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Vacuum fluctuation of conformally coupled scalar field in FLRW space-times

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The regularized vacuum fluctuation related to a conformally coupled massless scalar field defined on a space-time with dynamical horizon is computed with respect a radially moving observer in a generic flat Friedmann-Lemaître-Robertson-Walker space-time. Two simple measurement prescriptions are given in order to remove the ambiguity associated with the short distance singularity of the correlation function. In some cases, it turns out that one is dealing with a quantum thermometer, recovering a proposal due to Buchholz *et al.* in order to determine local temperature in the framework of quantum field theory. In general, by arranging the detector so that it does not register for inertial motion in flat space, the regularized quantum fluctuation may be used as a probe of space-time geometry and, in particular, may provide information on the Hubble parameter. As an aside, it is not possible in general to fully decouple the effect of the detector's motion from the Universe expansion, a fact that could be interpreted as a kind of Machian effect, which can be traced back to the global nature of the vacuum.

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I. INTRODUCTION

Relativistic theories of gravity on flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-times have become important in modern cosmology after the discovery of the current cosmic acceleration, the onset of the dark energy issue, and the confirmation of inflationary models. Among the several descriptions of the current accelerated expansion of the Universe, the simplest one is the introduction of a small positive cosmological constant in the framework of general relativity, so that one is dealing with a perfect fluid for which the equation-of-state parameter $\omega = -1$. This fluid model is able to describe the current cosmic acceleration, but also other forms of fluid (phantom, quintessence, inhomogeneous fluids,…) satisfying a suitable equation of state are not excluded, since the observed small value of the cosmological constant leads to several conceptual problems, such as the role of vacuum energy and the coincidence problem. For this reason, several different approaches to the dark energy issue have been proposed. Among them, the modified theories of gravity (see, for example, Refs. [1–5] and references therein) represent an interesting extension of Einstein's theory. Unfortunately, a large class of these modified models admit future singularities, the worst being the so-called big rip singularity [6,7].

With regard to quantum fields in curved space-time, the other *leitmotif* of this paper, one of its most important predictions is the Hawking radiation [8]. Several derivations of this effect can be found in literature [9–13], and recently the search for experimental verification making use of analog models was pursued by many investigators (see, for example, Refs. [14,15]).

In a seminal paper, Parikh and Wilczek [16] (see also Ref. [17]) introduced a further approach, the so-called tunneling method, for investigating corrections to the standard semiclassical treatment of Hawking radiation. A variant of their method has been also introduced and called the Hamilton–Jacobi tunneling method [18–20]. This method is covariant and enjoys the peculiar feature to admit a generalization to the dynamical case [21–23]. For a recent review, see Ref. [24] and the references therein, and for a rigorous quantum theoretical approach, see also Ref. [25].

Recall that in the tunneling approach the semiclassical emission rate reads

$$\Gamma \propto |\text{Amplitude}|^2 \propto e^{-2\frac{\Im I}{\hbar}}. \tag{1}$$

with $\Im I$ standing for the imaginary part of the classical action. The leading terms in the semi-classical approximation of the

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tunneling probability, taking into account energy conservation and the perturbed geometry, reads

$$\Gamma \propto e^{-\frac{2\pi}{\kappa_H}\omega(1-\omega/2M)}, \qquad (2)$$

in which an energy ω of the particle is released and the surface gravity evaluated at the horizon $\kappa_H = 1/4M$ appears. It is amusing that the exponent is the variation in the entropy of the black hole caused by the emission process. Also, by neglecting the small factor $\omega/2M$, one gets the Boltzmann tail of the thermal distribution.

From this asymptotic, one obtains the Hawking temperature by the identification $T_H = \frac{\kappa_H}{2\pi}$. The method is quite general and works for a generic stationary black hole. With appropriate mathematical notions of horizons and surface gravities, the above formula is still valid in the spherically symmetric dynamical case in which the mass and possibly other parameters depend on time, as shown in Refs. [21–23,26], but the interpretation of $\kappa_H/2\pi$ as an effective temperature parameter is more delicate[1] and can be answered in principle by asking what a local detector on a given trajectory can actually detect. However, there is another sense in which $\frac{\kappa_H}{2\pi}$ can be considered as an effective T_H. If the dynamical horizon really emits blackbody radiation in the surrounding vacuum, then by Liouville's theorem, the distribution function of the radiation is constant along phase space trajectories and therefore must have a temperature equal to that of the emitting horizon but redshifted as predicted by general relativity. If it does not emit as a blackbody, one can still define a local temperature by comparing the radiation density at each point in phase space with the equilibrium Planck density, but such a local temperature will generally depend on the frequency and direction.

With regard to this temperature issue, we are particularly interested in the cosmological scenario, and we would like to continue the investigation by making use of quantum field theory to evaluate the fluctuation of the simplest quantum probe at our disposal, namely, a conformally coupled scalar field defined on a spherically symmetric space-time with horizons (see Refs. [27–29] and references therein).

We mainly restrict our analysis to a flat FLRW space-time, which is a spherically symmetric dynamical space-time admitting in principle several past- and future-oriented dynamical horizons. We can now briefly review the general formalism [30–32] and the relevant quantities that will be used.

Any spherically symmetric metric can locally be expressed in the form

$$ds^2 = \gamma_{ij}(x^i)dx^i dx^j + R^2(x^i)d\Omega^2, \qquad i, j \in \{0, 1\}, \qquad (3)$$

[1]The radiating system is obviously not in global thermodynamic equilibrium.

where the two-dimensional metric

$$d\gamma^2 = \gamma_{ij}(x^i)dx^i dx^j \qquad (4)$$

is referred to as the normal metric, $\{x^i\}$ are associated coordinates, and $R(x^i)$ is the areal radius, considered as a scalar field in the two-dimensional normal space. A relevant scalar quantity in the reduced normal space is

$$\chi(x) = \gamma^{ij}(x)\partial_i R(x)\partial_j R(x), \qquad (5)$$

since the dynamical trapping horizon, if it exists, is defined by

$$\chi(x)|_H = 0, \qquad (6)$$

provided the condition $\partial_i \chi|_H \neq 0$ is satisfied. One significant scalar in the normal space is given by the interesting proposal due to Hayward [31],

$$\kappa_H = \frac{1}{2}\Box_\gamma R|_H, \qquad (7)$$

which is a generalization of the usual Killing surface gravity. This is the quantity that appears in the tunneling rate (1). But there is another one, still given by Hayward [32], that is defined by computing on the horizon the quantity

$$\mathcal{K}_H = \frac{1}{2}\sqrt{-n^\mu \nabla_\mu \theta}|_H, \qquad (8)$$

where θ is the expansion of an appropriately oriented null geodesic congruence with tangent vector l^μ and n^μ is another future-pointing null congruence such that $n \cdot l = -1$.

As an example, let us consider the flat FLRW space-time; the metric is usually written in the form

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2), \qquad (9)$$

the coordinates are $x = (t, r)$, the areal radius is $R = a(t)r$, and the normal metric simply reads

$$d\gamma^2 = -dt^2 + a^2(t)dr^2. \qquad (10)$$

Thus,

$$\chi = -(\partial_t R)^2 + \frac{1}{a^2(t)}(\partial_r R)^2 = 0; \qquad (11)$$

namely, the trapping horizon is located at $r_H = \frac{1}{\dot{a}}$ and in terms of the areal radius reads

$$R_H = a(t)r_H = \frac{1}{H(t)}, \qquad (12)$$

where the Hubble parameter is given by $H(t) = \frac{1}{a}\frac{da}{dt}$. The quantity R_H is known as the Hubble sphere, but we may also refer to it as the Hubble dynamical horizon in Hayward's terminology. Thus, it turns out that $\kappa_H = R_H(t)\mathcal{K}_H^2/2$, so that both definitions contain the same information.

Furthermore, in a spherical symmetric dynamical case, it also is possible to introduce the Kodama vector field K. Given the metric Eq. (3), the Kodama vector components are

$$K^i(x) = \frac{1}{\sqrt{-\gamma}}\varepsilon^{ij}\partial_j R, \qquad K^\theta = 0 = K^\varphi. \tag{13}$$

We may introduce the Kodama trajectories, and related Kodama observer, by means of integral lines of the Kodama vector,

$$\frac{dx^i}{d\lambda} = K^i = \frac{1}{\sqrt{-\gamma}}\varepsilon^{ij}\partial_j R. \tag{14}$$

As a result, $\frac{dR(x(\lambda))}{d\lambda} = 0$. Thus, in generic spherically symmetric space-times, the areal radius R is conserved along Kodama trajectories. In other words, a Kodama observer is characterized by the condition $R = R_0$. The operational interpretation goes as follows. Static observers in static black hole (BH) space-times become, in the dynamical case, Kodama observers for which the velocity is

$$v_K^i = \frac{K^i}{\sqrt{\chi}}, \qquad \gamma_{ij}v_K^i v_K^j = -1. \tag{15}$$

The energy measured by this Kodama observer at fixed areal radius R_0 is

$$E = -v_K^i\partial_i I = -\frac{K^i\partial_i I}{\sqrt{\chi_0}} = \frac{\omega}{\sqrt{\chi_0}}, \tag{16}$$

where I is the classical action of the relativistic particle, $\omega = -K^i\partial_i I$ and $\partial_i I$ being its momentum. As a consequence, the tunneling rate may be written also as

$$\Gamma \simeq e^{-\frac{2\pi}{\kappa_H}\sqrt{\chi_0}E} \simeq e^{-\frac{E}{T_0}}, \tag{17}$$

and the local quantity T_0,

$$T_0 = \frac{T_H}{\sqrt{\chi_0}}, \qquad T_H = \frac{k_H}{2\pi}, \tag{18}$$

evaluated at radial radius R_0 is also invariant, since it contains the invariant factor $\sqrt{\chi}$. In the static case $\chi = g^{rr} = -g_{00}$ and recalling Tolman's theorem, "for a gravitational system at thermal equilibrium in a static gravitational field, the local temperature satisfies $T\sqrt{-g_{00}} = \text{constant}$." As a consequence, $T_H = \frac{\kappa_H}{2\pi}$ is the intrinsic temperature of the BH: the Hawking temperature. In the general static case, we confirmed this result by making use of the Unruh–de Witt detector formalism [28]. In the dynamical case, the full interpretation is still missing, and one of the aims of this paper is to give a contribution in order to clarify this issue for cosmological horizons by making use of concepts in linear quantum field theory.

Thus, in this paper, we shall evaluate the regularized vacuum expectation value, i.e., the quantum vacuum fluctuation, given formally by $\langle\phi^2(x)\rangle$, where $\phi(x)$ is a conformally coupled quantum field defined on a flat FLRW space-time. This is an ill-defined quantity, and a regularization is necessary. The computation of the coincidence limit will be done along world lines parametrized by the proper time or arc length in the spacelike case. Therefore, the final result will also depend on the invariant acceleration (the norm of the corresponding 4-vector) of an arbitrary observer, and the role of Kodama observers will be investigated. The computation of $\langle\phi^2(x)\rangle$ in a black hole space-time was also used recently in Ref. [33] to discuss the reality of the firewall proposal around the black hole horizon, and it is conceivable that the analysis of these authors could be relevant for the case of cosmic horizons as well.

Along the way, we shall also discuss the physical meaning of the renormalization procedures by insisting that they have to correspond to measurement procedures and point out that many of them have no operational meaning in general. We stress that the obtained results are approximatively valid for all states for which the leading singularity is of Hadamard's type.

In some special cases, including the important static black holes, this renormalized fluctuation gives direct information on the temperature associated with the quantum field at thermal equilibrium, and according to Buchholz [34] (see also the recent paper of Ref. [35]), in these cases, one is dealing with a quantum thermometer. In general, the quantum fluctuation will still give information on FLRW space-times. We then show that the fluctuations as measured locally by a comoving observer are isotropic, but they do not take the form of a quantum thermal bath with some characteristic horizon temperature parameter, as might be expected from general thermodynamical arguments based on horizon physics [36].

The paper is organized as follows. In Sec. II, the vacuum fluctuation is introduced, including its formal renormalization. In Sec. III, the general formula for the renormalized vacuum fluctuation is derived, and a physical meaning is attached to the formal renormalization procedure. Some applications are also discussed in Sec. IV, where the peculiar class of Kodama observers is analyzed in relation to our problem. Conclusions are given in Sec. V.

II. $\langle\phi^2\rangle$ AS AN OBSERVABLE

In the present section, we discuss the role of the the quantum vacuum fluctuation, given formally by $\langle\phi^2(x)\rangle$, where $\phi(x)$ is a quantum field defined on a generic curved space-time. It has been calculated in a cosmological context in Refs. [37–39]. A related quantity is the off-diagonal Wigthman function, given by

$$W(x,x') = \langle \phi(x)\phi(x') \rangle \tag{19}$$

and for which the evaluation in the coincidence limit $x' \to x$ gives the fluctuation at space-time event x. However, on general grounds, $W(x,x')$ in this limit possesses the so called Hadamard singularity: in $D = 4$, one has (see, for example, Ref. [40])

$$W(x,x') = \langle \phi(x)\phi(x') \rangle = \frac{1}{4\pi^2}\frac{D(x,x')}{\sigma^2(x,x')} + U(x,x')\ln(\lambda\sigma^2(x,x')) + V(x,x'), \tag{20}$$

where $\sigma^2(x,x')$ is the geodesic distance between x and x'; λ is a characteristic dimensional parameter (a mass or a scalar curvature term); and D, U, and V are smooth functions, regular at the coincidence limit. We left understood the presence of the $i\epsilon$ factor, which allows one to deal with tempered distributions. This means that in any case, at the coincidence limit, $W(x,x)$ is singular.

One of the simplest regularizations consists of removing the related Hadamard singularity associated with a reference space-time, typically Minkoswki space. But in general, the presence of the logarithmic divergence introduces a finite logarithmic ambiguity in the form of a dimensional parameter μ^2. The structure of the Hadamard singularity depends on the geometry of the space-time and on the differential operator L associated with the equation of motion of the field $\phi(x)$, while the finite part depends on the chosen quantum state. In our case, for the sake of simplicity, we consider neutral quasifree scalar fields so that the operator L consists of the D'Alembertian operator plus a term that may depend on the gravitational coupling of the field and on its mass. The renormalized value of the fluctuation $\langle \phi^2(x) \rangle_R$ contains physical information, and in this sense, it is an observable much simpler than the renormalized vacuum expectation value of the stress-energy tensor.

As an illustrative example, let us consider a free massive scalar field defined on the Euclidean manifold $S_1 \times R^3$, obtained by "rotating" Minkowski time to imaginary values and then compactifying it with periodicity β. It is well known that in this case one is dealing with a massive quantum scalar field in thermal equilibrium at temperature $T = \frac{1}{\beta}$. The relevant operator is

$$L = -\partial_\tau^2 - \nabla^2 + M^2, \tag{21}$$

τ being the imaginary time with period β.

One may compute the regularized fluctuation by means of the zeta-function regularization (see, for example, Refs. [41–43] and references therein). The general formula is [44,45]

$$\langle \phi(x)^2 \rangle_R = \lim_{\varepsilon\to 0}\left[\frac{d}{d\varepsilon}(\varepsilon\zeta(1+\varepsilon,x)) + \varepsilon\zeta(1+\varepsilon,x)\ln\mu^2\right], \tag{22}$$

where $\zeta(z,x)$ is the local zeta function associated with L and μ^2 is an arbitrary mass scale present when there is a pole of the local zeta function at $z = 1$. We omit the details of the calculation, giving instead the result:

$$\langle \phi(x)^2 \rangle_R = \frac{M}{2\pi\beta}\sum_{n=1}^{\infty}\frac{K_1(n\beta M)}{n} + \frac{M^2}{8\pi^2}\ln\left(\frac{M^2}{\mu^2}\right). \tag{23}$$

Here, $K_1(x)$ is the modified Bessel function. If M is not vanishing, the thermal properties are not transparent. Furthermore, the ambiguity given by the arbitrary mass scale μ^2 is still present, and a physical renormalization prescription is needed. In the massless case, there is a drastic simplification, and it is easy to show that the logarithmic term (with its arbitrary mass scale) is absent in the Hadamard singularity: the regularized result reads

$$\langle \phi(x)^2 \rangle_R = \frac{1}{12\beta^2} = \frac{T^2}{12}. \tag{24}$$

Thus, in this particular case, the fluctuation acts as a quantum thermometer [34]. However, one should observe that in a generic curved space-time and for an arbitrary gravitational coupling a logarithmic term is always present in the Hadamard singularity, even though one is dealing with a massless scalar field: as a consequence of the regularization process, a finite logarithmic term with an arbitrary mass scale μ^2 is also present. However, if one restricts the analysis to the massless conformally coupled case, one may get rid of the logarithmic term (see, for example, Refs. [46,47]).

As we have just seen, the proposal put forward by Buchholz and collaborators seems to work for the massless scalar field at finite temperature on Minskoswki space-time. In Ref. [34], also the de Sitter space-time was investigated, and we shall revisit this important case. For a Schwarzschild black hole, the situation is not so simple, in the sense that the renormalized vacuum fluctuation still gives information on Hawking temperature but in a less direct way. In fact, the result for the renormalized vacuum fluctuation of a massless conformally coupled scalar field on a Hartle–Hawking state reads [46]

$$\langle \phi(x)^2 \rangle_R = \frac{T_H^2}{12V} - \frac{T_U^2}{12V} + T_H\Delta_H, \tag{25}$$

where $T_H = \frac{1}{8\pi M}$ is the Hawking temperature, $T_U = \frac{M}{2\pi r^2}$ is the Unruh temperature, $V = 1 - \frac{2M}{r}$ is the lapse function, and Δ_H a finite contribution that can be numerically evaluated. It should be noted that on the horizon, since one is working on the Hartle–Hawking state, the vacuum fluctuation is finite because the local (redshifted) Hawking and Unruh temperature divergences cancel. Conversely, the finiteness of the correlation implies the value of the Hawking temperature.

Motivated by these arguments, we would like to consider a conformally coupled scalar field in a FLRW conformally flat space-time. In this case, the off-diagonal Wigthman function is given by (see, for example, Refs. [27,28])

$$W(x,x') = \langle \phi(x)\phi(x')\rangle = \frac{1}{4\pi^2}\frac{1}{\sigma^2(x,x')}, \qquad (26)$$

where $\sigma^2(x,x') = a(t)a(t')(x-x')^2$ and $a(t)$ is the scale factor. In the coincidence limit $x \to x'$, one has the Hadamard singularity without the logarithmic term.

III. A LOCAL EXPANSION OF $\langle \phi^2(x)\rangle$ IN TERMS OF PARAMETRIZED WORLD LINES

As we argued in the previous section, massless conformally coupled fields could be a convenient probe to investigate at quantum level the properties of nontrivial space-times through the quantity $\langle \phi^2(x)\rangle$. To calculate this quantum fluctuation, one needs to evaluate to the inverse of the square geodesic distance between two events. For our purposes, the geodesic distance may be conveniently expressed in terms of the corresponding world line $x(s)$ parametrized by the proper length, which is the proper time τ along timelike trajectories and the usual arc length along spacelike curves. This approach is similar to the one described in detail in the monograph [10], and it is the core of the adiabatic point-splitting regularization method. It is also strictly related to Unruh–de Witt detector approach (see, for example, Ref. [10] and references therein and the references contained in the recent papers, Refs. [28,48]). For a rigorous recent approach, see also Ref. [49]. In the following, we shall discuss both timelike and spacelike correlations. We shall use both forms of the spatially flat FLRW metric

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2) \\ &= a^2(\eta)(-d\eta^2 + dr^2 + r^2 d\Omega^2), \end{aligned} \qquad (27)$$

where, by a slight abuse of notation, we denoted with the same symbol two really different functions, namely, $a(t)$ and $a(\eta) = a(t(\eta))$; here, $\eta = \int dt/a(t)$ is the conformal time.

A. Timelike correlations

The proper-time parametrized Wightman function reads

$$W(x(\tau), x'(\tau')) = \frac{1}{4\pi^2}\frac{1}{\sigma^2(\tau,\tau')}, \qquad (28)$$

where

$$\sigma^2(\tau,\tau') = a(\tau)a(\tau')(x(\tau) - x(\tau'))^2 \qquad (29)$$

is computed in the FLRW metric. Because of the isotropy of FLRW space-time, we may restrict the analysis to radial trajectories, namely, $x(\tau) = (\eta(\tau),\ r(\tau))$. To discuss the coincidence limit, we put $\varepsilon = \tau - \tau'$. An overdot will mean derivation with respect to proper time τ. We define $a(\tau) = a(\eta(\tau))$ so that $\dot a = a^2 H \dot\eta$ and so on.

It will be sufficient to make an expansion to fourth order in ε of Eq. (29),

$$\begin{aligned} \sigma^2(\tau,\tau-\varepsilon) \simeq{}& -\varepsilon^2 a^2(\tau)(\dot\eta^2 - \dot r^2) + \frac{1}{2}\varepsilon^3 \partial_\tau[a^2(\tau)(\dot\eta^2 - \dot r^2)] \\ &+ \frac{1}{12}\varepsilon^4[6a\ddot a(\dot r^2 - \dot\eta^2) + 12a\dot a(\dot r\,\ddot r - \dot\eta\,\ddot\eta) \\ &+ 3a^2(\ddot r^2 - \ddot\eta^2) + 4a^2(\dot r\,\dddot r - \dot\eta\,\dddot\eta)]. \end{aligned} \qquad (30)$$

To simplify the expression, we just need to enforce the following relations:

(i) $a^2\dot x^2 = -1$ (for timelike trajectories) and its derivatives with respect to τ.
(ii) the relation between cosmic time and conformal time $d\eta/dt = a^{-1}$.
(iii) $\dot r = a^{-1}\sqrt{\dot t^2 - 1}$.

Because of relation i, the coefficient of ε^2 is 1, while the coefficient of ε^3 is zero. As regards the detailed calculations of the ε^4 term, see Appendix A. The result in terms of $\dot t$ and $H(t) = \partial_t a(t)/a(t)$ is

$$\begin{aligned} \sigma^2(\tau,\varepsilon) ={}& -\varepsilon^2 - \frac{1}{12}\left[\frac{\ddot t^2}{\dot t^2 - 1} + 2\ddot t H + \dot t^2(H^2 + 2\partial_t H)\right]\varepsilon^4 \\ &+ O(\varepsilon^6). \end{aligned} \qquad (31)$$

Note that $\dot t$ fully determines the trajectory via iii, while $H(t)$ is determined by $a(t)$ (i.e., by the model with which one is dealing). Furthermore, the τ-dependent coefficient in square brackets can be rewritten in a more enlightening way,

$$\sigma^2(\tau,\varepsilon) = -\varepsilon^2 - \frac{1}{12}[A^2 + H^2 + 2\dot t^2\partial_t H]\varepsilon^4 + O(\varepsilon^6), \qquad (32)$$

where

$$A^2 = \left[\frac{\ddot t}{\sqrt{\dot t^2 - 1}} + H\sqrt{\dot t^2 - 1}\right]^2$$

is the square of the 4-acceleration along the trajectory.

At this point, we must discuss a crucial matter: the renormalization of the singular term in the Wightman function. We think that, in order to avoid any possible ambiguity, this should be done by making reference to the actual method of measurement. It seems there are many possibilities, but to keep the matter as plain as possible, we will consider two simple and sensible ways to define the finite part of the correlation. The first is to subtract the value for Minkowski space on a linear trajectory, a flat geodesics in fact; the second is to subtract the value on the actual trajectory as embedded in flat space. Thus, in the first case, we set the detector so that it registers no signal when it is at rest in a freely falling frame; in the second case, we set it so that it gives no signal when it is moving on the actual trajectory again in a freely falling frame. If the detector is small with respect to the curvature scale, for example, a pointlike monopole detector, well-known arguments based on the equivalence principle will imply that the detector in general will register a signal when it is moving or is at rest in a comoving frame. Since Minkowski space can be locally reached by passing to a freely falling frame, both prescriptions are in principle achievable. By contrast, any subtraction corresponding to a globally nonisometric space-time has no operational meaning in the given space-time and should not be used. For instance, we can give $\langle\phi^2\rangle$ any value we like by subtracting its unrenormalized value in a contracting Universe, but clearly there is no natural physical meaning in this entirely arbitrary procedure. In the case of inertial trajectories in Minkowski space-time, one has $\dot{x}^\mu = U^\mu\tau$, $U^2 = -1$, and $a(t) = 1$. Thus,

$$\sigma_M^2(\tau,\varepsilon) = -\varepsilon^2. \tag{33}$$

The renormalization following from the first prescription requires the subtraction of this contribution from the expression given by Eq. (32). The renormalized vacuum fluctuation is then given by

$$\langle\phi^2(x)\rangle_R = \frac{1}{48\pi^2}\left[\frac{\ddot{t}^2}{\dot{t}^2-1} + 2\ddot{t}H + \dot{t}^2(H^2+2\partial_t H)\right] \tag{34}$$

or the equivalent form

$$\langle\phi^2(x)\rangle_R = \frac{1}{48\pi^2}[A^2 + H^2 + 2\dot{t}^2\partial_t H]. \tag{35}$$

This result is in agreement with the one obtained by Obadia [27] within the Unruh–deWitt detector approach. Furthermore, in this form, it shows in a clear way the contribution coming from the proper motion along the trajectory (through A and hence $\dot{t}$) and the one coming from the dynamics of the cosmological model (through H). The third term in square brackets represents a mixed contribution, which vanishes for stationary cosmological models. Thus, adjusting the detector so that it does not register for inertial motion in flat space provides a probe of certain features of space-time geometry and of the actual motion. Most interesting, of course, would be the proper motion of our neighborhood relative to the Hubble flow.

Interesting is the case whereby $\dot{H} \ll H^2$: then, $\langle\phi^2\rangle$ has a thermal interpretation in terms of de Sitter temperature (even in the presence of some acceleration; see below). In the general case, there is not such an interpretation. For instance, in the Einstein–de Sitter model, $\dot{H} = -3H^2/2$ so that $\langle\phi^2\rangle_R = -H^2/24\pi^2$ despite the presence of a trapping horizon.

For a comoving observer (we will see that this is a special class of Kodama observers), Eq. (35) reduces to

$$\langle\phi^2(x)\rangle_R = \frac{1}{48\pi^2}[H^2 + 2\dot{H}]. \tag{36}$$

Evidently the correlation acts as a probe for testing both the Hubble parameter as well as its time derivative. In de Sitter space relative to inflationary coordinates, one has $\dot{H} = 0$, and the correlation measures the Gibbons–Hawking temperature of de Sitter space [50] for a geodesic observer [see Eq. (24)]. The renormalization prescription just adopted should be equivalent to a subtraction relative to the Bunch–Davies vacuum, the same to be used in inflation theory, and in this sense is the favorite one, although the conformal field is not very relevant in inflationary theory. But there are other possibilities: an appropriate one for massless fields is perhaps the Kirsten–Garriga vacuum [51], or the various de Sitter α vacua. We have not investigated this matter any more.

The second prescription consists of subtracting the fluctuation on the actual trajectory as embedded in flat Minkowski space, namely, Eq. (34) with $H = 0$. As a result,

$$\langle\phi^2(x)\rangle_R = \frac{1}{48\pi^2}[2\ddot{t}H + \dot{t}^2(H^2 + 2\partial_t H)]. \tag{37}$$

It is designed to separate the Unruh effect from the expansion, but a coupling $\ddot{t}H$ actually remains. It resembles a Machian effect, showing that subtracting the acceleration relative to absolute space, as would be expected in this case, leaves nonetheless a coupling with the whole Universe. It is clearly not possible to cancel this term by renormalization with respect to a freely falling frame because, in such a frame, the rest of the Universe disappears from the stage.

In spherical symmetry, the physical radial trajectory is identified by the variation of the areal radius $R(t) = ra(t)$ so that, making use of the line element Eq. (3) restricted on a radial path,

$$\dot{t} = \frac{1}{\sqrt{1-(\frac{dR}{dt}-\frac{R}{R_H})^2}}, \tag{38}$$

and defining $\mathcal{V} = \frac{dR}{dt} - \frac{R}{R_H}$, one can express Eq. (35) as

$$\langle\phi^2(x)\rangle_R = \frac{1}{48\pi^2}\left[H^2 + \frac{1}{1-\mathcal{V}^2}\left(\frac{\dot{\mathcal{V}}}{\sqrt{1-\mathcal{V}^2}} + H\mathcal{V}\right)^2 + 2\frac{\partial_t H}{1-\mathcal{V}^2}\right], \tag{39}$$

where the second term is the proper acceleration. This expression can also be rewritten in an alternative form,

$$\langle\phi^2(x)\rangle_R = \frac{1}{48\pi^2}\frac{1}{1-\mathcal{V}^2}\left[H^2 + 2\partial_t H + \frac{1}{\sqrt{1-\mathcal{V}^2}}\left(\frac{\dot{\mathcal{V}}^2}{\sqrt{1-\mathcal{V}^2}} + 2H\mathcal{V}\dot{\mathcal{V}}\right)\right], \tag{40}$$

in view of the discussion in the following sections.

B. Spacelike correlations

In this case, we use as a parameter the arc length s along the curve, and we will simplify matters by choosing a purely spatial trajectory within the cosmic time slices of the metric. A simple calculation gives now nothing interesting, namely, the exact result

$$\sigma^2(s,\epsilon) = \epsilon^2. \tag{41}$$

Evidently, we need to consider spacelike trajectories with some extension in time. Formally, we can get the spacelike result by taking imaginary proper time, say $\tau = is$, in Eq. (34); we obtain, isolating the acceleration from the expansion,

$$\sigma^2(s,\epsilon) = \frac{1}{48\pi^2}\left[\frac{t''^2}{t'^2+1} - 2t''H - t'^2(H^2 + 2\partial_t H)\right] + .;. \tag{42}$$

where a prime denotes d/ds. Adopting once more the renormalization with respect to Minkowski space, we see that the spacelike correlations are again a probe of the Universe expansion. But unlike the timelike case, this time there is no detector available and no obvious local method to measure them.[2]

IV. KODAMA TRAJECTORIES

The previous results did not specify any special trajectory. Here, we restrict our analysis to the class of Kodama observers, i.e., the observers characterized by the condition $a(t)r \equiv R(t) = R_0$. The importance of these observers is related to the properties of the Kodama vector field in spherical symmetry (see the introduction). In the static patch of de Sitter space or static spaces, they stay at constant r; in FLRW spaces, they are on top of a constant areal radius, namely, $R = r(t)a(t)$ is a constant, and become null on the Hubble sphere; in Rindler space, they correspond to uniformly accelerated observers in Minkowski space; and so on. We are interested in Kodama trajectories since they are the closest analog of the hyperbolic histories giving rise to characteristic thermal effects.

From Eq. (39), we find

$$\langle\phi^2(x)\rangle_R = \frac{1}{48\pi^2}\frac{1}{1-R_0^2H^2}\left(H^2 + 2\partial_t H + \frac{(\partial_t H)^2 R_0^2}{(1-R_0^2H^2)^2} + \frac{2H^2R_0^2\partial_t H}{\sqrt{1-H^2R_0^2}}\right), \tag{43}$$

which for the comoving observer at $R_0 = 0$ reduces to Eq. (36). Apparently, we see no natural thermal interpretation of this formula.

An expression of the characteristic surface gravity associated with a trapping horizon in cosmology is given by Eq. (7), which when redshifted to a Kodama trajectory takes the form

$$\kappa_H = \left(H + \frac{\dot{H}}{2H}\right)(1 - H^2R_0^2)^{-1/2}. \tag{44}$$

According to a possible interpretation, this time-dependent parameter controls the particle creation rate by the trapping horizon located at $r_H a = H^{-1}$, as given by Eq. (2), lending support to the interpretation of $\kappa_H/2\pi$ as an effective temperature parameter. But Eq. (43), although similar, is quite different from Eq. (24) at the base of Buchholz's proposal for this temperature. Evidently, the cosmological horizon does not create a thermal background radiation with a simple geometric description, as it happens to black holes, unless $\dot{H} = 0$ (this case will be considered in the next subsection).

It seems truly remarkable that a local measurement such as the one involved in the fluctuation has a geometrical interpretation in terms of a quantity that is associated to a very distant, observer-dependent Hubble sphere, and it seems at odds with general results concerning the effects of expansion on local systems. Is this another instance of a Machian effect in cosmology? More modestly, we could simply say that there is at least a renormalization prescription that produces a geometrically meaningful (i.e., tightly connected to space-time geometry) result. But we have to recall that the expectation value is not a truly local quantity because the vacuum state to which it refers actually probes a cosmic time slice in its entirety.

A. Stationary space-times

These observations bring us, as an important check, to consider de Sitter space, which has $H(t) = H_0$ constant in inflationary coordinates.[3] The expression simplifies to

$$\langle\phi^2(x)\rangle_R = \frac{1}{48\pi^2}\frac{H_0^2}{1-R_0^2H_0^2}; \tag{45}$$

[2] In cosmology, the power spectrum is first determined by observations in causal directions; then, the spatial correlations are reconstructed via a Fourier transform.

[3] These are the only ones with flat spatial sections to which our formalism applies.

thus, one has

$$\langle\phi^2(x)\rangle_R = \frac{1}{12}\frac{T_{\rm GH}^2}{(1-R_0^2H_0^2)}, \tag{46}$$

where $T_{\rm GH} = \frac{H_0}{2\pi}$ is the Gibbons–Hawking temperature [50], $(1-R_0^2H_0^2)$ being the kinematical redshift factor. One also has the known result [52]

$$\langle\phi^2(x)\rangle_R = \frac{1}{12}\left(\frac{A_0^2}{4\pi^2}+T_{\rm GH}^2\right), \tag{47}$$

where

$$A_0 = \frac{R_0H^2}{\sqrt{1-H^2R_0^2}}$$

is the invariant acceleration of the Kodama observer (needed to remain at a constant radial distance from the origin in the static frame).

One may obtain a confirmation of the above result by observing that the de Sitter space-time admits a static patch. With regard to this issue, we may discuss the black hole general case. In fact, a generic spherically symmetric static black hole metric reads

$$\begin{aligned} ds^2 &= -V(r)dt_S^2 + \frac{dr^2}{V(r)} + r^2 d\Omega^2 \\ &= V(r^*)[-dt_S^2 + (dr^*)^2] + r^2(r^*)d\Omega^2, \end{aligned} \tag{48}$$

where t_S is the time coordinate in the static patch and r^* is the tortoise coordinate given by $dr^*(r) = \frac{dr}{V(r)}$. Introducing the Kruskal-like coordinates defined by

$$X = \frac{1}{\kappa_H}e^{\kappa r^*}\cosh(\kappa_H t_S), \qquad T = \frac{1}{\kappa_H}e^{\kappa_H r^*}\sinh(\kappa_H t_S),$$

where $\kappa_H = \frac{V'_H}{2}$ is the usual Killing surface gravity, one obtains

$$ds^2 = e^{-2\kappa_H r^*}V(r^*)[-dT^2 + dX^2] + r^2(T,R)d\Omega^2, \tag{49}$$

where now $r^* = r^*(T,X)$. The key point to recall here is that in the Kruskal gauge the normal metric is conformally related to two-dimensional Minkoswki space-time. The second observation is that Kodama observers are defined by the integral curves associated with the Kodama vector; thus, the areal radius r and r^* are *constant*, say $r = r_0$. As a consequence, one is dealing with an effective flat FLRW space-time,

$$\begin{aligned} ds^2 &= V_0 e^{-2\kappa_H r_0^*}(-dT^2 + dX^2) \\ &= -dt^2 + a^2(r_0)dX^2, \end{aligned} \tag{50}$$

where $t = \sqrt{V_0}e^{-\kappa_H r_0^*}T$ is a new "cosmological" time and $a(r_0^*) = \sqrt{V_0}e^{-\kappa_H r_0^*}$ is the related constant scale factor. Furthermore, the proper time along Kodama trajectories reads

$$\begin{aligned} d\tau^2 &= V_0 dt_S^2 = V_0 e^{-2\kappa_H r_0^*}(dT^2 - dX^2) \\ &= dt^2 - a^2(r_0)dX^2. \end{aligned} \tag{51}$$

Finally one also has, as functions of the proper time,

$$\begin{aligned} X(\tau) &= \frac{1}{\kappa_H}e^{\kappa_H r_0^*}\cosh\left(\kappa_H\frac{\tau}{\sqrt{V_0}}\right) \\ T(\tau) &= \frac{1}{\kappa_H}e^{\kappa_H r_0^*}\sinh\left(\kappa_H\frac{\tau}{\sqrt{V_0}}\right). \end{aligned} \tag{52}$$

Since for the metric Eq. (50) one has $H \equiv \partial_t a/a = 0$, we have to apply the general formula (34), namely, the one associated with the first renormalization prescription, obtaining

$$\langle\phi^2(x)\rangle_R = \frac{1}{48\pi^2}\left[\frac{\ddot{t}^2}{\dot{t}^2-1}\right]. \tag{53}$$

For the static black hole, one obtains

$$\dot{t} = \cosh\left(\kappa_H\frac{\tau}{\sqrt{V_0}}\right), \qquad \ddot{t} = \frac{\kappa_H}{\sqrt{V_0}}\sinh\left(\kappa_H\frac{\tau}{\sqrt{V_0}}\right), \tag{54}$$

which gives for the quantum fluctuation

$$\langle\phi^2(x)\rangle_R = \frac{1}{48\pi^2}\frac{\kappa_H^2}{V_0} = \frac{T_H^2}{12V_0}. \tag{55}$$

As a result, also in this case, one has a Buchholz quantum thermometer with Hawking temperature at infinity $T_H = \frac{\kappa_H}{2\pi}$ redshifted by the usual Tolman factor V_0. For de Sitter, $V(r) = 1-H_0^2r^2$ and $T_H = \frac{|\kappa_H|}{2\pi} = \frac{H_0}{2\pi}$, thus recovering the previous result. We may conclude that this temperature is an intrinsic property of de Sitter space, not depending on the coordinates used.[4]

B. Nonstationary space-times: The big rip

For the big rip solution [6,7], one has

$$H(t) = \frac{c}{(t_s-t)^\alpha}, \qquad c>0 \quad \text{and} \quad \alpha>0. \tag{56}$$

As in stationary scenarios, for Kodama observers, the fluctuation diverges on the Hubble horizon $R_H = H^{-1}$. On the other hand, if $0 < R_0 < R_H$, with every $\alpha > 1/2$, the fluctuation at the big rip attains a finite value:

[4]From a canonical perspective, it is an observable in the Bergmann sense of a "gauge-invariant phase-space function."

$$\lim_{t\to t_s}\langle\phi^2(x)\rangle_R = -\frac{1}{48\pi^2}\frac{1}{R_0^2}. \tag{57}$$

If $\alpha = 1/2$, we have instead

$$\lim_{t\to t_s}\langle\phi^2(x)\rangle_R = -\frac{1}{192\pi^2 c^4 R_0^4} - \frac{1}{48\pi^2}\frac{1}{R_0^2}. \tag{58}$$

It is interesting to note that only in this latter case the limiting value depends on $(cR_0)^{-4}$, i.e., on the details of the model through c.

The finite, but negative, values of $\langle\phi^2(x)\rangle_R$ at the big rip are due to the fact that a Kodama observer with $R_0 \neq 0$ has a nonvanishing proper acceleration that diverges toward the singularity: this divergence offsets exactly the infinite contribution coming from $H(t)$.

V. CONCLUSIONS

We have computed at length the simplest local observable available in a conformal scalar field theory in a flat FLRW metric. Instead of throwing away the full short-distance singularity of its Wightman function, as one might have supposed to do, we retain some terms that look meaningful. We have given justifications for this procedure. In de Sitter space, it gives the result obtained in inflation theory for the field in the Bunch–Davies vacuum. The fluctuations as computed show no relation with the thermodynamics of cosmic horizons, except for de Sitter space or in a quasi-de Sitter regime. The Machian flavor of the results, namely, the connection of local measurements with the expansion of the Universe, is explained by the nonlocal character of the vacuum state.

APPENDIX: FOURTH-ORDER TERM IN THE EXPANSION OF σ^2

Let us consider the ϵ^4 term in Eq. (30),

$$\frac{1}{12}[6a\ddot{a}(-\dot{\eta}^2+\dot{r}^2) + 12a\dot{a}(-\dot{\eta}\,\ddot{\eta}+\dot{r}\,\ddot{r}) + 3a^2(-\ddot{\eta}^2+\ddot{r}^2) + 4a^2(-\dot{\eta}\,\dddot{\eta}+\dot{r}\,\dddot{r})]:$$

(1) the first term can be rewritten by using the condition $a^2\dot{x}^2 = -1$. The result is

$$6a\ddot{a}(-\dot{\eta}^2+\dot{r}^2) = -6\frac{\ddot{a}}{a} = -6\dot{t}(\dot{t}\partial_t H + \dot{t}).$$

(2) By using the fact that $\partial_\tau[a^2\dot{x}^2] = 0$, the second term becomes

$$12a\dot{a}(-\dot{\eta}\,\ddot{\eta}+\dot{r}\,\ddot{r}) = 12H^2\dot{t}^2.$$

(3) By deriving once more the condition $\partial_\tau^2[a^2\dot{x}^2] = 0$, the sum of the last two terms can be recast in the form

$$3a^2(-\ddot{\eta}^2+\ddot{r}^2) + 4a^2(-\dot{\eta}\,\dddot{\eta}+\dot{r}\,\dddot{r}) = -H^2\dot{t}^2 + 4H\ddot{t} - \frac{\ddot{t}}{\dot{t}^2-1} + 4\dot{t}^2\partial_t H.$$

Putting all together, the ϵ^4 term is given by

$$-\frac{1}{12}\left[\frac{\ddot{t}^2}{\dot{t}^2-1} + 2H\ddot{t} + \dot{t}^2(H^2 + 2\partial_t H)\right].$$

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