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Spontaneous creation of the universe from nothing

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An interesting idea is that the universe could be spontaneously created from nothing, but no rigorous proof has been given. In this paper, we present such a proof based on the analytic solutions of the Wheeler-DeWitt equation (WDWE). Explicit solutions of the WDWE for the special operator ordering factor $p = -2$ (or 4) show that, once a small true vacuum bubble is created by quantum fluctuations of the metastable false vacuum, it can expand exponentially no matter whether the bubble is closed, flat, or open. The exponential expansion will end when the bubble becomes large and thus the early universe appears. With the de Broglie–Bohm quantum trajectory theory, we show explicitly that it is the quantum potential that plays the role of the cosmological constant and provides the power for the exponential expansion of the true vacuum bubble. So it is clear that the birth of the early universe completely depends on the quantum nature of the theory.

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I. INTRODUCTION

With the Lambda–cold dark matter (ΛCDM) model and all available observations (cosmic microwave background, abundance of light elements), it has been widely accepted that the universe was created in a big bang. However, there are still some puzzles, such as problems of the flatness, the horizon, the monopole, and the singularity [1]. Quantum mechanics has been applied to cosmology to study the formation of the universe and its early evolution. In particular, inflation theories, which suggest that the universe experienced an exponential expansion period, were proposed to solve puzzles of the early universe [2–4]. In quantum cosmology theory, the universe is described by a wave function rather than the classical spacetime. The wave function of the universe should satisfy the Wheeler-DeWitt equation (WDWE) [5]. With the development of quantum cosmology theory, it has been suggested that the universe can be created spontaneously from nothing, where “nothing” means there is neither matter nor space or time [6], and the problem of singularity can be avoided naturally.

Although the picture of the universe created spontaneously from nothing has emerged for a long time, a rigorous mathematical foundation for such a picture is still missing. According to Heisenberg’s uncertainty principle, a small empty space, also called a small true vacuum bubble, can be created probabilistically by quantum fluctuations of the metastable false vacuum. But if the small bubble cannot expand rapidly, it will disappear soon due to quantum fluctuations. In this case, the early universe would disappear before it grows up. On the other side, if the small bubble expands rapidly to a large enough size, the universe can then be created irreversibly.

In this paper, we obtain analytic solutions of the WDWE of the true vacuum bubble. With the de Broglie–Bohm quantum trajectory theory, we prove that once a small true vacuum bubble is created, it has the chance to expand exponentially when it is very small, i.e., $a \ll 1$. The exponential expansion will end when the true vacuum bubble becomes very large, i.e., $a \gg 1$. It is the quantum potential of the small true vacuum bubble that plays the role of the cosmological constant and provides the power for its exponential expansion. This explicitly shows that the universe can be created spontaneously by virtue of a quantum mechanism.

II. WDWE FOR THE SIMPLEST MINISUPERSPACE MODEL

Heisenberg’s uncertainty principle indicates that a small true vacuum bubble can be created probabilistically in a metastable false vacuum. The small bubble has 1 degree of freedom, the bubble radius. We can assume that the bubble is nearly spherical, isotropic and homogeneous, since it is a small true vacuum bubble. As we will show below, the small bubble will expand exponentially after its birth and all asymmetries will be erased by the inflation.

Since the small true vacuum bubble is nearly spherical, it can be described by a minisuperspace model [7–9] with one single parameter of the scalar factor a. The action of the minisuperspace can be written as

$$S = \frac{1}{16\pi G}\int R\sqrt{-g}d^4x. \tag{1}$$

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Since the bubble is homogeneous and isotropic, the metric in the minisuperspace model is given by

$$ds^2 = \sigma^2[N^2(t)dt^2 - a^2(t)d\Omega_3^2]. \quad (2)$$

Here, $N(t)$ is an arbitrary lapse function, $d\Omega_3^2$ is the metric on a unit three-sphere, and $\sigma^2 = 2G/3\pi$ is a normalizing factor chosen for later convenience. Substituting Eq. (2) into Eq. (1), we obtain the Lagrangian

$$\mathcal{L} = \frac{N}{2} a\left(k - \frac{\dot{a}^2}{N^2}\right), \quad (3)$$

where the dot denotes the derivative with respect to the time, t, and the momentum

$$p_a = -a\dot{a}/N.$$

The Lagrangian (3) can be expressed in the canonical form,

$$\mathcal{L} = p_a\dot{a} - N\mathcal{H},$$

where

$$\mathcal{H} = -\frac{1}{2}\left(\frac{p_a^2}{a} + ka\right).$$

In quantum cosmology theory, the evolution of the universe is completely determined by its quantum state that should satisfy the WDWE. With $\mathcal{H}\Psi = 0$ and $p_a^2 = -a^{-p}\frac{\partial}{\partial a}(a^p\frac{\partial}{\partial a})$, we get the WDWE [6,10]

$$\left(\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - ka^2\right)\psi(a) = 0. \quad (4)$$

Here, $k = 1, 0, -1$ are for spatially closed, flat, and open bubbles, respectively. The factor p represents the uncertainty in the choice of operator ordering. For simplicity, we have set $\hbar = c = G = 1$.

III. QUANTUM TRAJECTORY FROM WDWE

The complex function $\psi(a)$ can be rewritten as

$$\psi(a) = R(a)\exp(iS(a)), \quad (5)$$

where R and S are real functions [11,12]. Inserting $\psi(a)$ into Eq. (4) and separating the equation into real and imaginary parts, we get two equations [11,12]:

$$S'' + 2\frac{R'S'}{R} + \frac{p}{a}S' = 0, \quad (6)$$

$$(S')^2 + V + Q = 0. \quad (7)$$

Here $V(a) = ka^2$ is the classical potential of the minisuperspace, the prime denotes derivatives with respect to a, and $Q(a)$ is the quantum potential,

$$Q(a) = -\left(\frac{R''}{R} + \frac{p}{a}\frac{R'}{R}\right). \quad (8)$$

In the minisuperspace model, the current is [13]

$$j^a = \frac{i}{2}a^p(\psi^*\partial_a\psi - \psi\partial_a\psi^*) = -a^pR^2S'.$$

From Eq. (6), we derive the following equations step by step:

$$\frac{p}{a}RS' + \frac{1}{R}(R^2S')' = 0,$$

$$\frac{d(R^2S')}{R^2S'} + p\frac{da}{a} = 0,$$

$$a^pR^2S' = \text{const}.$$

Then we have $\partial_a j^a = 0$. This implies that Eq. (6) is the continuity equation.

It should be pointed out that Eq. (7) is similar to the classical Hamilton-Jacobi equation, supplemented by an extra term called quantum potential $Q(a)$. R and S in Eq. (7) can be obtained conveniently from $\psi(a)$ by solving Eq. (4) with relations,

$$\psi(a) = U + iW = R(a)\exp(iS(a)), \quad (9)$$

$$R^2 = U^2 + W^2, \qquad S = \tan^{-1}(W/U). \quad (10)$$

Generally speaking, the wave function of the bubble should be complex. Specifically, if the wave function of the universe is pure real or pure imaginary ($W = 0$ or $U = 0$), we have $S' = 0$. That means the quantum potential Q will counteract the ordinary potential V at all times. Thus, the vacuum bubble would evolve at a constant speed, and the small bubble cannot grow up rapidly. In the following, we consider the general case for the vacuum bubble, i.e., both U and W are nonzero functions.

By analogy with cases of nonrelativistic particle physics and quantum field theory in flat space-time, quantum trajectories can be obtained from the guidance relation [7,14],

$$\frac{\partial\mathcal{L}}{\partial\dot{a}} = -a\dot{a} = \frac{\partial S}{\partial a}, \quad (11)$$

$$\dot{a} = -\frac{1}{a}\frac{\partial S}{\partial a}. \quad (12)$$

Equation (12) is a first order differential equation, so the three-metric for all values of the parameter t can be obtained by integration.

IV. INFLATION OF THE TRUE VACUUM BUBBLE ($p \neq 1$)

In the following, we solve the WDWE of the bubble with $k = 1, -1, 0$, respectively. When the ordering factor takes a special value $p = -2$ (or 4 for equivalence), exponential expansion of the small true vacuum bubble induced by quantum potential can be obtained no matter whether the bubble is closed, open, or flat.

A. The closed bubble

In this case, the analytic solution of Eq. (4) is

$$\psi(a) = a^{(1-p)/2}\left[ic_1 I_\nu\left(\frac{a^2}{2}\right) - c_2 K_\nu\left(\frac{a^2}{2}\right)\right], \qquad (13)$$

where I_ν's are modified Bessel functions of the first kind, K_ν's are the modified Bessel function of the second kind, the coefficients c_1 and c_2 are arbitrary constants that should be determined by the state of the bubble, and $\nu = |1-p|/4$. As discussed previously, the wave function of the bubble should be complex. For simplicity, we set c_1 and c_2 as real numbers to find the inflation solution.

Using Eqs. (9) and (10), we can get

$$S = \tan^{-1}\left[-\frac{c_1}{c_2}\frac{I_\nu\left(\frac{a^2}{2}\right)}{K_\nu\left(\frac{a^2}{2}\right)}\right],$$

and

$$R = a^{(1-p)/2}\sqrt{\left[c_1 I_\nu\left(\frac{a^2}{2}\right)\right]^2 + \left[c_2 K_\nu\left(\frac{a^2}{2}\right)\right]^2}.$$

Here, we omit the sign "$\pm$" in front of R, since it does not affect the value of $Q(a)$ in Eq. (8). For small arguments $0 < x \ll \sqrt{\nu+1}$, Bessel functions take the following asymptotic forms:

$$I_\nu(x) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$$

and

$$K_\nu(x) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu, \qquad \nu \neq 0.$$

where $\Gamma(z)$ is the Gamma function. It is easy to get

$$S(a \ll 1) \approx -\frac{2c_1}{c_{2\Gamma(\nu)\Gamma(\nu+1)}}\left(\frac{a^2}{4}\right)^{2\nu}, \qquad \nu \neq 0.$$

Using the guidance relation (12), we can get the general Bohmian trajectories for any small scale factor

$$a(t) = \begin{cases} \left[\frac{(3-4\nu)\lambda(\nu)}{3}(t+t_0)\right]^{\frac{1}{3-4\nu}} & \nu \neq 0, \frac{3}{4} \\ e^{\lambda(3/4)(t+t_0)}, & \nu = \frac{3}{4}, \end{cases}$$

where $\lambda(\nu) = 6c_1/(4^{2\nu}c_2\Gamma(\nu)\Gamma(\nu+1))$. For the case of $\nu = 0$ (i.e., $p = 1$), we will discuss it later.

It is clear that only the ordering factor takes the value $p = -2$ (or $p = 4$ for equivalence), i.e., $\nu = 3/4$, has the scale factor $a(t)$ an exponential behavior. $\lambda(3/4) > 0$ corresponds to an expansionary bubble, and $\lambda(3/4) < 0$ implies a contractive bubble that does not satisfy the evolution of the early universe. Therefore, with the condition $\lambda(3/4) > 0$, we draw the conclusion that, for a closed true vacuum bubble, it can expand exponentially, and then the early universe appears irreversibly.

The quantum mechanism of spontaneous creation of the early universe can be seen from the quantum potential of the bubble. For the case of $p = -2$ (or 4), the quantum potential of the small true vacuum bubble is

$$Q(a \to 0) = -a^2 - \lambda(3/4)^2 a^4. \qquad (14)$$

We find that the first term in quantum potential $Q(a \to 0)$ exactly cancels the classical potential $V(a) = a^2$. The effect of the second term $-\lambda(3/4)^2 a^4$ is quite similar to that of the scalar field potential in [15] or the cosmological constant in [16] for inflation. For the small true vacuum bubble, we have $H \equiv \dot{a}/a$ and $\Lambda = 3H^2$. Then we can get the effective "cosmological constant" Λ for the vacuum bubble as $\Lambda \approx 3\lambda(3/4)^2$. In this way, we can see that the quantum potential of the small true vacuum bubble plays the role of the cosmological constant and provides the power for the exponential expansion. It is the quantum mechanism (i.e., the quantum potential) that dominates the exponential expansion of the vacuum bubble.

B. The open bubble

For this case, the analytic solution of Eq. (4) is found to be

$$\psi(a) = a^{(1-p)/2}\left[ic_1 J_\nu\left(\frac{a^2}{2}\right) + c_2 Y_\nu\left(\frac{a^2}{2}\right)\right], \qquad (15)$$

where J_ν's are Bessel functions of the first kind, and Y_ν's are Bessel function of the second kind and $\nu = |1-p|/4$. Likewise, we get

$$S = \tan^{-1}\left[\frac{c_1}{c_2}\frac{J_\nu\left(\frac{a^2}{2}\right)}{Y_\nu\left(\frac{a^2}{2}\right)}\right],$$

and

$$R = a^{(1-p)/2}\sqrt{\left[c_1 J_\nu\left(\frac{a^2}{2}\right)\right]^2 + \left[c_2 Y_\nu\left(\frac{a^2}{2}\right)\right]^2}.$$

For small arguments $0 < x \ll \sqrt{\nu+1}$, Bessel functions take the following asymptotic forms: $J_\nu(x) \sim (x/2)^\nu/\Gamma(\nu+1)$, and $Y_\nu(x) \sim -\Gamma(\nu)2^{\nu-1}/x^\nu$ for $(\nu \neq 0)$. Then we find

$$S(a \ll 1) \approx -\frac{\pi c_1}{c_{2\Gamma(\nu)\Gamma(\nu+1)}}\left(\frac{a^2}{4}\right)^{2\nu}, \qquad v \neq 0.$$

and

$$a(t) = \begin{cases} \left[\frac{(3-4\nu)\bar{\lambda}(\nu)}{3}(t+t_0)\right]^{\frac{1}{3-4\nu}}, & \nu \neq 0, \frac{3}{4} \\ e^{\bar{\lambda}(3/4)(t+t_0)}, & \nu = \frac{3}{4}, \end{cases}$$

where $\bar{\lambda}(\nu) = 3\pi c_1/(4^{2\nu}c_2\Gamma(\nu)\Gamma(\nu+1))$.

It is interesting that the scale factor for the open bubble ($k = -1$) is quite similar to that of the closed one ($k = 1$). For the special case of $p = -2$ (or 4), the scale factor $a(t)$ has an exponential behavior like before. In this case, the quantum potential for the open bubble can be obtained as

$$Q(a \to 0) = a^2 - \bar{\lambda}(3/4)^2 a^4. \qquad (16)$$

Comparing with the case of the closed bubble, we find that the terms a^2 in quantum potential $Q(a \to 0)$ and classical potential $V(a)$ change sign simultaneously, so they can still cancel each other exactly. Thus, it is the term $-\bar{\lambda}(3/4)^2 a^4$ in quantum potential $Q(a \to 0)$ that causes the exponential expansion of the vacuum bubble. Likewise, we can get the effective cosmological constant for the small true vacuum bubble, $\Lambda \approx 3\bar{\lambda}(3/4)^2$.

C. The flat bubble

The analytic solution of Eq. (4) is

$$\psi(a) = ic_1\frac{a^{1-p}}{1-p} - c_2, \qquad (17)$$

where $p \neq 1$, and hence

$$S = \tan^{-1}\left[-\frac{c_1}{c_2}\frac{a^{1-p}}{1-p}\right], \qquad p \neq 1,$$

$$R = \sqrt{c_2^2 + \left(c_1\frac{a^{1-p}}{1-p}\right)^2}, \qquad p \neq 1.$$

Using the guidance relation (12), we can get the general form of the Bohmian trajectories as

$$a(t) = \begin{cases} \left[\frac{c_1}{c_2}(3-|1-p|)(t+t_0)\right]^{\frac{1}{3-|1-p|}}, & |1-p| \neq 0, 3, \\ e^{c_1(t+t_0)/c_2}, & |1-p| = 3. \end{cases}$$

Likewise, only conditions $p = -2$ (or 4) and $c_1/c_2 > 0$ are satisfied, will the small true vacuum bubble expand exponentially. For the case of exponential expansion, the quantum potential for the vacuum bubble can be obtained as $Q(a \to 0) = -(c_1/c_2)^2 a^4$, while the classical potential is $V(a) = 0$. This definitely indicates that quantum potential $Q(a)$ is the origin of exponential expansion for the small true vacuum bubble. Similarly, we can get the effective cosmological constant for the small true vacuum bubble as $\Lambda \approx 3(c_1/c_2)^2$.

V. THE BOHMIAN TRAJECTORIES FOR $p = 1$

Solutions of Eq. (4) for the $p = 1$ case are still Eq. (13) and Eq. (15) for the closed and open bubbles, respectively. For the flat bubble, the solution of Eq. (4) for $p = 1$ is

$$\psi(a) = ic_1 - c_2 \ln a.$$

It is clear that the quantum potential $Q(a)$ of the bubble approaches infinity when the bubble is very small $a \to 0$, no matter whether the small bubble is closed, open or flat. The requirement of a finite value of $Q(a \to 0)$ will result in $a(t) =$ constant for $k = 0, \pm 1$.

VI. THE BEHAVIOR OF LARGE VACUUM BUBBLES

Let us look at behaviors of our solutions for large vacuum bubbles [19]. For the closed bubble, we get $S(a \gg 1) = -\tan^{-1}(c_1 e^{a^2}/c_2)$, and hence $\dot{a} = 2c_1 e^{-a^2}/c_2 \to 0$. The quantum potential of the bubble is $Q(a \gg 1) \sim -a^2$. It is obvious that there is no classical limit for the closed bubble. For the open bubble, we have $S(a \gg 1) \sim -\tan^{-1}[c_1 \tan(a^2/2 + \pi/4 - \nu\pi/2)/c_2]$. When $|c_1/c_2| = 1$, we can get its classical limit $\dot{a}^2 = 1$ as $Q(a \gg 1) \to 0$. For the case of the flat bubble, we get $\dot{a} = c_1 a^{-|1-p|-2}/c_2$. When the bubble becomes large enough, it can reach the classical limit, $\dot{a}^2 \to 0$ with $Q(a \gg 1) \to 0$. When the vacuum bubble becomes very large, it will stop expanding for $k = 0$, 1, or it will expand with a constant velocity for $k = -1$. In one word, it turns out that the vacuum bubble will stop accelerating when it becomes very large, no matter whether it is closed, flat, or open.

VII. THE OPERATOR ORDERING FACTOR

Generally speaking, the factor p in Eq. (4) represents the uncertainty of the operator ordering. Different p gives a different rule of quantization for the classical system. From Eqs. (6) and (8), we get a general form of the quantum potential,

$$Q(a) = -\left[\frac{-p^2 + 2p}{4a^2} + \frac{3(S'')^2}{4(S')^2} - \frac{S'''}{2S'}\right]. \tag{18}$$

It is clear that the effect of the ordering factor p is important only to small bubbles, and different p will result in different quantum potential. In other words, for small bubbles (i.e., $a \ll 1$), the first term is significant to $Q(a)$, while for large bubbles (i.e., $a \gg 1$), it is negligible. So, the factor p represents quantum effects of the system described by the WDWE in Eq. (4).

It is interesting that only when the ordering factor p takes value -2 (or 4) can one get the exponential expansion for the small true vacuum bubble, no matter whether the bubble is closed, flat, or open. It is generally believed that the operator ordering factor p can be restricted by the quantum to classical transition of the system [17]. Maybe a more elegant treatment of the quantum to classical transition is needed to restrict the interesting values of p, since the classical limit is independent of p in the present treatment. A hint from loop quantum gravity (LQG) theory is that when one wants to remove the ambiguities from LQG, the ordering factor should take the value $p = -2$ [18].

VIII. DISCUSSION AND CONCLUSION

In summary, we have presented a mathematical proof that the universe can be created spontaneously from nothing. When a small true vacuum bubble is created by quantum fluctuations of the metastable false vacuum, it can expand exponentially if the ordering factor takes the value $p = -2$ (or 4). In this way, the early universe appears irreversibly. We have shown that it is the quantum potential that provides the power for the exponential expansion of the bubble. Thus, we can conclude that the birth of the early universe is completely determined by quantum mechanism.

One may ask the question when and how space, time and matter appear in the early universe from nothing. With the exponential expansion of the bubble, it is doubtless that space and time will emerge. Due to Heisenberg's uncertainty principle, there should be virtual particle pairs created by quantum fluctuations. Generally speaking, a virtual particle pair will annihilate soon after its birth. But, two virtual particles from a pair can be separated immediately before annihilation due to the exponential expansion of the bubble. Therefore, there would be a large amount of real particles created as vacuum bubble expands exponentially. A rigorous mathematical calculation for the rate of particle creation with the exponential expansion of the bubble will be studied in our future work.

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