

Enhancement of chiral symmetry breaking from the pion condensation at finite isospin chemical potential in a holographic QCD model

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We study the pion condensation at the finite isospin chemical potential using a holographic QCD model. By solving the equations of motion for the pion fields together with those for the isosinglet scalar and isotriplet vector meson fields, we show that the phase transition from the normal phase to the pion condensation phase is second order with the mean-field exponent, and that the critical value of the isospin chemical potential μ_I is equal to the pion mass, consistently with the result obtained by the chiral effective Lagrangian at $O(p^2)$. For a higher chemical potential, we find a deviation, which can be understood as a higher order effect in the chiral effective Lagrangian. We investigate the μ_I dependence of the chiral condensate defined by $\tilde{\sigma} \equiv \sqrt{\langle \sigma \rangle^2 + \langle \pi^a \rangle^2}$. We find that $\tilde{\sigma}$ is almost constant in the small μ_I region, while it grows with μ_I in the large μ_I region. This implies that the strength of the chiral symmetry breaking is not changed for small μ_I : The isospin chemical potential plays a role to rotate the “vacuum angle” of the chiral circle $\tan^{-1} \sqrt{\langle \pi^a \rangle^2 / \langle \sigma \rangle^2}$ with keeping the “radius” $\tilde{\sigma}$ unchanged for small μ_I . For the large μ_I region, on the other hand, the chiral symmetry breaking is enhanced by the existence of μ_I .

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I. INTRODUCTION

Quantum chromodynamics (QCD) at the finite isospin chemical potential is an interesting subject to study. When combined with the finite baryon number chemical potential it will provide a clue to understand the symmetry energy which is important to describe the equation of state inside neutron stars [1]. In addition, it may give some information on the structure of the chiral symmetry breaking [2].

When we turn on the isospin chemical potential μ_I at the zero baryon number density, the pion condensation is expected to occur at a critical point. Son and Stephanov [3] showed that, using the chiral Lagrangian at $O(p^2)$, the phase transition to the pion condensation phase is of the second order and the critical value of μ_I is equal to the pion mass. It was also shown [4] that $\langle \bar{q}\gamma_5 q \rangle$ condenses in the high isospin density limit. Then a conjecture of no phase transition from the pion condensation phase to $\langle \bar{q}\gamma_5 q \rangle$ condensation phase was made. The structure in the mid μ_I region is a highly nonperturbative issue, so that it is not easy to understand such a region.

A pure isospin matter with zero baryon density can be simulated by the lattice analysis. References [5–7] show that the phase transition is of the second order, and that the critical chemical potential is equal to the pion mass. Because of the existence of the sign problem, it is difficult to apply the lattice analysis for studying the hadron property at the finite baryon number density. In this sense, an analysis by models may give some clues to understand

the phase structure and the relevant phenomenon in the mid μ_I region. Actually, many analyses were done by using the Nambu-Jona-Lasinio model [8–15], the random matrix model [16], the strong coupling lattice analysis [17], the Ginzburg-Landau approach [18], the hadron resonance gas model [19], and the holographic QCD models [20–31].

Although there are so many works on the pion condensation at the finite isospin chemical potential, there are not many works for studying the strength of the chiral symmetry breaking. Namely, it is interesting to ask whether or not the chiral symmetry is partially restored in the isospin matter.

In Refs. [12–15], based on the Nambu-Jona-Lasinio model analysis, the authors seemed to conclude that the reduction of $\langle \bar{q}q \rangle$ in the isospin matter implies the partial chiral symmetry restoration. In Ref. [18], the Ginzburg-Landau approach is used to study the $\langle \bar{q}q \rangle$ condensate together with the pion condensation. However, the absolute strength of the chiral symmetry breaking, which is characterized by the chiral condensate $\tilde{\sigma} = \sqrt{\langle \sigma \rangle^2 + \langle \pi^a \rangle^2}$, is not clearly studied. The analysis using the strong coupling lattice in Ref. [17] shows that $\tilde{\sigma}$ decreases in the high isospin chemical potential associated with the decreasing pion condensation. The decreasing pion condensation might be a special feature in the strong coupling lattice analysis, so that it would be interesting to study the behavior of $\tilde{\sigma}$ using the various ways.

In this work, we study the pion condensation phase in a holographic QCD model [32,33] by solving the equations of motion for mean fields corresponding to π , σ and the time component of the ρ meson. Our results show that the phase transition is of the second order, which is consistent

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with the one obtained in the $O(p^2)$ chiral Lagrangian [3], as well as the one in the holographic QCD model [27]. It is remarkable that the chiral condensate defined by $\tilde{\sigma} \equiv \sqrt{\langle \sigma \rangle^2 + \langle \pi^a \rangle^2}$ is almost constant in the small μ_I region, while it grows with μ_I in the large μ_I region. This implies that the chiral symmetry breaking is enhanced by the existence of the isospin chemical potential.

This paper is organized as follows: In Sec. II we introduce some basic point of the model which we use in the present analysis. Section III is devoted to the main part of this paper, where we study the pion condensation together with the chiral condensation. In Sec. IV we analyze our result in terms of the chiral Lagrangian at $O(p^4)$. Finally, we give a summary and discussions in Sec. V

II. MODEL

In the present analysis we use the hard-wall holographic QCD model given in Refs. [32,33]. The action is given by

$$S_5 = \int d^4x \int_{\varepsilon}^{z_m} dz \mathcal{L}_5, \quad (2.1)$$

where ε and z_m are the UV and the IR cutoffs. The five-dimensional Lagrangian is

$$\mathcal{L}_5 = \sqrt{g} \text{Tr} \left[|DX|^2 + 3|X|^2 - \frac{1}{4g_5^2} (F_L^2 + F_R^2) \right] + \mathcal{L}_5^{BD}, \quad (2.2)$$

where the metric is given by

$$ds^2 = a^2(z) (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2) \quad (2.3)$$

with $a(z) = 1/z$. The covariant derivative and the field strength are given by

$$D_M X = \partial_M X - iL_M X + iX R_M, \quad (2.4)$$

$$F_{MN}^L = \partial_M L_N - \partial_N L_M - i[L_M, L_N], \quad (2.5)$$

where $M = (\mu, 5)$ are the fifth-dimensional indices. \mathcal{L}_5^{BD} in Eq. (2.2) is the boundary term introduced as [34]

$$\mathcal{L}_5^{BD} = -\sqrt{g} \text{Tr} \{ \lambda z_m |X|^4 - m^2 z_m |X|^2 \} \delta(z - z_m), \quad (2.6)$$

where z_m in the coefficients of the $|X|^4$ term and the $|X|^2$ term are introduced in such a way that λ and m^2 carry no dimension. This model has a chiral symmetry corresponding to $U(2)_L \times U(2)_R$. There exists the Chern-Simons (CS) term in addition to the above term. However, the CS term does not contribute to the pion condensation when the spatial rotational symmetry is assumed as in this paper.

The scalar field X and the gauge fields L_M and R_M transform under the $U(2)_L \times U(2)_R$ as

$$X \rightarrow X' = g_L X g_R^\dagger, \quad (2.7)$$

$$L_M \rightarrow L'_M = g_L L_M g_L^\dagger + i g_L \partial_M g_L^\dagger, \quad (2.8)$$

where $g_{L,R} \in U(2)_{L,R}$ are the transformation matrices of the chiral $U(2)_L \times U(2)_R$ symmetry. In the following analysis we adopt the $L_5 = R_5 = 0$ gauge, and use the IR-boundary condition $F_{5\mu}^L|_{z_m} = F_{5\mu}^R|_{z_m} = 0$.

In the vacuum, the chiral symmetry is spontaneously broken down to $U(2)_V$ by the vacuum expectation value of X . This is given by solving the equation of motion as [32,33]

$$X_0(z) = \frac{1}{2} (m_q z + \sigma z^3) \equiv \frac{1}{2} v(z), \quad (2.9)$$

where m_q corresponds to the current quark mass and σ to the quark condensate. They are related with each other by the IR-boundary condition:

$$z_m \partial_5 v|_{\text{IR}} = -\frac{v}{2} (\lambda v^2 - 2m^2)|_{\text{IR}}. \quad (2.10)$$

The fields are parametrized as

$$X = \frac{1}{2} (S^0 \sigma^0 + S^a \sigma^a) e^{i\pi^b \sigma^b + i\eta}, \quad (2.11)$$

$$V_\mu = \frac{L_\mu + R_\mu}{2}, \quad (2.12)$$

$$A_\mu = \frac{L_\mu - R_\mu}{2}, \quad (2.13)$$

$$V_\mu^A = \text{Tr}[V_\mu \sigma^A], \quad (2.14)$$

$$A_\mu^A = \text{Tr}[A_\mu \sigma^A], \quad (2.15)$$

where σ^a ($a = 1, 2, 3$) are Pauli matrices and $\sigma^0 = 1$, and the superscript index A runs over 0, 1, 2, and 3. The isosinglet scalar part S^0 is separated into a background field part and a fluctuation part as $S^0 = v + \tilde{S}^0$. A parameter g_5^2 is determined by matching with QCD as

$$g_5^2 = \frac{12\pi^2}{N_c}. \quad (2.16)$$

The pion is described as a linear combination of the lowest eigenstate of π^a and the longitudinal mode of A_μ^a , and the ρ meson is the lowest eigenstate of V_μ . The values of the m_q and z_m together with that of σ are fixed by fitting them to the pion mass $m_\pi = 139.6$ MeV, the ρ -meson mass $m_\rho = 775.8$ MeV, and the pion decay constant $f_\pi = 92.4$ MeV as [32,33]

$$\begin{aligned} m_q &= 2.29 \text{ MeV}, & z_m &= 1/(323 \text{ MeV}), \\ \sigma &= (327 \text{ MeV})^3. \end{aligned} \quad (2.17)$$

By using this value of σ and a scalar meson mass as inputs, the values of the parameters m^2 and λ in the boundary potential are fixed. [34] It was shown [34] that there is an upper bound for the scalar meson mass as 1.2 GeV, but the dependence on the mass on the value of λ is small. So in the present analysis, we use the a_0 meson mass $m_{a_0} = 980$ MeV as a reference value, which fixes $m^2 = 5.39$ and $\lambda = 4.4$, and see the dependence of our results on the scalar meson mass.

III. PION CONDENSATION PHASE

In this section we study the pion condensation for the finite isospin chemical potential μ_I in the holographic QCD model explained in the previous section. Since the pion mass exists in the present model. We will have a phase transition from the normal phase to the pion condensation phase for increasing μ_I . In the present paper we are interested in the pion condensation phase for the small isospin chemical potential, so that we assume that the rotational symmetry $O(3)$ is not broken by, e.g., the ρ -meson condensation. We also assume the time-independent condensate; then the vacuum structure is determined by studying the mean fields of five-dimensional fields which do not depend on the four-dimensional coordinate.

We introduce the isospin chemical potential μ_I as a UV-boundary value of the time component of the gauge field of $SU(2)_V$ symmetry as

$$V_0^3(z)|_e = \mu_I, \quad (3.1)$$

where the superscript 3 indicates the third component of the isospin corresponding to the neutral ρ meson. The assumption of rotational invariance implies $L_i = R_i = 0$. Then the wave functions of $V_0^a(z)$ ($a = 1, 2, 3$), $\pi_0^a(z)$, and $A_0^a(z)$ are determined by solving the equations of motion.

In the present analysis, the CS term, given as

$$S_{CS} \propto \int d^4x \int_{e}^{z_m} dz \times e^{MNPQS} L_M^0 \text{Tr}[F_{NP}^L F_{QS}^L] - (L \rightarrow R), \quad (3.2)$$

vanishes because this term must be proportional to L_i^a or R_i^a , where L_M^0 is a gauge field corresponding to $U(1)_L$,

i.e., $L_M^0 = \text{Tr}[L_M]$. The gauge field corresponding to $U(1)_V$, which includes the ω meson and its radial excitations, does not show up in our analysis, because it couples to other fields only through the CS term.

The X field consists of 8 degrees of freedom, which include η and S^a ($a = 1, 2, 3$). Since the η is an isosinglet, it does not condense by itself. However, the existence of S^a condensation (a_0 meson condensation) together with the pion condensation triggers the η condensation. The a_0 meson condensation will occur for $\mu_I \geq m(a_0)$. Since we study the region of $\mu_I \leq m_\rho$, we expect that both η and a_0 condensations vanish in this region. Actually, we can check that $\eta = S^a = 0$ is a solution of the equations of motion in the following way: By using the parametrization of Eq. (2.11), the terms including η and S^a of Lagrangian (2.2) are written as

$$\begin{aligned} \mathcal{L}_5 \sim & \frac{a^3}{4} \text{Tr}[SS\{(U\partial_5 U^\dagger)^2 + L_0 L_0 + R_0 R_0 + a^2\} \\ & - ((S^0)^2 + SS)(\partial_5 \eta)^2 + 4iS^0(\partial_5 \eta)S(U\partial_5 U^\dagger) \\ & - [S, (\partial_5 S)](U\partial_5 U^\dagger) - (\partial_5 S)^2 - 2SL_0 SUR_0 U^\dagger \\ & + 8S^0 A_0^0 S(L_0 - UR_0 U^\dagger)], \end{aligned} \quad (3.3)$$

where $S = S^a \sigma^a$ and $U = e^{i\pi^a \sigma^a}$. It is easy to confirm that $\eta = S^a = 0$ together with $A_0^0 = 0$ is a solution of the equations of motion for them. Then, in the following analysis, we take $\eta = S^a = 0$.

For writing the equations of motion for mean fields, it is convenient to express

$$e^{i\pi^a \sigma^a} = \cos b 1 + i \sin b (n^a \sigma^a), \quad (3.4)$$

where

$$b = \sqrt{\pi^b \pi^b}, \quad n^a = \frac{\pi^a}{b}. \quad (3.5)$$

We include the condition $(n^a)^2 = 1$ into the Lagrangian using a Lagrange multiplier $\tilde{\lambda}$.

Now, the coupled equations of motion are given as

$$\begin{aligned} \partial_5(-a^3(S^0)^2 \sin^2 b \partial_5 n^3) - a^3(S^0)^2 [\sin^2 b n^3 (n^b V_0^b) + \sin^2 b A_0^3 (n^b A_0^b) + \sin b \cos b \epsilon^{bc3} A_0^b V_0^c] + 2\tilde{\lambda} n^3 &= 0, \\ \partial_5 \left(\frac{a}{g_5^2} \partial_5 A_0^3 \right) - a^3(S^0)^2 [\cos^2 b A_0^3 + \sin^2 b n^3 (n^b A_0^b) - \sin b \cos b \epsilon^{bc3} V_0^b n^c] &= 0, \\ \partial_5 \left(\frac{a}{g_5^2} \partial_5 V_0^1 \right) - a^3(S^0)^2 [\sin^2 b V_0^1 - \sin^2 b n^1 (n^b V_0^b) + \sin b \cos b \epsilon^{bc1} A_0^b n^c] &= 0, \\ \partial_5 \left(\frac{a}{g_5^2} \partial_5 V_0^2 \right) - a^3(S^0)^2 [\sin^2 b V_0^2 - \sin^2 b n^2 (n^b V_0^b) + \sin b \cos b \epsilon^{bc2} A_0^b n^c] &= 0, \end{aligned} \quad (3.6)$$

where the summation over the indices b and c are understood.

We can easily check that $\pi^3 = A_0^3 = 0$ together with $V_0^1 = V_0^2 = 0$ gives a set of solutions for the above coupled equations of motion, which implies that the neutral pion does not condense. Then, in the following we assume that this set of solutions is physically realized, and take $\pi^3 = A_0^3 = 0$ and $V_0^1 = V_0^2 = 0$.

At the finite isospin chemical potential, this theory has the U(1) symmetry which is a subgroup of the isospin SU(2)_V. Using this U(1) symmetry, we rotate away the condensation of the π^2 field to keep only the π^1 condensation.

By setting $\pi^3 = A_0^3 = 0$ and $V_0^1 = V_0^2 = 0$ and writing

$$e^{i\pi^a \sigma^a} = \cos b 1 + i \sin b \sigma^1,$$

$$A_0^a = \theta(\cos \zeta, \sin \zeta, 0), \quad V_0^3 = \varphi + \mu_I,$$

$$\begin{aligned} \partial_5(-a^3 \partial_5 S^0) + a^3 S^0 (\partial_5 b)^2 - 3a^5 S^0 - a^3 S^0 [\sin^2 b (\varphi + \mu_I)^2 - \theta \sin 2b \sin \zeta (\varphi + \mu_I) + \theta^2 - \theta^2 \sin^2 b \sin^2 \zeta] &= 0, \\ \partial_5(-a^3 (S^0)^2 \partial_5 b) - \frac{a^3 (S^0)^2}{2} [\sin 2b \{(\varphi + \mu_I)^2 - \theta^2 \sin^2 \zeta\} - 2\theta \cos 2b \sin \zeta (\varphi + \mu_I)] &= 0, \\ \partial_5 \left(\frac{a}{g_5^2} \partial_5 \theta \right) - \frac{a}{g_5^2} \theta (\partial_5 \zeta)^2 - \frac{a^3 (S^0)^2}{2} [-\sin 2b \sin \zeta (\varphi + \mu_I) + 2\theta \{1 - \sin^2 b \sin^2 \zeta\}] &= 0, \\ \partial_5 \left(\frac{a}{g_5^2} \theta^2 \partial_5 \zeta \right) + \frac{a^3 (S^0)^2}{2} [-\theta \sin 2b \cos \zeta (\varphi + \mu_I) - \theta^2 \sin 2\zeta] &= 0, \\ \partial_5 \left(\frac{a}{g_5^2} \partial_5 \varphi \right) - \frac{a^3 (S^0)^2}{2} [2\sin^2 b (\varphi + \mu_I) - \theta \sin 2b \sin \zeta] &= 0. \end{aligned} \quad (3.8)$$

Using the boundary conditions listed in Table I, we solve the above coupled equations of motion to determine the isospin chemical potential as an eigenvalue. Then, we calculate the isospin number density from the following formula obtained from the Lagrangian (3.7):

$$\begin{aligned} n_I &= \int dz \frac{\partial \mathcal{L}_5}{\partial \mu_I} \\ &= \int dz \frac{a^3 (S^0)^2}{2} [2\sin^2 b (\varphi + \mu_I) - \theta \sin 2b \sin \zeta]. \end{aligned} \quad (3.9)$$

We show the resultant relation between the isospin chemical potential and the isospin density in Fig. 1 for $\lambda = 1, 4.4,$ and 100 corresponding to $m_{a_0} = 610, 980,$ and 1210 MeV. This shows that the phase transition is of the

TABLE I. Boundary conditions of variables.

Variables	UV	IR
S^0	$\frac{S^0}{z} _e = m_q$	$\partial_5 S^0 _{z_m} = -\frac{S^0}{2z_m} (\lambda (S^0)^2 - 2m^2) _{z_m}$
b	$b _e = 0$	$\partial_5 b _{z_m} = 0$
θ	$\theta _e = 0$	$\partial_5 \theta _{z_m} = 0$
ζ	$\zeta _e = \frac{\pi}{2}$	$\partial_5 \zeta _{z_m} = 0$
φ	$\varphi _e = 0$	$\partial_5 \varphi _{z_m} = 0$

the Lagrangian is rewritten as

$$\begin{aligned} \mathcal{L}_5 &= \frac{a^3}{2} [-(\partial_5 S^0)^2 - (S^0)^2 (\partial_5 b)^2] + \frac{a^3 (S^0)^2}{2} [\sin^2 b (\varphi + \mu_I)^2 \\ &\quad - \theta \sin 2b \sin \zeta (\varphi + \mu_I) + \theta^2 - \theta^2 \sin^2 b \sin^2 \zeta] \\ &\quad + \frac{3a^5}{2} (S^0)^2 + \frac{a}{2g_5^2} [(\partial_5 \varphi)^2 + (\partial_5 \theta)^2 + \theta^2 (\partial_5 \zeta)^2]. \end{aligned} \quad (3.7)$$

From the above Lagrangian, the equations of motion are obtained as

second order and the critical chemical potential is predicted to be equal to the pion mass. This is consistent with the result obtained by the chiral Lagrangian approach in Ref. [3] and the one from the holographic QCD model [27]. Furthermore, our result on the relation between the isospin number density and isospin chemical potential for small μ_I agrees with the following one obtained by O(p^2) chiral Lagrangian [3]:

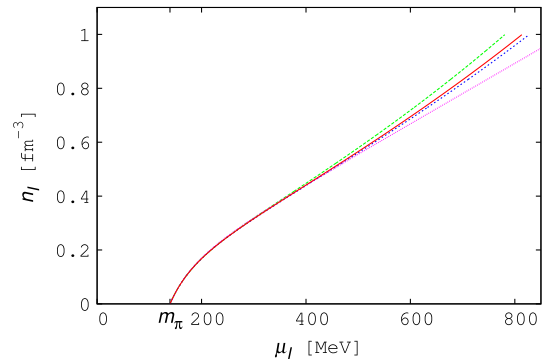


FIG. 1 (color online). Relation between the isospin number density n_I and the isospin number chemical potential μ_I . The green, red, and blue curves show our results for $\lambda = 1, 4.4,$ and $100,$ respectively. The pink dashed curve shows the result given by the chiral Lagrangian in Ref. [3]. Each choice of λ corresponds to $m_{a_0} = 610, 980,$ and 1210 MeV, respectively.

$$n_I = f_\pi^2 \mu_I \left(1 - \frac{m_\pi^4}{\mu_I^4} \right). \quad (3.10)$$

For $\mu_I > 500$ MeV, there is a difference between our predictions and the one from the $O(p^2)$ chiral Lagrangian, which can be understood as the higher order contribution as we will show in the next section.

We next study the μ_I dependences of the “ σ ” condensate corresponding to $\langle \bar{q}q \rangle$ and the π condensate to $\langle \bar{q}\gamma_5 \sigma^a q \rangle$. For obtaining these condensates through the AdS/CFT correspondence, we introduce the scalar source s and the pseudoscalar source p^a as

$$\frac{\delta X}{z} = \frac{\delta X^\dagger}{z} = \frac{s}{2} 1, \quad \frac{\delta X}{z} = -\frac{\delta X^\dagger}{z} = \frac{ip^a}{2} \sigma^a. \quad (3.11)$$

The UV-boundary term of X is written as

$$\begin{aligned} \delta S^{UV} &= \int d^4x \text{Tr} \left[\frac{\delta X^\dagger}{z} a^2 (\partial_5 X) + \text{H.c.} \right]_e \\ &= \int d^4x \text{Tr} \left[\frac{\delta X^\dagger}{z} a \left(\partial_5 \frac{X}{z} + \frac{X}{z^2} \right) + \text{H.c.} \right]_e. \end{aligned} \quad (3.12)$$

We neglect the second term in the last line of the above equation. Then the π condensate and σ condensate are defined as¹

$$\begin{aligned} \langle \bar{q}\gamma_5 \sigma^a q \rangle &\equiv \frac{1}{2} \text{Tr} \left[i\sigma^a a \left(\partial_5 \frac{X}{z} \right) + \text{H.c.} \right]_e = \langle \pi^a \rangle, \\ \langle \bar{q}q \rangle &\equiv \frac{1}{2} \text{Tr} \left[a \left(\partial_5 \frac{X}{z} \right) + \text{H.c.} \right]_e = \langle \sigma \rangle. \end{aligned} \quad (3.13)$$

We show the μ_I dependences of these condensates in Fig. 2, where $\langle \sigma \rangle_0$ is the σ condensate at $\mu_I = 0$. This shows that the σ condensate decreases rapidly after the phase transition where the π condensate grows rapidly. The σ condensate becomes very small for $\mu_I \gtrsim 400$ MeV, while the π condensate keeps increasing. Using the form $\langle \pi^a \rangle \propto (\mu_I - \mu_I^c)^\nu$ near the phase transition point, we fit the critical exponent ν to obtain $\nu = \frac{1}{2}$. This implies that the phase transition here is the mean-field type.

We also show the “chiral circle” in Fig. 3. It is remarkable that the value of the “chiral condensate” defined by

$$\langle \tilde{\sigma} \rangle = \sqrt{\langle \sigma \rangle^2 + \langle \pi^a \rangle^2} \quad (3.14)$$

is constant for increasing the isospin chemical potential μ_I for $\mu_I \lesssim 300$ MeV, and that it grows rapidly in the large μ_I region.

¹Note that we use $\langle \bar{q}q \rangle = \langle \bar{u}u \rangle + \langle \bar{d}d \rangle$.

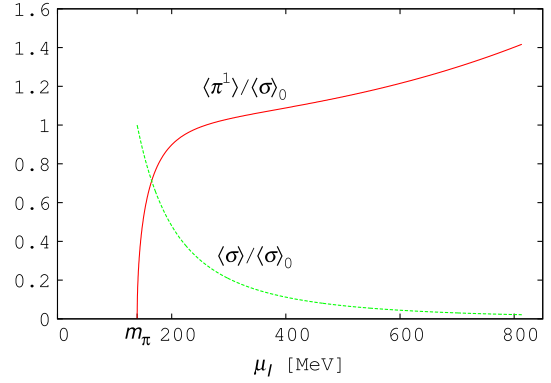


FIG. 2 (color online). μ_I dependences of the π condensate (red curve) and the σ condensate (green curve).

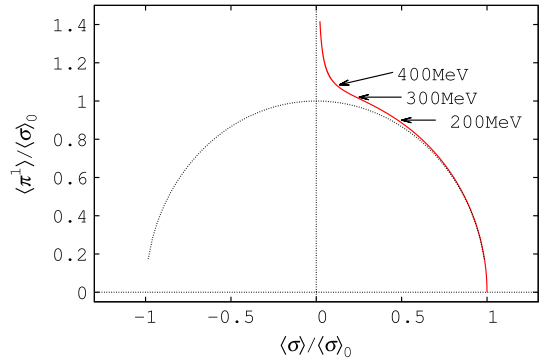


FIG. 3 (color online). The chiral circle is shown as the red curve. The black curve is an unit circle.

IV. COMPARISON WITH THE CHIRAL LAGRANGIAN

In this section, we compare our prediction on the relation between the isospin number density and the isospin chemical potential shown in Fig. 1 as well as the μ_I dependences of the π condensate and the σ condensate in Fig. 2, with the ones from the chiral Lagrangian including the $O(p^4)$ terms. Here we use the following chiral Lagrangian for two flavor case [35,36]:

$$\begin{aligned} \mathcal{L}^{\text{ChPT}} &= \frac{1}{4} F_0^2 \text{Tr} [D_\mu U D^\mu U^\dagger] + \frac{1}{4} F_0^2 \text{Tr} [\chi^\dagger U + U^\dagger \chi] \\ &+ L_1^{(2)} (\text{Tr} [D^\mu U^\dagger D_\mu U])^2 + L_2^{(2)} \text{Tr} [D_\mu U^\dagger D_\nu U] \\ &\times \text{Tr} [D^\mu U^\dagger D^\nu U] + L_4^{(2)} \text{Tr} [D^\mu U^\dagger D_\mu U] \text{Tr} [\chi^\dagger U + U^\dagger \chi] \\ &+ L_6^{(2)} (\text{Tr} [\chi^\dagger U + U^\dagger \chi])^2 + L_7^{(2)} (\text{Tr} [\chi^\dagger U - U^\dagger \chi])^2 \\ &+ L_8^{(2)} \text{Tr} [\chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger] + iL_9^{(2)} \text{Tr} [F_{\mu\nu}^R D^\mu U^\dagger D^\nu U \\ &+ F_{\mu\nu}^L D_\mu U D^\mu U^\dagger] + L_{10}^{(2)} \text{Tr} [U^\dagger F_{\mu\nu}^L U F^{R\mu\nu}] \\ &+ H_1^{(2)} \text{Tr} [F^{L\mu\nu} F_{\mu\nu}^L + F^{R\mu\nu} F_{\mu\nu}^R] + H_2^{(2)} \text{Tr} [\chi^\dagger \chi], \end{aligned} \quad (4.1)$$

where U is parametrized by the pseudoscalar meson fields as

$$U = e^{i2\pi/f_\pi}, \quad \pi = \pi^a T_a. \quad (4.2)$$

χ includes the scalar and pseudoscalar source fields, and the covariant derivative $D_\mu U$ is expressed as

$$D_\mu U \equiv \partial_\mu U - i\mathcal{L}_\mu U + iU\mathcal{R}_\mu, \quad (4.3)$$

where \mathcal{L}_μ and \mathcal{R}_μ are external gauge fields corresponding to $SU(2)_{L,R}$.

In the following analysis, we will study the relation between the isospin number density and the isospin chemical potential as well as the condensates using the above chiral Lagrangian at tree level. In the ordinary chiral perturbation theory, the tree-level contribution from $O(p^4)$ terms are of the same order as the one-loop contribution of $O(p^2)$, so that both contributions should be included. However, because one-loop corrections are counted as next to leading order in the $1/N_c$ expansion, we neglect the one-loop corrections in the present analysis. Then, one can simply introduce the isospin chemical potential μ_I as vacuum expectation values of these external gauge fields as $\langle \mathcal{L}_\mu^3 \rangle = \langle \mathcal{R}_\mu^3 \rangle = \frac{\mu_I}{2} \delta_{0\mu}$. In this case, μ_I appears only through the covariant derivative as

$$D_0 U = \partial_0 U - i\frac{\mu_I}{2} [\sigma^3, U], \quad (4.4)$$

where σ^3 is the third component of the Pauli matrices. Parametrizing U as $U = \cos \alpha + i \sin \alpha (\sigma^1 \cos \phi + \sigma^2 \sin \phi)$, we get the effective potential as

$$\begin{aligned} V_{\text{eff}} &= -\mathcal{L}^{\text{ChPT}} \\ &= -\frac{f_\pi^2}{2} \mu_I^2 (2 - \beta) \beta - f_\pi^2 m_\pi^2 (1 - \beta) - 4A\mu_I^4 (2 - \beta)^2 \beta^2 \\ &\quad + 8Cm_\pi^2 \mu_I^2 (2 - \beta) \beta^2 - 8Bm_\pi^4 \beta^2 + (\text{const}), \end{aligned} \quad (4.5)$$

where $\beta \equiv 1 - \cos \alpha$ and

$$\begin{aligned} A &\equiv L_1^{(2)} + L_2^{(2)}, \\ B &\equiv 2L_6^{(2)} + L_8^{(2)}, \\ C &\equiv L_4^{(2)}. \end{aligned} \quad (4.6)$$

We determine the value of β by minimizing the above V_{eff} , and then calculate the isospin number density, the σ condensate and the π condensate through

TABLE II. The best fitted values of $A \equiv L_1^{(2)} + L_2^{(2)}$, $B \equiv 2L_6^{(2)} + L_8^{(2)}$, and $C \equiv L_4^{(2)}$ compared with a set of empirical values.

	$A \times 10^3$	$B \times 10^3$	$C \times 10^3$
Fitting result	0.093	1.01	0.63
Empirical values	0.4 ± 1.2	1.2 ± 0.9	1.1 ± 0.6

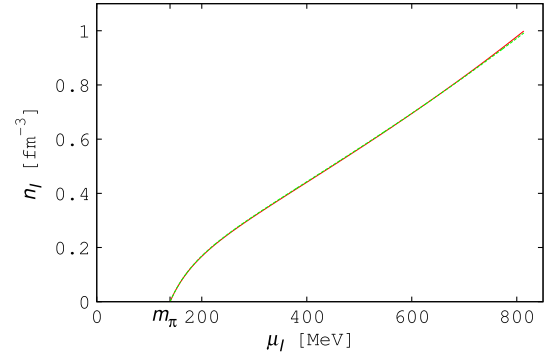


FIG. 4 (color online). μ_I dependence of n_I obtained from the $O(p^4)$ chiral Lagrangian for the best fitted values of A , B , and C (green curve) compared with our result (red curve).

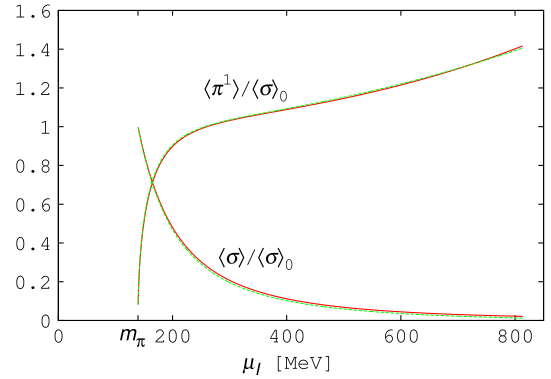


FIG. 5 (color online). μ_I dependence of $\frac{\langle \sigma \rangle}{\langle \sigma \rangle_0}$ and $\frac{\langle \pi^1 \rangle}{\langle \sigma \rangle_0}$ obtained from the $O(p^4)$ chiral Lagrangian for the best fitted values of A , B , and C (green curve) compared with our result (red curve).

$$\begin{aligned} n_I &= -\frac{\partial V_{\text{eff}}}{\partial \mu_I} \\ &= f_\pi^2 \mu_I (2 - \beta) \beta + 16A\mu_I^3 (2 - \beta)^2 \beta^2 - 16Cm_\pi^2 \mu_I (2 - \beta) \beta^2, \end{aligned} \quad (4.7)$$

$$\frac{\langle \sigma \rangle}{\langle \sigma \rangle_0} = 1 - \kappa \left[1 - \left(8C \frac{\mu_I^2}{f_\pi^2} (2 - \beta) - 16B \frac{m_\pi^2}{f_\pi^2} \right) (1 - \beta) \right] \beta, \quad (4.8)$$

$$\frac{\langle \pi \rangle}{\langle \sigma \rangle_0} = \kappa \left[1 + \left(8C \frac{\mu_I^2}{f_\pi^2} (2 - \beta) - 16B \frac{m_\pi^2}{f_\pi^2} \right) \beta \right] \sqrt{\beta(2 - \beta)}, \quad (4.9)$$

where

$$\kappa = \frac{f_\pi^2 m_\pi^2}{m_q \langle \sigma \rangle_0}. \quad (4.10)$$

We set $\kappa \approx 1.04$,² which is given by using $m_\pi = 139.6$ MeV, $f_\pi = 92.4$ MeV, and (2.17).

²The deviation of κ from 1 expresses the deviation from the Gell-Mann-Oakes-Renner relation due to the contribution of $H_2^{(2)} - 2L_8^{(2)}$.

We fit the values of the coefficients A , B , and C to our result on the μ_I dependence of n_I , $\frac{\langle\sigma\rangle}{\langle\sigma\rangle_0}$ and $\frac{\langle\pi^1\rangle}{\langle\sigma\rangle_0}$ shown in Figs. 1 and 2, by minimizing

$$\sum_{\text{date}} \left[\left(\frac{n_I}{f_\pi^2 \mu_I} \Big|_{\text{result}} - \frac{n_I(A, B, C)}{f_\pi^2 \mu_I} \Big|_{\text{ChPT}} \right)^2 + \left(\frac{\langle\sigma\rangle}{\langle\sigma\rangle_0} \Big|_{\text{result}} - \frac{\langle\sigma\rangle}{\langle\sigma\rangle_0} \Big|_{\text{ChPT}} \right)^2 + \left(\frac{\langle\pi^1\rangle}{\langle\sigma\rangle_0} \Big|_{\text{result}} - \frac{\langle\pi^1\rangle}{\langle\sigma\rangle_0} \Big|_{\text{ChPT}} \right)^2 \right]. \quad (4.11)$$

We show the best fitted values of A , B , and C for $\lambda = 4.4$ in Table II together with a set of empirical values.³ We show the μ_I dependence of n_I in Fig. 4 and the μ_I dependence of $\frac{\langle\sigma\rangle}{\langle\sigma\rangle_0}$ and $\frac{\langle\pi^1\rangle}{\langle\sigma\rangle_0}$ in Fig. 5. These figures show that the deviation of our result from the one obtained from the $O(p^2)$ chiral Lagrangian is actually explained by including the effects of $O(p^4)$ terms.

V. A SUMMARY AND DISCUSSIONS

We studied the phase transition to the pion condensation phase for finite isospin chemical potential using the holographic QCD model given in Refs. [32,33]. We introduced the isospin chemical potential μ_I as a UV-boundary value of the time component of the gauge field of the $SU(2)_V$ symmetry as $V_0^3(z)|_e = \mu_I$. We assumed the nonexistence of vector meson condensates since we are interested in studying the small μ_I region. Furthermore, we assumed that the neutral pion does not condense. We solved the coupled equations of motion for the π condensate and σ condensate together V_0^3 to determine μ_I as an eigenvalue.

Our result shows that the phase transition is of the second order and the critical chemical potential is predicted to be equal to the pion mass. This is consistent with the result obtained by the chiral Lagrangian approach in Ref. [3] and the one from the holographic QCD model [27]. Furthermore, our result on the relation between the isospin number density and the isospin chemical potential for small μ_I agrees with the one obtained by the $O(p^2)$ chiral Lagrangian [3]. For large $\mu_I (> 500 \text{ MeV})$, there is a

difference between our predictions and the one from the $O(p^2)$ chiral Lagrangian, which is shown to be understood as the $O(p^4)$ contributions.

We also studied the μ_I dependence of the π condensate and σ condensate. Our result shows that, at the phase transition point, the π condensate increases from zero as $\langle\pi^a\rangle \propto (\mu_I - \mu_I^c)^\nu$ with $\nu = 1/2$ consistently with the mean-field type of the phase transition. Furthermore, we find that the σ condensate decreases rapidly after the phase transition where the π condensate grows rapidly, while the value of the ‘‘chiral condensate’’ defined by $\langle\sigma\rangle = \sqrt{\langle\sigma\rangle^2 + \langle\pi^a\rangle^2}$ is constant for $\mu_I \lesssim 300 \text{ MeV}$, and that it grows rapidly in the large μ_I region, which is contrary to the result by the strong coupling lattice shown in Ref. [17]. This indicates that the chiral symmetry restoration at finite baryon density and/or finite temperature will be delayed when the nonzero isospin chemical potential is turned on.⁴

In the present analysis, we did not include the effect of CS term. When the CS term is included, an additional contribution from the ω -type gauge field should be included. However, as far as the $O(3)$ spatial rotation is kept unbroken, the result given in the present analysis will not be changed.

It is interesting to study the ρ -meson condensation by extending the present analysis. In such a case, the ω -type gauge field will give a contribution through the CS term. It is also interesting to include the explicit degrees of nucleons, by which we can study the phase structure, including the baryon number chemical potential as well as the isospin chemical potential. This will be done by using the ‘‘holographic mean-field’’ approach done in Refs. [38,39]. We leave these analyses for future publications.

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³We calculate the empirical values of A , B , and C through Eq. (4.6), where $L_i^{(2)}$ are determined from the values of the low energy constant in the two flavor ChPT [37] with the renormalization scale equal to M_η .

⁴Our result of the enhancement of the chiral symmetry breaking indicates that the critical point for the chiral phase transition may be shifted to a higher chemical potential due to the existence of the isospin chemical potential. This is consistent with the result of Ref. [18].

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