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Vacuum alignment and lattice artifacts: Staggered fermions

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In confining lattice gauge theories in which part of the flavor group is coupled weakly to additional gauge fields, both the dynamics of the weak gauge fields as well as lattice artifacts may have nontrivial effects on the orientation of the vacuum in flavor space. Here we discuss this issue for lattice gauge theories employing staggered fermions. Staggered fermions break flavor symmetries to a much smaller group on the lattice, and orientations in flavor space that are equivalent in the continuum may be distinct on the lattice. Assuming universality, we show that in the continuum limit the weakly gauged flavor symmetries are always vectorlike, disproving a recent claim in the literature.

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I. INTRODUCTION

Recently, there has been a growing interest in the nonperturbative study of gauge theories with both strong and weak gauge interactions—with the latter being weak at the scale where the former is strong—encompassing both Standard Model and beyond the Standard Model physics. An example of the first is the inclusion of electromagnetic effects in lattice QCD [1], and an example of the second is the study of composite Higgs models [2,3], where a relatively light Higgs particle is assumed to arise as a Nambu-Goldstone boson (NGB) of some new strong dynamics. In such models, some of the flavor symmetries of the new strong sector are weakly coupled to additional gauge fields, turning the Higgs particle into a pseudo-NGB, which then induces electroweak symmetry breaking dynamically.

On the lattice, most discretizations of the fermion action typically have a much reduced flavor symmetry in comparison to the corresponding continuum theory. However, for commonly used fermion discretizations the full continuum symmetry gets restored in the continuum limit with minimal or no fine-tuning. Indeed a large variety of lattice fermion actions gives rise to the same theory in the continuum limit, a phenomenon known as universality.

In order to make the discussion more concrete, we will limit ourselves here to confining $SU(N_c)$ gauge theories with $N_c \geq 3$, coupled to an even number N_f of Dirac fermions in the fundamental representation.[1] Any such theory can be formulated on the lattice using reduced staggered fermions [4–8]. If N_f is a multiple of 4, standard staggered fermions can be used.[2] In the massless limit, the flavor symmetry of the continuum theory is $SU(N_f)_L \times SU(N_f)_R$. The use of reduced or standard staggered fermions leaves intact only a rather small subgroup of the flavor symmetry, but the remaining flavor symmetries are automatically restored in the continuum limit [7,8].

There is a less well-known aspect of staggered fermions. The physical role of the continuous global symmetries of the massless lattice theory, or lattice flavor symmetries for short, depends on the choice of lattice mass terms. The unit cell of the staggered action is a 2^4 hypercube, and mass terms coupling any two lattice sites within the unit cell can be written down [7,8]. For the most common, same-site mass term, some of the lattice flavor symmetries become axial symmetries in the continuum limit. However, for mass terms that couple pairs of lattice sites separated by an odd number of links, all the lattice flavor symmetries become vector symmetries in the continuum limit. Thus, even if all fermions have equal masses, the embedding of the lattice flavor symmetries in the continuum flavor group $SU(N_f)_L \times SU(N_f)_R$ depends on the choice of lattice mass terms.

So far, these observations were essentially just technical. The situation changes if a subgroup of the continuum flavor symmetry is weakly coupled to new dynamical gauge fields ("weak gauge fields," for short). The new dynamics will typically distinguish between different orientations of the mass terms, or, in the massless limit, of the fermion

[1]We anticipate that the generalization to other groups and representations is relatively straightforward in many cases.

[2]Large-scale numerical simulations of QCD make use of three standard staggered fields (one for each of the up, down and strange quarks), but the fourth root of the fermion determinant is taken in order to reduce the number of fermion species from 12 to 3 (see, e.g., Refs. [9,10] and references therein). In this paper we only consider local lattice theories, avoiding any fractional powers of the staggered-fermion determinant.

condensate. Orientations of the fermion condensate that do not break spontaneously any of the weakly gauged symmetries will be energetically favorable, a phenomenon known as vacuum alignment [11]. Away from the chiral limit, there will be competing effects between the explicit mass terms on the one hand, and the effective potential induced by the weak gauge fields on the other hand. The outcome—the orientation of the vacuum—will depend on the details.

A third source of dynamical effects is provided by the discretization itself. The reduced symmetry of the lattice theory allows for the dynamical generation of an effective potential at order a^2, where a is the lattice spacing. This effective potential, too, can give rise to a nontrivial phase diagram. The most familiar example of this sort is the so-called Aoki phase encountered for Wilson fermions [12–14], where some of the lattice (vector) flavor symmetries undergo spontaneous breaking. An order-a^2 effective potential gets generated for staggered fermions as well [15,16], leading to the possibility of similar phases [17].

When studying composite-Higgs models, or any other model involving the dynamical breaking of electroweak symmetry, we have to take the combined continuum and chiral limit. The relevant phase diagram is therefore controlled by the dynamics of the weak gauge fields only. However, realistic lattice simulations are carried out away from both limits. In the lattice simulation, all three sources—explicit mass terms, weak gauge fields, and discretization effects—will in general compete, leading to a potentially very complicated outcome. Both the continuum and the chiral limits will have to be studied with great care, in order to determine whether we have arrived close enough to the combined limit such that the weak gauge field dynamics has taken over.

In this paper we study these questions using chiral Lagrangian techniques [18]. After a brief review of relevant facts about staggered fermions in Sec. II, we turn in Sec. III to the eight-flavor theory. This theory can be formulated on the lattice using two standard staggered fields, or, equivalently, four reduced staggered fields. Starting with the case that none of the flavor symmetries are gauged we compare two different choices: the same-site and the one-link mass terms. While the continuum limit is the same for both choices, only in the case of the same-site mass term do some of the lattice flavor symmetries turn into axial symmetries of the continuum theory.

We then study what happens when the lattice flavor symmetries are weakly gauged. Using Witten's inequality [19] we prove that, after taking the continuum and chiral limits, the vacuum state orients itself along the one-link mass term. Therefore all of the weakly gauged symmetries are vectorial, and none of them are broken spontaneously, in agreement with the Vafa-Witten theorem [20]. This result refutes a claim recently made in the literature [21]. In Sec. IV we study the six-flavor theory, with the new element that in this case the reduced staggered formalism is indispensable, and we arrive at similar conclusions. We conclude in Sec. V. In Appendix A we rederive the continuum effective potential, while Appendix B contains some simple observations which follow from the structure of the order-a^2 effective potential for the eight-flavor theory.

II. STAGGERED-FERMIONS BASICS

In this section, we review some of the basic properties of staggered fermions. For a comprehensive treatment, we refer to Refs. [7,8], and to the reviews in Refs. [10,18].

The Lagrangian for a single massless staggered fermion $\chi(x)$ coupled to a gauge field $U_\mu(x)$ is

$$S = \frac{1}{2}\sum_{x\mu}\eta_\mu(x)\bar{\chi}(x)(U_\mu(x)\chi(x+\mu) - U_\mu^\dagger(x-\mu)\chi(x-\mu)), \tag{2.1}$$

in which the phase factors

$$\eta_\mu(x) = (-1)^{x_1+\cdots+x_{\mu-1}}, \qquad \mu = 1,\ldots,4 \tag{2.2}$$

take over the role of the Dirac matrices. Along with a suitable pure-gauge action, the staggered-fermion action (2.1) gives rise to a gauge theory with four massless Dirac flavors all in the same representation of the gauge group in the continuum limit.[3] The continuum theory thus has an $SU(4)_L \times SU(4)_R$ flavor symmetry.

Apart from fermion number, the lattice action (2.1) has only one continuous symmetry, $U(1)_\epsilon$, given by [5]

$$\chi(x) \to e^{i\alpha\epsilon(x)}\chi(x), \qquad \bar{\chi}(x) \to \bar{\chi}(x)e^{i\alpha\epsilon(x)}, \tag{2.3}$$

with

$$\epsilon(x) = (-1)^{x_1+x_2+x_3+x_4}. \tag{2.4}$$

This symmetry is usually interpreted as an axial symmetry, but this interpretation actually depends on the mass terms that are added to the lattice theory. In most applications, a single-site mass term $m\bar{\chi}(x)\chi(x)$ is chosen. This breaks $U(1)_\epsilon$ softly, signifying that $U(1)_\epsilon$ is indeed an axial symmetry in this case.

However, one may choose different mass terms. For instance, another gauge-invariant mass term is given by

$$S_{\text{1-link}} = \frac{1}{2}\sum_{x\mu}m_\mu\zeta_\mu(x)\bar{\chi}(x)(U_\mu(x)\chi(x+\mu) + U_\mu^\dagger(x-\mu)\chi(x-\mu)), \tag{2.5}$$

[3]In the context of QCD, usually these four flavors are referred to as "tastes," but here we will choose to refer to them as flavors.

with a new set of phase factors

$$\zeta_\mu(x) = (-1)^{x_{\mu+1}+\cdots+x_4}. \tag{2.6}$$

These phase factors ensure that $S_{\text{1-link}}$ is invariant under hypercubic rotations if m_μ is treated as a vector spurion. Since $S_{\text{1-link}}$ couples fermion and antifermion fields that are one link apart, it is invariant under $U(1)_\epsilon$, which implies that the $U(1)_\epsilon$ symmetry ends up as a vector symmetry in the continuum limit [7,8]. In this limit the four flavors are degenerate, and their common mass is proportional to $m_{\text{1-link}} = \sqrt{\sum_\mu m_\mu^2}$. The one-link mass term (2.5) is particularly relevant in the case of reduced staggered fermions, which we introduce next.

Let us project the field $\chi(x)$ onto the even sites, and the independent field $\bar{\chi}(x)$ onto the odd sites:

$$\chi^+(x) = \frac{1}{2}(1+\epsilon(x))\chi(x), \qquad \bar{\chi}^-(x) = \frac{1}{2}(1-\epsilon(x))\bar{\chi}(x), \tag{2.7}$$

thereby thinning out the number of degrees of freedom by a factor 2. Applying this projection to the action (2.1) gives rise to the (massless) reduced staggered fermion action

$$S^+ = \frac{1}{2}\sum_{x\mu}\eta_\mu(x)\bar{\chi}^-(x)(U_\mu(x)\chi^+(x+\mu) - U_\mu^\dagger(x-\mu)\chi^+(x-\mu)). \tag{2.8}$$

Instead of four, this action gives rise to two Dirac flavors in the continuum limit, with flavor symmetry group $SU(2)_L \times SU(2)_R$ [4,7,8].

A different reduced staggered action is obtained by reversing the projections in Eqs. (2.7) and (2.8), namely, by choosing

$$\chi^-(x) = \frac{1}{2}(1-\epsilon(x))\chi(x), \qquad \bar{\chi}^+(x) = \frac{1}{2}(1+\epsilon(x))\bar{\chi}(x). \tag{2.9}$$

We may take two reduced staggered fields, one of each type, and reassemble them into a single standard staggered fermion. The same-site mass term we have discussed for the standard case decomposes as

$$m[\bar{\chi}^+(x)\chi^+(x) + \bar{\chi}^-(x)\chi^-(x)] = m\bar{\chi}(x)\chi(x), \tag{2.10}$$

showing that the two reduced-staggered types defined by the projections (2.7) and (2.9) are coupled to each other. In contrast, the one-link mass term of Eq. (2.5) involves no coupling between the two reduced staggered types.

Let us elaborate on this observation. Given a single reduced staggered field, it is evidently not possible to construct a same-site mass term. The simplest mass term is the one-link mass term obtained from Eq. (2.5) above via the relevant projection

$$S^\pm_{\text{1-link}} = \frac{1}{2}\sum_{x\mu} m_\mu \zeta_\mu(x)\bar{\chi}^\mp(x)(U_\mu(x)\chi^\pm(x+\mu) + U_\mu^\dagger(x-\mu)\chi^\pm(x-\mu)). \tag{2.11}$$

An independent mass term can be constructed by coupling fermion and antifermion fields that are three links apart.[4] Either way, the number of links separating the reduced fermion and antifermion fields has to be odd, and therefore any mass term in the reduced case is invariant under $U(1)_\epsilon$.

It follows that whenever the lattice theory does not involve bilinear couplings between reduced staggered fields of different types, all the lattice flavor symmetries necessarily turn into vector symmetries in the continuum limit. Indeed, considering the standard staggered action (2.1) let us denote the generators of fermion number and of $U(1)_\epsilon$ by Q_s and Q_ϵ respectively. It is easily seen that the linear combinations $Q_s + Q_\epsilon$ and $Q_s - Q_\epsilon$ generate the fermion number symmetries associated with the projections (2.7) and (2.9), respectively. Provided that the chosen mass terms respect the individual fermion number symmetries, these symmetries are, therefore, vectorial.

One can construct theories with an arbitrary even number of flavors, N_f, using $N^+ \leq N_f/2$ reduced staggered fields of type (2.7), together with $N^- = N_f/2 - N^+$ reduced fields of type (2.9). The same set of fields can also be regarded as consisting of $N_s = \min(N^+, N^-)$ standard staggered fields, with the remaining reduced staggered fields being all of the same type. In the massless case, the lattice flavor symmetry is $U(N^+) \times U(N^-)$, generalizing the fermion number symmetries of the individual reduced fields. The flavor symmetry remains intact if one-link mass terms with the same vector $m_\mu^\pm$ are introduced for all reduced fields of a given type, consistent with the fact that in this case, all the lattice flavor symmetries are vectorial. For other choices of mass terms, some of the lattice flavor symmetries may be softly broken.

As a final comment we note that, thanks to additional discrete symmetries, the renormalization of all mass terms for both standard and reduced staggered fields is multiplicative [8]. The chiral limit is therefore well defined at nonzero lattice spacing, and corresponds to the vanishing of all bare mass terms.

In the next two sections, we will employ these observations in the context of the eight-flavor and six-flavor theories. The eight-flavor theory we will consider corresponds to the choice $N^+ = N^- = 2$, whereas the six-flavor theory corresponds to $N^+ = 2$, $N^- = 1$.

[4]Replacing the one-link mass terms by three-link mass terms does not change our conclusions. We will therefore limit the discussion to one-link mass terms.

III. EIGHT FLAVORS

In this section, we will consider an eight-flavor theory coupled to a strong gauge field $U_\mu(x)$. The lattice theory is constructed using two standard staggered fields, or, equivalently, four reduced staggered fields, two of each type. For clarity, we will denote by χ_i, $\bar{\chi}_i$, $i = 1, 2$, the two reduced staggered fields of type (2.7), and by λ_i, $\bar{\lambda}_i$, $i = 1, 2$, the two reduced fields of type (2.9). According to the discussion in the previous section, if we disregard $U(1)$ factors, the non-Abelian global symmetry of the massless lattice theory is $SU(2)_\chi \times SU(2)_\lambda$. In the continuum limit, the flavor symmetry enlarges to $SU(8)_L \times SU(8)_R$, which we will assume to be spontaneously broken to the diagonal $SU(8)_V$ subgroup.

We will consider two choices for the lattice mass term, as well as the corresponding orientations of the fermion condensate in the chiral limit. While both choices give rise to the same continuum limit, the embedding of the lattice flavor symmetries inside the continuum symmetry group is different. For one of these choices, some of the lattice flavor symmetries become spontaneously broken axial symmetries in the continuum limit; for the other choice, all the lattice flavor symmetries are vectorial in the continuum limit. We will then weakly couple all the (non-Abelian) lattice flavor symmetries to additional gauge fields, and prove that in this case all of them become unbroken vectorial symmetries of the continuum theory.

In the continuum limit, each reduced staggered field gives rise to two Dirac fields, according to

$$\chi_1 \to \psi_1, \psi_2, \qquad \chi_2 \to \psi_3, \psi_4,$$
$$\lambda_1 \to \psi_5, \psi_6, \qquad \lambda_2 \to \psi_7, \psi_8. \tag{3.1}$$

Alternatively, viewing the fermion content as two standard staggered fields $\chi_i + \lambda_i$, the continuum flavors ψ_1, ψ_2, ψ_5 and ψ_6 emerge from $\chi_1 + \lambda_1$, while ψ_3, ψ_4, ψ_7 and ψ_8 emerge from $\chi_2 + \lambda_2$.

Our first choice for the mass terms is to use the one-link mass term (2.11) for each reduced staggered fermion, always with the same parameters m_μ. In the continuum theory, the resulting mass term is

$$\sum_{i=1}^{2} S_{\text{1-link}}(\chi_i, \lambda_i; m_\mu) \to m \int d^4x \sum_{k=1}^{8} \bar{\psi}_k \psi_k, \tag{3.2}$$

where $m \geq 0$ is given by[5]

$$m^2 = \sum_\mu m_\mu^2. \tag{3.3}$$

If we arrange the continuum Dirac fields into a vector, the continuum mass matrix is proportional to the identity matrix,

$$M_1 = m I_8, \tag{3.4}$$

where I_n denotes the $n \times n$ identity matrix.[6]

Alternatively, we can use the single-site mass term (2.10), obtaining

$$m \sum_x (\bar{\lambda}_1 \chi_1 + \bar{\lambda}_2 \chi_2 + \bar{\chi}_1 \lambda_1 + \bar{\chi}_2 \lambda_2)$$
$$\to m \int d^4x (\bar{\psi}_5 \psi_1 + \bar{\psi}_6 \psi_2 + \bar{\psi}_7 \psi_3 + \bar{\psi}_8 \psi_4 + \bar{\psi}_1 \psi_5$$
$$+ \bar{\psi}_2 \psi_6 + \bar{\psi}_3 \psi_7 + \bar{\psi}_4 \psi_8). \tag{3.5}$$

The corresponding mass matrix can be written in the form

$$M_0 = m\tau_1 \otimes I_2 \otimes I_2 = m\tau_1 \otimes I_4. \tag{3.6}$$

In this notation, any 8×8 matrix is expressed as a sum of tensor products. Each tensor product consists of three terms, each of which can be one of the Pauli matrices τ_a, $a = 1, 2, 3$, or the identity matrix I_2. The index of the first 2×2 matrix in the tensor product identifies the reduced staggered type (χ or λ) from which the continuum flavor originates, and the associated projectors are

$$P_\chi = \frac{1}{2}(I_2 + \tau_3) \otimes I_4 \equiv \tilde{P}_\chi \otimes I_4,$$
$$P_\lambda = \frac{1}{2}(I_2 - \tau_3) \otimes I_4 \equiv \tilde{P}_\lambda \otimes I_4. \tag{3.7}$$

The index of the second factor in the tensor product is the flavor index of the corresponding reduced staggered type, while the index of the last factor runs over the two continuum flavors that emerge from a given reduced staggered field.

In the continuum limit, the two choices for the mass term are equivalent. Indeed, one can rotate M_0 to the standard form (3.4) by a nonanomalous transformation $U \in SU(8)_L \times SU(8)_R$, under which

$$\psi \to U\psi, \qquad \bar{\psi} \to \bar{\psi}\gamma_0 U^\dagger \gamma_0. \tag{3.8}$$

To this end, we first apply the purely vectorial transformation

$$P = \frac{1}{\sqrt{2}}(I_2 - i\tau_2) \otimes I_4, \tag{3.9}$$

so that now

[5]We disregard the (multiplicative) renormalization of the mass parameters.

[6]In the continuum limit, a basis can always be chosen for the two Dirac fields originating from a given reduced staggered field such that the mass matrix takes the form (3.4) by construction [7,8].

$$P^\dagger M_0 P = m\tau_3 \otimes I_4 = m\begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}. \tag{3.10}$$

In order to rotate the lower-right block from $-I_4$ into $+I_4$ we apply the nonanomalous axial rotation

$$Q = P_\chi + i\gamma_5 \tilde{P}_\lambda \otimes \tau_3 \otimes I_2 = \begin{pmatrix} I_4 & 0 \\ 0 & i\gamma_5\tau_3 \otimes I_2 \end{pmatrix}. \tag{3.11}$$

Using Eq. (3.8) we arrive at

$$QP^\dagger M_0 PQ = mI_8, \tag{3.12}$$

thereby reproducing Eq. (3.4), but now for the same-site mass term.

A. Global lattice flavor symmetry

Let us now discuss the interplay of the $SU(2)_\chi \times SU(2)_\lambda$ lattice flavor symmetry and the two mass terms. We are interested in the fate of these symmetries after taking the continuum limit, followed by the chiral limit where the mass term is turned off (after the infinite-volume limit has been taken).

The one-link mass term (3.2) respects the full lattice flavor symmetry. On the basis of continuum fields introduced in Eq. (3.1), the resulting mass matrix (3.4) is proportional to the identity matrix, whereas the $SU(2)_\chi \times SU(2)_\lambda$ generators take the form

$$T_a^\chi = \tilde{P}_\chi \otimes \tau_a \otimes I_2, \tag{3.13a}$$

$$T_a^\lambda = \tilde{P}_\lambda \otimes \tau_a \otimes I_2. \tag{3.13b}$$

All six generators are vectorial in this case, as they are proportional to the identity matrix in Dirac space.

We next turn to the same-site mass term (3.5). The diagonal subgroup generated by $T_a^\chi + T_a^\lambda$ commutes with this mass term. The other three generators, $T_a^\chi - T_a^\lambda$, are proportional to the phase factor $\epsilon(x)$ of Eq. (2.4), and are broken by the same-site mass term. On the same continuum basis, these linear combinations take the form

$$T_a^+ = T_a^\chi + T_a^\lambda = I_2 \otimes \tau_a \otimes I_2, \tag{3.14a}$$

$$T_a^- = T_a^\chi - T_a^\lambda = \tau_3 \otimes \tau_a \otimes I_2. \tag{3.14b}$$

As on the lattice, the T_a^+ commute with the mass matrix (3.6), whereas the T_a^- do not. Applying the basis transformation that brings the mass matrix (3.6) to the diagonal form (3.12), the generators become

$$Q^\dagger P^\dagger T_a^+ PQ = (\tilde{P}_\chi \otimes \tau_a + \tilde{P}_\lambda \otimes \tau'_a) \otimes I_2, \tag{3.15a}$$

$$Q^\dagger P^\dagger T_a^- PQ = \gamma_5(\epsilon_{ab3}\tau_1 \otimes \tau_b - \delta_{a3}\tau_2 \otimes I_2) \otimes I_2, \tag{3.15b}$$

where $\tau'_a = \tau_3\tau_a\tau_3$. While the T_a^+ still generate a vectorial symmetry, the T_a^- now generate an axial symmetry.

The fact that we can rotate the mass matrix (3.6) to the diagonal form (3.12) using an $SU(8)_L \times SU(8)_R$ transformation implies that the two mass matrices are equivalent. So are the corresponding orientations of the fermion condensate in the chiral limit. Indeed, with the restriction to a degenerate mass for all eight flavors, all possible choices for the lattice mass terms are equivalent in that, in the continuum limit, the resulting symmetry-breaking pattern is always $SU(8)_L \times SU(8)_R \to SU(8)_V$, with the unbroken $SU(8)_V$ commuting with the mass matrix. Any violation of this observation would constitute a violation of universality.

But the fate of the lattice flavor symmetries is not the same. In the case of the one-link mass term (3.2), all of them become unbroken vectorial symmetries of the continuum theory, whereas in the case of the same-site mass term (3.6), this is true only for half of the lattice symmetries, while the other half turn into axial symmetries, which are spontaneously broken in the chiral limit.

B. Gauging $SU(2)_\chi \times SU(2)_\lambda$

We now introduce a new element, by promoting the lattice global symmetry group $SU(2)_\chi \times SU(2)_\lambda$ to a local symmetry. We introduce a dynamical gauge field $V_{\mu a}$ minimally coupled to the conserved currents of $SU(2)_\chi$ with coupling constant g_χ, and, similarly, a gauge field $W_{\mu a}$ with coupling g_λ for $SU(2)_\lambda$. We will assume that both of the new couplings are weak at the scale Λ where the original strong dynamics of the gauge field U_μ is confining.[7]

The effective low-energy theory depends on a nonlinear field $\Sigma(x) \in SU(8)$. We may think of $\Sigma_{k\ell}(x)$ as representing the composite operator $\mathrm{tr}((1-\gamma_5)\psi_k(x)\bar{\psi}_\ell(x))$, where the trace is over Dirac and strong gauge group indices. The interaction with the dynamical weak gauge fields induces an effective potential for the continuum theory. To lowest nontrivial order in the weak gauge couplings, the effective potential is [11]

$$V_{\mathrm{weak}}(\Sigma) = -g_\chi^2 C \sum_a \mathrm{tr}(\Sigma T_a^\chi \Sigma^\dagger T_a^\chi) - g_\lambda^2 C \sum_a \mathrm{tr}(\Sigma T_a^\lambda \Sigma^\dagger T_a^\lambda). \tag{3.16}$$

We have restricted the nonlinear field to a constant value $\Sigma(x) = \Sigma$ representing the vacuum. In the chiral limit of the

[7]For $N_c = 3$ it is not clear whether or not the eight-flavor theory is confining [22]. The eight-flavor theory confines in the large N_c limit, and we will assume that N_c is large enough that this is the case.

continuum theory there are no other effects of similar magnitude, and the vacuum state is determined by minimizing $V_{\mathrm{weak}}(\Sigma)$.

Let us now compare the vacua Σ_0, defined by the orientation of the same-site mass matrix (3.6), and Σ_1, defined by the orientation of the one-link mass matrix (3.4). On the continuum basis introduced in Eq. (3.1), these vacua are

$$\Sigma_0 = \tau_1 \otimes I_4, \qquad \Sigma_1 = I_8. \tag{3.17}$$

We find that

$$V_{\mathrm{weak}}(\Sigma_0) = 0, \tag{3.18a}$$

$$V_{\mathrm{weak}}(\Sigma_1) = -12(g_\chi^2 + g_\lambda^2)C. \tag{3.18b}$$

A key observation is that the low-energy constant C is positive [19]. The Σ_1 vacuum wins, and, in fact, since Eq. (3.18b) is the minimum value V_{weak} can take, Σ_1 is the correct vacuum of the continuum theory. The full vacuum manifold consists of all $\Sigma \in SU(8)$ where V_{weak} retains the value (3.18b), and therefore any representative of the true vacuum must commute with all $T_a^{\chi,\lambda}$.

In accordance with Ref. [11], the weak gauge-field dynamics aligns the vacuum such that the corresponding gauge fields remain massless, and the subgroup $SU(2)_\chi \times SU(2)_\lambda$ is unbroken; no dynamical Higgs mechanism is taking place. This disproves recent claims in the literature [21].

The inequality of Ref. [19], which guarantees that $C > 0$ in the broken phase, makes this a rigorous result. We stress that the eight-flavor theory can be regularized such that all the conditions of Ref. [19] are fulfilled. According to universality, the resulting effective potential, Eq. (3.16), must be independent of all details of the lattice regularization.

In order to keep this paper self-contained we have included a rederivation of the most general order-g^2 continuum effective potential in Appendix A (for a recent review, see Ref. [3]). As explained in the Appendix, when applying the master formula (A3) we have to treat differently those generators that are proportional to γ_5 on the basis we are using, and those that are not. In the Appendix we illustrate this by working out explicitly the case where an Abelian gauge field is weakly coupled to the generator T_3^-. We show that, even though this generator is axial with respect to the basis that diagonalizes the same-site mass matrix [see Eq. (3.15b)], the true vacuum realigns itself along the one-link mass term, and the Abelian symmetry ends up being vectorial and unbroken.

C. Comments on the lattice theory

The result of the previous subsection was derived after taking the continuum and chiral limits. In a numerical lattice computation, there are usually practical considerations dictating the use of nonvanishing mass terms. In addition, discretization effects are unavoidable.

Let us momentarily turn off the weak gauge couplings g_χ and g_λ and thus the associated continuum effective potential (3.16), as well as any mass terms. In other words, let us consider the chiral limit at nonzero lattice spacing, with $SU(2)_\chi \times SU(2)_\lambda$ a global symmetry group. The order-a^2 staggered effective potential corresponding to this situation is recorded in Appendix B.[8] It should be clear from the complicated form of this effective potential that many orientations of the vacuum will be inequivalent on the lattice, even if they become equivalent in the massless continuum theory for $g_\chi = g_\lambda = 0$. As discussed in Appendix B, one can easily envisage values for the order-a^2 low-energy constants (LECs) that would prefer the vacuum Σ_0, and others that would prefer the vacuum Σ_1.[9]

We do not know the actual values of the LECs of the eight-flavor staggered theory for a given number of colors. There is a clear message, however, that does not require this knowledge. In general, the vacuum of the theory will be influenced by all sources: discretization effects, explicit mass terms, and weak gauge fields. In the region where they are comparable,

$$m/\Lambda \sim a^2\Lambda^2 \sim g_\chi^2 \sim g_\lambda^2, \tag{3.19}$$

one would expect a complicated phase diagram. In the previous subsection we considered the limiting case where

$$m/\Lambda, a^2\Lambda^2 \ll g_\chi^2, g_\lambda^2. \tag{3.20}$$

By contrast, the opposite limit

$$m/\Lambda, a^2\Lambda^2 \gg g_\chi^2, g_\lambda^2 \tag{3.21}$$

will be dominated by discretization effects that in general will have nothing to do with the continuum physics we are after. According to one example we give in Appendix B, the discretization effects prefer the Σ_0 vacuum associated with the same-site mass term. Close enough to the continuum limit, we would then expect a crossover from the vacuum $\Sigma \sim \Sigma_0$ to the true continuum vacuum Σ_1 of the weakly gauged theory. Thus, ensuring that a lattice study is conducted with the correct parameter hierarchy (3.20), as opposed to the hierarchy (3.21), may not be an easy task.

IV. SIX FLAVORS

In this section we consider a six-flavor theory. As before, the fermion fields reside in the fundamental representation

[8]The symmetries of staggered fermions forbid order-a terms in the effective potential.

[9]Note the change of basis of the Σ field performed in Appendix B.

of a strongly interacting $SU(N_c)$ gauge group with $N_c \geq 3$, that confines at a scale Λ. In the absence of additional weak gauge fields, the massless continuum theory has an $SU(6)_L \times SU(6)_R$ flavor symmetry, which is spontaneously broken to the diagonal, vectorial subgroup $SU(6)_V$.

The lattice theory is constructed using three reduced staggered fields, with two fields χ_i, $i = 1, 2$, defined by the projection (2.7), and a single field λ defined by the alternative projection (2.9). The non-Abelian flavor symmetry is therefore $SU(2) = SU(2)_\chi$. Notice that the same lattice theory can also be viewed as composed of one standard staggered fermion, say $\chi_2 + \lambda$, and a single reduced staggered fermion, χ_1.

Our interest in this particular discretization arises because the model with $N_c = 3$ was recently investigated numerically in Ref. [21]. As in the eight-flavor case we will first study two choices for the mass matrix, while keeping the $SU(2)$ flavor symmetry global. We will then study the weak coupling of $SU(2)$ to an additional gauge field, again finding that the vacuum aligns such that this symmetry is unbroken.

The six Dirac flavors of the continuum theory emerge from the lattice fields according to

$$\chi_1 \to \psi_1, \psi_2, \qquad \chi_2 \to \psi_3, \psi_4, \qquad \lambda \to \psi_5, \psi_6. \tag{4.1}$$

We first choose one-link mass terms (2.11) for all reduced staggered fields. On the continuum basis above the resulting mass term will be proportional to the 2×2 identity matrix for each reduced staggered field. However, we will now allow the Dirac fields originating from χ_1 to have a different mass from the rest, namely,

$$\int d^4x (m'(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) + m(\bar{\psi}_3\psi_3 + \bar{\psi}_4\psi_4 + \bar{\psi}_5\psi_5 + \bar{\psi}_6\psi_6)). \tag{4.2}$$

Alternatively, we may introduce a single-site mass term (2.10) for the standard staggered field $\chi_2 + \lambda$, and a one-link mass term only for the remaining reduced staggered fermion χ_1. In the continuum limit, the mass term is now

$$\int d^4x (m'(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) + m(\bar{\psi}_5\psi_3 + \bar{\psi}_6\psi_4 + \bar{\psi}_3\psi_5 + \bar{\psi}_4\psi_6)), \tag{4.3}$$

which corresponds to the mass matrix

$$M_0 = m\begin{pmatrix} \xi & 0 \\ 0 & \tau_1 \end{pmatrix} \otimes I_2 = m\begin{pmatrix} \xi & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.4}$$

In the block form in the middle, τ_1 is the first Pauli matrix, the upper-left entry is $\xi = m'/m$, and the off-diagonal entries represent 1×2 and 2×1 blocks of zeros.

The mass matrix (4.4) can be rotated to a positive diagonal matrix by a nonanomalous $SU(6)_L \times SU(6)_R$ basis transformation. Using the vectorial $SU(2)$ transformation

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}}(I_2 - i\tau_2) \end{pmatrix} \otimes I_2, \tag{4.5}$$

which rotates ψ_3 and ψ_4 into ψ_5 and ψ_6, the mass matrix is first brought to the form

$$P^\dagger M_0 P = m\begin{pmatrix} \xi & 0 \\ 0 & \tau_3 \end{pmatrix} \otimes I_2 = m \;\mathrm{diag}(\xi, \xi, 1, 1, -1, -1). \tag{4.6}$$

We then apply the nonanomalous chiral rotation

$$Q = \mathrm{diag}(1, 1, 1, 1, i\gamma_5, -i\gamma_5) \tag{4.7}$$

arriving, analogous to the eight-flavor case, at

$$QP^\dagger M_0 P Q = m \;\mathrm{diag}(\xi, \xi, 1, 1, 1, 1). \tag{4.8}$$

A. Global lattice flavor symmetry

The lattice $SU(2)$ flavor symmetry that rotates χ_1 into χ_2 will, in the continuum limit, rotate ψ_1 into ψ_3 and ψ_2 into ψ_4. Using the basis introduced in Eq. (4.1), the $SU(2)$ generators are

$$T_a = \begin{pmatrix} \tau_a & 0 \\ 0 & 0 \end{pmatrix} \otimes I_2. \tag{4.9}$$

We see that relative to the continuum basis where the one-link mass term takes the form (4.2), the $SU(2)$ transformations are vectorial, and unbroken provided that $m' = m$.

In contrast, the mass term (4.3) softly breaks the $SU(2)$ symmetry. We may recast the $SU(2)$ generators T_a of Eq. (4.9) on the basis in which the mass matrix (4.4) takes the form (4.8), obtaining

$$T_1' = Q^\dagger P^\dagger T_1 PQ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & I_2 & -i\gamma_5\tau_3 \\ I_2 & 0 & 0 \\ i\gamma_5\tau_3 & 0 & 0 \end{pmatrix}, \quad (4.10a)$$

$$T_2' = Q^\dagger P^\dagger T_2 PQ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -iI_2 & -\gamma_5\tau_3 \\ iI_2 & 0 & 0 \\ -\gamma_5\tau_3 & 0 & 0 \end{pmatrix}, \quad (4.10b)$$

$$T_3' = Q^\dagger P^\dagger T_3 PQ = \frac{1}{2} \begin{pmatrix} 2I_2 & 0 & 0 \\ 0 & -I_2 & i\gamma_5\tau_3 \\ 0 & -i\gamma_5\tau_3 & -I_2 \end{pmatrix}. \quad (4.10c)$$

We see that, relative to the "canonical" basis defined by the mass matrix (4.8), the $SU(2)$ group of Eq. (4.9) has turned into an admixture of vectorial and axial transformations.

B. Gauging $SU(2)$

As was done in Ref. [21], one may promote the $SU(2)$ lattice flavor symmetry to a local symmetry by introducing a new gauge field $V_{\mu a}$, with a coupling g that is weak at the confinement scale Λ. Integrating out the weak gauge field gives rise to the continuum effective potential

$$V_{\text{weak}}(\Sigma) = -g^2 C \sum_a \text{tr}(\Sigma T_a \Sigma^\dagger T_a), \quad (4.11)$$

with, now, $\Sigma \in SU(6)$. Again C is a positive low-energy constant [19].

Let us now compare the vacua Σ_0 and Σ_1 defined by taking the chiral limit with the mass terms (4.3) and (4.2), respectively. Explicitly, these vacua are

$$\Sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & \tau_1 \end{pmatrix} \otimes I_2, \qquad \Sigma_1 = I_6. \quad (4.12)$$

Notice that these vacua have a larger symmetry than the mass terms from which they have emerged, because the chiral limit does not depend on the ratio $\xi = m'/m$. We find that

$$V_{\text{weak}}(\Sigma_0) = -2g^2 C, \quad (4.13a)$$

$$V_{\text{weak}}(\Sigma_1) = -12g^2 C. \quad (4.13b)$$

The conclusion is analogous to the previous section. The true vacuum is Σ_1. It is the orientation that was selected by choosing one-link mass terms for all reduced staggered fields. Once again the vacuum aligns such that the gauged $SU(2)$ flavor group is unbroken. In terms of the continuum theory, we have therefore gauged a subgroup of the unbroken diagonal $SU(6)_V$ flavor symmetry group.

It follows that the apparent "Higgsing" of the weak gauge fields claimed in Ref. [21] must be a lattice artifact, caused by contributions to the effective potential that vanish in the continuum limit.

V. CONCLUSION

A strongly coupled theory with multiple standard or reduced staggered fermions has a lattice flavor symmetry group which is smaller than the flavor symmetry group of its continuum limit. If all fermions are massless, some of the lattice flavor symmetries will have generators of the form $T_a^\epsilon = T_a \epsilon(x)$, where T_a is an element of some Lie algebra, and $\epsilon(x)$ is defined in Eq. (2.4). Whether such a symmetry should be interpreted as a vector or an axial symmetry in the continuum limit depends on the mass terms that may be added to the theory, as explained in Sec. II. If the massless limit is taken after the continuum limit, the embedding of the flavor symmetry of the lattice theory into the larger flavor symmetry of the continuum theory will depend on the mass terms originally chosen on the lattice. Of course, in the continuum limit this is irrelevant, because the flavor symmetry emerging in that limit will always be the same. In both concrete examples considered in this article, the emerging symmetry is $SU(N_f)_L \times SU(N_f)_R$, with $N_f = 8$ or $N_f = 6$, spontaneously broken to the diagonal subgroup $SU(N_f)$ in the massless limit.

The situation changes if one chooses to gauge the lattice flavor symmetry group, or a subgroup of it. With staggered fermions, global symmetries with generators T_a^ϵ may also be gauged. Since it is customary to interpret these symmetries as axial symmetries, this raises the intriguing prospect of obtaining an exact chiral gauge group from the lattice. Moreover, since the strong dynamics spontaneously breaks axial symmetries in the massless limit, naturally a Higgs mechanism would take place, with the weak gauge fields coupled to the T_a^ϵ acquiring a mass.[10] Reference [21] claims to find evidence for this mechanism from numerical studies of the six-flavor and eight-flavor theories we discussed in this article.

However, the analysis of Ref. [11] of the effective potential generated by the weak gauge fields, combined with the rigorous inequality of Ref. [19], implies that this cannot happen in the continuum limit. In making this statement we are, of course, invoking universality in that we assume that the form of the continuum effective potential must be independent of all details of the lattice regularization.

It is the dynamics of the weak gauge fields themselves that gives rise to vacuum alignment. The true vacuum aligns such that all the lattice flavor symmetries that have

[10] Undoing this Higgs mechanism by simply turning off the strong interactions would then lead to a genuine chiral gauge theory with unbroken gauge symmetry on the lattice.

been weakly gauged, including those generated by the T_a^ϵ, become unbroken vector symmetries in the continuum limit. Indeed, as we have explained in detail, it is always possible to choose mass terms for the staggered fields such that all fermions will be massive while none of the lattice flavor symmetries are broken by these mass terms. It follows that none of these symmetries will be spontaneously broken when these mass terms are taken to zero. In other words, the flavor structure of the lattice theory will always make it possible for "complete" vacuum alignment to take place, so that all the gauge fields that were coupled weakly to lattice flavor currents remain massless.

It follows that the numerical evidence presented in Ref. [21] must be the consequence of lattice artifacts. Indeed, away from the continuum limit, lattice artifact contributions to the effective potential for the vacuum may compete with the contribution generated by the dynamical flavor gauge fields. A more detailed study of the effective potential along the lines of Ref. [17] is possible, but outside the scope of this article.

A competition between lattice artifacts and the dynamics of weak gauge fields may arise for other fermion formulations as well. For a study of these effects with Wilson fermions, we refer to Ref. [23].

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APPENDIX A: THE CONTINUUM EFFECTIVE POTENTIAL

In this Appendix we rederive the continuum effective potential for the nonlinear Σ field induced by a single weak gauge-boson exchange in the underlying theory. This can be done via an elegant spurion trick [24].[11]

As usual we will take the strong sector to be an $SU(N_c)$ gauge theory with $N_c \geq 3$, coupled to N_f Dirac fields in the fundamental representation.[12] The global symmetry of the massless theory is $SU(N_f)_L \times SU(N_f)_R$. We introduce global flavor spurions $Q^L = Q_a^L T_a$, $Q^R = Q_a^R T_a$, where $a = 1, 2, \ldots, N_f^2 - 1$. Under $g_{L,R} \in SU(N_f)_{L,R}$ they transform as $Q^{L,R} \to g_{L,R} Q^{L,R} g_{L,R}^\dagger$. The partition function is

$$Z(Q^L, Q^R) = \int d[A]d[W]d[\psi]d[\bar\psi] \exp[-S_\mathrm{S}(A_\mu, \psi_i, \bar\psi_i) - S_\mathrm{W}(W_\mu, \psi_i, \bar\psi_i, Q^L, Q^R)], \tag{A1}$$

where S_S is the action for the strong dynamics, with A_μ the $SU(N_c)$ gauge field, and $\psi_i, \bar\psi_i$, $i = 1, 2, \ldots, N_f$, the quark fields. The weakly coupled dynamics is accounted for by

$$\begin{aligned} S_\mathrm{W} &= \frac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 + g W_\mu (Q_a^L J_{\mu a}^L + Q_a^R J_{\mu a}^R), \\ J_{\mu a}^R &= \frac{1}{2}\bar\psi_i \gamma_\mu (1 + \gamma_5) T_{aij} \psi_j, \\ J_{\mu a}^L &= \frac{1}{2}\bar\psi_i \gamma_\mu (1 - \gamma_5) T_{aij} \psi_j. \end{aligned} \tag{A2}$$

The partition function $Z(Q^L, Q^R)$ is invariant under global $SU(N_f)_L \times SU(N_f)_R$ transformations. The flavor indices are carried by the spurions Q^L, Q^R, while the W_μ is a single gauge field, inert under the flavor transformations. We get away with not having a full set of flavored gauge fields because we are only aiming to extract the effect of a single weak gauge-boson exchange.

To order g^2, the most general effective potential consistent with the flavor symmetry is

$$V_\mathrm{eff}(\Sigma) = g^2 C_{RR} \mathrm{tr}(Q^R Q^R) + g^2 C_{LL} \mathrm{tr}(Q^L Q^L) - g^2 C_{LR} \mathrm{tr}(Q^L \Sigma Q^R \Sigma^\dagger). \tag{A3}$$

The only part that depends on the nonlinear field is the last term. The corresponding LEC, C_{LR}, may be isolated by assuming that the vacuum state is the identity matrix I_{N_f}, so that[13]

$$\frac{\partial}{\partial Q_a^L} \frac{\partial}{\partial Q_b^R} V_\mathrm{eff}(I_{N_f}) = -\frac{g^2}{2} \delta_{ab} C_{LR}. \tag{A4}$$

In order to relate C_{LR} to the microscopic theory we apply the same differentiations to the partition function $Z(Q^L, Q^R)$, finding

$$C_{LR} = \frac{1}{16\pi^2} \int_0^\infty dq^2 q^2 \Pi_{LR}(q^2), \tag{A5}$$

where ($P_{\mu\nu}^\perp$ is the transverse projector)

$$\frac{1}{2}\delta_{ab} q^2 P_{\mu\nu}^\perp \Pi_{LR}(q^2) = -\int d^4x e^{iqx} \langle J_{\mu a}^L(x) J_{\nu b}^R(0) \rangle. \tag{A6}$$

According to Ref. [19], $\Pi_{LR}(q^2) \geq 0$, and so is $C_{LR} \geq 0$.

We next explain how to use the master formula (A3) when various subgroups of the flavor symmetry group are

[11] For early discussions of the continuum effective potential, see for example Ref. [25].

[12] The derivation in this Appendix applies to any $N_f \geq 2$.

[13] Here we assume the standard orthogonality relation $\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$. Notice that the generators discussed in the main text are normalized differently.

weakly gauged. As a first example, let us weakly gauge the $SU(2)_\chi$ of Sec. III. We obtain the contributions of the weak gauge fields $V_{\mu a}$, $a = 1, 2, 3$, to the effective potential of Eq. (3.16), one at a time, as follows. On the continuum basis (3.1), the weak gauge field $V_{\mu a}$ couples to a vector current $J^L_{\mu a} + J^R_{\mu a}$, with a generator given explicitly in Eq. (3.13a). We therefore set $Q^L_a = Q^R_a = 1$ for the left and right spurions associated with this particular generator, while setting to zero all other spurions. With the obvious identification $g^2 C_{LR} \to g^2_\chi C$, after summing over the three generators, we obtain the first term on the right-hand side of Eq. (3.16). The same argument applies to the second term.

As another example, suppose that we weakly gauge only the $U(1)$ symmetry generated by T^-_3 of Eq. (3.14b), with coupling constant e. We will work out the vacuum energies for the two vacua $\Sigma_{0,1}$ of Sec. III. We first do the calculation using, as before, the basis (3.1). On this basis, the Abelian gauge field couples to a vector current whose associated generator is given explicitly in Eq. (3.14b). Following the same steps, the effective potential is

$$V_{\rm weak} = -e^2 C_{LR} {\rm tr}(\Sigma T^-_3 \Sigma^\dagger T^-_3), \qquad \text{one-link basis.} \tag{A7}$$

The vacuum energies are

$$V_{\rm weak}(\Sigma_0) = +8e^2 C_{LR}, \tag{A8a}$$

$$V_{\rm weak}(\Sigma_1) = -8e^2 C_{LR}. \tag{A8b}$$

As expected, the vacuum aligns with Σ_1, so that the $U(1)$ symmetry is vectorial and unbroken.

Let us repeat the calculation, but now using the basis in which the same-site mass term is diagonal, Eq. (3.12). According to Eq. (3.15b), on this basis the generator T^-_3 is axial, which implies that we now have $Q^L = -Q^R \equiv \tilde{T}^-_3$ in Eq. (A3). Therefore, this time we find

$$V_{\rm weak} = +e^2 C_{LR} {\rm tr}(\Sigma \tilde{T}^-_3 \Sigma^\dagger \tilde{T}^-_3), \qquad \text{same-site basis.} \tag{A9}$$

The actual value $\tilde{T}^-_3$ of the spurions can be read off from the flavor matrix that multiplies γ_5 in Eq. (3.15b) for $a = 3$, leading to

$$\tilde{T}^-_3 = \tau_2 \otimes I_4. \tag{A10}$$

We next reevaluate $V_{\rm weak}$ on the two vacua. Now we must use the expressions for $\Sigma_{0,1}$ appropriate for the basis (3.12). The vacuum oriented along the same-site mass term is $\Sigma_0 = I_8$, and plugging this into Eq. (A9) reproduces Eq. (A8a). Analogous to Eq. (3.12), the vacuum oriented along the one-link mass term is now

$$\Sigma_1 = QP^\dagger I_8 PQ = Q^2 = \tau_3 \otimes I_4, \tag{A11}$$

and plugging this into Eq. (A9) reproduces Eq. (A8b).

As it must be, the vacuum energies are independent of the basis we choose. This example demonstrates explicitly that, even if the weakly gauged (Abelian) generator looks axial on some basis, the true vacuum will reorient itself such that, relative to it, that generator is vectorial and unbroken.

APPENDIX B: STAGGERED EFFECTIVE POTENTIAL AT ORDER a^2

When writing down the staggered low-energy effective theory it is customary to use a basis for the Σ field in which the same-site mass term is diagonal in flavor (or taste) space. Applying the change of basis $\Sigma \to QP^\dagger \Sigma PQ$ to the nonlinear field introduced in Sec. III B [cf. Eq. (3.12)], the order-a^2 staggered effective potential for the eight-flavor theory is [15–17]

$$\mathcal{V} = \mathcal{U} + \mathcal{U}', \tag{B1}$$

where

$$\begin{aligned} -\mathcal{U} = {} & C_1 {\rm tr}(\xi^{(2)}_5 \Sigma \xi^{(2)}_5 \Sigma^\dagger) \\ & + \frac{C_3}{2} \sum_\nu [{\rm tr}(\xi^{(2)}_\nu \Sigma \xi^{(2)}_\nu \Sigma) + {\rm H.c.}] \\ & + \frac{C_4}{2} \sum_\nu [{\rm tr}(\xi^{(2)}_{\nu 5} \Sigma \xi^{(2)}_{5\nu} \Sigma) + {\rm H.c.}] \\ & + C_6 \sum_{\mu<\nu} {\rm tr}(\xi^{(2)}_{\mu\nu} \Sigma \xi^{(2)}_{\nu\mu} \Sigma^\dagger), \end{aligned} \tag{B2}$$

$$\begin{aligned} -\mathcal{U}' = {} & \frac{C_{2V}}{4} \sum_\nu [{\rm tr}(\xi^{(2)}_\nu \Sigma) {\rm tr}(\xi^{(2)}_\nu \Sigma) + {\rm H.c.}] \\ & + \frac{C_{2A}}{4} \sum_\nu [{\rm tr}(\xi^{(2)}_{\nu 5} \Sigma) {\rm tr}(\xi^{(2)}_{5\nu} \Sigma) + {\rm H.c.}] \\ & + \frac{C_{5V}}{2} \sum_\nu [{\rm tr}(\xi^{(2)}_\nu \Sigma) {\rm tr}(\xi^{(2)}_\nu \Sigma^\dagger)] \\ & + \frac{C_{5A}}{2} \sum_\nu [{\rm tr}(\xi^{(2)}_{\nu 5} \Sigma) {\rm tr}(\xi^{(2)}_{5\nu} \Sigma^\dagger)]. \end{aligned} \tag{B3}$$

Here

$$\xi^{(2)}_B = \begin{pmatrix} \xi_B & 0 \\ 0 & \xi_B \end{pmatrix}, \tag{B4}$$

and

$$\{\xi_B\} = \{I, \xi_\mu, \xi_{\mu<\nu}, \xi_{\mu 5}, \xi_5\} \tag{B5}$$

is a basis for 4×4 Hermitian matrices in flavor space, constructed in the usual way from the matrices ξ_μ satisfying the Dirac algebra.

Depending on the actual values of the LECs, the vacuum state will have different orientations. A sufficient condition that the vacuum be oriented along the same-site mass term (in the basis used here, this is the identity matrix I_8) is that C_1, C_3, C_4 and C_6 are all positive, while $C_{2A,V} = C_{5A,V} = 0$. A different parameter range, where the vacuum is oriented with the one-link mass term, is when C_{2V} and C_{5V} are positive, and the remaining LECs vanish.

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