Connecting leptonic unitarity triangle to neutrino oscillation

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The leptonic unitarity triangle (LUT) provides a geometric description of CP violations in the lepton-neutrino sector and is directly measurable in principle. In this paper, we reveal that the angles in the LUT have definite physical meaning, and demonstrate the exact connection of the LUT to neutrino oscillations. *For the first time*, we prove that these leptonic angles act as phase shifts in neutrino oscillations, by shifting $\Delta m^2 L/2E$ to $\Delta m^2 L/2E + \alpha$, where (L, E, α) denote the baseline length, neutrino energy and corresponding angle of the LUT. Each LUT has three independent parameters and contains only partial information of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. We demonstrate that the partial information in each LUT can describe the corresponding neutrino oscillation. Hence, for the first time, we uncover that any given kind of neutrino oscillations contains at most three (rather than four) independent degrees of freedom from the PMNS matrix, and this may provide a cleaner way for fitting the corresponding oscillation data.

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I. INTRODUCTION

Discovering leptonic CP violation has vital importance for neutrino physics, as it may provide the origin of the observed matter-antimatter asymmetry in the Universe. For the quark sector, the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix [1,2] generates six unitarity triangles (UT) [3]. The angles of each triangle have clear physical meaning, and their nonzero values directly prove the CP violation. For instance, the most commonly used $d - b$ triangle is given by the relation,

$$V_{ud}^* V_{ub} + V_{cd}^* V_{cb} + V_{td}^* V_{tb} = 0. \tag{1}$$

Its three angles (α, β, γ) can be directly measured in CP violation experiments such as B meson decays.

In parallel, the lepton-neutrino sector has the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix [4] in charged currents. The neutrino oscillations are crucial for testing the PMNS matrix, including its Dirac CP angle. Measuring the CP asymmetry of neutrino oscillations, $P[\nu_\ell \to \nu_{\ell'}] - P[\bar{\nu}_\ell \to \bar{\nu}_{\ell'}]$ $(\ell \neq \ell')$, is a direct probe of Dirac CP violation [5,6], and poses a major challenge to particle physics today. An alternative and complementary method is to measure the leptonic unitarity triangles (LUT) from neutrino oscillations.

Hence, our natural question is: in the leptonic sector, *what is the physical meaning of those angles in the LUT and how do they exactly connect to neutrino oscillations?*

In this paper, we reveal that the angles of the LUT have definite physical meaning, and demonstrate the exact connection of LUT to neutrino oscillations. We note that the LUT has only three independent parameters and does not contain the full information of the PMNS matrix; but we will demonstrate that the three parameters of each LUT are enough to describe the corresponding neutrino oscillations. Especially, we will prove that *the angles of the LUT act as the phase shifts in the corresponding neutrino oscillation probabilities.* Thus, for the long baseline oscillation experiments with enough precision to measure the distortion of energy spectrum, the angles in the LUT may be directly extracted from the shift of the maximal appearance point in the spectrum. We also note that some other nice features of the LUT and their tests were studied in the recent literature [7].

II. CONNECTING LUT TO NEUTRINO OSCILLATION

Neutrinos are normally produced and detected in their flavor eigenstates $|\nu_\ell\rangle$ with $\ell = e, \mu, \tau$, which are mixtures of their mass eigenstates $|\nu_j\rangle$ with $j = 1, 2, 3$. The $|\nu_\ell\rangle$ and $|\nu_j\rangle$ are connected by the PMNS matrix U [4], $|\nu_\ell\rangle = \sum_{j=1}^{3} U_{\ell j} |\nu_j\rangle$. Thus, a flavor state $|\nu_\ell\rangle$ can oscillate into $|\nu_{\ell'}\rangle$ after flying a distance L. The vacuum transition probability is [5,6],

$$P_{\ell \to \ell'} = \sum_{j=1}^{3} |U_{\ell' j} U_{\ell j}|^2 + 2 \sum_{j<k} |U_{\ell' j} U_{\ell j} U_{\ell k} U_{\ell' k}| \cos(2\Delta_{jk} \mp \phi_{\ell'\ell;jk}), \tag{2}$$

where $\Delta_{jk} \equiv L \Delta m_{jk}^2/(4E)$, Δm_{jk}^2 is the mass-squared difference between $|\nu_j\rangle$ and $|\nu_k\rangle$, E denotes neutrino

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energy, and the "$\mp$" signs correspond to $\nu_\ell/\bar{\nu}_\ell$ oscillations. The phase angle $\phi_{\ell'\ell;jk}$ is defined as [3,5]

$$\phi_{\ell'\ell;jk} \equiv \arg\left(U_{\ell' j}U^*_{\ell j}U_{\ell k}U^*_{\ell' k}\right). \tag{3}$$

Thus we have, $\phi_{\ell'\ell;jk} = -\phi_{\ell\ell';jk} = -\phi_{\ell'\ell;kj}$ and $\Delta_{jk} = -\Delta_{kj}$. Equation (2) is a precise oscillation formula without approximations [3,5]. It also holds for $\ell = \ell'$, which gives the survival probability of $\nu_\ell \to \nu_\ell$ ($\bar{\nu}_\ell \to \bar{\nu}_\ell$) with phase angle $\phi_{\ell\ell;jk} = 0$. This survival probability ($\ell = \ell'$) depends only on three parameters $(\tilde{a}_\ell, \tilde{b}_\ell, \tilde{c}_\ell) \equiv (|U_{\ell 1}|^2, |U_{\ell 2}|^2, |U_{\ell 3}|^2)$, which obey the unitarity constraint of the matrix U, $\tilde{a}_\ell + \tilde{b}_\ell + \tilde{c}_\ell = 1$. Hence, under $\ell = \ell'$, Eq. (2) actually contains only *two independent degrees of freedom* among all four parameters in the PMNS matrix. For instance, we can express the disappearance probability in terms of $(\tilde{a}_\ell, \tilde{b}_\ell)$, apart from Δ_{jk},

$$\begin{aligned} P_{\text{dis}} &= 1 - P_{\ell\to\ell} = 2\sum_{j<k}|U_{\ell j}|^2|U_{\ell k}|^2[1-\cos(2\Delta_{jk})] \\ &= 4\tilde{a}_\ell\tilde{b}_\ell\sin^2\Delta_{12} + 4(1-\tilde{a}_\ell-\tilde{b}_\ell) \\ &\quad\times(\tilde{a}_\ell\sin^2\Delta_{31} + \tilde{b}_\ell\sin^2\Delta_{23}). \end{aligned} \tag{4}$$

This clearly shows that the disappearance oscillations do not directly measure the LUT parameters (cf. Fig. 1), especially the LUT angles for *CP* violation. Hence, we will focus on the appearance oscillations ($\ell \neq \ell'$), which contain nontrivial phase shift $\phi_{\ell'\ell;jk} \neq 0$. Our key finding is to quantitatively connect the appearance oscillations to LUT's.

We note that the oscillation formulas [(2) and (3)] mainly depend on the absolute values such as $|U_{\ell j}|$, and $\phi_{\ell'\ell;jk}$ is the *only place* where complex phases of $U_{\ell j}$ enter and generate observable *CP* violation in ν oscillations. We stress that, in contrast to the Dirac *CP* phase δ in the conventional PMNS matrix [6], the phase angle $\phi_{\ell'\ell;jk}$ has the advantage of being parametrization independent. Furthermore, $\phi_{\ell'\ell;jk}$ explicitly appears as the phase angle shift in Eq. (2), and may be directly read out from the shift of the maximal transition point in the ν energy spectrum once the measurements become precise enough.

Then, we wish to ask: what is the physical meaning of the phase-angle-shift $\phi_{\ell'\ell;jk}$ and how is it connected to the LUT? Strikingly, we find that *the phase angle shift $\phi_{\ell'\ell;jk}$ in neutrino oscillations is just one of the exterior angles in the LUT*. This will be proven as follows.

The unitarity conditions of the PMNS matrix, $U^\dagger U = UU^\dagger = 1$, will result in two sets of LUT's, $\sum_j U_{\ell j}U^*_{\ell' j} = 0$ with $\ell \neq \ell'$ (row triangles or "Dirac triangles") and $\sum_\ell U^*_{\ell j}U_{\ell j'} = 0$ with $j \neq j'$ (column triangles or "Majorana triangles"). For studying the flavor neutrino oscillations, we consider the Dirac triangles,

$$U_{\ell 1}U^*_{\ell' 1} + U_{\ell 2}U^*_{\ell' 2} + U_{\ell 3}U^*_{\ell' 3} = 0, \qquad (\ell \neq \ell'). \tag{5}$$

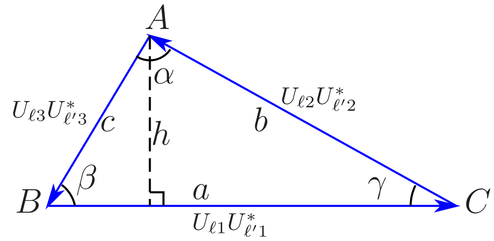

FIG. 1 (color online). The leptonic unitarity triangle (LUT), where $\ell \neq \ell'$, (a, b, c) denote lengths of the three sides, (α, β, γ) denote the three angles, and h denotes the height.

This forms a triangle in the complex plane, as shown in Fig. 1. Its three sides have lengths

$$(a, b, c) \equiv (|U_{\ell 1}U_{\ell' 1}|, |U_{\ell 2}U_{\ell' 2}|, |U_{\ell 3}U_{\ell' 3}|). \tag{6}$$

The three angles are expressed as

$$\begin{aligned} \alpha &= \arg\left(-\frac{U_{\ell 2}U^*_{\ell' 2}}{U_{\ell 3}U^*_{\ell' 3}}\right) \\ \beta &= \arg\left(-\frac{U_{\ell 3}U^*_{\ell' 3}}{U_{\ell 1}U^*_{\ell' 1}}\right) \\ \gamma &= \arg\left(-\frac{U_{\ell 1}U^*_{\ell' 1}}{U_{\ell 2}U^*_{\ell' 2}}\right). \end{aligned} \tag{7}$$

In order to make exact connections to the phase angle (3), we compute a generic arc angle of Eq. (7),

$$\begin{aligned} \arg\left(-\frac{U_{\ell j}U^*_{\ell' j}}{U_{\ell k}U^*_{\ell' k}}\right) &= \arg(U_{\ell j}U^*_{\ell' j}) - \arg(-U_{\ell k}U^*_{\ell' k}) \\ &= \arg(U_{\ell j}U^*_{\ell' j}) + \arg(-U^*_{\ell k}U_{\ell' k}) \\ &= \arg(U_{\ell j}U^*_{\ell' j}) + \arg(U^*_{\ell k}U_{\ell' k}) + \pi \\ &= \arg(U^*_{\ell' j}U_{\ell j}U^*_{\ell k}U_{\ell' k}) + \pi \\ &= \pi - \phi_{\ell'\ell;jk}, \end{aligned} \tag{8}$$

where we have used the identities, $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$, $-\arg(z) = \arg(z^*)$, $\arg(-z) = \arg(z) + \pi$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. Note that all these equalities hold modulo $2n\pi (n \in \mathbb{Z})$. Hence, we conclude that $\phi_{\ell'\ell;jk}y$ just equals one of the exterior angles in the LUT,

$$\alpha = \pi - \phi_{\ell'\ell;23}, \quad \beta = \pi - \phi_{\ell'\ell;31}, \quad \gamma = \pi - \phi_{\ell'\ell;12}. \tag{9}$$

We further present a geometrical proof of the identities (9). Let us first consider the β angle. From Fig. 1, we have

$$S_\triangle = \frac{1}{2}ah, \qquad \sin\beta = \frac{h}{c} = \frac{2S_\triangle}{ac}, \tag{10}$$

where $S_\triangle$ is the area of the triangle and h denotes the height. The Jarlskog invariant J [8] is the rephasing-invariant measure of CP violation and equals $J = \mathrm{Im}(U_{\ell' j}U^*_{\ell j}U_{\ell k}U^*_{\ell' k})$, where $\ell \neq \ell'$ and $j \neq k$. Hence, we can derive the phase angle from (3),

$$\sin\phi_{\ell'\ell;31} = \frac{J}{|U_{\ell' 3}U^*_{\ell 3}U_{\ell 1}U^*_{\ell' 1}|} = \frac{J}{ac}. \tag{11}$$

Because each UT has its area equal half of the Jarlskog invariant, $S_\Delta = J/2$ [9], the right-hand sides of (11) and the second relation of (10) are equal. Hence, we deduce

$$\sin\beta = \sin\phi_{\ell'\ell;31}. \tag{12}$$

Similarly, we derive

$$\sin\alpha = \sin\phi_{\ell'\ell;23}, \qquad \sin\gamma = \sin\phi_{\ell'\ell;12}. \tag{13}$$

These elegantly reprove our result (9) *in a geometrical way*. It invokes the Jarlskog invariant, and also reveals a clear picture for the relation between (α,β,γ) and J. We present this in Fig. 2, which demonstrates that the productions of any two sides of the triangle in the complex plane share the same imaginary part, i.e., the same height in Fig. 2,

$$bc\sin\alpha = ca\sin\beta = ab\sin\gamma = J. \tag{14}$$

Using (6) and (9), we can express (2) fully in terms of the geometrical parameters in the corresponding LUT,

$$\begin{aligned} P_{\ell\to\ell'} = {} & a^2 + b^2 + c^2 - 2ab\cos(2\Delta_{12} \pm \gamma) \\ & - 2bc\cos(2\Delta_{23} \pm \alpha) - 2ca\cos(2\Delta_{31} \pm \beta). \end{aligned} \tag{15}$$

Note that $P_{\ell\to\ell'}(L=0) = 0$ holds as expected, since the source neutrinos have no time to oscillate. Thus, we can simplify the form of (15) by subtracting $P_{\ell\to\ell'}(L=0)$,

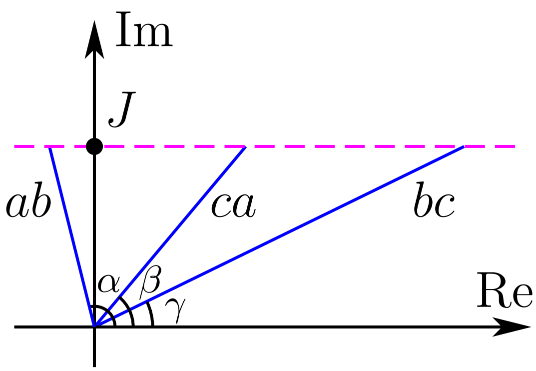

FIG. 2 (color online). Relation between (α,β,γ) and Jarlskog invariant J. The productions of any two sides of the triangle in the complex plane share the same imaginary part.

$$\begin{aligned} P_{\ell\to\ell'} = {} & 4ab\sin(\Delta_{12} \pm \gamma)\sin\Delta_{12} \\ & + 4bc\sin(\Delta_{23} \pm \alpha)\sin\Delta_{23} \\ & + 4ac\sin(\Delta_{31} \pm \beta)\sin\Delta_{31}. \end{aligned} \tag{16}$$

Equations (15) and (16) demonstrate the quantitative connection between (α,β,γ) of the LUT and the oscillation probabilities. Hence, we have explicitly proven that *the physical meanings of (α,β,γ) are just the phase angle shifts in the neutrino oscillations.* It is striking to see that a flavor-changing oscillation ($\ell \neq \ell'$) is *fully determined by the geometrical parameters of the LUT,* for the given Δm^2_{jk} and experimental setup (E, L) [10].

This has an important implication. Apart from two possible Majorana phases, the PMNS matrix has *four independent parameters* (three mixing angles and one Dirac CP angle), which would all appear in the standard oscillation formula (2). But, a LUT has only *three independent geometrical parameters* and thus only contains partial information in the PMNS matrix. Impressively, we have proven that this partial information of the PMNS matrix, as contained in a given LUT (5), is enough to determine the corresponding oscillation probability, for the inputs Δm^2_{jk} and (E, L).

This feature is important for fitting an oscillation experiment when higher experimental precision is reached such that all four parameters of the PMNS matrix have observable effects. In this case, we may suggest a three-parameter fit based on each given LUT, rather than the conventional four-parameter fit in terms of $(\theta_{12}, \theta_{13}, \theta_{23}, \delta)$ which contain a redundant degree of freedom that cannot be determined independently in a given kind of appearance experiments. This has two advantages: (i) the simplicity of (16) in terms of the geometric parameters of LUT; (ii) the extra redundant degree of freedom in the conventional four-parameter fit of the PMNS matrix is automatically removed for a given kind of oscillation experiments ($\ell \neq \ell'$).

Finally, since combining two different LUT's will provide the full information of the PMNS matrix [11], making two kinds of oscillation experiments can fit the two corresponding LUT's, and thus give a full reconstruction of the PMNS matrix.

III. PROBING THE LUT VIA NEUTRINO OSCILLATIONS

We further study how to test the LUT parameters via neutrino oscillations. To determine a LUT, we can choose two sides plus one angle, say (a, b, γ), as the three independent geometrical parameters. Then, all other parameters in this LUT can be expressed in terms of (α, β, γ),

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ \tan\alpha &= \frac{a\sin\gamma}{b - a\cos\gamma} \\ \tan\beta &= \frac{b\sin\gamma}{b - a\cos\gamma}. \end{aligned} \tag{17}$$

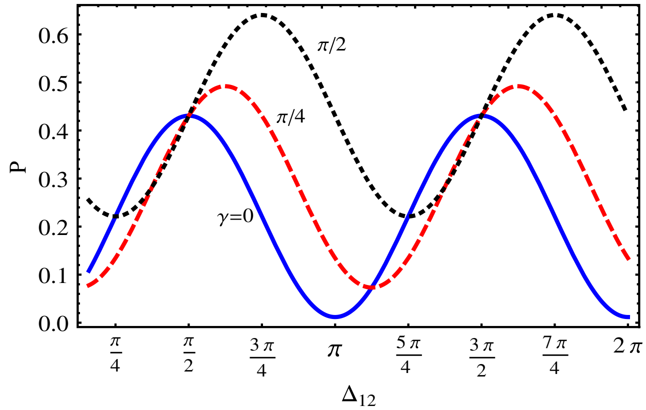

FIG. 3 (color online). Phase-shift effects of γ on neutrino oscillation probability $P[\bar{\nu}_\ell \to \bar{\nu}_{\ell'}]$. For illustration, we plot three curves for $\gamma = 0$ (blue solid), $\frac{\pi}{4}$ (red dashed), and $\frac{\pi}{2}$ (black dotted).

Hence, we can reexpress (15) or (16) fully in terms of (a, b, γ). From the current oscillation data [12,13], $\Delta m^2 \equiv |\Delta m^2_{13}| \simeq 2.4 \times 10^{-3}\ \mathrm{eV}^2$ and $\delta m^2 \equiv |\Delta m^2_{12}| \simeq 7.5 \times 10^{-5}\ \mathrm{eV}^2$, and considering the case of $E/L \sim \delta m^2$, we find, $\Delta_{12} = O(1)$ and $|\Delta_{23}|, |\Delta_{31}| \gg 1$. Thus, the last two terms in (15) will be averaged out due to integration over the neutrino production region and the energy resolution function, etc. [3]. So, we have

$$\begin{aligned} P_{\ell\to\ell'} &\simeq a^2 + b^2 + c^2 - 2ab\cos(2\Delta_{12} \pm \gamma) \\ &= 2(a^2 + b^2) - 4ab\cos(\Delta_{12} \pm \gamma)\cos\Delta_{12}. \end{aligned} \tag{18}$$

In Eq. (18), if $\gamma = 0$, the $\nu_{\ell'}$ maximal appearance point is $\Delta_{12} = \frac{\pi}{2}$. For a nonzero γ, the maximal appearance point is shifted to

$$\Delta^\star_{12} = \frac{\pi}{2} \mp \frac{\gamma}{2}, \tag{19}$$

and its corresponding appearance probability is

$$P^{\max}_{\ell\to\ell'} \simeq 2(a^2 + b^2) + 4ab\sin^2\frac{\gamma}{2}. \tag{20}$$

This phase-shift effect is shown in Fig. 3 for $\bar{\nu}_\ell \to \bar{\nu}_{\ell'}$ oscillations, where we use (18) with sample inputs $(a, b) = (0.29, 0.36)$. The three curves in Fig. 3 correspond to $\gamma = (0, \frac{\pi}{4}, \frac{\pi}{2})$, and have their first maximal appearance points located at $\Delta^\star_{12} = (\frac{\pi}{2}, \frac{5\pi}{8}, \frac{3\pi}{4})$, in accord with (19). Figure 3 also shows that the curves move upward with the increase of γ. This can be understood from (20) which monotonously rises with the increase of $\gamma \in (0, \pi)$. We have made similar analyses for choosing other input parameters of the LUT.

In addition, using Eq. (18), we can derive the probability difference between the neutrino and antineutrino oscillations,

$$\begin{aligned} P_{\ell\to\ell'} - P_{\bar{\ell}\to\bar{\ell}'} &\simeq 4ab\sin\gamma\sin(2\Delta_{12}) \\ &= 8S_\Delta\sin(2\Delta_{12}) \\ &= 4J\sin(2\Delta_{12}). \end{aligned} \tag{21}$$

This *CP* asymmetry provides the net measure of *CP* violation in terms of the area of the LUT, $S_\Delta = J/2$, as expected.

From Eqs. (18)–(20), we see that the angle γ plays the physical role of phase shift in neutrino oscillations with $E/L \sim \delta m^2$. In principle, we can change either L or E to detect how the maximal appearance point is shifted, and thus directly measure γ. In practice, it is much easier to vary E since moving around a large detector would be hard.

Actually, a more realistic method is to measure the distortion of neutrino energy spectrum. For instance, we may produce many ν_μ with different energies which can be measured or are already known. Then, at the far detector with $L \sim E/\delta m^2$, we will measure the ν_e appearance with a different energy spectrum. Then, we may use (18) to fit the distortion of the spectrum and infer (a, b, γ) in the $e - \mu$ LUT. The current $\nu_\mu \to \nu_e$ experiments cannot reach such a small $E/L \sim \delta m^2$. For instance, the MINOS experiment [14] has E/L about 3 GeV/735 km $\simeq 8 \times 10^{-4}\ \mathrm{eV}^2$ which is insensitive to the oscillations via Δ_{12}. The situation of the NOνA experiment [15] is similar, which has $E/L \simeq 2$ GeV/810 km $\simeq 5 \times 10^{-4}\ \mathrm{eV}^2$. The Super Beam Project [16] creates 300 MeV muon neutrinos and has a 130 km baseline, with $E/L \simeq 4.5 \times 10^{-4}\ \mathrm{eV}^2$ also at the same order as MINOS. The future Neutrino Factory [17]

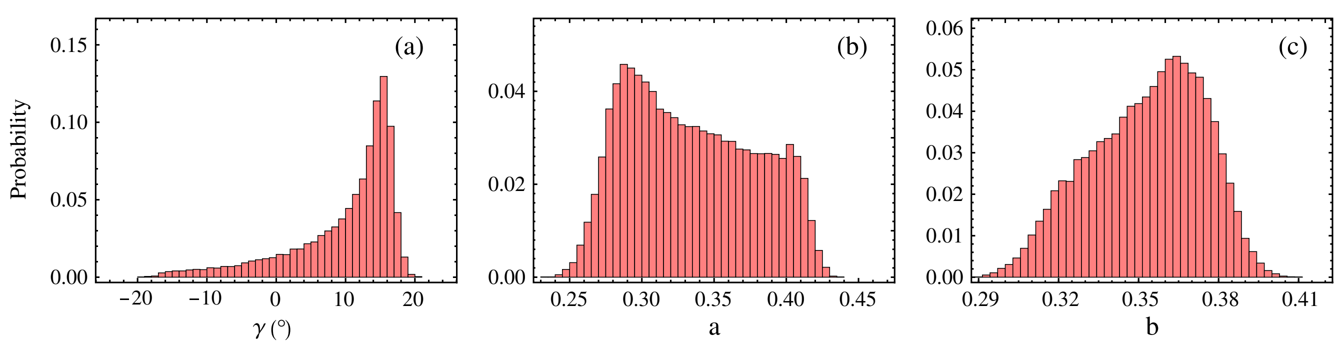

FIG. 4 (color online). Probability distributions of the geometric parameters in the $e - \mu$ LUT, γ [plot (a)], a [plot (b)], and b [plot (c)], based on the current neutrino global fit [12]. We have simulated 30000 samples in each plot.

will have $L = 2000 - 7500$ km and $E = O(1 - 10)$ GeV, which is possible to realize $E/L \sim \delta m^2$.

The $\nu_\mu \to \nu_e$ oscillation experiments measure the probability $P_{\mu\to e}(E)$ in (18) as a function of E in a long baseline $L \sim E/\delta m^2$. To inspect the sensitivity of $P_{\mu\to e}(E)$ to γ, we first evaluate the ranges of (γ, a, b) in the $e-\mu$ LUT from the present oscillation data. The new global fit of the PMNS matrix gives [12]

$$\begin{aligned} s_{12}^2 &= (3.08 \pm 0.17) \times 10^{-1}, \\ s_{23}^2 &= (4.25 \pm 0.28) \times 10^{-1}, \\ s_{13}^2 &= (2.34 \pm 0.20) \times 10^{-2}, \\ \delta &= (1.39 \pm 0.30)\pi, \end{aligned} \tag{22}$$

where $s_{ij}^2 \equiv \sin^2\theta_{ij}$ and $\pm 1\sigma$ errors are included. The PMNS matrix can be expressed as $U = U_0 U'$, with

$$U_0 = \begin{pmatrix} c_{31}c_{12} & c_{31}s_{12} & s_{31}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{31}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{31}e^{i\delta} & s_{23}c_{31} \\ s_{12}s_{23} - c_{12}c_{23}s_{31}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{31}e^{i\delta} & c_{23}c_{31} \end{pmatrix}. \tag{23}$$

The Majorana phase matrix $U' = \mathrm{d}iag(1, e^{i\varphi_2}, e^{i\varphi_3})$ does not affect the Dirac triangles (5) and is irrelevant to the oscillation analyses. Using Eqs. (6) and (7) and the mixing matrix (23), we can reconstruct the LUT parameters (γ, a, b) from the neutrino data (22).

We present the probability distributions of (γ, a, b) in Fig. 4. We have simulated 30000 samples in each plot and normalized the total area of each histogram as a unit. We find that γ falls into a narrow range,

$$-20^\circ \lesssim \gamma \lesssim 20^\circ, \tag{24}$$

with a most probable value $\gamma \simeq 15.5^\circ$. Figure 4 further constrains,

$$0.25 \lesssim a \lesssim 0.45, \qquad 0.29 \lesssim b \lesssim 0.42 \tag{25}$$

with the most probable values $(a, b) \simeq (0.29, 0.36)$.

In Fig. 5, we plot the oscillation probability $P_{\mu\to e}(E)$ as a function of E/L, based upon (18), where we vary γ values in the range $[-20^\circ, 20^\circ]$ with steps by 2°. We also set the sample inputs $(a, b) = (0.29, 0.36)$ from their most probable values in Fig. 4. Figure 5 shows that with γ changing from -20° to 20°, the maximum point of $P_{\mu\to e}$ shifts from left to right, and its tail on the right-hand side lifts up. Figure 5 clearly illustrates how γ plays the physical role of phase shift.

We note that observing the phase-shift effects is based on a premise that L/E is *variable* in the experiments. Hence, to probe the phase-shift effects requires experiments to reach a relatively high resolution on the neutrino energy and the energy spectrum, so the shape of the distribution in Fig. 5 (by varying energy E) can be measured. Thus, the LUT parameters (γ, a, b) can be inferred from fitting the measured energy spectrum.

In passing, we note that determining CP violation in an experiment with *fixed* L/E also involves the well-known parameter degeneracy problem [18], such as the (θ_{13}, δ) degeneracy, implying that the oscillation probability for one pair of inputs (θ_{13}, δ) may equal that for another pair (θ'_{13}, δ'). This problem is inherent in the three-neutrino oscillations and cannot be removed by simply enhancing the accuracy. It may be resolved by varying L/E, e.g., combining data from experiments with different baselines and channels, or making use of the energy spectrum.

Finally, we comment on the matter effects [3,19]. To effectively measure the phase-shift effects, we should check the required size of E/L. According to the above discussions, a relatively small $E/L \sim \delta m^2$ is needed, which is beyond the current experimental setup. For instance, the 735 km baseline of MINOS [14] would need a neutrino beam energy $E \sim 100$ MeV for a sensitive probe. In such a case, the matter effect is only about 1/30 of that involved in the current MINOS setup (with $E \simeq 3$ GeV), and thus negligible. The case of NOνA [15] (with $E/L \simeq 2$ GeV/810 km) is similar. Besides, the Super Beam Project [16] has $E/L \simeq 300$ MeV/130 km, whose neutrino energy is about a factor 1/10 lower than the current MINOS setup, so its matter effect will be insignificant. This means that our formula (18) would give a fine

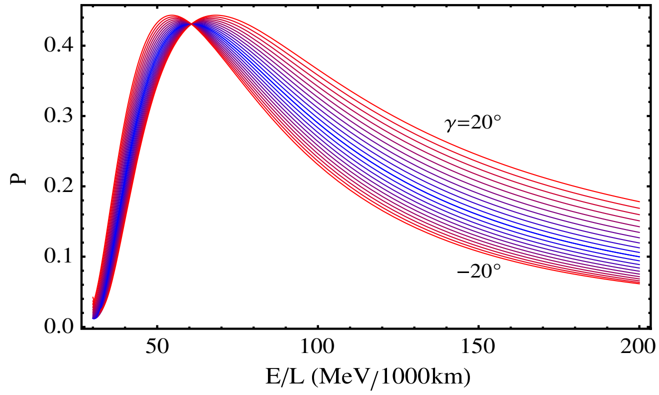

FIG. 5 (color online). The $\nu_\mu \to \nu_e$ oscillation probability versus E/L, for illustration of long baseline oscillations with $E/L \sim \delta m^2$. Different curves (from bottom to top) correspond to varying γ within $[-20^\circ, 20^\circ]$ and with steps of 2°.

approximation for δm^2 dominated oscillations with $L < 1000$ km. Only for experiments with very long baselines (well above 1000 km) and high precision, the matter effect would become sizable for probing the LUT's; but this is fully beyond our current scope and we will pursue such elaborated applications elsewhere.

IV. CONCLUSIONS

Probing the leptonic CP violation poses a major challenge to particle physics today. It may provide the origin of the observed matter-antimatter asymmetry in the Universe [20]. The leptonic unitarity triangle (LUT) gives a geometric description of CP violations in the lepton-neutrino sector and is directly measurable. Finding any nonzero angle of the LUT will be a direct proof of the leptonic CP violation [9].

In this paper, we revealed that the angles in the LUT have definite physical meaning, and they act as the phase shifts of neutrino oscillations. *For the first time*, we proved that the oscillation phases $\phi_{\ell'\ell;jk}$ in the conventional formula (2) exactly equal the corresponding exterior angles of the LUT, as in (9). Our proof uncovers that a given kind of appearance oscillations can be described by the corresponding LUT with only three independent geometric parameters. This may provide a cleaner way for fitting oscillation data, since each kind of long baseline oscillation (2) is traditionally described by four independent parameters in the PMNS matrix (23).

Without losing generality, we considered the $\nu_\mu \to \nu_e$ oscillations with a long baseline $L = E/\delta m^2$, and studied one of the LUT angles γ for illustration. We demonstrated that the oscillation formula takes a simple form (18), depending only on the three independent geometric parameters $(\gamma,\ a,\ b)$ of the unitarity triangle. We explicitly analyzed how the maximal appearance point of $\nu_\mu \to \nu_e$ oscillations gets shifted when γ changes, as shown in Eq. (19) and Figs. 3,5. We will apply this LUT method to study concrete long baseline oscillation experiments elsewhere.

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[10] For disappearance oscillations ($\ell = \ell'$) with $\phi_{\ell\ell;jk} = 0$, the survival probability actually depends only on two independent degrees of freedom among all four parameters in the PMNS matrix, as shown in Eq. (4). It does not directly measure the LUT parameters (Fig. 1), especially the LUT angles for CP violation.

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