Dissipative collapse of axially symmetric, general relativistic sources: A general framework and some applications

L. Herrera[*] and A. Di Prisco[†]

Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas 1050, Venezuela

J. Ibáñez[‡]

Departamento de Física Teórica e Historia de la Ciencia, Universidad del País Vasco, Bilbao 48993, Spain

J. Ospino[§]

Departamento de Matemática Aplicada, Universidad de Salamanca, Salamanca 37007, Spain
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We carry out a general study on the collapse of axially (and reflection-)symmetric sources in the context of general relativity. All basic equations and concepts required to perform such a general study are deployed. These equations are written down for a general anisotropic dissipative fluid. The proposed approach allows for analytical studies as well as for numerical applications. A causal transport equation derived from the Israel-Stewart theory is applied, to discuss some thermodynamic aspects of the problem. A set of scalar functions (the structure scalars) derived from the orthogonal splitting of the Riemann tensor are calculated and their role in the dynamics of the source is clearly exhibited. The characterization of the gravitational radiation emitted by the source is discussed.

I. INTRODUCTION

In a recent paper we have presented a general framework for studying axially symmetric static sources [1]. The physical arguments supporting the study of axially symmetric sources were clearly exposed there; accordingly, we shall not repeat them here.

We intend in this work to extend the above-mentioned study to the fully dynamic case. The reasons to undertake such an endeavor are easy to understand.

Indeed, the static (and quasistatic) approximation is very sensible because the hydrostatic time scale is very small for many phases of the life of a star. Thus, it is of the order of 27 minutes for the sun, 4.5 seconds for a white dwarf and 10^{-4} seconds for a neutron star of 1 solar mass and a radius of 10 km [2–4]. However, during their evolution, self-gravitating objects may pass through phases of intense dynamical activity, with time scales of the order of magnitude of (or even smaller than) the hydrostatic time scale, and for which the static (quasistatic) approximation is clearly not reliable, e.g., the collapse of very massive stars [5], and the quick collapse phase preceding neutron star formation; see for example Ref. [6] and references therein. In these cases it is mandatory to take into account terms which describe departure from equilibrium, i.e. a full dynamic description has to be used.

Analytical approaches to describe the evolution of axially (and reflection-)symmetric self-gravitating fluids have been proposed before [7–9]. However in these latter references only perfect fluids were considered, and furthermore the source was described in Bondi null coordinates [10,11]. However, the perfect-fluid condition seems to be a too stringent restriction for axially symmetric sources, even in the static case [1,12]. On the other hand, Bondi coordinates are known to be very useful for the treatment of gravitational radiation in vacuum, but are not particularly suitable within the source. An analytical approach, which shares some similarities with ours, although restricted to the perfect-fluid case, may be found in Ref. [13].

Therefore, here, we propose a $1+3$ approach, and the source under consideration is as general as possible. It includes all nonvanishing stresses compatible with the symmetry of the problem, as well as dissipative phenomena.

The relevance of dissipative processes in the study of gravitational collapse cannot be overemphasized. Indeed, dissipation due to the emission of massless particles (photons and/or neutrinos) is a characteristic process in the evolution of massive stars. In fact, it seems that the only plausible mechanism to carry away the bulk of the binding energy of the collapsing star, leading to a neutron star or black hole, is neutrino emission [14].

We shall describe dissipation in the diffusion approximation. This assumption is in general very sensible, since the mean free path of particles responsible for the

[*]lherrera@usal.es
[†]adiprisc@fisica.ciens.ucv.ve
[‡]j.ibanez@ehu.es
[§]jhozcrae@usal.es

propagation of energy in stellar interiors is in general very small as compared with the typical length of the object. Thus, for a main sequence star such as the sun, the mean free path of photons at the center, is of the order of 2 cm. Also, the mean free path of trapped neutrinos in compact cores with densities of about 10^{12} g cm^{-3} becomes smaller than the size of the stellar core [15,16].

Furthermore, the observational data collected from supernova 1987 A indicates that the regime of radiation transport prevailing during the emission process, is closer to the diffusion approximation than to the streaming-out limit [17].

On the other hand, the inclusion of pressure anisotropy is based on the fact that the local anisotropy of pressure may be caused by a large variety of physical phenomena, of the kind we expect in compact objects (see Refs. [18–25] and references therein for an extensive discussion on this point).

Among all possible sources of anisotropy, there are two that are particularly related to our primary interest. The first one is the intense magnetic field observed in compact objects, such as white dwarfs, neutron stars, or magnetized strange-quark stars (see, for example, Refs. [26–30] and references therein). Indeed, it is a well-established fact that a magnetic field acting on a Fermi gas produces pressure anisotropy (see Refs. [31–34] and references therein). In some way, the magnetic field can be addressed as a fluid anisotropy.

Another source of anisotropy expected to be present in neutron stars and, in general, in highly dense matter, is the viscosity (see Refs. [35–42] and references therein).

To carry out the program sketched above, we shall apply the $1 + 3$ formalism developed in Refs. [43–51] (not to confound with the $3 + 1$ formalism used in numerical general relativity), coupled to the Israel-Stewart transport equation, within the context of axial symmetry. However, in spite of its advantages (e.g. coordinate independence and completeness [47]), we shall not follow here a frame formalism but rather a coordinate basis approach in which the orthonormal frame is only used to identify frame components of proper vectors as scalars that can have a covariant interpretation. The reason for proceeding in this way is not related to any specific advantage of our approach, with respect to the tetrad formalism, but rather by the simple fact that having been working in the past, with the former [52,53], we are more familiar with it.

Besides the great complexity of the equations, the setup of the presented framework, faces another important challenge, namely the fact that the source should emit gravitational radiation. Indeed, the gravitational collapse even if only slightly aspherical, will lead to copious gravitational-wave emission [54]. This implies (for the case of bounded sources) that the exterior spacetime should in principle describe such a radiation. However, as is well known, no exact solution, describing gravitational radiation from bounded sources, is available in closed analytical

form. The best we have is perhaps the Bondi approach [10,11] which provides expressions for the metric functions in terms of inverse power series of the null coordinate, and whose convergence is only assured very far from the source. In other words, there is not any explicit exterior metric, to which we could match our interior fluid distribution (in the most general case). In spite of this drawback, we shall be able to provide a formal characterization for the emitted (gravitational) radiation within the source, together with the flow of super-energy associated to the vorticity of the fluid.

An important role in this study is played by a set of scalar functions known as structure scalars. These are obtained from the orthogonal splitting of the Riemann tensor [52]. They have been shown to be related to fundamental properties of the fluid distribution [53,55–61]. We shall calculate them for our problem. There will be 12 of them, in contrast with the cylindrically symmetric case which is characterized by eight [53] or the spherically symmetric case, where there is only five [52]. We shall relate them to specific physical aspects of the source, and we shall write down for them the relevant equations. A systematic, though nonexhaustive, study of these equations is carried out.

Dissipative processes will be treated by means of a causal transport equation derived from the Israel-Stewart theory [62–65]. This allows for discussing some interesting thermodynamic aspects of the problem. Also, its coupling with the generalized "Euler" equation, will illustrate the decreasing of the effective inertial mass density, due to thermal effects, and which may lead to the occurrence of the thermoinertial bounce. These effects have already been discussed, in spherically and cylindrically symmetric systems (see Refs. [53,66–74] and references therein).

Finally, in the last section, the results will be summarized and a list of issues deserving further attention will be presented.

II. THE METRIC AND THE SOURCE: BASIC DEFINITIONS AND NOTATION

We shall consider, axially (and reflection-)symmetric sources. For such a system the most general line element may be written in "Weyl spherical coordinates" as

$$ds^2 = -A^2 dt^2 + B^2(dr^2 + r^2 d\theta^2) + C^2 d\phi^2 + 2G d\theta dt,$$
(1)

where A, B, C, G are positive functions of t, r and θ. We number the coordinates $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$.

We shall assume that our source is filled with an anisotropic and dissipative fluid. We are concerned with either bounded or unbounded configurations. In the former case we should further assume that the fluid is bounded by a timelike surface Σ, and junction (Darmois) conditions should be imposed there.

The energy-momentum tensor may be written in the "canonical" form, as

$$T_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + P g_{\alpha\beta} + \Pi_{\alpha\beta} + q_\alpha V_\beta + q_\beta V_\alpha. \quad (2)$$

The above is the canonical, algebraic decomposition of a second-order symmetric tensor with respect to a unit timelike vector, which has the standard physical meaning when $T_{\alpha\beta}$ is the energy-momentum tensor describing some energy distribution, and V^μ is the four-velocity assigned by a certain observer.

With the above definitions it is clear that μ is the energy density (the eigenvalue of $T_{\alpha\beta}$ for the eigenvector V^α), and q_α is the heat flux, whereas P is the isotropic pressure, and $\Pi_{\alpha\beta}$ is the anisotropic tensor. We emphasize that we are considering an Eckart frame where fluid elements are at rest.

Thus, it is immediate to see that

$$\mu = T_{\alpha\beta} V^\alpha V^\beta, \qquad q_\alpha = -\mu V_\alpha - T_{\alpha\beta} V^\beta, \quad (3)$$

$$P = \frac{1}{3} h^{\alpha\beta} T_{\alpha\beta}, \qquad \Pi_{\alpha\beta} = h_\alpha^\mu h_\beta^\nu (T_{\mu\nu} - P h_{\mu\nu}), \quad (4)$$

with $h_{\mu\nu} = g_{\mu\nu} + V_\nu V_\mu$.

Since, we choose the fluid to be comoving in our coordinates, then

$$V^\alpha = \left(\frac{1}{A}, 0, 0, 0\right), \qquad V_\alpha = \left(-A, 0, \frac{G}{A}, 0\right). \quad (5)$$

Next, let us introduce the unit spacelike vectors $\mathbf{K}, \mathbf{L}, \mathbf{S}$, with components

$$K_\alpha = (0, B, 0, 0), \qquad L_\alpha = \left(0, 0, \frac{\sqrt{A^2 B^2 r^2 + G^2}}{A}, 0\right), \quad (6)$$

$$S_\alpha = (0, 0, 0, C), \quad (7)$$

satisfying the following relations:

$$V_\alpha V^\alpha = -K^\alpha K_\alpha = -L^\alpha L_\alpha = -S^\alpha S_\alpha = -1, \quad (8)$$

$$V_\alpha K^\alpha = V^\alpha L_\alpha = V^\alpha S_\alpha = K^\alpha L_\alpha = K^\alpha S_\alpha = S^\alpha L_\alpha = 0. \quad (9)$$

The unitary vectors $V^\alpha, L^\alpha, S^\alpha, K^\alpha$ form a canonical orthonormal tetrad (say $e_\alpha^{(a)}$), such that

$$e_\alpha^{(0)} = V_\alpha, \qquad e_\alpha^{(1)} = K_\alpha, \qquad e_\alpha^{(2)} = L_\alpha, \qquad e_\alpha^{(3)} = S_\alpha,$$

with $a = 0, 1, 2, 3$ (with latin indices labeling different vectors of the tetrad). The dual vector tetrad $e^\alpha_{(a)}$ is easily computed from the condition

$$\eta_{(a)(b)} = g_{\alpha\beta} e^\alpha_{(a)} e^\beta_{(b)}.$$

The anisotropic tensor may be expressed in the form

$$\Pi_{\alpha\beta} = \frac{1}{3}(2\Pi_I + \Pi_{II})\left(K_\alpha K_\beta - \frac{h_{\alpha\beta}}{3}\right)$$
$$+ \frac{1}{3}(2\Pi_{II} + \Pi_I)\left(L_\alpha L_\beta - \frac{h_{\alpha\beta}}{3}\right) + 2\Pi_{KL} K_{(\alpha} L_{\beta)}, \quad (10)$$

with

$$\Pi_{KL} = K^\alpha L^\beta T_{\alpha\beta}, \quad (11)$$

$$\Pi_I = (2K^\alpha K^\beta - L^\alpha L^\beta - S^\alpha S^\beta) T_{\alpha\beta}, \quad (12)$$

$$\Pi_{II} = (2L^\alpha L^\beta - S^\alpha S^\beta - K^\alpha K^\beta) T_{\alpha\beta}. \quad (13)$$

This specific choice of these scalars is justified by the fact, that the relevant equations used to carry out this study, become more compact and easier to handle, when expressed in terms of them.

Finally, we observe that from the condition $q^\mu V_\mu = 0$, and the fact that due to the symmetry of the problem, the Einstein equations imply $T_{03} = 0$, it follows that

$$q_\mu = q_I K_\mu + q_{II} L_\mu \quad (14)$$

or, in coordinate components

$$q^\mu = \left(\frac{q_{II} G}{A\sqrt{A^2 B^2 r^2 + G^2}}, \frac{q_I}{B}, \frac{A q_{II}}{\sqrt{A^2 B^2 r^2 + G^2}}, 0\right), \quad (15)$$

$$q_\mu = \left(0, B q_I, \frac{\sqrt{A^2 B^2 r^2 + G^2} q_{II}}{A}, 0\right). \quad (16)$$

Of course, all the above quantities depend, in general, on t, r, θ.

III. KINEMATICAL VARIABLES

The kinematical variables play an important role in the description of a self-gravitating fluid. Here, besides the four-acceleration, the expansion scalar and the shear tensor, we have a component of vorticity.

Thus we obtain, respectively, for these variables (see for example Ref. [51])

$$a_\alpha = V^\beta V_{\alpha;\beta} = a_I K_\alpha + a_{II} L_\alpha$$
$$= \left(0, \frac{A_{,r}}{A}, \frac{G}{A^2}\left[-\frac{A_{,t}}{A} + \frac{G_{,t}}{G}\right] + \frac{A_{,\theta}}{A}, 0\right), \quad (17)$$

$$\Theta = V^{\alpha}_{;\alpha}$$

$$= \frac{AB^2}{r^2A^2B^2 + G^2}\left[r^2\left(2\frac{B_{,t}}{B} + \frac{C_{,t}}{C}\right)\right.$$

$$\left. + \frac{G^2}{A^2B^2}\left(\frac{B_{,t}}{B} - \frac{A_{,t}}{A} + \frac{G_{,t}}{G} + \frac{C_{,t}}{C}\right)\right], \quad (18)$$

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta h_{\alpha\beta}. \quad (19)$$

The nonvanishing components of the shear tensor are

$$\sigma_{11} = -\frac{1}{3}\frac{1}{r^2A^2B^2 + G^2}\frac{B^2}{A}\left[r^2A^2B^2\left(-\frac{B_{,t}}{B} + \frac{C_{,t}}{C}\right)\right.$$

$$\left. + G^2\left(-2\frac{B_{,t}}{B} - \frac{A_{,t}}{A} + \frac{G_{,t}}{G} + \frac{C_{,t}}{C}\right)\right], \quad (20)$$

$$\sigma_{22} = -\frac{1}{3}\frac{1}{A^3}\left[r^2A^2B^2\left(-\frac{B_{,t}}{B} + \frac{C_{,t}}{C}\right)\right.$$

$$\left. + G^2\left(2\frac{A_{,t}}{A} + \frac{B_{,t}}{B} - 2\frac{G_{,t}}{G} + \frac{C_{,t}}{C}\right)\right], \quad (21)$$

$$\sigma_{33} = \frac{1}{3}\frac{1}{r^2A^2B^2 + G^2}\frac{C^2}{A}\left[2r^2A^2B^2\left(-\frac{B_{,t}}{B} + \frac{C_{,t}}{C}\right)\right.$$

$$\left. + G^2\left(2\frac{C_{,t}}{C} - \frac{B_{,t}}{B} - \frac{G_{,t}}{G} + \frac{A_{,t}}{A}\right)\right]. \quad (22)$$

However they are not independent, and therefore the shear tensor may be defined through two scalar functions, as

$$\sigma_{\alpha\beta} = \frac{1}{3}(2\sigma_I + \sigma_{II})\left(K_\alpha K_\beta - \frac{1}{3}h_{\alpha\beta}\right)$$

$$+ \frac{1}{3}(2\sigma_{II} + \sigma_I)\left(L_\alpha L_\beta - \frac{1}{3}h_{\alpha\beta}\right). \quad (23)$$

Using Eqs. (20), (21) and (22) the above scalars may be written in terms of the metric functions and their derivatives as

$$2\sigma_I + \sigma_{II} = \frac{3}{A}\left(\frac{B_{,t}}{B} - \frac{C_{,t}}{C}\right), \quad (24)$$

$$2\sigma_{II} + \sigma_I = \frac{3}{A^2B^2r^2 + G^2}\left[AB^2r^2\left(\frac{B_{,t}}{B} - \frac{C_{,t}}{C}\right)\right.$$

$$\left. + \frac{G^2}{A}\left(-\frac{A_{,t}}{A} + \frac{G_{,t}}{G} - \frac{C_{,t}}{C}\right)\right], \quad (25)$$

where the comma and the semicolon denote derivatives and covariant derivatives, respectively. Once again, this specific choice of scalars, is justified by the very conspicuous way, in which they appear in the relevant equations (see the Appendix).

Finally, for the vorticity vector defined as

$$\omega_\alpha = \frac{1}{2}\eta_{\alpha\beta\mu\nu}V^{\beta;\mu}V^\nu = \frac{1}{2}\eta_{\alpha\beta\mu\nu}\Omega^{\beta\mu}V^\nu, \quad (26)$$

where $\Omega_{\alpha\beta} = V_{[\alpha;\beta]} + a_{[\alpha}V_{\beta]}$ and $\eta_{\alpha\beta\mu\nu}$ denote the vorticity tensor and the Levi-Civita tensor, respectively, we find a single component different from zero, producing

$$\Omega_{\alpha\beta} = \Omega(L_\alpha K_\beta - L_\beta K_\alpha), \quad (27)$$

and

$$\omega_\alpha = -\Omega S_\alpha, \quad (28)$$

with the scalar function Ω given by

$$\Omega = \frac{G\left(\frac{G_{,r}}{G} - \frac{2A_{,r}}{A}\right)}{2B\sqrt{A^2B^2r^2 + G^2}}. \quad (29)$$

We observe that from Eq. (29) and regularity conditions at the center, it follows that $G = 0 \Leftrightarrow \Omega = 0$.

IV. THE ORTHOGONAL SPLITTING OF THE RIEMANN TENSOR AND STRUCTURE SCALARS

In this section we shall introduce a set of scalar functions, known as structure scalars, which are obtained from the orthogonal splitting of the Riemann tensor (see Refs. [52,53,55–58] for details). The reason for doing this is that we shall express the set of the basic equations deployed in the Appendix, in terms of these scalars.

Thus, using the Einstein equations, the Riemann tensor can be decomposed as

$$R^{\alpha\beta}_{\ \ \nu\delta} = R^{\alpha\beta}_{(F)\ \nu\delta} + R^{\alpha\beta}_{(Q)\ \nu\delta} + R^{\alpha\beta}_{(E)\ \nu\delta} + R^{\alpha\beta}_{(H)\ \nu\delta}, \quad (30)$$

with

$$R^{\alpha\beta}_{(F)\ \nu\delta} = \frac{16\pi}{3}(\mu + 3P)V^{[\alpha}V_{[\nu}h^{\beta]}_{\delta]} + \frac{16\pi}{3}\mu h^\alpha_{[\nu}h^\beta_{\delta]}, \quad (31)$$

$$R^{\alpha\beta}_{(Q)\ \nu\delta} = -16\pi V^{[\alpha}h^{\beta]}_{[\nu}q_{\delta]} - 16\pi V_{[\nu}h^{[\alpha}_{\delta]}q^{\beta]} - 16\pi V^{[\alpha}V_{[\nu}\Pi^{\beta]}_{\delta]}$$

$$+ 16\pi h^{[\alpha}_{[\nu}\Pi^{\beta]}_{\delta]}, \quad (32)$$

$$R^{\alpha\beta}_{(E)\ \nu\delta} = 4V^{[\alpha}V_{[\nu}E^{\beta]}_{\delta]} + 4h^{[\alpha}_{[\nu}E^{\beta]}_{\delta]}, \quad (33)$$

$$R^{\alpha\beta}_{(H)\ \nu\delta} = -2\epsilon^{\alpha\beta\gamma}V_{[\nu}H_{\delta]\gamma} - 2\epsilon_{\nu\delta\gamma}V^{[\alpha}H^{\beta]\gamma}, \quad (34)$$

where $E_{\alpha\beta}$ and $H_{\alpha\beta}$ are the electric and magnetic parts, respectively, of the Weyl tensor $C_{\alpha\beta\gamma\delta}$, defined as usual by

$$E_{\alpha\beta} = C_{\alpha\nu\beta\delta}V^{\nu}V^{\delta},$$
$$H_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\nu\epsilon\rho}C^{\epsilon\rho}_{\beta\delta}V^{\nu}V^{\delta}, \tag{35}$$

where $\epsilon_{\alpha\beta\rho} = \eta_{\nu\alpha\beta\rho}V^{\nu}$ and the subscripts F, Q, E, H have an obvious meaning.

The electric part of the Weyl tensor has only three independent nonvanishing components, whereas only two components define the magnetic part. Thus we may also write

$$E_{\alpha\beta} = \frac{1}{3}(2\mathscr{E}_{I} + \mathscr{E}_{II})\left(K_{\alpha}K_{\beta} - \frac{1}{3}h_{\alpha\beta}\right)$$
$$+ \frac{1}{3}(2\mathscr{E}_{II} + \mathscr{E}_{I})\left(L_{\alpha}L_{\beta} - \frac{1}{3}h_{\alpha\beta}\right)$$
$$+ \mathscr{E}_{KL}(K_{\alpha}L_{\beta} + K_{\beta}L_{\alpha}), \tag{36}$$

and

$$H_{\alpha\beta} = H_{1}(S_{\alpha}K_{\beta} + S_{\beta}K_{\alpha}) + H_{2}(S_{\alpha}L_{\beta} + S_{\beta}L_{\alpha}). \tag{37}$$

The orthogonal splitting of the Riemann tensor is carried out by means of three tensors $Y_{\alpha\beta}$, $X_{\alpha\beta}$ and $Z_{\alpha\beta}$ defined as

$$Y_{\alpha\beta} = R_{\alpha\nu\beta\delta}V^{\nu}V^{\delta}, \tag{38}$$

$$X_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\nu}{}^{\epsilon\rho}R^{\star}_{\epsilon\rho\beta\delta}V^{\nu}V^{\delta}, \tag{39}$$

and

$$Z_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\epsilon\rho}R_{\delta\beta}{}^{\epsilon\rho}V^{\delta}, \tag{40}$$

where $R^{\star}_{\alpha\beta\nu\delta} = \frac{1}{2}\eta_{\epsilon\rho\nu\delta}R_{\alpha\beta}{}^{\epsilon\rho}$.
Using Eqs. (30)–(33) and (36), we obtain

$$Y_{\alpha\beta} = \frac{1}{3}Y_{T}h_{\alpha\beta} + \frac{1}{3}(2Y_{I} + Y_{II})\left(K_{\alpha}K_{\beta} - \frac{1}{3}h_{\alpha\beta}\right)$$
$$+ \frac{1}{3}(2Y_{II} + Y_{I})\left(L_{\alpha}L_{\beta} - \frac{1}{3}h_{\alpha\beta}\right)$$
$$+ Y_{KL}(K_{\alpha}L_{\beta} + K_{\beta}L_{\alpha}), \tag{41}$$

with

$$Y_{T} = 4\pi(\mu + 3P), \tag{42}$$

$$Y_{I} = \mathscr{E}_{I} - 4\pi\Pi_{I}, \tag{43}$$

$$Y_{II} = \mathscr{E}_{II} - 4\pi\Pi_{II}, \tag{44}$$

$$Y_{KL} = \mathscr{E}_{KL} - 4\pi\Pi_{KL}. \tag{45}$$

In a similar way the tensor $X_{\alpha\beta}$ can be written as

$$X_{\alpha\beta} = \frac{1}{3}X_{T}h_{\alpha\beta} + \frac{1}{3}(2X_{I} + X_{II})\left(K_{\alpha}K_{\beta} - \frac{1}{3}h_{\alpha\beta}\right)$$
$$+ \frac{1}{3}(2X_{II} + X_{I})\left(L_{\alpha}L_{\beta} - \frac{1}{3}h_{\alpha\beta}\right)$$
$$+ X_{KL}(K_{\alpha}L_{\beta} + K_{\beta}L_{\alpha}), \tag{46}$$

with

$$X_{T} = 8\pi\mu, \tag{47}$$

$$X_{I} = -\mathscr{E}_{I} - 4\pi\Pi_{I}, \tag{48}$$

$$X_{II} = -\mathscr{E}_{II} - 4\pi\Pi_{II}, \tag{49}$$

$$X_{KL} = -\mathscr{E}_{KL} - 4\pi\Pi_{KL}. \tag{50}$$

Once again, the specific choice of all the scalars above has the purpose of rendering the basic equations in the Appendix in the simpler form.

Finally, from Eqs. (30)–(33), (35) and ([13]) we obtain

$$Z_{\alpha\beta} = H_{\alpha\beta} + 4\pi q^{\rho}\epsilon_{\alpha\beta\rho}, \tag{51}$$

or

$$Z_{\alpha\beta} = Z_{I}K_{\beta}S_{\alpha} + Z_{II}K_{\alpha}S_{\beta} + Z_{III}L_{\alpha}S_{\beta} + Z_{IV}L_{\beta}S_{\alpha}, \tag{52}$$

where

$$Z_{I} = (H_{1} - 4\pi q_{II}), \qquad Z_{II} = (H_{1} + 4\pi q_{II}),$$
$$Z_{III} = (H_{2} - 4\pi q_{I}), \qquad Z_{IV} = (H_{2} + 4\pi q_{I}). \tag{53}$$

Before ending this section it would be useful to introduce a relevant quantity defined in terms of tensors $Y_{\alpha\beta}, X_{\alpha\beta}, Z_{\alpha\beta}$. This is the super-Poynting vector defined by

$$P_{\alpha} = \epsilon_{\alpha\beta\gamma}(Y^{\gamma}_{\delta}Z^{\beta\delta} - X^{\gamma}_{\delta}Z^{\delta\beta}), \tag{54}$$

which can be written as

$$P_{\alpha} = P_{I}K_{\alpha} + P_{II}L_{\alpha}, \tag{55}$$

with

$$P_I = \frac{H_2}{3}(2Y_{II} + Y_I - 2X_{II} - X_I) + H_1(Y_{KL} - X_{KL}) + \frac{4\pi q_I}{3}[2Y_T + 2X_T - X_I - Y_I] - 4\pi q_{II}(X_{KL} + Y_{KL}),$$

$$P_{II} = \frac{H_1}{3}(2X_I + X_{II} - Y_{II} - 2Y_I) + H_2(X_{KL} - Y_{KL}) - 4\pi q_I(Y_{KL} + X_{KL}) + \frac{4\pi q_{II}}{3}[2Y_T + 2X_T - X_{II} - Y_{II}]. \quad (56)$$

Three comments are in order at this point:

(i) The super-Poynting vector may be defined in terms of the Riemann tensor [as in Eq. (54)], or in terms of the Weyl tensor [75–77]. Obviously they coincide in vacuum, but are different within the fluid distribution.

(ii) Both components P_I, P_{II} have terms not containing heat dissipative contributions. It is reasonable to associate these with gravitational radiation.

(iii) Both components of the super-Poynting vector have contributions of both components of the heat flux vector.

We shall come back to these points, later.

V. THE HEAT TRANSPORT EQUATION

We shall need a transport equation derived from a causal dissipative theory (e.g. the Müller-Israel-Stewart second-order phenomenological theory for dissipative fluids [62–65]).

Indeed, the Maxwell-Fourier law for radiation flux leads to a parabolic equation (diffusion equation) which predicts the propagation of perturbations with infinite speed (see Refs. [78]–[80] and references therein). This simple fact is at the origin of the pathologies [81] found in the approaches of Eckart [82] and Landau [83] for relativistic dissipative processes. To overcome such difficulties, various relativistic theories with nonvanishing relaxation times have been proposed in the past [62–65,84,85]. The important point is that all these theories provide a heat transport equation which is not of Maxwell-Fourier type but rather of Cattaneo type [86], leading thereby to a hyperbolic equation for the propagation of thermal perturbations.

A fundamental parameter in these theories is the relaxation time τ of the corresponding dissipative process. This positive-definite quantity has a distinct physical meaning, namely the time taken by the system to return spontaneously to the steady state (whether of thermodynamic equilibrium or not) after it has been suddenly removed from it. Therefore, when studying transient regimes, i.e., the evolution between two steady-state situations, τ cannot be neglected. In fact, leaving aside that parabolic theories are necessarily noncausal, it is obvious that whenever the time scale of the problem under consideration becomes of the order of (or smaller) than the relaxation time, the latter cannot be ignored, since neglecting the relaxation time amounts—in this situation—to disregarding the whole problem under consideration.

Thus, the transport equation for the heat flux reads [63–65,79],

$$\tau h_\nu^\mu q_{;\beta}^\nu V^\beta + q^\mu = -\kappa h^{\mu\nu}(T_{,\nu} + Ta_\nu) - \frac{1}{2}\kappa T^2 \left(\frac{\tau V^\alpha}{\kappa T^2}\right)_{;\alpha} q^\mu, \quad (57)$$

where τ, κ, T denote the relaxation time, the thermal conductivity and the temperature, respectively.

Contracting Eq. (57) with L_μ, we obtain

$$\frac{\tau}{A}(q_{II,t} + Aq_I\Omega) + q_{II}$$
$$= -\frac{\kappa}{A}\left(\frac{GT_{,t} + A^2 T_{,\theta}}{\sqrt{A^2 B^2 r^2 + G^2}} + ATa_{II}\right) - \frac{\kappa T^2 q_{II}}{2}\left(\frac{\tau V^\alpha}{\kappa T^2}\right)_{;\alpha}, \quad (58)$$

where Eq. (29), has been used.

On the other hand, by contracting Eq. (57) with K_μ, we find

$$\frac{\tau}{A}(q_{I,t} - Aq_{II}\Omega) + q_I$$
$$= -\frac{\kappa}{B}(T_{,r} + BTa_I) - \frac{\kappa T^2 q_I}{2}\left(\frac{\tau V^\alpha}{\kappa T^2}\right)_{;\alpha}. \quad (59)$$

It is worth noting that the two equations above are coupled through the vorticity. We shall discuss this point further in Sec. VII.

VI. BASIC EQUATIONS

The relevant equations (besides the transport equation shown in the previous section) for describing the evolution of our axially and reflection-symmetric dissipative fluid, are obtained by applying the $1 + 3$ formalism [43–51] to axial symmetry. Accordingly, they are not new (see for example Ref. [47]), but are exhibited here in the form explicitly adapted to the problem under consideration. The equivalent set of equations for the spherically symmetric case was obtained in Ref. [52], whereas in Ref. [53], they were obtained for the cylindrically symmetric case. Obviously, not all of them are independent; however, depending on the problem under consideration, it may be more advantageous to use one subset instead of the other, and therefore here we present them all. They are presented, with brief comments about their origins, in Appendix A. The scalar equations obtained by projecting them on all possible combinations of tetrad vectors **V**, **K**, **L**, **S**, are deployed in Appendix B.

In what follows we shall extract and discuss part of the information contained in these equations.

VII. SOME THERMODYNAMIC ASPECTS OF THE PROBLEM

The thermodynamics of fluids endowed with vorticity may be quite complicated even in Newtonian theory (e.g. see Ref. [87] for a discussion on this point). However, even at this level of generality, some interesting conclusions may be drawn from the study of the transport equation (57) and the generalized "Euler" equation (A7).

Thus, as we shall see, the combination of the two above-mentioned equations lead to a decreasing of the "effective" inertial mass density. This is a known effect, with important implications on the evolution of the object. On the other hand, the fact that both components of Eq. (57) are coupled (through the vorticity), produces a result which recalls the well-known von Zeipel's theorem [3]. Let us analyze these two issues in some detail.

A. The effective inertial mass density of the dissipative fluid

In classical dynamics the inertial mass is defined as the factor of proportionality between the three-force applied to a particle (a fluid element) and the resulting three-acceleration, according to Newton's second law. In relativistic dynamics a similar relation only holds (in general) in the instantaneous rest frame (i.r.f.), since the three-acceleration and the force that causes it are not (in general) parallel, except in the i.r.f. (see for example Ref. [88]). However, under a variety of circumstances, this factor of proportionality does not coincide with the mass (density) of the particle (fluid element) in the absence of interactions. In such cases we refer to this proportionality factor as the "effective inertial mass" (e.i.m.). Thus for example the e.i.m. of an electron moving under a given force through a crystal, differs from the value corresponding to an electron moving under the same force in free space, and may even become negative (see Refs. [89,90]).

In our case, by combining Eqs. (A7) and (57) we obtain

$$
(\mu + P)\left[1 - \frac{\kappa T}{\tau(\mu + P)}\right]a_\alpha
$$
$$
= -h_\alpha^\beta \Pi^\mu_{\beta;\mu} - \nabla_\alpha P - (\sigma_{\alpha\beta} + \Omega_{\alpha\beta})q^\beta
$$
$$
+ \frac{\kappa}{\tau}\nabla_\alpha T + \left\{\frac{1}{\tau} + \frac{1}{2}D_t\left[ln\left(\frac{\tau}{\kappa T^2}\right)\right] - \frac{5}{6}\Theta\right\}q_\alpha, \quad (60)
$$

where $\nabla_\alpha P \equiv h_\alpha^\beta P_{,\beta}$ and $D_t f \equiv f_{,\beta}V^\beta$.

In the above equation we have on the right-hand side, besides some dissipative terms, terms representing the hydrodynamic "forces" acting on any fluid element. On the left-hand side, it is clear that the factor multiplying the four-acceleration vector represents the effective inertial mass density. Thus, the obtained expression for the e.i.m. density contains a contribution from dissipative variables, which reduces its value with respect to the

nondissipative situation. Such a decreasing of the e.i.m. density was pointed out for the first time in Ref. [66], and since then, it has been shown to appear in a great variety of scenarios (see Refs. [59,67,70,72] and references therein).

The potential consequences of the above-mentioned effect, on the evolution of the self-gravitating object, should be seriously considered. Indeed, from the equivalence principle it follows that the "passive" gravitational mass density should be reduced too, by the same factor as the e.i.m. density. This in turn might lead, in some critical cases when such diminishing is significative, to a bouncing of the collapsing object (see Ref. [69] for a specific numerical example).

B. Vorticity and heat transport

As we mentioned earlier, the two components of the transport equation (58) and (59), are coupled through the vorticity. This fact entails an interesting thermodynamic consequence. Indeed, let us assume that at some initial time (say $t = 0$) and before it, there is thermodynamic equilibrium in the θ direction; this implies $q_{II} = 0$, and also that the corresponding Tolman's temperature [91] is constant, which in turns implies that the term within the round bracket in the first term on the right-hand side of Eq. (58) vanishes. Then it follows at once from Eq. (58) that

$$
q_{II,t} = -A\Omega q_I, \quad (61)
$$

implying that the propagation in time of the vanishing of the meridional flow, is subject to the vanishing of the vorticity and/or the vanishing of heat flow in the r direction.

Inversely, by repeating the same argument for Eq. (59) we obtain at the initial time when we assume thermodynamic equilibrium,

$$
q_{I,t} = A\Omega q_{II}. \quad (62)
$$

Thus, it appears that the vanishing of the radial component of the heat flux vector at some initial time, will propagate in time if only, the vorticity and/or the meridional heat flow are different from zero.

In other words, time propagation of the thermal equilibrium condition, in either direction r, θ, is assured only in the absence of vorticity. Otherwise, it requires initial thermal equilibrium in both directions.

This result is a clear reminiscence of von Zeipel's theorem [3].

VIII. EVOLUTION OF THE EXPANSION SCALAR AND THE SHEAR

Let us now consider Eqs. (B1)–(B4). In order to elucidate the significance of these equations, we shall, for simplicity, restrict ourselves to the geodesic fluid ($a_\mu = 0$). The first of these equations, describes the evolution of the expansion scalar (Raychaudhuri equation).

First of all we observe that the evolution of the expansion scalar is controlled not only by the scalar Y_T and σ as in the cylindrically [53] and the spherically symmetric [52] cases, but it also depends on the vorticity vector. It is worth mentioning that, as is apparent from Eqs. (B5)–(B9), in the nongeodesic case there is a coupling between $H_1, H_2, \Omega, q_I, q_{II}$, implying that in the general case all these factors also affect the evolution of Θ.

For the shear we have two equations, Eqs. (B2) and (B4) (for the two independent scalars defining the shear tensor).

We observe that even in the geodesic case, unlike the cylindrically symmetric case, Eqs. (B2) and (B4) are coupled through the $2\sigma^2 + \Omega^2$ term. Thus, assuming that the fluid is initially shear free, the system will deviate from such a condition even if we keep Y_I, Y_{II} vanishing all along the evolution. In order to keep the fluid shear free, we need to also keep it, vorticity free. This last condition implies because of Eq. (B3) that Y_{KL} should also vanish all along the evolution. Thus, the evolution of the shear is now controlled by three structure scalars: Y_I, Y_{II}, Y_{KL}. In other words all the information about the stability of the shear-free condition is encrypted in Y_I, Y_{II}, Y_{KL}. Once again it should be emphasized that this conclusion is true only for the geodesic case. In the general case, because of Eqs. (B5)–(B9), we see that the magnetic part of the Weyl tensor and the heat flux vector also affect the stability of the shear-free condition.

IX. EVOLUTION OF THE VORTICITY

Let us now turn to Eqs. (B5)–(B9). If we restrain to the geodesic case, then it seems from Eq. (B5), that an initially vorticity-free configuration, will remain vorticity free during the evolution. The same situation happens for the shear-free case.

Indeed, from Eqs. (24) and (25), it follows that the shear-free condition implies

$$G = ACf(r, \theta), \tag{63}$$

where $f(r, \theta)$ is an arbitrary function of its arguments. Since, neither A nor C can vanish during the evolution, it follows at once from Eq. (63) that a shear-free configuration, which is initially vorticity free, will remain vorticity free during the evolution.

However such conclusions have to be taken with caution. Indeed, as it follows from Eq. (B3), the vorticity-free condition implies, in the geodesic case $Y_{KL} = 0$. On the other hand as it follows from Eq. (B6) (remember that the metric is nondiagonal and therefore $L^0 \neq 0$), the vorticity-free condition is unstable in the presence of dissipative fluxes, as a result of which it appears that the geodesic condition and the shear-free condition, are too restrictive, and the stability of the vorticity-free condition depends on the above-mentioned factors. This fact is in turn, in full

agreement with earlier works, where it was shown that vorticity generation is sourced by entropy gradients (see Refs. [92–96] and references therein).

Finally, we observe that if the fluid is shear free, the vanishing of the vorticity implies, as it follows from Eqs. (B8) and (B9), that the magnetic part of the Weyl tensor vanishes, too. Also, as it follows from Eq. (B16), the inverse is true for nondissipative fluids. This is in full agreement with a result by Glass [97], indicating that a necessary and sufficient condition for a shear-free perfect fluid to be irrotational is that the Weyl tensor is purely electric. Thus we have extended the Glass result, to anisotropic fluids. In the case of dissipative fluids, the vanishing of the magnetic part of the Weyl tensor does not necessarily imply the vanishing of the vorticity.

X. THE DENSITY INHOMOGENEITY FACTORS AND THEIR EVOLUTION

The density inhomogeneity factors (in Refs. [52,53,98] they are referred to as inhomogeneity factors), are specific combinations of physical and geometrical variables (say Ψ_i), such that their vanishing is a sufficient and necessary condition for the homogeneity of the energy density i.e. $\nabla_\alpha \mu \equiv h_\alpha^\beta \mu_{,\beta} = 0$. Of course these latter conditions are necessary but not sufficient for the system to be homogeneous in the broad sense (i.e. a system where spatial gradients of the Hubble scalar, the pressure, etc, also vanish).

In the spherically symmetric case, in the absence of dissipation, the density inhomogeneity factor is the scalar associated to the trace-free part of $X_{\alpha\beta}$. If dissipation is present then additional terms including dissipative flux appear (see Refs. [52,98] for a detailed discussion).

In the cylindrically symmetric case, it was not possible to identify explicitly the density inhomogeneity factors; nevertheless, it was easy to check that the trace-free part of $X_{\alpha\beta}$, besides the magnetic part of the Weyl tensor and the dissipative flux determine the energy density inhomogeneity.

In the static axially symmetric case it was possible to identify the density inhomogeneity factors; they are the structure scalars associated to the trace-free part of $X_{\alpha\beta}$.

In the present case however, the situation is quite complicated and we were not able to explicitly identify the density inhomogeneity factors. However we can identify the structure scalars these factors are made of, and their evolution.

Indeed, it follows at once from Eqs. (B14) and (B15) that the vanishing of $X_I, X_{II}, X_{KL}, Z_I, Z_{II}, Z_{III}, Z_{IV}$ implies the homogeneity of the energy density (in the sense defined above). On the other hand, the evolution of the above mentioned scalars is determined by Eqs. (58), (59), (B10), (B11), (B12), (B13), (B17), and (B18).

XI. THE SUPER-POYNTING VECTOR AND GRAVITATIONAL RADIATION

In the theory of the super-Poynting vector, a state of gravitational radiation is associated to a nonvanishing component of the latter (see Refs. [75–77]). This is in agreement with the established link between the super-Poynting vector and the news functions [99], in the context of the Bondi-Sachs approach [10,11]. Furthermore, as it was shown in Ref. [99], there is always a nonvanishing component of P^μ, on the plane orthogonal to a unit vector along which there is a nonvanishing component of vorticity (the θ-r plane). Inversely, P^μ vanishes along the ϕ direction since there are no motions along this latter direction, because of the reflection symmetry.

Therefore we can identify three different contributions in Eq. (56). On the one hand we have contributions from the heat transport process. These are in principle independent of the magnetic part of the Weyl tensor, which explains why they remain in the spherically symmetric limit. However the intriguing fact is the appearance of both components of the four-vector **q** in both components of **P**. We observe that this is achieved through the $X_{KL} + Y_{KL}$ terms in Eq. (56), or using Eqs. (45) and (50), through Π_{KL}. But we have also seen that both components of the heat flux vector are coupled through the vorticity, in the transport equation. Thus, the vorticity acts as a coupling factor between the two components of the heat flux vector in the transport equation, whereas Π_{KL} couples the two components of the super-Poynting vector, with the two components of the heat flux vector.

On the other hand we have contributions from the magnetic part of the Weyl tensor. These are of two kinds: on the one hand contributions associated with the propagation of gravitational radiation within the fluid, and on the other, contributions of the flow of super-energy associated with the vorticity on the plane orthogonal to the direction of propagation of the radiation. Both contributions are intertwined, and it appears to be impossible to disentangle them through two independent scalars.

It is worth noticing that the factors multiplying the H terms in Eq. (56), are $\mathscr{E}_I, \mathscr{E}_{II}, \mathscr{E}_{KL}$, implying that purely magnetic or purely electric sources, do not produce gravitational radiation. This is consistent with the result obtained in vacuum for the Bondi metric [100], which states that purely electric Bondi metrics are static, whereas purely magnetic ones, are just Minkowski.

XII. CONCLUSIONS AND SUMMARY OF RESULTS

We have carried out a general study on axially (and reflection-)symmetric relativistic fluids. An important role in this study is played by the structure scalars.

We have defined the complete set of such scalars corresponding to our problem. It turns out that there are 12 structure scalars ($X_{T,I,II,KL}, Y_{T,I,II,KL}, Z_{I,II,III,IV}$) in contrast with the spherically symmetric case where there are only five, and the cylindrically symmetric case where there are only eight. Besides, two scalars defining the shear tensor ($\sigma_{I,II}$), one scalar defining the vorticity (Ω), and five scalars defining the electric and magnetic parts of the Weyl tensor ($\mathscr{E}_{I,II,KL}, H_{1,2}$) were also introduced.

Next we have identified and deployed, the set of equations governing the structure and evolution of the system under consideration and brought out the role of structure scalars in these equations, in order to exhibit the physical relevance of the former.

We have first considered the dynamical equation (A7) derived from conservation laws, and coupled it with a transport equation derived from a causal dissipative theory. The resulting equation exhibits the decreasing of the effective inertial mass term due to thermal effects. It is worth noticing that such a decreasing is described by the term in the square bracket on the left-hand side of Eq. (60), which in turn is produced by the first term on the left-hand side and the second term on the right-hand side, of Eq. (57) (see Ref. [70] for a detailed discussion on this point). But these two terms should enter into any causal and relativistic theory of dissipation. Therefore the effect under consideration is not exclusive to the Israel-Stewart theory, but rather must be present in any other reasonable theory of dissipation.

We have also pointed out the coupling of both components of the heat flux vector, through the vorticity. The resulting situation recalls the picture described by von Zeipel's theorem.

Next, we have studied the evolution of the expansion scalar, the shear and the vorticity. For simplicity we have considered the geodesic case. Thus we have seen that the evolution of the expansion scalar is controlled by the scalar Y_T. However the appearance of the vorticity in the corresponding equation, together with the fact that in the nongeodesic case there is a coupling between Ω and $Z_{I,II,III,IV}$, leads us to conclude that the latter scalars also affect the evolution of Θ, if the fluid is not geodesic.

For the shear the situation is similar: in the geodesic case the evolution is controlled by $Y_{I,II,KL}$; however, in the nongeodesic case (for the same reason as in the case of the scalar expansion), the four scalars $Z_{I,II,III,IV}$ are also expected to affect the evolution of the shear.

For the vorticity, it appears that the geodesic condition may be too stringent. In the general case the evolution of the vorticity depends upon Y_{KL} and $Z_{I,II,III,IV}$.

Next we have considered the density inhomogeneity factors. Although we were unable to identify these factors explicitly, it was shown that the scalars $X_{I,II,KL}, Z_{I,II,III,IV}$ are the basic constituents of such factors.

Finally, we analyzed the super-Poynting vector. It contains three types of contributions. On the one hand we have contributions from the dissipative processes associated to the heat flux vector. Next, we have contributions from

gravitational radiation, associated to the magnetic parts of the Weyl tensor. Finally, we have contributions from the flow of super-energy, which in turn, acts as the source of the vorticity.

However, while the pure dissipative contribution is trivially identified, we could not do the same for the other two contributions, since the factors multiplying the H_1, H_2 terms in Eq. (56), do not vanish if $\Omega = 0$. On the other hand the coupling of both components of the super-Poynting vector with the two components of the heat flux vector, through Π_{KL}, appears explicitly in Eq. (56).

Before ending, we would like to make some final remarks and to present a partial list of issues, which remain unanswered in this manuscript, but should be addressed in the future.

(i) We have considered some particular cases, where some variables (e.g. the shear) were considered to vanish. We did so, on the one hand for simplicity, and on the other hand, in order to bring out the role of some specific variables. However, it should be kept in mind that such kinds of "suppressions" may lead to inconsistencies in the set of equations. This is for example the case of "silent" universes [101,102], where dust sources have a vanishing magnetic Weyl tensor, and lead to a system of $1 + 3$ constraint equations that do not seem to be integrable in general [103]. In other words for any specific modeling, the possible occurrence of these types of inconsistencies should be carefully considered.

(ii) In the case of specific modeling, another important question arises, namely: what additional information is required to close the system of equations? It is clear that information about local physical aspects of the source (e.g. equations of state and/or information about energy production) are not included in the set of deployed equations and therefore should be given, in order that metric and matter functions could be solved for in terms of initial data.

(iii) From Eq. (56) it follows that either one of the "gravitational" terms vanish, not only if $H_1 = 0$ or $H_2 = 0$, but also if, either $\mathscr{E}_I$, $\mathscr{E}_{II}$, or $\mathscr{E}_{KL}$ vanish. What else do these latter conditions imply?

(iv) Could it be possible to find the exact solution corresponding to nondissipative dust with shear (the analog of the Lemaitre-Tolman-Bondi solution)? Would this solution have a nonvanishing magnetic part of the Weyl tensor?

(v) We observe that the shear-free condition can be easily integrated from Eqs. (24) and (25). Could it be possible to provide a comprehensive specific description of shear-free fluids?

(vi) We have identified the subset of equations which should determine the density inhomogeneity factors and their evolution, but we were unable to isolate such factors in the general case. Is this possible?

(vii) How could one describe the "cracking" (splitting) of the configurations as described in Refs. [25,104] (and references therein)?

(viii) As mentioned in the Introduction, we do not have an exact solution (written down in closed analytical form) describing gravitational radiation in vacuum, from bounded sources. Furthermore, we do not harbor the hope to find exact analytical solutions, for evolving axially symmetric sources (except perhaps in very restricted situations, e.g. dust). Accordingly, any specific modeling of such a source should be done numerically.

(ix) It could be useful to introduce the concept of a mass function, similar to the one existing in the spherically symmetric case. This could be relevant, in particular, in the case of matching the source to a specific exterior. With respect to this point, it should be mentioned that in this work we have not considered in detail such a problem, since no specific solution has been presented. However, for any specific model, the correct treatment of such a matching, would be mandatory.

ACKNOWLEDGMENTS

APPENDIX A: THE BASIC EQUATIONS

1. Ricci identities

From the Ricci identities for the vector V_α the following set of equations are obtained.

The time-propagation equation for the expansion Θ is

$$\Theta_{;\alpha}V^\alpha + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \Omega^2) - a^\alpha_{;\alpha} + 4\pi(\mu + 3P) = 0, \tag{A1}$$

where $2\sigma^2 = \sigma_{\alpha\beta}\sigma^{\alpha\beta}$.

The time-propagation equation for the shear $\sigma_{\alpha\beta}$ is

$$h^\mu_\alpha h^\nu_\beta \sigma_{\mu\nu;\delta}V^\delta + \sigma^\mu_\alpha \sigma_{\beta\mu} + \frac{2}{3}\Theta\sigma_{\alpha\beta}$$
$$- \frac{1}{3}(2\sigma^2 + \Omega^2 - a^\delta_{;\delta})h_{\alpha\beta} + \omega_\alpha\omega_\beta - a_\alpha a_\beta$$
$$- h^\mu_{(\alpha} h^\nu_{\beta)} a_{\nu;\mu} + E_{\alpha\beta} - 4\pi\Pi_{\alpha\beta} = 0, \tag{A2}$$

and the time-propagation equation for $\Omega_{\alpha\beta}$ is

$$h_\alpha^\mu h_\beta^\nu \Omega_{\mu\nu;\delta} V^\delta + \frac{2}{3}\Theta\Omega_{\alpha\beta} + 2\sigma_{\mu[\alpha}\Omega_{\beta]}^\mu - h_{[\alpha}^\mu h_{\beta]}^\nu a_{\mu;\nu} = 0. \tag{A3}$$

Besides, the following constraint equations follow:

$$h_\alpha^\beta\left(\frac{2}{3}\Theta_{;\beta} - \sigma_{\beta;\mu}^\mu + \Omega_{\beta;\mu}^\mu\right) + (\sigma_{\alpha\beta} + \Omega_{\alpha\beta})a^\beta = 8\pi q_\alpha, \tag{A4}$$

$$2\omega_{(\alpha}a_{\beta)} + h_{(\alpha}^\mu h_{\beta)\nu}(\sigma_{\mu\delta} + \Omega_{\mu\delta})_{;\gamma}\eta^{\nu\kappa\gamma\delta}V_\kappa = H_{\alpha\beta}. \tag{A5}$$

2. Conservation laws

The conservation law $T_{\beta;\alpha}^\alpha = 0$ leads to the following equations:

$$\mu_{;\alpha}V^\alpha + (\mu + P)\Theta + \frac{1}{9}(2\sigma_I + \sigma_{II})\Pi_I + \frac{1}{9}(2\sigma_{II} + \sigma_I)\Pi_{II}$$
$$+ q_{;\alpha}^\alpha + q^\alpha a_\alpha = 0, \tag{A6}$$

$$(\mu + P)a_\alpha + h_\alpha^\beta(P_{;\beta} + \Pi_{\beta;\mu}^\mu + q_{\beta;\mu}V^\mu)$$
$$+ \left(\frac{4}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \Omega_{\alpha\beta}\right)q^\beta = 0. \tag{A7}$$

The first of these equations is the "continuity" equation, whereas the second one is the "generalized Euler" equation.

3. Differential equations for the Weyl tensor derived from Bianchi identities

From the Bianchi identities and Einstein equations, the following set of equations are obtained:

$$h_{(\alpha}^\mu h_{\beta)}^\nu E_{\mu\nu;\delta}V^\delta + \Theta E_{\alpha\beta} + h_{\alpha\beta}E_{\mu\nu}\sigma^{\mu\nu} - 3E_{\mu(\alpha}\sigma_{\beta)}^\mu + h_{(\alpha}^\mu\eta_{\beta)}^{\delta\gamma\kappa}V_\delta H_{\gamma\mu;\kappa} - E_{\delta(\alpha}\Omega_{\beta)}^\delta - 2H_{(\alpha}^\mu\eta_{\beta)\delta\kappa\mu}V^\delta a^\kappa$$
$$= -4\pi(\mu + P)\sigma_{\alpha\beta} - \frac{4\pi}{3}\Theta\Pi_{\alpha\beta} - 4\pi h_{(\alpha}^\mu h_{\beta)}^\nu\Pi_{\mu\nu;\delta}V^\delta - 4\pi\sigma_{\mu(\alpha}\Pi_{\beta)}^\mu$$
$$- 4\pi\Omega^\mu_{(\alpha}\Pi_{\beta)\mu} - 8\pi a_{(\alpha}q_{\beta)} + \frac{4\pi}{3}(\Pi_{\mu\nu}\sigma^{\mu\nu} + a_\mu q^\mu + q_{;\mu}^\mu)h_{\alpha\beta} - 4\pi h_{(\alpha}^\mu h_{\beta)}^\nu q_{\nu;\mu}, \tag{A8}$$

$$h_\alpha^\mu h^{\nu\beta}E_{\mu\nu;\beta} - \eta_\alpha^{\delta\nu\kappa}V_\delta\sigma_\nu^\gamma H_{\kappa\gamma} + 3H_{\alpha\beta}\omega^\beta = \frac{8\pi}{3}h_\alpha^\beta\mu_{;\beta} - 4\pi h_\alpha^\beta h^{\mu\nu}\Pi_{\beta\nu;\mu} - 4\pi\left(\frac{2}{3}\Theta h_\alpha^\beta - \sigma_\alpha^\beta + 3\Omega_\alpha^\beta\right)q_\beta, \tag{A9}$$

$$(\sigma_{\alpha\delta}E_\beta^\delta + 3\Omega_{\alpha\delta}E_\beta^\delta)\epsilon_\kappa^{\ \alpha\beta} + a^\nu H_{\nu\kappa} - H^{\nu\delta}_{\ ;\delta}h_{\nu\kappa} = +4\pi(\mu + P)\Omega_{\alpha\beta}\epsilon_\kappa^{\ \alpha\beta} + 4\pi[q_{\alpha;\beta} + \Pi_{\nu\alpha}(\sigma^\nu_{\ \beta} + \Omega^\nu_{\ \beta})]\epsilon_\kappa^{\ \alpha\beta}, \tag{A10}$$

$$2a_\beta E_{\alpha\kappa}\epsilon_\gamma^{\ \alpha\beta} - E_{\nu\beta;\delta}h_\kappa^\nu\epsilon_\gamma^{\ \delta\beta} + E_{\beta;\delta}^\delta\epsilon_{\gamma\kappa}^{\ \ \beta} + \frac{2}{3}\Theta H_{\kappa\gamma} + H_{\nu;\delta}^\mu V^\delta h_\kappa^\nu h_{\mu\gamma} - (\sigma_{\kappa\delta} + \Omega_{\kappa\delta})H_\gamma^\delta + (\sigma_{\beta\delta} + \Omega_{\beta\delta})H_\alpha^\mu\epsilon_{\kappa\ \mu}^{\ \delta}\epsilon_\gamma^{\ \alpha\beta} + \frac{1}{3}\Theta H_\alpha^\mu\epsilon_{\kappa\ \mu}^{\ \delta}\epsilon_\gamma^{\ \alpha}_{\ \ \delta}$$
$$= \frac{4\pi}{3}\mu_{,\beta}\epsilon_{\gamma\kappa}^{\ \ \beta} + 4\pi\Pi_{\alpha\nu;\beta}h_\kappa^\nu\epsilon_\gamma^{\ \alpha\beta} + 4\pi\left[q_\kappa\Omega_{\alpha\beta} + q_\alpha\left(\sigma_{\kappa\beta} + \Omega_{\kappa\beta} + \frac{1}{3}\Theta h_{\kappa\beta}\right)\right]\epsilon_\gamma^{\ \alpha\beta}. \tag{A11}$$

Projecting the above equations, on all possible combinations of the tetrad vectors **V**, **K**, **L**, **S**, we find a set of scalar equations, which are deployed in Appendix B.

APPENDIX B: SUMMARY OF SCALAR EQUATIONS

From our basic equations, by projecting on all possible combinations of the tetrad vectors **V**, **K**, **L**, **S**, we find the following scalar equations.

For Eq. (A1),

$$\Theta_{;\alpha}V^\alpha + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \Omega^2) - a_{;\alpha}^\alpha + Y_T = 0. \tag{B1}$$

By contracting Eq. (A2) with **KK**, **KL** and **LL** we obtain, respectively,

$$\sigma_{I,\delta}V^\delta + \frac{1}{3}\sigma_I^2 + \frac{2}{3}\Theta\sigma_I - (2\sigma^2 + \Omega^2 - a_{;\delta}^\delta)$$
$$- 3(K^\mu K^\nu a_{\nu;\mu} + a_I^2) + Y_I = 0, \tag{B2}$$

$$\frac{1}{3}(\sigma_I - \sigma_{II})\Omega - a_I a_{II} - K^{(\mu}L^{\nu)}a_{\nu;\mu} + Y_{KL} = 0, \tag{B3}$$

$$\sigma_{II,\delta}V^\delta + \frac{1}{3}\sigma_{II}^2 + \frac{2}{3}\Theta\sigma_{II} - (2\sigma^2 + \Omega^2 - a_{;\delta}^\delta)$$
$$- 3(L^\mu L^\nu a_{\nu;\mu} + a_{II}^2) + Y_{II} = 0. \tag{B4}$$

By contracting Eq. (A3) with **KL**, we obtain

$$\Omega_{,\delta}V^\delta + \frac{1}{3}(2\Theta + \sigma_I + \sigma_{II})\Omega + K^{[\mu}L^{\nu]}a_{\mu;\nu} = 0. \tag{B5}$$

By contracting Eq. (A4) with **K** and **L** we obtain, respectively,

$$\frac{2}{3B}\Theta_{,r} - \Omega_{;\mu}L^\mu + \Omega(L_{\beta;\mu}K^\mu K^\beta - L^\mu_{;\mu}) + \frac{1}{3}\sigma_I a_I - \Omega a_{II} - \frac{1}{3}\sigma_{I;\mu}K^\mu$$

$$-\frac{1}{3}(2\sigma_I + \sigma_{II})\left(K^\mu_{;\mu} - \frac{a_I}{3}\right) - \frac{1}{3}(2\sigma_{II} + \sigma_I)\left(L_{\beta;\mu}L^\mu K^\beta - \frac{a_I}{3}\right) = 8\pi q_I, \tag{B6}$$

$$\frac{1}{3\sqrt{A^2 B^2 r^2 + G^2}}\left(\frac{2G}{A}\Theta_{,t} + 2A\Theta_{,\theta}\right) + \frac{a_{II}\sigma_{II}}{3} + \Omega_{;\mu}K^\mu + \Omega(K^\mu_{;\mu} + L^\mu K^\beta L_{\beta;\mu}) + \Omega a_I - \frac{1}{3}\sigma_{II;\mu}L^\mu$$

$$+\frac{1}{3}(2\sigma_I + \sigma_{II})\left(L_{\beta;\mu}K^\beta K^\mu + \frac{a_{II}}{3}\right) - \frac{1}{3}(2\sigma_{II} + \sigma_I)\left(L^\mu_{;\mu} - \frac{a_{II}}{3}\right) = 8\pi q_{II}. \tag{B7}$$

By contracting Eq. (A5) with **KS** and **LS** we obtain, respectively,

$$-\Omega a_I - \frac{1}{2}(K^\mu S_\nu + S^\mu K_\nu)(\sigma_{\mu\delta} + \Omega_{\mu\delta})_{;\gamma}\epsilon^{\nu\gamma\delta} = H_1, \tag{B8}$$

$$-\Omega a_{II} - \frac{1}{2}(L^\mu S_\nu + S^\mu L_\nu)(\sigma_{\mu\delta} + \Omega_{\mu\delta})_{;\gamma}\epsilon^{\nu\gamma\delta} = H_2. \tag{B9}$$

Finally, by contracting Eq. (A8) with **KK**, **KL**, **LL** and **SS** we obtain

$$-\frac{1}{3}(X_I - 4\pi\mu)_{,\delta}V^\delta + \frac{1}{9}\mathscr{E}_I(3\Theta + \sigma_{II} - \sigma_I) + \frac{1}{9}(2\sigma_{II} + \sigma_I)\mathscr{E}_{II}$$

$$- K_\nu\epsilon^{\nu\gamma\kappa}[H_{1,\kappa}S_\gamma + H_1 S_{\gamma;\kappa} + H_2(S_{\mu;\kappa}L_\gamma K^\mu + L_{\mu;\kappa}S_\gamma K^\mu)] + \Omega X_{KL}$$

$$= 2a_{II}H_1 - \frac{4\pi}{3}\left(\mu + P + \frac{1}{3}\Pi_I\right)(\sigma_I + \Theta) - 8\pi a_I q_I - \frac{4\pi}{B}(q_I)_{,r} - \frac{4\pi q_{II}A}{\sqrt{A^2 B^2 r^2 + G^2}}\left(\frac{GB_{,t}}{A^2 B} + \frac{B_{,\theta}}{B}\right), \tag{B10}$$

$$-X_{KL,\delta}V^\delta + \frac{1}{6}\Omega(X_{II} - X_I) - \frac{1}{2}X_{KL}(2\Theta - \sigma_I - \sigma_{II}) + a_I H_1 - H_2 a_{II}$$

$$-\frac{1}{2}[(H_{1,\kappa}S\gamma + H_1(S_{\gamma;\kappa} + S_{\mu;\kappa}K_\gamma K^\mu) + H_2 S_\gamma L_{\mu;\kappa}K^\mu)\epsilon^{\beta\gamma\kappa}L_\beta - (H_1 K^\mu S_\gamma + H_2 S^\mu L_\gamma)L_{\mu;\kappa}\epsilon^{\beta\gamma\kappa}K_\beta] - \frac{1}{2}(H_{2,\kappa}S_\gamma + H_2 S_{\gamma;\kappa})\epsilon^{\beta\gamma\kappa}K_\beta$$

$$= \frac{8\pi}{3}\Pi_{KL}(\Theta - \sigma_I - \sigma_{II}) - 4\pi a_{II}q_I - 2\pi(K^\mu L^\nu + K^\nu L^\mu)q_{\nu;\mu} - 4\pi a_I q_{II}, \tag{B11}$$

$$\frac{1}{3}(-X_{II} + 4\pi\mu)_{;\delta}V^\delta + \frac{1}{9}\mathscr{E}_{II}(3\Theta + \sigma_I - \sigma_{II}) + \frac{1}{9}(2\sigma_I + \sigma_{II})\mathscr{E}_I - \Omega X_{KL}$$

$$- [H_{2;\kappa}S_\gamma - H_1(S_\gamma L_{\mu;\kappa}K^\mu + L_{\mu;\kappa}S^\mu K_\gamma) + H_2 S_{\gamma;\kappa}]\epsilon^{\beta\gamma\kappa}L_\beta + 2a_I H_2$$

$$= -\frac{4\pi}{3}\left(\mu + P + \frac{1}{3}\Pi_{II}\right)(\sigma_{II} + \Theta) - 8\pi a_{II}q_{II} - 4\pi L^\mu L_\nu q^\nu_{;\mu}, \tag{B12}$$

$$\frac{1}{3}(X_I + X_{II} + 4\pi\mu)_{;\delta}V^\delta + \frac{1}{3}(X_I + X_{II})(\Theta + \sigma_I + \sigma_{II}) + \frac{1}{9}(2\sigma_I + \sigma_{II})\mathscr{E}_I + \frac{1}{9}(2\sigma_{II} + \sigma_I)\mathscr{E}_{II}$$

$$- (H_{1,\kappa}K_\gamma + H_{2,\kappa}L_\gamma + H_1 K_{\gamma;\kappa} + H_2 L_{\gamma;\kappa})\epsilon^{\beta\gamma\kappa}S_\beta + 2(H_1 a_{II} - H_2 a_I)$$

$$= \frac{4\pi}{3}(\mu + P)(\sigma_I + \sigma_{II} - \Theta) - \frac{8\pi}{9}(\Theta + 2\sigma_I + 2\sigma_{II})(\Pi_I + \Pi_{II}) - 4\pi q_I\frac{C_{,r}}{BC} - \frac{4\pi q_{II}A}{\sqrt{A^2 B^2 r^2 + G^2}}\left(\frac{GC_{,t}}{A^2 C} + \frac{C_{,\theta}}{C}\right). \tag{B13}$$

Contracting Eq. (A9) with **K** and **L** produces

$$-\frac{1}{3}X_{I,\beta}K^{\beta} - X_{KL,\beta}L^{\beta} - \frac{1}{3}(2X_I + X_{II})(K^{\beta}_{;\beta} - a_{\nu}K^{\nu}) - \frac{1}{3}(X_I + 2X_{II})L_{\mu;\beta}L^{\beta}K^{\mu}$$

$$- X_{KL}(L_{\mu;\beta}K^{\mu}K^{\beta} + L^{\beta}_{;\beta} - a_{\beta}L^{\beta}) - \frac{1}{3}H_2(\sigma_I + 2\sigma_{II}) - 3\Omega H_1$$

$$= \frac{8\pi}{3}\mu_{;\beta}K^{\beta} - \frac{4\pi}{3}q_I(2\Theta - \sigma_I) + 12\pi\Omega q_{II}, \tag{B14}$$

$$-\frac{1}{3}X_{II,\beta}L^{\beta} - X_{KL,\beta}K^{\beta} - \frac{1}{3}(X_I + 2X_{II})(L^{\beta}_{;\beta} - a_{\beta}L^{\beta}) - \frac{1}{3}(2X_I + X_{II})K_{\mu;\beta}L^{\mu}K^{\beta}$$

$$- X_{KL}(K_{\mu;\beta}L^{\mu}L^{\beta} + K^{\beta}_{;\beta} - a_{\beta}K^{\beta}) + \frac{1}{3}H_1(2\sigma_I + \sigma_{II}) - 3\Omega H_2$$

$$= \frac{8\pi}{3}\mu_{;\beta}L^{\beta} - 12\pi\Omega q_I - \frac{4\pi q_{II}}{3}(2\Theta - \sigma_{II}). \tag{B15}$$

Contracting Eq. (A10) with **S** yields

$$-\frac{1}{3}X_{KL}(\sigma_{II} - \sigma_I) + a_I H_1 + a_{II}H_2 - H_{1,\delta}K^{\delta} - H_{2,\delta}L^{\delta} - H_1(K^{\delta}_{;\delta} + K^{\nu}_{;\delta}S^{\delta}S_{\nu}) - H_2(L^{\delta}_{;\delta} + S^{\delta}S_{\nu}L^{\nu}_{;\delta})$$

$$= \left\{8\pi\left[\mu + P - \frac{1}{3}(\Pi_I + \Pi_{II})\right] - Y_I - Y_{II}\right\}\Omega - \frac{4\pi A(q_I B)_{,\theta}}{B\sqrt{A^2 B^2 r^2 + G^2}}$$

$$+ \frac{4\pi A}{B\sqrt{A^2 B^2 r^2 + G^2}}\left[\frac{q_{II}\sqrt{(A^2 B^2 r^2 + G^2)}}{A}\right]_{,r}, \tag{B16}$$

whereas by contracting Eq. (A11) with **SK** and **SL** we obtain

$$-\frac{2}{3}a_{II}\mathcal{E}_I + 2a_I\mathcal{E}_{KL} - E^{\delta}_{2;\delta}L^2 - \frac{AY_{I,\theta}}{3\sqrt{A^2 B^2 r^2 + G^2}} + \frac{Y_{KL,r}}{B}$$

$$- \left[\frac{1}{3}(2Y_I + Y_{II})K_{\beta;\delta} + \frac{1}{3}(2Y_{II} + Y_I)K^{\nu}L_{\nu;\delta}L_{\beta} + Y_{KL}(L_{\nu;\delta}K^{\nu}K_{\beta} + L_{\beta;\delta})\right]\epsilon^{\gamma\delta\beta}S_{\gamma}$$

$$+ H_{1,\delta}V^{\delta} + \frac{1}{3}H_1(3\Theta + \sigma_{II} - \sigma_I) + \Omega H_2 = -\frac{4\pi}{3}\mu_{,\theta}L^2 + 12\pi\Omega q_I + \frac{4\pi q_{II}}{3}(\sigma_I + \Theta), \tag{B17}$$

$$\frac{2a_I}{3}\mathcal{E}_{II} - 2a_{II}\mathcal{E}_{KL} + E^{\delta}_{\beta;\delta}K^{\beta} + \frac{Y_{II,r}}{3B} - \frac{AY_{KL,\theta}}{\sqrt{A^2 B^2 r^2 + G^2}}$$

$$- \left[-\frac{1}{3}(2Y_I + Y_{II})L_{\nu;\delta}K^{\nu}K_{\beta} + \frac{1}{3}(2Y_{II} + Y_I)L_{\beta;\delta} + Y_{KL}(K_{\beta;\delta} - K^{\nu}L_{\beta}L_{\nu;\delta})\right]\epsilon^{\gamma\delta\beta}S_{\gamma}$$

$$+ H_{2,\delta}V^{\delta} + \frac{1}{3}H_2(3\Theta + \sigma_I - \sigma_{II}) - \Omega H_1 = \frac{4\pi}{3}\mu_{,\beta}K^{\beta} - \frac{4\pi q_I}{3}(\sigma_{II} + \Theta) + 12\pi\Omega q_{II}. \tag{B18}$$

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