{ C_{\text{grav}} \sqrt{\frac{R_0}{C_{\text{int}}^{3/2}}} \sqrt{C_{\text{grav}} R_0^2 - C_{\text{int}}} \right. \\ \left. - C_{\text{int}}^{5/4} [\sqrt{2}\sqrt{3} - F(\text{csc}^{-1}(\sqrt[3]{3}) \middle| -1)] \right. \\ \left. - C_{\text{int}}^{5/4} F \left(\sin^{-1} \left(\frac{\sqrt{\frac{C_{\text{int}}}{C_{\text{grav}}}}}{\sqrt{R_0}} \right) \middle| -1 \right) \right\}. \quad (87)$$

The numerical values of the elliptic integral of the first kind are of the order of unity. By neglecting the numerical factors of the order of unity, and by considering that the initial radius R_0 of the dark matter halo is enough high so that the condition $C_{\text{grav}} R_0^2 \gg C_{\text{int}}$ is satisfied, the collapse time can be obtained in the simple form of

$$t_{\text{coll}} \approx \frac{\sqrt{2m_{\text{eff}}}}{3} C_{\text{grav}}^{-1/2} R_0^{3/2}, \quad E_0 \approx 0, R_0 \gg \sqrt{\frac{C_{\text{int}}}{C_{\text{grav}}}}. \quad (88)$$

With the use of the expression of C_{grav} we obtain

$$t_{\text{coll}} \approx \frac{1}{3} \sqrt{\frac{3}{5}} \frac{R_0^{3/2}}{\sqrt{GM}} = 3.821 \times 10^{18} \times \left(\frac{R_0}{100 \text{ kpc}} \right)^{3/2} \\ \times \left(\frac{M}{10^6 M_\odot} \right)^{-1/2}, \quad E_0 \approx 0, \\ R_0 \gg \left(\frac{a}{10^{-7} \text{ cm}} \right) \left(\frac{m_\chi}{10^{-32} \text{ g}} \right)^{-3} \text{ kpc}. \quad (89)$$

In this approximation the collapse time of the dark matter halo is determined only by the gravitational properties of the system, and it is independent of the zero-point kinetic energy and the interaction energy.

The total mass M_{st} of the stable condensate configuration with radius R_{st} , at the end of the collapse process, can be approximated as

$$M_{st} = \frac{4}{\pi} R_{st}^3 \rho_c \approx \frac{108}{\pi} \left(\frac{7}{2} \right)^{3/2} \frac{\hbar^3}{m_\chi^{9/2}} \left(\frac{a}{G} \right)^{3/2} \rho_c \\ = 2.419 \times 10^8 \times \left(\frac{m_\chi}{10^{-32} \text{ g}} \right)^{-9/2} \left(\frac{a}{10^{-7} \text{ cm}} \right)^{3/2} \\ \times \left(\frac{\rho_c}{10^{-24} \text{ g/cm}^3} \right) M_\odot, \quad (90)$$

where ρ_c is the central density of the dark matter halo.

At the equilibrium point $R = R_{st}$ we can easily obtain the relations

$$R \frac{dU}{dR} \Big|_{R=R_{st}} = 2E_{zp}(R_{st}) + E_{\text{grav}}(R_{st}) + 3E_{\text{int}}(R_{st}) = 0, \quad (91)$$

and

$$R^2 \frac{d^2U}{dR^2} \Big|_{R=R_{st}} = 6E_{zp}(R_{st}) + 2E_{\text{grav}}(R_{st}) + 12E_{\text{int}}(R_{st}), \quad (92)$$

respectively.

VI. THE STABILITY OF THE COLLAPSED BOSE-EINSTEIN CONDENSATE DARK MATTER HALOS

For condensed dark matter halos with masses of the order of $M = 10^8 M_\odot$ and radii of the order of $R = 10 \text{ kpc}$, the ‘‘relativity parameter’’ $2GM/c^2 R \approx 10^{-9} \ll 1$, which shows that in this range of the physical parameters general relativistic effects can be neglected, and the dynamics of the halos can be described with a very good approximation in the framework of Newtonian physics. Hence stable configurations are possible, and the halo would not collapse to a black hole, ending in a singular state. The collapse ends when the interaction energy equals the gravitational energy,

$$|E_{\text{grav}}| = E_{\text{int}}, \quad (93)$$

a condition which allows the approximate determination of the radius of the stable configuration as $R_{st} = \sqrt{C_{\text{int}}/C_{\text{grav}}}$, a result which is consistent with the radius of the static configuration given by Eq. (78) obtained by solving the full set of dynamical evolution equations. In the case of the gravitationally confined Bose-Einstein condensates the interaction energy plays the same role as the electron and neutron quantum degenerate pressures in the cases of the white dwarfs and of the neutron stars, and it supports the dark matter halo against gravitational collapse.

The total energy E of the collapsed Bose-Einstein condensates dark matter halo can be written as $E = E_{zp} + E_{\text{int}} + E_{\text{grav}}$, where E_{zp} , E_{int} and E_{grav} are the zero-point kinetic energy, the interaction energy, and the gravitational energy, respectively. In the following we will consider an approximate estimate of the energy, by following the approach considered in [15]. The kinetic energy per particle is $\hbar^2/2m_\chi R_{st}^2$, and therefore the total kinetic energy of the system is given by $E_{kin} = N\hbar^2/2m_\chi R_{st}^2$, where N is the total particle number. The interaction energy can be obtained as $E_{\text{int}} = (1/2)(N^2/V)mu_0$, where V is the volume of the condensate, while the gravitational potential energy is $E_{\text{grav}} = -GM^2/R_{st}$ (for simplicity we neglect in the following the factor 3/4 in the expression of the gravitational potential energy). Therefore the total energy of the condensate is given by

$$E = N \frac{\hbar^2}{2m_\chi R_{st}^2} + \frac{3}{2} N^2 \frac{\hbar^2 a}{m_\chi R_{st}^3} - \frac{GM^2}{R_{st}}. \quad (94)$$

By taking into account the explicit expressions of the radius and mass of the static condensate, given by Eqs. (3) and (4), respectively, we obtain for the total energy the expression

$$E = \frac{2\sqrt{a}\hbar^3 [4\pi(3-2\pi^2)a^2\hbar^2\rho_c + Gm_\chi^4]}{\sqrt{G^3 m_\chi^{15}}} \rho_c. \quad (95)$$

For $a = 0$ we have $E = 0$, but this case is not relevant for the study of the condensate dark matter halos. If the parameters of the BEC dark matter halo satisfy the constraint

$$210.351 \frac{a^2 \hbar^2}{Gm_\chi^4} \rho_c > 1, \quad (96)$$

then the total energy E of the halo satisfies the stability condition $E < 0$. The condition given by Eq. (96) can be reformulated as

$$3.50 \times 10^{45} \times \frac{(a/10^{-7} \text{ cm})^2}{(m_\chi/10^{-32} \text{ g})^4} \times \frac{\rho_c}{10^{-24} \text{ g/cm}^3} > 1, \quad (97)$$

and it is obvious that it is satisfied by most of the realistic BEC dark matter halo models.

A. The frequency of the small oscillations about the equilibrium state

Equations (91) and (92) allow the study of the frequency of the small oscillations about the equilibrium state of the condensed dark matter halos. Expanding the potential $U(R)$ to second order in $R - R_{st}$ we find

$$U(R) = U(R_{st}) + \frac{1}{2} K_{\text{eff}} (R - R_{st})^2, \quad (98)$$

where $K_{\text{eff}} = U''(R_{st})$. The equation of motion of the perturbed dark matter halo is

$$m_{\text{eff}} \ddot{R} + K_{\text{eff}} (R - R_{st}) = 0, \quad (99)$$

and therefore the frequency ω of the small oscillations is obtained as

$$\omega^2 = \frac{K_{\text{eff}}}{m_{\text{eff}}} = \frac{R_{st}^2 U''(R_{st})}{R_{st}^2 m_{\text{eff}}} = \frac{3E_{\text{int}}(R_{st}) - E_{\text{grav}}(R_{st})}{R_{st}^2 m_{\text{eff}}}. \quad (100)$$

With the use of Eq. (78) for the equilibrium radius, and of the expression of m_{eff} , the oscillation frequency can be expressed as

$$\omega^2 = \frac{14C_{\text{grav}}^{5/2}}{9\sqrt{3}C_{\text{int}}^{3/2}M} > 0, \quad (101)$$

or

$$\omega^2 = \sqrt{\frac{2}{7}} \frac{320}{567} \pi^{3/2} G^{5/2} \frac{m_\chi^3}{u_0^{3/2}} M = \sqrt{\frac{2}{7}} \frac{40}{567} \sqrt{\frac{G^5 m_\chi^9}{a^3 \hbar^6}} M > 0. \quad (102)$$

Equivalently, we obtain

$$\omega = 4.837 \times 10^{-17} \times \left(\frac{m_\chi}{10^{-32} \text{ g}} \right)^{9/4} \left(\frac{a}{10^{-7} \text{ cm}} \right)^{-3/4} \times \left(\frac{M}{10^6 M_\odot} \right)^{1/2} \text{ s}^{-1}. \quad (103)$$

VII. Discussions and final remarks

In this paper we have analyzed a simple model for the collapse of BEC dark matter halos, based on the dynamical properties of the Gross-Pitaevskii equation. The present model does not include damping nor a microscopic mechanism for particle ejection. The rotational effects have also been ignored, as well as the possible presence of

vortices in the condensate. We also do not include any decay mechanism of the vortices (or turbulence) during the collapse, since these effects can be neglected because of the rapid increase of the density. Starting from the general description of a time-dependent Bose-Einstein gravitationally confined condensate, we focus our attention on a simple limiting case, in order to obtain some intuitive understanding of the physical properties of the gravitational collapse of the dark matter halos, and the formation of stable astrophysical systems.

In order to obtain a simple mathematical description of the collapse process we have used a variational approach, in which a trial form of the condensate wave function was adopted, with all the physical parameters of the dark matter halos assumed to have a $r/R(t)$ dependence. With the help of the trial wave function, the equation of motion and dynamic properties of the time-dependent condensate can be obtained from an effective time-dependent Lagrangian, which describes the time evolution of the condensate radius R . If the condensate wave function depends on one or more parameters, the resulting Lagrangian functional yields approximate Lagrangian equations of motion for these parameters. With the help of the trial wave function one minimizes the action with respect to the free parameter (the Rayleigh-Ritz method). The choice of the trial wave function is not unique, and different choices may lead to different results. The precision of the method depends on the number of free trial parameters, and on how physically realistic the trial function is. The continuity and Poisson equations can be solved exactly, and the density (square of the wave function) and the gravitational potential of the dark matter halo can be explicitly obtained in an analytical form, thus allowing a complete description of the dynamical evolution of the condensate. The motion of the collapsing dark matter halos can be described as the motion of a single point particle with mass m_{eff} in the force field generated by the effective potential $U(R)$, which incorporates the effects of the zero-point kinetic energy E_{zp} , of the interaction energy of the condensate particles E_{int} , and of the gravitational energy E_{grav} . The variational procedure allows us to express these energies as a function of the radius $R(t)$ only. However, their explicit functional form also depends on the scattering length a , the mass of the dark matter particles, and the total mass of the dark matter halo. The trial wave function depends on a single parameter $R(t)$, the time-dependent radius of the condensate. The time-dependent density profile, as well as the gravitational potential, are similar to the first order approximations of the static density and gravitational potential profiles, given by Eqs. (5) and (7), respectively, with the static radius of the condensate R_{BE} substituted by the time-dependent radius $R(t)$, and with a time-dependent central density $\rho_{\text{BE}}^{(c)}(t)$. The variational procedure used can be extended and significantly improved by the choice of a trial wave function depending on several physical parameters. The adopted

approach allows a complete exact analytical treatment of the gravitational collapse of the condensed dark matter halos. Other choices of the trial wave function, or the increase of the number of free parameters would require the extensive use of numerical methods for the integration of the evolution equations. The expected error in this variational approach may be of the order of a few percent, when compared to the exact numerical solution.

The study of the equation of motion of the collapsing condensate shows that the collapse process ends with the formation of a stable configuration, with radius R_{st} and mass M_{st} . The resulting configuration can be of stellar nature, or of galactic nature, depending on the physical processes and the initial mass of the dark matter halo. During the cosmological evolution such a collapse process could have played an important role in the formation of the galactic structure, and of the dark matter halos. On the other hand local perturbations of the condensed dark matter could lead to the formation of smaller mass condensate stars. At the end of the collapse the density distribution of the formed stable static structure is given by

$$\rho_{\text{BE}}(r) = \frac{15M_{st}}{8\pi} \frac{1}{R_{st}^3} \left(1 - \frac{r^2}{R_{st}^2}\right) = \rho_{\text{BE}}^{(c)} \left(1 - \frac{r^2}{R_{st}^2}\right), \quad (104)$$

where the central density $\rho_{\text{BE}}^{(c)}$ of the condensate is given by

$$\begin{aligned} \rho_{\text{BE}}^{(c)} &= \frac{15}{8\pi} \left(\frac{2}{63}\right)^{3/2} \frac{G^{3/2} m_{\chi}^{9/2}}{\hbar^3 a^{3/2}} M_{st} = 3.141 \times 10^{-27} \\ &\times \left(\frac{m_{\chi}}{10^{-32} \text{ g}}\right)^{9/2} \left(\frac{a}{10^{-7} \text{ cm}}\right)^{-3/2} \left(\frac{M_{st}}{10^6 M_{\odot}}\right) \text{ g/cm}^3. \end{aligned} \quad (105)$$

Equation (105) gives the mass-central density relation for stable gravitationally confined condensed astrophysical objects. This central density is of the same order of magnitude as the central dark matter density of galactic dark matter halos. On the other hand, Eq. (104) is consistent with Eq. (5), which gives an approximate representation of the density profile of the static density profile of the condensed Bose-Einstein dark matter halos. In the first order approximation the density profile of the static condensate is $\rho_{\text{BE}}(r) \approx \rho_{\text{BE}}^{(c)} [1 - (\pi^2/6)(r^2/R_{\text{BE}}^2)]$. With the start of the collapse, the radius of the halo becomes time dependent, $R_{\text{BE}} \rightarrow R(t)$, while the central density of the initial static halo changes in time as $\rho_{\text{BE}}^{(c)} \rightarrow M/R^3(t)$. This shows that the choice of the trial function for the time-dependent case is consistent with the static case.

An interesting question is the possibility of formation of dark dense Bose-Einstein condensed stars, having astrophysical properties (mass and radius) similar to those of the standard neutron stars. The radius and the mass of the dark star are determined by the mass of the dark matter particle, and by the scattering length. The radius of a neutron star is

of the order of $R_{\text{NS}} \approx 10^6$ cm. A scaling of the mass and of the scattering length of the form $a = \alpha \times 10^{-7}$ cm, $m_\chi = \beta \times 10^{-32}$ g will give a radius of the same order of magnitude as the radius of a neutron star if the coefficients α and β satisfy the condition $\alpha/\beta^3 = 10^{-15}$. For $\alpha = 10^{-12}$, giving $a = 10^{-19}$ cm, we obtain $\beta = 10$, which implies a dark matter particle mass of $m_\chi \approx 10^{-31}$ g ≈ 55 eV. On the other hand, for realistic dark matter densities the corresponding mass of the star will exceed the general relativistic stability limit. Hence the realistic description of the dark stars requires the inclusion of general relativistic effects in the study of their structure [31].

The details of the collapse of the Bose-Einstein condensate, as well as the numerical values of the physical parameters of the resulting stable configuration are strongly dependent on the numerical values of the two parameters describing the physical properties of the condensate, the dark matter particle mass m_χ , and the scattering length a . The numerical values of these physical quantities are poorly known. We have discussed a number of observational constraints (galactic radii and Bullet Cluster data) that provide some limits for m_χ and a . Within the framework of the Bose-Einstein condensed dark matter model these astrophysical constraints point towards a dark matter particle with mass in the range of meV to a few eV,

and a scattering length of the order of 10^{-19} cm. However, in the present paper most of the numerical results are normalized for a scattering length of 10^{-7} cm, and a mass of the dark matter particle of the order of 10^{-32} g. By a simple scaling all the numerical values corresponding to other choices of m_χ and a can be obtained easily. We have also considered the stability properties of the stable dark matter halos with respect to small oscillations, and the oscillations frequencies of the halos have also been obtained. These results show that the stable configuration, formed from the collapse of the condensed dark matter halos are stable with respect to small perturbations.

A large number of astrophysical observations, including the flat galactic cores, or the constant density surfaces points towards the possibility that dark matter may exist in the Universe in the form of a Bose-Einstein condensate, and that this possibility cannot be excluded *a priori*. The confirmation of this hypothesis by further observations on both galactic and cosmological scales would lead to a major change in our understanding of the basic principles of cosmology and astrophysics. In the present paper we have developed some theoretical tools that can help in the better understanding of the structure formation in the presence of condensed dark matter.

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