Brownian motion in strongly coupled, anisotropic Yang-Mills plasma: A holographic approach

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We employ methods of gauge/string duality to analyze the nonrelativistic Brownian motion and the concomitant Langevin equation of a heavy quark in a strongly coupled, thermal, anisotropic Yang-Mills plasma in the low anisotropy limit. We consider fluctuations both along and perpendicular to the direction of anisotropy and study the effects of anisotropy on the drag coefficient, the diffusion constant, and the Langevin coefficient for both the directions. We also verify the fluctuation-dissipation theorem for Brownian motion in an anisotropic medium.

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I. INTRODUCTION

A particle immersed in a hot fluid exhibits an incessant, random dynamics known as Brownian motion [1]. The Brownian motion originates from the collisions experienced by the particle with the constituents of the fluid undergoing a random thermal motion. The consideration of these random collisions requires the fact that the fluid medium is not a continuum but made of finite-size constituents. Hence, the Brownian motion actually offers a better understanding of the underlying microscopic physics of the medium. The random dynamics of a Brownian particle is encoded in the Langevin equation describing the total force acting on the particle as a sum of dissipative and random forces. Although both of these forces have the same microscopic origin, phenomenologically the dissipative force describes the in-medium frictional effect, and the random force stands for a source of random kicks from the medium.

Brownian motion is a universal phenomenon for all finite temperature systems. Therefore, a heavy probe quark immersed in a strongly coupled hot quark-gluon plasma (QGP), which is believed to be created in the relativisitic heavy ion collider (RHIC) and LHC experiments [2], undergoes the same thermal motion [3]. From field theoretical standpoint, the random motion in the QGP phase is hard to study due to nonperturbative strong coupling effects. However, the AdS/CFT correspondence [4–7] seems to be a good theoretical tool in this regard, since it has been extensively used to study a large class of strongly coupled plasma having well-defined gravity duals. In spite of intensive efforts, to date, the gravity dual of the strongly coupled QGP phase remains elusive, and the gauge theories having well-defined gravity duals are different from QGP in several aspects. Nonetheless, it is remarkably found in some instances that many strong coupling features extracted holographically from known geometric duals for UV conformal theories agree with the thermal QGP phase. For example, in Refs. [8,9], the AdS/CFT correspondence has been used to show that the shear viscosity to the entropy ratio of four-dimensional $SU(N_c)$ (N_c being the number of colors) Yang-Mills theory with $\mathcal{N} = 4$ supersymmetries is $1/4\pi$. This low viscosity is also speculated from the estimation of the RHIC data for QGP [10]. Later the ratio was found to be universal for all the strongly coupled gauge theories, in the $N_c \to \infty$ limit, having a gravity dual [11]. Subsequently, it was found that there are other physical quantities, such as R-charge conductivity to charge susceptibility ratio, a certain combination of thermal conductivity, temperature, and chemical potential, that show universal behavior, too [11,12]. Motivated by these universal outcomes, there has been a substantial amount of holographic analysis of dissipative physics of various types of thermal plasma having dual gravity to understand the dynamical feature of QGP phase in a better way; see, for example, Refs. [13–33]. Recently, as an important improvement in this direction, the Brownian motion of a probe particle has been successfully studied using the framework of the AdS/CFT correspondence [34,35].

The bulk interpretation of the Brownian motion of a heavy probe quark immersed in a $SU(N_c)$ Yang-Mills theory with $\mathcal{N} = 4$ supersymmetries emerges from the consideration of a probe fundamental string in the dual anti-de Sitter (AdS) black hole background, stretching between the AdS boundary and the horizon. The end point of the string attached to the boundary is holographically mapped to the boundary probe quark. The transverse modes of the probe string are thermally excited by the black hole environment. This excitation propagates up to boundary and holographically incorporates the Brownian motion of the boundary quark. In an intuitive way, the fact that, semiclassically, the transverse string modes are thermally

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excited by Hawking radiation reflects the bulk interpretation of random force in the boundary Langevin equation. On the other hand, the fact that the string excitation is absorbed by the black hole environment stands for the bulk realization of boundary frictional force. In the detailed course of computation, we need to quantize the transverse string modes. As explained in Ref. [36], the Hawking radiation associated with the string excitations occurs upon quantizing these modes. Once these modes are quantized, using holographic prescription, the erratic motion of string end point attached to the boundary can be realized as the Brownian motion.

There are two independent approaches available in the literature to obtain these results. In the first approach, the state of the quantized scalar fields are identified with the Hartle–Hawking vacuum representing the black hole at thermal equilibrium [34]. In the second approach, the Gubser-Klebanov-Polyakov-Witten (GKPW) prescription [5,6] of computing the retarded Green function is used. The computation of Langevin equation is done by exploring the correspondence between the Kruskal extension of the AdS black hole geometry and the Schwinger–Keldysh formalism [35]. The detailed comparison between the two independent approaches is given in Ref. [37]. There are further generalizations in this direction. Holographic Brownian motion has been studied in the case of charged plasma [38], rotating plasma [39–41], non-Abelian super Yang-Mills (SYM) plasma [42], nonconformal plasma [43], and $(1+1)$-dimensional strongly coupled conformal field theory at finite temperature [44]. It has also been studied in the low temperature domain (near criticality) [45,46]. The relativistic formulation of holographic Langevin dynamics was successfully addressed in Ref. [47]. Moreover, some important universality related issues regarding the Langevin coefficients computed along the longitudinal as well as the transverse directions to the probe quark's motion was studied in Ref. [48].

In our paper, we study the holographic Brownian motion of a heavy probe quark moving in a strongly coupled *anisotropic* plasma at finite temperature. For simplicity, we only consider the nonrelativistic limit; i.e., we take $v \ll 1$ where v is the velocity of the heavy quark that undergoes Brownian motion. We also take the medium to have small anisotropy and consider only the low-lying modes of the string fluctuations. These conditions are imposed only to facilitate analytical computation. The anisotropic thermal plasma we are interested in is a spatially deformed four-dimensional $\mathcal{N}=4$ $SU(N_c)$ SYM plasma at finite temperature [49,50]. The deformation in the gauge theory has been achieved by adding a topological Yang-Mills coupling where the coupling parameter has a functional dependence on one of the three spatial boundary coordinates signifying the anisotropic direction. The dual bulk geometry develops an anisotropic black hole horizon and behaves as a regular solution embedded in type IIB string theory. The motivation for studying the Brownian motion in the context of anisotropic $\mathcal{N}=4$ SYM plasma comes from experimental observations at the RHIC signifying the possible existence of a locally anisotropic phase of QGP at thermal equilibrium. In the heavy ion collisions, right after the plasma is formed, it is anisotropic and also far away from equilibrium for a time $t < \tau_{\rm out}$. Further, in the temporal window $\tau_{\rm out} < t < \tau_{\rm iso}$, it settles down into an equilibrium state but still does not achieve isotropy. Thus, if one wishes to probe the early time dynamics of the plasma, it is essential to take into consideration this intrinsic anisotropy. In the regime $\tau_{\rm out} < t < \tau_{\rm iso}$, the plasma has a significant momentum anisotropy that leads to an unequal expansion of the plasma in the beam direction and the transverse directions. Although the anisotropic plasma we are interested in does not incorporate the dynamical anisotropy as in QGP, it can be a good toy model since it has a well-defined gravity dual.

With this gravity background, following Ref. [34], we study the bulk interpretation of the boundary Brownian motion. In particular, we explicitly compute the friction coefficient, the diffusion constant, and the random force correlator from a holographic perspective when the thermal background has an inherent anisotropy and verify the fluctuation-dissipation theorem and the Einstein–Sutherland relation. In our bulk analysis, we include fluctuations of the probe string modes along both isotropic as well as anisotropic directions. We systematically study the effect of anisotropy in the low-frequency limit of the thermal fluctuation.

The paper is organized as follows. In Sec. II we briefly review the field theoretic aspects of Brownian motion and follow it, in Sec. III, with the holographic description of Brownian motion in the anisotropic medium. Section III is divided into four subsections. In Sec. III A we describe the gauge theory and its supergravity dual that we are interested in. In Sec. III B we discuss some generic features of the holographic formulation of the problem. In Sec. III C we perform the holographic computation for the anisotropic direction, and from there the computation for the isotropic direction follows in a special limit which is discussed in Sec. III D. Finally, we conclude with a discussion of our results in Sec. IV.

II. BROWNIAN MOTION IN THE BOUNDARY

We begin by presenting a brief review of the field theoretic aspect of the problem following Refs. [34,38,42]. The simplest phenomenological model which attempts to explain the Brownian motion of a nonrelativistic particle of mass m immersed in a thermal bath is given by the Langevin equation along the ith spatial direction,[1]

$$\dot{p}_i(t) = -\gamma_o^{(i)} p_i(t) + R_i(t), \tag{1}$$

where $p_i(t) = m\dot{x}_i$ is the nonrelativistic momentum of the Brownian particle along the ith direction. The model, though

[1] We shall explicitly keep track of the direction index i in our discussion since we need to distinguish between the anisotropic direction and the directions transverse to it.

simple, is capable of capturing the salient features of a particle undergoing Brownian motion. The particle is acted upon by a random force $R_i(t)$ arising out of its interaction with the thermal bath, and, at the same time, it is suffering energy dissipation due to the presence of the frictional term with $\gamma_0^{(i)}$ being the friction coefficient. Under the effect of these two competing forces, the particle undergoes random thermal motion. The interaction between the Brownian particle and the fluid particles at a temperature T allows for an exchange of energy between the Brownian particle and the fluid leading to the establishment of a thermal equilibrium. In an isotropic medium, the friction coefficient does not depend upon the particular space direction under consideration. However, if the medium in which the particle is immersed has an anisotropy, then we expect the drag coefficient along the anisotropic direction $\gamma_0^{\parallel}$ to be different from that in the isotropic plane $\gamma_0^{\perp}$.

The random force $R_i(t)$ can be approximated by a sequence of independent impulses, each of random sign and magnitude, such that the average vanishes. Each such impulse is an independent random event; i.e., $R_i(t)$ is independent of $R_i(t')$ for $t \neq t'$. Such a noise source goes by the name of white noise. These considerations imply

$$\langle R_i(t)\rangle = 0, \qquad \langle R_i(t)R_j(t')\rangle = \kappa_0^{(i)}\delta_{ij}\delta(t-t'), \tag{2}$$

where we call $\kappa_0^{(i)}$ the Langevin coefficient. Again, the presence of anisotropy inflicts a directional dependence upon $\kappa_0^{(i)}$. Note that, in particular, the random forces at two different instants are not correlated. The two parameters $\gamma_0^{(i)}$ and $\kappa_0^{(i)}$ completely characterize the Langevin equation [Eq. (1)]. As we shall see, $\gamma_0^{(i)}$ and $\kappa_0^{(i)}$ are not independent, which is not unexpected since they are related by the fluctuation-dissipation theorem,[2]

$$\gamma_0^{(i)} = \frac{\kappa_0^{(i)}}{2mT}. \tag{3}$$

Assuming the theorem of equipartition of energy which states that each degree of freedom contributes $\frac{1}{2}T$ to the energy (T being the temperature, and we have set the Boltzmann constant $k_B = 1$), it is possible to derive the temporal variation of the displacement squared of the particle [34]

$$\langle s^i(t)^2\rangle = \langle (x^i(t) - x^i(0))^2\rangle = \frac{2D^{(i)}}{\gamma_0^{(i)}}(\gamma_0^{(i)}t - 1 + e^{-\gamma_0^{(i)}t}), \tag{4}$$

[2]The relation between the two quantities has its root in the fact that both the frictional force and the random force have the same origin—microscopically, they arise due to the interaction of the particle with the thermal medium. In this sense, the separation of the rhs of Eq. (1) in two parts is *ad hoc* from the microscopic point of view, being only dictated by considerations of phenomenological simplicity.

where $D^{(i)}$ is defined to be the diffusion constant. It is related to the friction coefficient $\gamma_0^{(i)}$ through the Einstein–Sutherland relation,

$$D^{(i)} = \frac{T}{\gamma_0^{(i)}m}. \tag{5}$$

The solution to Eq. (1) has a homogeneous part determined by the initial conditions and an inhomogeneous part proportional to the random force. The homogeneous part will decay to zero in a time of order $t_{\text{relax}}^{(i)} = 1/\gamma_0^{(i)}$ and the long-time dynamics will be governed entirely by the inhomogeneous part, independent of the initial conditions. Based on these considerations, one can distinguish between two different temporal domains: $t \ll 1/\gamma_0^{(i)}$ whence $s^i \sim \sqrt{T/mt}$ showing that the particle moves under inertia as if no force is acting upon it. The speed in this case is fixed by the equipartition theorem. In the opposite regime $t \gg 1/\gamma_0^{(i)}$, one obtains $s^i \sim \sqrt{2D^{(i)}t}$, which is reminiscent of the random walk problem. In this time domain, the Brownian particle loses its memory of the initial value of the velocity. The transition from one regime to another occurs at the critical value of

$$t_{\text{relax}}^{(i)} \sim \frac{1}{\gamma_0^{(i)}}, \tag{6}$$

which represents a characteristic time scale of the theory, called the relaxation time, beyond which the system thermalizes.

The model we have considered above is based on two assumptions: i) the friction to be instantaneous and ii) the random forces at two different instants to be uncorrelated. The validity of these assumptions holds well only when the Brownian particle is very heavy compared to the constituents of the medium. However, this does not give the correct picture when the Brownian particle and the constituents of the medium have comparable masses. To overcome these pitfalls, the Langevin equation is generalized such that the friction now depends upon the past history of the particles, and also the random forces at different instants are correlated. To incorporate these effects, we modify Eq. (1) to the generalized Langevin equation,

$$\dot{p}_i(t) = -\int_{-\infty}^{t} dt'\gamma^{(i)}(t-t')p_i(t') + R_i(t) + K_i(t). \tag{7}$$

Note that now the history of the particle is encoded in the function $\gamma^{(i)}(t-t')$, and we have also included the possibility of an external force impressed upon the particle through the term $K_i(t)$. $R_i(t)$ now obeys

$$\langle R_i(t)\rangle = 0, \qquad \langle R_i(t)R_i(t')\rangle = \kappa^{(i)}(t-t'). \tag{8}$$

At this stage it is convenient to go over to the Fourier space representation of the generalized Langevin equation,

$$p_i(\omega) = \frac{R_i(\omega) + K_i(\omega)}{-i\omega + \gamma^{(i)}[\omega]}, \tag{9}$$

where $p_i(\omega)$, $R_i(\omega)$, and $K_i(\omega)$ are the Fourier transforms of $p_i(t)$, $R_i(t)$ and $K_i(t)$, respectively, i.e.,

$$p_i(\omega) = \int_{-\infty}^{\infty} dt p_i(t) e^{i\omega t} \tag{10}$$

and so on. On the other hand, causality restricts $\gamma^{(i)}(t) = 0$ for $t < 0$ so that $\gamma^{(i)}[\omega]$ is the Fourier–Laplace transform

$$\gamma^{(i)}[\omega] = \int_0^{\infty} dt \gamma^{(i)}(t) e^{i\omega t}. \tag{11}$$

Upon taking statistical average in Eq. (9), one finds

$$\langle p_i(\omega)\rangle = \mu^{(i)}(\omega) K_i(\omega), \tag{12}$$

where we have made use of Eq. (8).

$$\mu^{(i)}(\omega) \equiv \frac{1}{-i\omega + \gamma^{(i)}[\omega]} \tag{13}$$

is called the admittance, and since it depends upon $\gamma^{(i)}$, it inherits the anisotropic effect. The admittance is a measure of the response of the Brownian particle to external perturbations. In particular, if the external force is taken as

$$K_i(t) = K_i^{(0)} e^{-i\omega t}, \tag{14}$$

then the response is

$$\langle p_i(t)\rangle = \mu^{(i)}(\omega) K_i^{(0)} e^{-i\omega t}. \tag{15}$$

If the memory kernel $\gamma^{(i)}(t - t')$ is sharply peaked around $t' = t$, then

$$\int_0^{\infty} dt' \gamma^{(i)}(t-t') p_i(t') \approx \int_0^{\infty} dt' \gamma^{(i)}(t') p_i(t) = \frac{1}{t_{\text{relax}}^{(i)}} p_i(t). \tag{16}$$

Thus, for the generalized Langevin equation, described by Eq. (7), the generalization of the relaxation time is

$$t_{\text{relax}}^{(i)} \sim \left(\int_0^{\infty} dt \gamma^{(i)}(t)\right)^{-1} = \frac{1}{\gamma^{(i)}[\omega=0]} = \mu^{(i)}(\omega=0). \tag{17}$$

The Wiener–Khintchine theorem relates the power spectrum $I_{\mathcal{O}}(\omega)$ of any quantity $\mathcal{O}$ with its two-point function as

$$\langle \mathcal{O}(\omega)\mathcal{O}(\omega\prime)\rangle = 2\pi\delta(\omega + \omega) I_{\mathcal{O}}(\omega), \tag{18}$$

where the power spectrum $I_{\mathcal{O}}(\omega)$ is defined as

$$I_{\mathcal{O}}(\omega) = \int_{-\infty}^{\infty} dt \langle \mathcal{O}(t_0)\mathcal{O}(t_0 + t)\rangle e^{i\omega t}. \tag{19}$$

For stationary systems this does not depend upon the choice of t_0, and hence we can as well set $t_0 = 0$. Now if we turn off the external force $K_i(t)$, then from Eq. (9) we get

$$p_i(\omega) = \frac{R_i(\omega)}{-i\omega + \gamma^{(i)}[\omega]} = \mu^{(i)}(\omega) R_i(\omega), \tag{20}$$

which leads to the obvious result

$$I_{p_i}(\omega) = \frac{I_{R_i}(\omega)}{|\gamma^{(i)}[\omega] - i\omega|^2} = |\mu^{(i)}(\omega)|^2 I_{R_i}(\omega). \tag{21}$$

Making use of Eqs. (8) and (21), we are led to the result

$$\kappa^{(i)} = I_{R_i} = \frac{I_{p_i}(\omega)}{|\mu^{(i)}(\omega)|^2}. \tag{22}$$

The random force correlator $\kappa^{(i)}$ provides yet another time scale involved in the Brownian motion. If we take $\kappa^{(i)}$ to be of the form

$$\kappa^{(i)}(t) = \kappa^{(i)}(0) e^{-\frac{t}{t_{\text{col}}}}, \tag{23}$$

then t_{col} is the width of the correlator. It is the temporal span over which the random forces are correlated and gives the time scale for the duration of a collision.

In the next section, following holographic techniques prescribed in Ref. [34], we investigate the bulk realization of the boundary Brownian motion of a heavy probe moving in an anisotropic thermal plasma. In doing so, we first describe the profile of the probe string stretching between the AdS boundary and the horizon as well as the black hole background dual to the anisotropic plasma. Then we describe how to compute bulk correlators of the transverse fluctuations of the probe string.

III. HOLOGRAPHIC STORY

To incorporate the heavy dynamical probe quark in the boundary theory, one introduces N_f $D7$-flavor branes located at $r = r_m$. We work within probe approximation meaning $N_f \ll N_c$ and neglect the backreaction of the flavor brane on the background (for simplicity we take $N_f = 1$). On the gauge theory side, this is tantamount to working in the quenched approximation. The probe string stretches from the boundary at $r = r_m$ to the black hole horizon $r = r_h$. The flavor brane spans the four gauge theory directions, the radial direction, and also a 3-sphere $S^3 \subset S^5$. We take the boundary gauge theory to live at the radial coordinate $r = r_m$. We assume that the source of the fluctuations of the string modes is purely Hawking radiation. Moreover, keeping the string coupling g_s small ensures that we can ignore the interaction between the transverse fluctuation modes and the closed string modes in the bulk.

A. Anisotropic supergravity dual

In this subsection we briefly provide the details of the gauge theory we are interested in and its supergravity dual. The gauge theory under consideration is a spatially deformed $\mathcal{N}=4$, $SU(N_c)$ SYM plasma at large t'Hooft coupling $\lambda = g_{YM}^2 N_c$. The deformation is achieved by introducing a θ parameter in our theory that depends linearly upon any one of the three spatial directions, which we take to be x^3 in our case. Consequently, we can write the gauge theory action as

$$S_{\text{gauge}} = S_{\text{SYM}} + \delta S, \tag{24}$$

where

$$\delta S = \frac{1}{8\pi^2}\int \theta(x^3)\text{Tr}\, F\wedge F. \tag{25}$$

The presence of $\theta(= 2\pi n_{D7} x^3)$ reduces the $SO(3)$ rotational symmetry of the original theory down to a $SO(2)$ symmetry in the x^1-x^2 plane (where we have taken $\{t, x^1, x^2, x^3\}$ to be the gauge theory coordinates) and is responsible for making the theory anisotropic. Here n_{D7} is a constant with energy dimension. In the context of heavy ion collisions, x^3 will correspond to the direction of beam, whereas the x^1, x^2 directions span the transverse plane. In heavy ion collisions, the plasma will expand and cool down gradually, and the anisotropy parameter will also decay with time. However, here we shall restrict ourselves to a time domain where such temporal variation can be neglected. The type IIB supergravity dual to this gauge theory was given in Refs. [49,50] inspired by Ref. [51] and reads in the string frame

$$ds^2 = r^2\left(-\mathcal{F}\mathcal{B}dt^2 + (dx^1)^2 + (dx^2)^2 + \mathcal{H}(dx^3)^2 + \frac{dr^2}{r^4\mathcal{F}}\right) + e^{\frac{1}{2}\phi} d\Omega_5^2, \tag{26}$$

$$\chi = a x^3, \qquad \phi = \phi(r), \tag{27}$$

where the axion χ is proportional to the anisotropic direction x^3, the proportionality constant a being the anisotropy parameter. The theory also has a running dilaton $\phi(r)$. r is the AdS radial coordinate with the boundary at $r = \infty$ and the horizon at $r = r_h$, and $d\Omega_5^2$ is the metric on the 5-sphere S^5. We have suppressed the common radius R of the AdS space and S^5 set $R = 1$. There is also a Ramond-Ramond (RR) self-dual 5-form which will not play any role in our discussion here. The axion, which is dual to the gauge theory θ term, is responsible for making the background anisotropic. It turns out [49] that the anisotropy parameter a is proportional to n_{D7}, the number density of $D7$-branes along the x^3 direction, $a = \lambda n_{D7}/4\pi N_c$. The $D7$-branes, which source the axion, wrap around S^5 and extend along the transverse directions, x^1, x^2. However, the $D7$-branes do not span the radial direction and hence do not reach the boundary. So they do not contribute any new degrees of freedom to the theory. $\mathcal{F}$, $\mathcal{B}$, $\mathcal{H}$ are all functions of the radial coordinate r and are known analytically only in the limiting cases when the anisotropy is very high or low (with respect to the temperature). In the intermediate regime, they are known only numerically. $\mathcal{F}$ is the usual "blackening factor" that vanishes at the horizon, i.e., $\mathcal{F}(r_h) = 0$. The presence of anisotropy implies that the dual theory develops an anisotropic horizon. The strength of anisotropy can be tuned by varying the parameter a. In this paper we shall consider only weakly anisotropic plasma (the small a or high temperature T limit, whence $a/T \ll 1$). In this regime the functions $\mathcal{F}$, $\mathcal{B}$, $\mathcal{H}$ can be expanded to leading order in a around the black $D3$-brane solution,

$$\begin{aligned}
\mathcal{F}(r) &= 1 - \frac{r_h^4}{r^4} + a^2\mathcal{F}_2(r) + \mathcal{O}(a^4),\\
\mathcal{B}(r) &= 1 + a^2\mathcal{B}_2(r) + \mathcal{O}(a^4),\\
\mathcal{H}(r) &= e^{-\phi(r)}\\
\text{with } \phi(r) &= a^2\phi_2(r) + \mathcal{O}(a^4),
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\mathcal{F}_2(r) &= \frac{r_h^2}{24r^4}\left[\frac{8(r^2 - r_h^2)}{r_h^2} - 10\log 2 + \frac{3r^4 + 7r_h^4}{r_h^4}\log\left(1 + \frac{r_h^2}{r^2}\right)\right],\\
\mathcal{B}_2(r) &= -\frac{1}{24r_h^2}\left[\frac{10r_h^2}{r_h^2 + r^2} + \log\left(1 + \frac{r_h^2}{r^2}\right)\right],\\
\phi_2(r) &= -\frac{1}{4r_h^2}\log\left(1 + \frac{r_h^2}{r^2}\right).
\end{aligned} \tag{29}$$

The Hawking temperature of the above solution is

$$T = \frac{r_h}{\pi} + \frac{a^2}{r_h}\frac{(5\log 2 - 2)}{48\pi} + \mathcal{O}(a^4), \tag{30}$$

which is identified as the temperature of the deformed SYM theory. The horizon position can be obtained in terms of the temperature, which, in the limit $a/T \ll 1$, reads

$$r_h \sim \pi T\left[1 - a^2\frac{5\log 2 - 2}{48\pi^2 T^2}\right] + \mathcal{O}(a^4). \tag{31}$$

B. Bulk view of Brownian motion

To study the dynamics of the fundamental string in the background given by Eq. (26), we need to evaluate the Nambu–Goto string world sheet action,

$$S_{\rm NG} = \frac{1}{2\pi\alpha'}\int d\sigma d\tau \sqrt{-\det g_{\alpha\beta}}, \tag{32}$$

where $g_{\alpha\beta}$ is the induced metric on the string world sheet,

$$g_{\alpha\beta} = G_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta}. \tag{33}$$

Here $\xi^{\alpha,\beta}$ are the coordinates on the string world sheet Σ; $\xi^0 = \tau$ and $\xi^1 = \sigma$, $G_{\mu\nu}$ is the 10-dimensional metric as given in Eq. (26), and $\{X^\mu(\tau,\sigma)\}$ are the 10-dimensional coordinates which specify the string embedding in the full 10-dimensional spacetime. We choose the static gauge for evaluating Eq. (32) as $\tau = t, \sigma = r$. The trivial solution that satisfies the equation of motion obtained by variation of $S_{\rm NG}$ is given by $X^m = \{t, \vec{0}, r\}$. This corresponds to a quark that is in equilibrium in a thermal bath and in the bulk picture to a string hanging straight down radially. We now wish to consider fluctuations around this classical solution. We want to see the effects of anisotropy both along the anisotropic direction as well as in the isotropic plane. To this end we consider fluctuations of the form $X^m = \{t, X_1(t,r), 0, X_3(t,r), r\}$, where $X_1(t,r)$ is a fluctuation in a isotropic direction while $X_3(t,r)$ is a perturbation along the anisotropic direction. The position of the quark is given by $x^\mu = \{t, X_1(t,r_m), 0, X_3(t,r_m)\}$. Using this parametrization we find out the components of the world sheet metric as

$$\begin{aligned} g_{\tau\tau} &= r^2(-\mathcal{F}\mathcal{B} + (\dot{X}_1)^2 + \mathcal{H}(\dot{X}_3)^2),\\ g_{\sigma\sigma} &= r^2\left((X_1')^2 + \mathcal{H}(X_3')^2 + \frac{1}{r^4\mathcal{F}}\right),\\ g_{\tau\sigma} &= r^2(\mathcal{H}\dot{X}_1X_3' + \mathcal{H}X_1'\dot{X}_3), \end{aligned} \tag{34}$$

where $X_i' \equiv \partial_\sigma X_i$ and $\dot{X}_i \equiv \partial_\tau X^i$. From now on, we suppress the explicit r dependence of the metric elements $\mathcal{F}$, $\mathcal{B}$, $\mathcal{H}$. If we restrict ourselves to small perturbation around the classical solution, we can safely leave out terms higher than quadratic order in the fluctuations whence the action reduces to[3]

$$S_{\rm NG} = \frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{\mathcal{B}}[\mathcal{F}r^4((X_1')^2 + \mathcal{H}(X_3')^2) - \frac{1}{\mathcal{F}\mathcal{B}}((\dot{X}_1)^2 + \mathcal{H}(\dot{X}_3)^2)]. \tag{35}$$

While writing Eq. (35), we have omitted a constant factor that is independent of X_i. Variation of the above action yields the equation of motion for the fluctuation X_3,

$$\ddot{X}_3 - \frac{\mathcal{F}\sqrt{\mathcal{B}}}{H} r_h^2\partial_y(\sqrt{\mathcal{B}}\mathcal{H}\mathcal{F}y^4X_3') = 0, \tag{36a}$$

where we have used the new scaled coordinate, $y = r/r_h$, and now the prime $'$ denotes a derivative with respect to y. The equation of motion for X_1 is obtained in a similar fashion,

$$\ddot{X}_1 - \mathcal{F}\sqrt{\mathcal{B}}r_h^2\partial_y(\sqrt{\mathcal{B}}\mathcal{F}y^4X_1') = 0, \tag{36b}$$

which is the same as Eq. (36a) with $\mathcal{H} = 1$. Later on, we shall also consider forced motion of the quark under the effect of an electromagnetic field. This is simply achieved by switching on a $U(1)$ electromagnetic field on the flavor $D7$-brane. Since the string end point on the boundary represents a quark, it is charged and hence will couple to the electromagnetic field. Consequently, we need to incorporate this effect at the level of the action. The action $S_{\rm NG}$ is then generalized to $S = S_{\rm NG} + S_b$, where

$$S_b = \int_{\partial\Sigma}(A_t + A_i\dot{X}_i)dt. \tag{37}$$

Since it is just a boundary term, it will not affect the dynamics of the string in the bulk. However, it will modify the boundary conditions that we need to impose upon the string end point. We need to find solutions to Eqs. (36a) and (36b) near the boundary, which we shall do by employing the matching technique. The solutions are, in general, quite complicated. However, they are readily obtained near the horizon. So before finding out the actual solutions, let us see how these solutions behave in the vicinity of $y \to 1$. First of all, we inflict a coordinate transformation $r \to r_*$, which takes us to the tortoise coordinates so that

$$\frac{d}{dr} = \frac{1}{r^2\mathcal{F}\sqrt{\mathcal{B}}}\frac{d}{dr_*} \tag{38}$$

and

$$dr = r^2\mathcal{F}\sqrt{\mathcal{B}}dr_*. \tag{39}$$

In this new coordinate system, the Nambu–Goto action assumes the form

$$S_{\rm NG} = \frac{1}{4\pi\alpha'}\int d\tau dr_* r^2[((\partial_{r_*}X_1)^2 - (\dot{X}_1)^2) + \mathcal{H}((\partial_{r_*}X_3)^2 - (\dot{X}_3)^2)]. \tag{40}$$

Near the horizon it simplifies to

$$S_{\rm NG} = \frac{1}{4\pi\alpha'}r_h^2\int d\tau dr_*[((\partial_{r_*}X_1)^2 - (\dot{X}_1)^2) + \mathcal{H}(r_h)((\partial_{r_*}X_3)^2 - (\dot{X}_3)^2)]. \tag{41}$$

The equation of motion for both X_1 and X_3 obtained by varying this action turns out to be the same,

$$(\partial_{r_*}^2 - \partial_\tau^2)X_{1,3} = 0. \tag{42}$$

So near the boundary, the fluctuations are governed by a Klein–Gordon equation for massless scalars. From now on,

[3]This essentially means that we are in the regime $|\partial_t X_i| \ll 1$, which, in turn, implies taking the nonrelativistic limit. Hence, on the gauge theory side, the dual picture will also be nonrelativistic.

in this section, we shall refer to the fluctuations as X_i, it being understood that everything we discuss here holds true for both X_1 as well as X_3. From Eq. (26) it is clear that t is an isometry of the background, and hence we can try solutions of the form

$$X_i(t,r) \sim e^{-i\omega t} g_\omega(r). \tag{43}$$

Equation (42) has two independent solutions corresponding to ingoing and outgoing waves, respectively, which we write as

$$X_i^{\rm out}(r) = e^{-i\omega t} g_i^{\rm out}(r) \sim e^{-i\omega(t-r_*)} \tag{44a}$$

$$X_i^{\rm in}(r) = e^{-i\omega t} g_i^{\rm in}(r) \sim e^{-i\omega(t+r_*)}. \tag{44b}$$

To find r_* we need to solve Eq. (39), which yields

$$r_* = \frac{1}{4r_h}\log\left(\frac{r}{r_h}-1\right)\left[1-\frac{\tilde{a}^2}{48}(5\log 2-2)\right], \tag{45}$$

where we have defined $\tilde{a} = \frac{a}{r_h} \sim \frac{a}{\pi T}$. Hence,

$$g_i^{\rm out/in}(r) = \left(\frac{r}{r_h}-1\right)^{\pm\frac{i\nu}{4}(1-\frac{\tilde{a}^2}{48}(5\log 2-2))}, \tag{46}$$

where $\nu = \frac{\omega}{r_h}$. One thus finds that $g_i^{\rm out} = (g_i^{\rm in})^*$.

Following standard quantization techniques of scalar fields in curved spacetime, we can perform a mode expansion of the fluctuations as

$$X_i(t,r) = \int_0^\infty \frac{d\omega}{2\pi}[a_\omega u_\omega(t,r) + a_\omega^\dagger u_\omega(t,r)^*]. \tag{47}$$

Here $u_\omega(t,r)$ is a set of positive frequency basis. These modes can, in turn, be expressed as a linear combination of the ingoing and the outgoing waves:

$$u_\omega(t,r) = A[g^{\rm out}(r) + B g^{\rm in}(r)]e^{-i\omega t}. \tag{48}$$

The constant B is determined by imposing a boundary condition at $r = r_m$, i.e., $y = 1$. However, as we shall later see, B turns to be a pure phase. This implies that the outgoing and the ingoing modes have the same amplitude. This signifies that the black hole environment which can emit Hawking radiation is in a state of thermal equilibrium. One is then left with determining the constant A, which is fixed by demanding normalization of the modes through the conventional Klein–Gordon inner product defined via

$$(f_i, g_j)_\sigma = -\frac{i}{2\pi\alpha'}\int_\sigma \sqrt{\tilde{g}} n^\mu G_{ij}(f_i\partial_\mu g_j^* - \partial_\mu f_i g_j^*). \tag{49}$$

Here, σ defines a Cauchy surface in the (t,r) subspace of the 10-dimensional spacetime metric, $\tilde{g}$ is the induced metric on the surface σ, and n^μ denotes a unit normal to σ in the future direction. Without any loss of generality, we can take the surface σ to be a constant t surface since the inner product does not depend upon the exact choice of the surface in the (t,r) plane [52]. Following Ref. [38] we argue that the primary contribution to the above integral arises from the IR region. Of course, regions away from the horizon do contribute, but since the horizon is semi-infinite in the tortoise coordinate, the normalization is completely fixed by the near-horizon regime. For the anisotropic direction, this gives

$$(f_i, g_j)_\sigma = -\frac{i\delta_{ij} r_h^2 \mathcal{H}(r_h)}{2\pi\alpha'}\int_{r_*\to-\infty} dr_*(f_i\dot{g}_j^* - \dot{f}_i g_j^*), \tag{50}$$

from which we can extract A to be

$$A = \sqrt{\frac{\pi\alpha'}{\omega r_h^2 \mathcal{H}(r_h)}}. \tag{51}$$

On the other hand, for fluctuations along the isotropic direction, we have

$$(f_i, g_j)_\sigma = -\frac{i\delta_{ij} r_h^2}{2\pi\alpha'}\int_{r_*\to-\infty} dr_*(f_i\dot{g}_j^* - \dot{f}_i g_j^*), \tag{52}$$

which fixes A as

$$A = \sqrt{\frac{\pi\alpha'}{\omega r_h^2}}. \tag{53}$$

The normalization ensures that the inner product $(u_\omega, u_\omega) = 1$, which, in turn, guarantees that the canonical commutation relations are satisfied,

$$[a_\omega, a_{\omega'}] = [a_\omega^\dagger, a_{\omega'}^\dagger] = 0, \quad [a_\omega, a_{\omega'}^\dagger] = 2\pi\delta(\omega+\omega'). \tag{54}$$

In the semiclassical approximation, the string modes are thermally excited by the Hawking radiation of the world sheet horizon and obey the Bose–Einstein distribution,

$$\langle a_\omega a_\omega^\dagger \rangle = \frac{2\pi\delta(\omega+\omega')}{e^{\beta\omega}-1}. \tag{55}$$

Equipped with this much machinery, we are now ready to compute the displacement squared for the test quark in the boundary. This is required if we wish to find out an expression for the diffusion constant. Recalling that the position of the Brownian particle is specified by $x_i(t) = X_i(t, r_m)$, we have

$$\langle x_i(t)x_i(0)\rangle = \int_0^\infty \frac{d\omega d\omega'}{(2\pi)^2}[\langle a_\omega a_{\omega'}^\dagger\rangle u_\omega(t,r_m)u_{\omega'}(0,r_m)^* + \langle a_\omega^\dagger a_{\omega'}\rangle u_\omega(t,r_m)^* u_{\omega'}(0,r_m)]. \tag{56}$$

However, this is afflicted by a divergence that can be attributed to the zero point energy, which persists even when we go to the zero temperature limit. The way to bypass this catastrophe is to invoke the normal ordering of products

$$\langle : x_i(t)x_i(0) : \rangle = \int_0^\infty \frac{d\omega}{2\pi} \frac{2|A|^2 \cos\omega t}{e^{\beta\omega}-1} |g^{\rm out}(r_m) + B g^{\rm in}(r_m)|^2. \tag{57}$$

Finally, after a little algebra, we arrive at the expression for displacement squared,

$$s_i^2(t) \equiv \langle : [x_i(t) - x_i(0)]^2 : \rangle = \frac{4}{\pi}\int_0^\infty d\omega |A|^2 \frac{\sin^2 \omega t/2}{e^{\beta\omega}-1} |g^{\rm out}(r_m) + B g^{\rm in}(r_m)|^2. \tag{58}$$

With the general formalism in place, we are now in a position to take up the problem of analyzing Brownian motion in an anisotropic strongly coupled plasma from the holographic point of view. In Sec. III C we study the case of Brownian motion in the plasma along the anisotropic direction. We discuss this case in detail. Later in Sec. III D we consider Brownian motion along one of the isotropic directions.

C. Brownian motion along anisotropic direction

Our first job will be to solve Eq. (36a) in the asymptotic limit. Making use of Eq. (43), we recast Eq. (36a) as

$$\nu^2 g(y) + \frac{\mathcal{F}\sqrt{\mathcal{B}}}{\mathcal{H}} \partial_y [\sqrt{\mathcal{B}}\mathcal{H}\mathcal{F} y^4 g'(y)] = 0. \tag{59}$$

Inserting the explicit expressions of the various functions, this can be written as

$$g''(y) + 4\frac{y^3}{y^4-1}[1+\tilde{a}^2\Psi(y)]g'(y) + \frac{y^4\nu^2}{(y^4-1)^2}[1+\tilde{a}^2\Upsilon(y)]g(y) = 0, \tag{60}$$

where

$$\Psi(y) = \frac{1}{96y^4(y^4-1)}\left[3 - 9y^2 - 23y^6 + y^4(29 + 40\log 2) - 40y^4 \log\left(1+\frac{1}{y^2}\right)\right]$$
$$\Upsilon(y) = \frac{1}{24(y^4-1)}\left[6 - 6y^2 + 20\log 2 - 5(3+y^4)\log\left(1+\frac{1}{y^2}\right)\right]. \tag{61}$$

We need to find a solution to this equation. However, as it turns out, obtaining an analytic solution is a notoriously difficult problem for any arbitrary frequency ν. To circumvent this difficulty, we work only in the low-frequency approximation and then attempt to solve the equation by the "matching technique." Since we only require the solution near the boundary, we just give here the expression of the required solution. The interested reader is referred to Appendix A for the details of the solution. We shall have two solutions corresponding to the ingoing and outgoing waves,

$$g^{\rm out/in} = k_1^{\rm out/in}\left[1 + \frac{\nu^2}{2y^2} + \mathcal{O}\left(\frac{1}{y^4}\right)\right] + k_3^{\rm out/in}\left[\frac{1}{y^3} + \mathcal{O}\left(\frac{1}{y^5}\right)\right], \tag{62}$$

where

$$k_1^{\rm out/in} = 1 \mp \frac{i\nu}{8}(\pi - 2\log 2) \pm \frac{i\nu\tilde{a}^2}{768}[28 - 16\beta(2) - 20(\log 2)^2 + \pi(-8 + \pi + 14\log 2) + 8\log 2] + \mathcal{O}(\nu^2)$$
$$k_3^{\rm out/in} = \mp\frac{i\nu}{3}\left(1 + \frac{\tilde{a}^2}{4}\log 2\right) + \mathcal{O}(\nu^2), \tag{63}$$

where $\beta(2)$[4] ~0.915966. We find that the relation, $g^{\rm out} = g^{\rm in*}$, obtained earlier in the near-horizon analysis, continues to hold true in the asymptotic limit.

We can now use these solutions, supplemented by the appropriate boundary conditions, to find out various quantities of interest. However, before going into the intricacies of the actual computation, let us digress a little bit to clarify the boundary conditions involved in the problem.

Although we are interested in the world sheet theory of the probe string, the choice of the static gauge implies that the characteristics of the background spacetime are

[4] $\beta(s)$ is the Dirichlet beta function given by $\beta(s) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^s}$.

encoded in the induced metric. Hence, we can exploit the rules of the AdS/CFT correspondence to understand the boundary conditions. When working in the Lorentzian AdS/CFT, it is customary to choose normalizable boundary conditions [53] for the modes. In the present scenario, this amounts to pushing the boundary all the way up to $y \to \infty$. However, the AdS/CFT dictionary tells us that the radial distance is mapped holographically to the mass of the probe quark so that placing the boundary at $y \to \infty$ essentially means that we are considering our probe quark to be infinitely massive. Of course, this at once rules out any possibility of the quark undergoing Brownian motion. The problem can be solved if, instead, we impose a UV cutoff in our theory. More specifically, we introduce a UV cutoff surface and identify it with the boundary where the gauge theory lives. In fact, this is exactly the location of the flavor brane y_m to which the endpoint of the string is attached. The relation between the position of the UV cutoff and the mass of the probe can be read off easily as

$$m = \frac{1}{2\pi\alpha'} \int_{r_h}^{r_m} dr \sqrt{-g_{tt} g_{rr}} = \frac{1}{2\pi\alpha'} \left[y_m - 1 + \frac{\tilde{a}^2}{24} (\log 2 - 3\pi) \right], \tag{64}$$

and the world sheet metric elements g_{tt}, g_{rr} are written for the classical string configuration, i.e., omitting the contribution arising out of the fluctuations. On this surface we can impose Neumann boundary condition[5] $\partial_r X_i = 0$. However, this works only when we consider the free Brownian motion of the particle in the absence of any external force. In the case of forced motion, this is modified to

$$\Pi_i^y|_{\partial\Sigma} \equiv \frac{\partial \mathcal{L}}{\partial X_i{}'} = K_i = K_i^{(0)} e^{-i\omega t}, \tag{65}$$

where we have assumed a fluctuating external force.

Now the general solution X_i is a linear combination of the outgoing and the ingoing modes at the horizon,

$$X_i = A^{\rm out} X_i^{\rm out} + A^{\rm in} X_i^{\rm in}. \tag{66}$$

where $X_i^{\rm out/in} = e^{-i\omega t} g^{\rm out/in}$ and $g^{\rm out/in}$ is given in Eq. (62). In the semiclassical approximation, the outgoing modes are thermally excited by the Hawking radiation emanating from the black hole, whereas the ingoing modes can be arbitrary. Since the Hawking radiation is a random phenomena, the phase of $A^{\rm out}$ takes random values, and its average $\langle A^{\rm out} \rangle$ vanishes. So we can omit the first term in Eq. (66) and need to consider only the ingoing wave. When one plugs the form of the Lagrangian into Eq. (65), one finds that, like the equations of motion, the boundary conditions along the anisotropic direction and the isotropic directions decouple, which allows us to treat each direction separately. Coming back to the particular case of the anisotropic direction, the boundary condition given in Eq. (65) assumes the form

$$\frac{1}{2\pi\alpha'} \mathcal{HF}\sqrt{\mathcal{B}} y^4 r_h^3 X_3{}'|_{y=y_m} = K_3 = K_3^{(0)} e^{-i\omega t}. \tag{67}$$

This yields

$$A^{\rm in} = \frac{2\pi\alpha' K_3^{(0)}}{\mathcal{HF}\sqrt{\mathcal{B}} y^4 r_h^3 g'(y)}\bigg|_{y=y_m}, \tag{68}$$

where $g(y)$ represents the ingoing solution in Eq. (62). So, on the boundary the average position of the Brownian quark is given by

$$\langle x_3(t) \rangle = \langle X_3(t, y_m) \rangle = K_3^{(0)} e^{-i\omega t} \frac{2\pi\alpha' g}{\mathcal{HF}\sqrt{\mathcal{B}} y^4 r_h^3 g'}\bigg|_{y=y_m}. \tag{69}$$

The average momentum is

$$\langle p_3(t) \rangle = m\langle \dot{x_3} \rangle = -K_3 \frac{2i\pi\alpha' m\nu g}{\mathcal{HF}\sqrt{\mathcal{B}} y^4 r_h^2 g'}\bigg|_{y=y_m}. \tag{70}$$

Comparison with Eq. (12) results in

$$\mu^{\parallel}(\nu) \equiv \mu^{(3)}(\nu) = -\frac{2i\pi\alpha' m\nu g}{\mathcal{HF}\sqrt{\mathcal{B}} y^4 r_h^2 g'}\bigg|_{y=y_m}. \tag{71}$$

Here we have used the superscript "∥" to denote quantities along the anisotropic direction (the x_3 direction). Reinstating the expressions for the various functions and expanding up to $\mathcal{O}(\tilde{a}^2)$ in the low-frequency regime, we obtain the relaxation time for the heavy quark diffusing along the anisotropic direction,

$$\mu^{\parallel}(0) = t_{\rm relax}^{\parallel} = \frac{2m}{\pi\sqrt{\lambda} T^2} \left[1 - \frac{a^2}{24\pi^2 T^2} (2 + \log 2) \right], \tag{72}$$

from which one gets the drag coefficient along the anisotropic direction

$$\gamma^{\parallel}[0] = \frac{\pi\sqrt{\lambda} T^2}{2m} \left[1 + \frac{a^2}{24\pi^2 T^2} (2 + \log 2) \right] = \gamma_{\rm iso} \left[1 + \frac{a^2}{24\pi^2 T^2} (2 + \log 2) \right], \tag{73}$$

where $\gamma_{\rm iso}$ represents the drag coefficient when the quark moves in a isotropic SYM plasma. Here we have used the standard AdS/CFT dictionary, $R^4 = (\alpha')^2 \lambda$, with $R = 1$ in our convention. Our expression for the friction coefficient

[5]One cannot impose the Dirichlet condition since it implies no fluctuation on the boundary at all.

$\gamma^{\parallel}$ matches exactly with that obtained in Ref. [17] in the nonrelativistic limit $v \ll 1$ along the anisotropic direction. Note that the drag force increases compared to its isotropic counterpart when the quark moves along the anisotropic direction. Next we turn toward computing the displacement squared for the Brownian particle from which we can extract the expression for the diffusion constant $D^{\parallel}$. We have already provided a generic expression for s_i^2 in Eq. (58). The details of the calculation will depend upon the background metric. Let us again return to the boundary condition, Eq. (65), but now with the gauge fields turned off. Equation (65) then reads for the anisotropic direction

$$\frac{\partial \mathcal{L}}{\partial X_3{}'} = \frac{1}{2\pi\alpha'} \mathcal{H}\mathcal{F}\sqrt{\mathcal{B}} y^4 r_h^3 X_3{}' \Big|_{y=y_m} = 0, \tag{74}$$

which translates to $X_3{}' = 0$ at the boundary. The fluctuations $X_i(t, y)$ can be expressed as the sum of outgoing and ingoing modes as

$$X_i(t, y) = A[g^{\mathrm{out}}(y) + B g^{\mathrm{in}}(y)] e^{-i\omega t}. \tag{75}$$

It then easily follows that $X_3{}' = 0$ implies

$$B = -\frac{g^{\mathrm{out}\prime}}{g^{\mathrm{in}\prime}}\Big|_{y=y_m} = 1 + \mathcal{O}(\nu), \tag{76}$$

which gives

$$|g^{\mathrm{out}}(y_m) + B g^{\mathrm{in}}(y_m)|^2 = 4 + \mathcal{O}(\nu). \tag{77}$$

Using Eqs. (51) and (77) in Eq. (58), one then has

$$s_3^2 = \frac{4t}{\pi T\sqrt{\lambda}} \left[1 - \frac{a^2}{24\pi^2 T^2}(2 + \log 2)\right]. \tag{78}$$

Hence, the diffusion constant along the anisotropic direction is

$$D^{\parallel} = \frac{2}{\pi T\sqrt{\lambda}} \left[1 - \frac{a^2}{24\pi^2 T^2}(2 + \log 2)\right] = \frac{T}{m\gamma^{\parallel}}. \tag{79}$$

This is nothing but the Einstein–Sutherland relation [Eq. (5)] mentioned earlier. We have thus performed an explicit verification of the relation from the bulk point of view. Finally, we proceed to verify the fluctuation-dissipation theorem for which we need to know the random force correlator. First of all, we compute the two-point correlator of the momentum along the ith direction,

$$\begin{aligned} \langle : p_i(t) p_i(0) : \rangle &\equiv -m^2 \partial_t^2 \langle : x_i(t) x_i(0) : \rangle \\ &= \int_0^\infty \frac{d\omega}{2\pi} \frac{2m^2\omega^2 |A|^2 \cos \omega t}{e^{\beta\omega} - 1} |g^{\mathrm{out}}(y_m) + B g^{\mathrm{in}}(y_m)|^2. \end{aligned} \tag{80}$$

Invoking the Wiener–Khintchine theorem [Eq. (18)] and the expression for A [Eq. (51)] and specializing to the anisotropic direction, we find

$$I_{p_3}(\omega) = 4 \frac{m^2 \pi}{r_h^2 \alpha' \mathcal{H}(y=1)\beta} \frac{\beta\omega}{e^{\beta\omega} - 1}. \tag{81}$$

Expanding in ω and keeping only the leading-order term, one has

$$\begin{aligned} I_{p_3}(\omega) &= \frac{4m^2}{\sqrt{\lambda}\pi T}\left(1 + \frac{a^2}{24\pi^2 T^2}(5 \log 2 - 2)\right) \\ &\quad \times \left(1 - \frac{a^2}{4\pi^2 T^2} \log 2\right) + \mathcal{O}(\omega). \end{aligned} \tag{82}$$

Now, the Langevin coefficient along the direction of anisotropy is

$$\begin{aligned} \kappa^{\parallel} &= I_{R_3} = \frac{I_{p_3}(\omega)}{|\mu^{\parallel}(\omega)|^2} \\ &= 2mT \frac{\pi\sqrt{\lambda} T^2}{2m}\left[1 + \frac{a^2}{24\pi^2 T^2}(2 + \log 2)\right] = 2mT\gamma^{\parallel} \\ &= \kappa_{\mathrm{iso}}\left[1 + \frac{a^2}{24\pi^2 T^2}(2 + \log 2)\right] \end{aligned} \tag{83}$$

(where κ_{iso} is the Langevin coefficient in isotropic plasma), which is nothing but the statement of the fluctuation-dissipation theorem. We thus observe that the strength of the autocorrelator along the anisotropic direction increases in the presence of anisotropy. Thus, we explicitly check the validity of the fluctuation-dissipation theorem for a heavy test quark executing Brownian motion in a strongly coupled, anisotropic plasma when the fluctuations are aligned with the direction of anisotropy.

D. Brownian motion transverse to the anisotropic direction

In this subsection we discuss the case of the Brownian motion in the isotropic plane. For definiteness, we take the motion to be along the X_1 direction. The calculations in this case proceed in almost the same way as in Sec. III C. As is evident upon comparing Eqs. (36a) and (36b), the equation of motion in the isotropic direction can be simply obtained by setting $\mathcal{H} = 1$ in the anisotropic case. This can also be understood by looking at the metric in Eq. (26). So we shall be brief in our discussion here. The equation to solve is

$$\nu^2 g(y) + \mathcal{F}\sqrt{\mathcal{B}}\partial_y(\sqrt{\mathcal{B}}\mathcal{F} y^4 g') = 0, \tag{84}$$

which can be recast as

$$\begin{aligned} g''(y) &+ 4\frac{y^3}{y^4 - 1}[1 + \tilde{a}^2\tilde{\Psi}(y)]g'(y) \\ &+ \frac{y^4\nu^2}{(y^4 - 1)^2}[1 + \tilde{a}^2\tilde{\Upsilon}(y)]g(y) = 0, \end{aligned} \tag{85}$$

where

$$\tilde{\Psi}(y) = \frac{1}{96y^4(y^4-1)}\left[15 - 21y^2 - 11y^6 + y^4(17 + 40\log 2) - 40y^4 \log\left(1 + \frac{1}{y^2}\right)\right]$$
$$\tilde{\Upsilon}(y) = \frac{1}{24(y^4-1)}\left[6 - 6y^2 + 20\log 2 - 5(3 + y^4)\log\left(1 + \frac{1}{y^2}\right)\right]. \tag{86}$$

As in the anisotropic version, here, too, we look for solutions by resorting to the matching technique. Here we present only the final form of the solution in the asymptotic limit,

$$g^{\mathrm{out/in}} = \tilde{k}_1^{\mathrm{out/in}}\left[1 + \frac{\nu^2}{2y^2} + \mathcal{O}\left(\frac{1}{y^4}\right)\right] + \tilde{k}_3^{\mathrm{out/in}}\left[\frac{1}{y^3} + \mathcal{O}\left(\frac{1}{y^5}\right)\right], \tag{87}$$

where

$$\tilde{k}_1^{\mathrm{out/in}} = 1 \mp \frac{i\nu}{8}(\pi - 2\log 2) \mp \frac{i\nu\tilde{a}^2}{768}[-80\beta(2) + \pi(8 + 5\pi) - 4(7 + 2\log 2) + 10(\pi + 2\log 2)\log 2] + \mathcal{O}(\nu^2)$$
$$\tilde{k}_3^{\mathrm{out/in}} = \mp\frac{i\nu}{3}. \tag{88}$$

We thus find that while the y dependence is the same as in its anisotropic counterpart, only the coefficients $\tilde{k}_1$ and $\tilde{k}_3$ are different. Note that, in particular, the coefficient $\tilde{k}_3$ does not pick up any contribution from anisotropy. The boundary condition now reads in the presence of the gauge field on the boundary

$$\frac{1}{2\pi\alpha'}\mathcal{F}\sqrt{\mathcal{B}}y^4 r_h^3 X_1'|_{y=y_m} = K_1 = K_1^{(0)}e^{-i\omega t}, \tag{89}$$

which fixes the normalization factor

$$A^{\mathrm{in}} = \left.\frac{2\pi\alpha' K_1^{(0)}}{\mathcal{F}\sqrt{\mathcal{B}}y^4 r_h^3 g'}\right|_{y=y_m}. \tag{90}$$

One can now easily obtain expressions for the position and hence the momentum of the Brownian quark from which follows the expression for the admittance,

$$\mu^{\perp}(\nu) = -\left.\frac{2i\pi\alpha'\nu m g}{\mathcal{F}\sqrt{\mathcal{B}}y^4 r_h^2 g'}\right|_{y=y_m}, \tag{91}$$

with $g(y)$ now being the ingoing solution in Eq. (87). Here we denote the direction transverse to the anisotropic one as "$\perp$." Reinstating the expressions for the various functions and expanding up to $\mathcal{O}(\tilde{a}^2)$ in the low-frequency domain, we obtain the relaxation time for fluctuations in the transverse plane,

$$\mu^{\perp}(0) = t_{\mathrm{relax}}^{\perp} = \frac{2m}{\pi\sqrt{\lambda}T^2}\left[1 + \frac{a^2}{24\pi^2T^2}(5\log 2 - 2)\right], \tag{92}$$

from which one gets the drag coefficient along the isotropic direction,

$$\gamma^{\perp}[0] = \frac{\pi\sqrt{\lambda}T^2}{2m}\left[1 - \frac{a^2}{24\pi^2T^2}(5\log 2 - 2)\right]$$
$$= \gamma_{\mathrm{iso}}\left[1 - \frac{a^2}{24\pi^2T^2}(5\log 2 - 2)\right]. \tag{93}$$

This expression for the friction coefficient $\gamma^{\perp}$ in the isotropic direction agrees with that obtained in Ref. [17] in the nonrelativistic limit $v \ll 1$. It is to be observed that the isotropic direction also picks up correction from anisotropy; i.e., even the isotropic plane can "feel" the presence of anisotropy in the normal direction. Moreover, while the presence of anisotropy increases the drag force along the anisotropic direction, it leads to a suppression in the drag force in the isotropic plane. The computation for the displacement squared for the Brownian particle proceeds in exactly a similar fashion as in the previous subsection. Switching off the external field, we impose the free Neumann condition,

$$\frac{\partial\mathcal{L}}{\partial X_i'} = \left.\frac{1}{2\pi\alpha'}\mathcal{F}\sqrt{\mathcal{B}}y^4 r_h^3 X_1'\right|_{y=y_m} = 0, \tag{94}$$

which translates to $X_1' = 0$ at the boundary that furnishes

$$B = -\left.\frac{g^{\mathrm{out}\prime}}{g^{\mathrm{in}\prime}}\right|_{y=y_m} = 1 + \mathcal{O}(\nu), \tag{95}$$

which implies

$$|g^{\mathrm{out}}(y_m) + Bg^{\mathrm{in}}(y_m)|^2 = 4 + \mathcal{O}(\nu). \tag{96}$$

Using Eqs. (96) and (53) in Eq. (58), one then has

$$s_1^2 = \frac{4t}{\pi T\sqrt{\lambda}}\left[1 + \frac{a^2}{24\pi^2T^2}(5\log 2 - 2)\right]. \tag{97}$$

We can now easily read off the diffusion constant to be

$$D^{\perp} = \frac{2}{\pi T\sqrt{\lambda}}\left[1 + \frac{a^2}{24\pi^2T^2}(5\log 2 - 2)\right]. \tag{98}$$

A comparison of Eqs. (93) and (98) reveals the relation,

$$D^{\perp} = \frac{T}{m\gamma^{\perp}}, \tag{99}$$

which verifies the validity of the Einstein–Sutherland relation in the isotropic plane. Next we find the random force correlator $\kappa^{\perp}$ along the isotropic direction. We have

$$I_{p_1}(\omega) = 4\frac{m^2\pi}{r_h^2\alpha'\beta}\frac{\beta\omega}{e^{\beta\omega}-1} = \frac{4m^2}{\sqrt{\lambda}\pi T}\left(1+\frac{a^2}{24\pi^2T^2}(5\log 2-2)\right)+\mathcal{O}(\omega). \tag{100}$$

Now,

$$\kappa^{\perp} = I_{R_1} = \frac{I_{p_1}(\omega)}{|\mu^{\perp}(\omega)|^2} = 2mT\frac{\pi\sqrt{\lambda}T^2}{2m}\left[1-\frac{a^2}{24\pi^2T^2}(5\log 2-2)\right] = 2mT\gamma^{\perp} = \kappa^{\rm iso}\left[1-\frac{a^2}{24\pi^2T^2}(5\log 2-2)\right]. \tag{101}$$

Hence, we find that the fluctuation-dissipation theorem continues to hold true in the isotropic plane, too, and also the random forces are less correlated in the isotropic plane due to the presence of anisotropy in the perpendicular direction.

IV. CONCLUSION

In this work we have studied the holographic Brownian motion of a nonrelativistic heavy probe quark immersed in an weakly anisotropic, strongly coupled hot plasma. Our computation in the bulk theory involves an explicit solution of the transverse fluctuation modes of the probe string in the low-frequency regime along anisotropic as well as isotropic directions. The above restrictions are imposed to have an analytic handle upon the computations. One might try to relax some of these restrictions, like considering general values of the parameter a/T. For large values of a/T, or small values of T, the gravity background is known analytically, and one might try to perform a similar computation. However, in that regime of the parameter space, the quantum fluctuations will dominate over the random fluctuations. For intermediate values of a/T, no analytical results are available, and one will have to fall back upon numerical means right from the outset. It might also be possible that some of the results obtained in this paper get modified away from these limits we have considered. Hence, it might be interesting to investigate the Brownian motion in more general scenarios.[6] The analytic solution is obtained using matching boundary techniques. Recently it has been shown that in the presence of a background electric field in the bulk the world sheet of the probe open string develops an induced horizon structure. This is also true when the probe string possesses a nontrivial velocity profile [55,56]. Since we have not considered such configurations, our solution smoothly interpolates from the boundary to the black hole horizon. The only horizon structure is embedded in the black hole background. It is important to note that if we could precisely measure the Brownian dynamics in the boundary it would be a very promising step toward learning the quantum dynamics of black hole physics. However, that requires the knowledge of nonperturbative gauge theory correlators, which is beyond the scope of this paper. In this work, using the holographic prescription, we have computed the drag coefficient, the diffusion constant, and the strength of the random force in a low-frequency as well as nonrelativistic limits. The expressions for the drag coefficient and the Langevin coefficient along the anisotropic direction clearly signify an enhancement over the corresponding isotropic counterparts. The fluctuations along the isotropic direction also respond to the anisotropy in the bulk. As a result, in the boundary theory, we observe that both the drag coefficient and the coefficient of autocorrelator take lower values compared to the case of ordinary SYM plasma. We have also checked that even in the presence of anisotropy the fluctuation-dissipation theorem is still valid for random variation along both isotropic and anisotropic directions. Moreover, we have computed the diffusion constant and reproduced the Einstein–Sutherland relation in a holographic sense. Before closing let us also observe an interesting qualitative agreement of our result with those obtained in the case of noncommutative Yang-Mills (NCYM) plasma, which also has an inherent anisotropy built into it. In Ref. [42] the drag force, the diffusion constant, and the Langevin coefficient were holographically computed for strongly coupled NCYM. In the case of NCYM, an unbroken $SO(2)$ symmetry is confined to the noncommutative plane, whereas for spatially deformed anisotropic YM plasma, the unbroken $SO(2)$ symmetry lives on the isotropic plane (x_1-x_1 plane). Therefore it is reasonable to compare the result in the isotropic plane in the present paper with the NCYM result. Within the small anisotropy approximation, it is observed that in both cases the drag force coefficient is weaker than the one computed in the context of ordinary Yang-Mills plasma. This observation is also true for the relevant Langevin coefficient. It is important to check the validity of this comparison for the arbitrary strength of anisotropy. However, this is beyond the scope of analytic computation and is left for a future work.

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[6]In a recent paper [54], the authors studied the relativistic Langevin diffusion of a heavy quark in strongly coupled, anisotropic Yang-Mills plasma for both small and large values of the anisotropy parameter.

APPENDIX A: DETAILS OF THE SOLUTION ALONG ANISOTROPIC DIRECTION BY THE MATCHING TECHNIQUE

In this appendix we present the details of the solution [Eq. (62)] referred to in Sec. III C. We employ the so-called matching technique. The solution to Eq. (60) is extremely difficult to obtain analytically for any frequency. To make the problem tractable, we focus only on the behavior of the solution in the low-frequency domain. In this frequency domain, we resort to the matching technique whereby we find the solutions in three different regimes and then match these solutions to the leading in the frequency at the interface of two domains. To be more specific, we find solutions to Eq. (60) in the following three limiting cases: (A) near the horizon, i.e., $y \to 1$ for arbitrary frequency and then taking the low-frequency limit; (B) throughout the bulk (i.e., arbitrary y) but for low frequency $\nu \ll 1$ and then taking the near-horizon limit [we match this solution with the low-frequency limit of the solution obtained in (A)]; (C) we solve the equation in the asymptotic limit ($y \to \infty$) for arbitrary ν, and then taking the low-frequency limit, we match it with the solution of (B). Below we elucidate the details of the solutions for each regime.

1. Near-horizon limit

In this regime we solve Eq. (60) near the horizon, i.e., in the limit $y \to 1$. In the near-horizon regime, Eq. (60) simplifies to

$$g_A''(y) + \frac{1}{y-1} g_A'(y) + \frac{\nu^2}{16(y-1)^2}\left[1 + \frac{\tilde{a}^2}{24}(5\log 2 - 2)\right] g_A(y) = 0. \tag{A1}$$

This has a solution

$$g_A(y) = A^{\rm out}(y-1)^{\frac{i\nu}{4}\left[1-\frac{\tilde{a}^2}{48}(5\log 2-2)\right]} + A^{\rm in}(y-1)^{-\frac{i\nu}{4}\left[1-\frac{\tilde{a}^2}{48}(5\log 2-2)\right]}, \tag{A2}$$

where the coefficients $A^{\rm out/in}$ correspond to outgoing and ingoing modes, respectively. We normalize these modes according to Eq. (46) and expand for low frequencies to obtain

$$g_A^{\rm out/in}(y) \sim 1 \pm \frac{i\nu}{4}\log(y-1)\left[1 - \frac{\tilde{a}^2}{48}(5\log 2 - 2)\right] + \mathcal{O}(\nu^2). \tag{A3}$$

2. Low-frequency limit

Next we attempt to solve Eq. (60) in the low-frequency limit but for arbitrary y, i.e., throughout the bulk. We can perform a series expansion in powers of ν to write the solution in the generic form,

$$g_B(y) = g_0(y) + \nu g_1(y) + \nu^2 g_2(\nu) + \cdots \tag{A4}$$

Inserting this ansatz in Eq. (60), setting the coefficient of each power of ν to zero, and solving the resulting equations, we can find g_0, g_1, g_2. At the zeroth order, the equation to solve is

$$g_0''(y) + \frac{4y^3}{y^4-1}[1 + \tilde{a}^2\Psi(y)]g_0'(y) = 0, \tag{A5}$$

with $\Psi(y)$ being given in Eq. (61). The solution to the equation for general y is quite complicated and is given by

$$\begin{aligned}
g_0(y) = {} & \frac{1}{2}C_1(\tan^{-1}y + \tanh^{-1}y) + C_2 + \frac{\tilde{a}^2}{768(y^4-1)}C_1\Bigg\{-16y + 16y^3 + 80y\log 2 + \log\left(1+\frac{1}{y^2}\right)(-80y - 51(y^4-1) \\
& \times \log(1-y) - 9(y^4-1)\log(y-1) + 60(y^4-1)\log(1+y)) - (y^4-1)\Bigg[\log(y-1)\left[-17 - 9\log\left(1+\frac{1}{y^2}\right)\right] \\
& + \log(1-y)(25 + 102\log y + 17\log(y^2+1)) - 8(1+17\log y)\log(1+y) - 8\log(y^2+1)\log(1+y) \\
& + 4\log(-i+y)\left[2i\log(1-iy) + 2\log\left(i\frac{y+1}{y-1}\right) - i\log(4(-i+y))\right] \\
& + 4\log(i+y)\left[-2i\log(1+iy) + 2\log\left(i\frac{y+1}{1-y}\right) + i\log(4(i+y))\right]\Bigg] \\
& - 8(y^4-1)\tanh^{-1}y(15\log 2 + 17\log(1+y^2)) + 8(y^4-1)\tan^{-1}y(4 - 15\log 2 + 4\log y) \\
& + 8(y^4-1)\Bigg[2\mathrm{Li}_2(1-y) - i\mathrm{Li}_2\left(\frac{1}{2}(1+iy)\right) + 2\mathrm{Li}_2(-y) - 2i\mathrm{Li}_2(-iy) + 2i\mathrm{Li}_2(iy) - \mathrm{Li}_2\left(\frac{1}{2}(-1+i)(y-i)\right) \\
& + \mathrm{Li}_2\left(\frac{1}{2}(1+i)(-i+y)\right) - \mathrm{Li}_2\left(\frac{1}{2}(-1-i)(i+y)\right) + i\mathrm{Li}_2\left(\frac{1}{2}(1-iy)\right) + \mathrm{Li}_2\left(\frac{1}{2}(1-i)(i+y)\right)\Bigg]\Bigg\} + \mathcal{O}(\tilde{a}^4),
\end{aligned} \tag{A6}$$

with C_1 and C_2 being the constants of integration and $\mathrm{Li}_n(z)$ is the polylogarithm function. Upon taking the near-horizon limit, it reduces to[7]

$$g_0(y) = C_2 + C_1\left[\left(\frac{1}{8}-\frac{i}{4}\right)\pi + \frac{\log 2}{4} - \frac{\log(y-1)}{4}\right] + \frac{\tilde{a}^2 C_1}{2304}[84 - 48\beta(2) + (24-75i-\pi)\pi - 90(1-2i)\pi\log 2 \\ -204(\log 2)^2 + 24\log 2 - 24\log(y-1) + 204\log 2\log(y-1)]. \quad \text{(A7)}$$

Upon comparison with Eq. (A3), we can extract the coefficients C_1 and C_2 as

$$C_1 = 0, \qquad C_2 = 1 \quad \text{(A8)}$$

for both outgoing and ingoing waves. Next we proceed to find $g_1(y)$. Now note that $g_1(y)$ satisfies the same equation as g_0 and so has the same solution [Eqs. (A6) and (A7))], albeit with different constants of integration, but now the matching has to be done with the coefficient of ν in Eq. (A3)). Replacing C_1 and C_2 in Eq. (A7)) with $\tilde{C}_1$ and $\tilde{C}_2$, respectively, and comparing with Eq. (A3), we can extract the constants for both the outgoing and ingoing waves as

$$\tilde{C}_1^{\text{out/in}} = \mp i\left[1 + \frac{\tilde{a}^2}{4}\log 2\right] + \mathcal{O}(\tilde{a}^4),$$
$$\tilde{C}_2^{\text{out/in}} = \pm\left(\frac{1}{4}+\frac{i}{8}\right)\pi \pm \frac{1}{4}i\log 2 \mp i\frac{\tilde{a}^2}{2304}[-84+48\beta(2)-(24-75i-\pi)\pi+(18(1-2i)\pi+60\log 2-24)\log 2] + \mathcal{O}(\tilde{a}^4). \quad \text{(A9)}$$

The constants so evaluated can now be used in the full solution for $g_B(y)$ and not just in the near-horizon limit (the restriction to low-frequency regime still holds, though), which now reads

$$g_B(y) = 1 + \nu\left[\frac{1}{2}\tilde{C}_1(\tan^{-1}y + \tanh^{-1}y) + \tilde{C}_2\right] + \frac{\nu\tilde{a}^2}{768(y^4-1)}\tilde{C}_1\Bigg\{-16y + 16y^3 + 80y\log 2 \\ + \log\left(1+\frac{1}{y^2}\right)(-80y - 51(y^4-1)\log(1-y) - 9(y^4-1)\log(y-1) + 60(y^4-1)\log(1+y)) \\ -(y^4-1)\left[\log(y-1)\left[-17 - 9\log\left(1+\frac{1}{y^2}\right)\right] + \log(1-y)(25 + 102\log y + 17\log(y^2+1))\right. \\ -8(1+17\log y)\log(1+y) - 8\log(y^2+1)\log(1+y) + 4\log(-i+y)\left[2i\log(1-iy) + 2\log\left(i\frac{y+1}{y-1}\right)\right. \\ \left.\left. - i\log(4(-i+y))\right] + 4\log(i+y)\left[-2i\log(1+iy) + 2\log\left(i\frac{y+1}{1-y}\right) + i\log(4(i+y))\right]\right] \\ -8(y^4-1)\tanh^{-1}y(15\log 2 + 17\log(1+y^2)) + 8(y^4-1)\tan^{-1}y(4 - 15\log 2 + 4\log y) \\ +8(y^4-1)\left[2\mathrm{Li}_2(1-y) - i\mathrm{Li}_2\left(\frac{1}{2}(1+iy)\right) + 2\mathrm{Li}_2(-y) - 2i\mathrm{Li}_2(-iy) + 2i\mathrm{Li}_2(iy) - \mathrm{Li}_2\left(\frac{1}{2}(-1+i)(y-i)\right)\right. \\ \left. + \mathrm{Li}_2\left(\frac{1}{2}(1+i)(-i+y)\right) - \mathrm{Li}_2\left(\frac{1}{2}(-1-i)(i+y)\right) + i\mathrm{Li}_2\left(\frac{1}{2}(1-iy)\right) + \mathrm{Li}_2\left(\frac{1}{2}(1-i)(i+y)\right)\right]\Bigg\} + \mathcal{O}(\tilde{a}^4). \quad \text{(A10)}$$

Next we can take the asymptotic limit of the full solution to arrive at

[7]While taking the near-horizon limit, we have let $y \to 1 + \epsilon$ and used the following expansion:

$$\mathrm{Li}_n(z + (a+ib)\epsilon) = \mathrm{Li}_n(z) + \epsilon\frac{a+ib}{z}\mathrm{Li}_{n-1}(z) + \mathcal{O}(\epsilon^2).$$

$$g_B^{\text{out/in}} \sim 1 \mp \frac{i\nu}{8}(\pi - 2\log 2) \pm \frac{i\nu\tilde{a}^2}{768}[28 - 16\beta(2) - 20(\log 2)^2 + \pi(-8 + \pi + 14\log 2) + 8\log 2] + \mathcal{O}(\nu^2)$$
$$\mp \frac{1}{y^3}\left[\frac{i\nu}{3}\left(1 + \frac{\tilde{a}^2}{4}\log 2\right) + \mathcal{O}(\nu^2)\right] + \mathcal{O}(1/y^4). \tag{A11}$$

3. Asymptotic limit

Finally, we are to solve Eq. (60) in the asymptotic limit, i.e., near the boundary where the gauge theory lives. We attempt a power series in the form

$$g_C(y) = k_0 + k_1/y + k_2/y^2 + k_3/y^3. \tag{A12}$$

It turns out the only the constants k_0 and k_3 are independent, and the solution assumes the form

$$g_C(y) = k_0\left[1 + \frac{\nu^2}{2y^2} + \mathcal{O}(1/y^4)\right] + k_3\left[\frac{1}{y^3} + \mathcal{O}(1/y^5)\right]. \tag{A13}$$

Matching the coefficients with Eq. (A11) in the low-frequency limit furnishes the two undetermined constants k_0 and k_3 as follows:

$$k_0^{\text{out/in}} = 1 \mp \frac{i\nu}{8}(\pi - 2\log 2) \pm \frac{i\nu\tilde{a}^2}{768}[28 - 16\beta(2) - 20(\log 2)^2 + \pi(-8 + \pi + 14\log 2) + 8\log 2] + \mathcal{O}(\nu^2)$$
$$k_3^{\text{out/in}} = \mp\left[\frac{i\nu}{3}\left(1 + \frac{\tilde{a}^2}{4}\log 2\right) + \mathcal{O}(\nu^2)\right]. \tag{A14}$$

The final result is then given in Eq. (62).

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