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Generalized dualities in one-time physics as holographic predictions from two-time physics

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In the conventional formalism of physics, with one time, systems with different Hamiltonians or Lagrangians have different physical interpretations and are considered to be independent systems unrelated to each other. However, in this paper we construct explicitly canonical maps in one-time (1T) phase space (including timelike components, specifically the Hamiltonian) to show that it is appropriate to regard various 1T physics systems, with different Lagrangians or Hamiltonians, as being duals of each other. This concept is similar in spirit to dualities discovered in more complicated examples in field theory or string theory. Our approach makes it evident that such generalized dualities are widespread. This suggests that, as a general phenomenon, there are hidden relations and hidden symmetries that conventional 1T physics does not capture, implying the existence of a more unified formulation of physics that naturally supplies the hidden information. In fact, we show that two-time (2T) physics in $(d+2)$ dimensions is the generator of these dualities in 1T physics in d dimensions by providing a holographic perspective that unifies all the dual 1T systems into one. The unifying ingredient is a gauge symmetry in phase space. Via such dualities it is then possible to gain new insights toward new physical predictions not suspected before, and suggest new methods of computation that yield results not obtained before. As an illustration, we will provide concrete examples of 1T systems in classical mechanics that are solved analytically for the first time via our dualities. These dualities in classical mechanics have counterparts in quantum mechanics and field theory, and in some simpler cases they have already been constructed in field theory. We comment on the impact of our approach on the meaning of space-time and on the development of new computational methods based on dualities.

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I. INTRODUCTION

Symmetry concepts and computational techniques that emerged from two-time (2T) physics in $4+2$ dimensions were successfully applied recently in $3+1$ dimensional cosmology, to obtain for the first time analytically the full set of homogeneous cosmological solutions of the standard model of particle physics coupled to gravity [1], and to propose a new cyclic cosmology driven only by the Higgs field with no recourse to an inflaton [2], in a geodesically complete Universe [3]. The underlying $4+2$ dimensions predicts the presence of a local conformal Weyl symmetry in $3+1$ dimensions with restrictions on how to couple the Higgs field to gravity such that the new conformally invariant standard model is geodesically complete through cosmological singularities in a cyclic universe. This Weyl symmetry carries information and imposes properties related to the extra $1+1$ space and time dimensions [4]. Unprecedented analytic control in these computations emerged from some very simple duality concepts that amounted to making Weyl gauge transformations between different fixed Weyl gauges of the same conformal standard model. Such gauge transformations, or dualities, amount to simple changes of the perspective of the $3+1$ dimensional phase space within the $4+2$ dimensional phase space, which is what we will study more generally in this paper.

Two crucial observations in M theory in 1995–1996 provided the initial hints for constructing 2T physics in 1998 based on phase space gauge symmetry [5]. These were (i) U-dualities in M theory appeared to be discrete phase space gauge transformations between various fixed gauges of a mysterious gauge symmetry in M theory [6], and (ii) there was a hint of an extra time dimension in M theory because the 11 dimensional extended supersymmetry of M theory is really a 12 dimensional SO(10,2) covariant supersymmetry written in the disguise of 11 dimensions [7]. Exploration of these notions [8] raised the question of whether the unknown M theory might be a two-time theory with a global supersymmetry $OSp(1|64)$ whose Bogomol'nyi-Prasad-Sommerfield sectors that explained the five dual corners of M theory [8] could naturally arise from the constraints of an underlying gauge symmetry. So what could the underlying gauge symmetry be? And how could a theory with two timelike dimensions be unitary?

A ghost-free unitary theory in a target space with two timelike dimensions could not be viable without the presence of a new type of more powerful gauge symmetry that could eliminate the problems of causality and ghosts from both timelike dimensions. After figuring out that such a gauge symmetry does not exist in position space, but it does exist in phase space [5], it became evident that the same phase space framework could also provide a natural connection to dualities. Starting in 1998, 2T physics was developed in phase space for particles in the worldline formalism with a target space in $d+2$ dimensions with two times, progressively including spin [9–12], background fields [11,13], supersymmetry [14], and twistors [15], [12] (for a recent overview see [16]). That M theory could be formulated naturally in $11+2$ dimensions, with an $\mathrm{OSp}(1|64)$ global supersymmetry and a gauge symmetry in the phase space of branes, was illustrated with a toy M model [17]. 2T physics was also extended to the framework of field theory [10,11], including the standard model in $4+2$ dimensions [18], gravity in $d+2$ dimensions [4], SUSY field theory with $N=1,2,4$ supersymmetries in $4+2$ dimensions [19], SUSY Yang-Mills in $10+2$ dimensions in 2010 [20], and finally supergravity [21]. It is still under construction for strings and branes [22,23] and M theory [17], and it is expected that the most powerful eventual form of 2T physics will be in the framework of field theory in phase space as initiated in [24]. By now it is evident that an underlying $4+2$ dimensional phase space, with appropriate extra gauge symmetry, fits all known physics in $3+1$ dimensions, from classical and quantum dynamics of particles, to field theory including the realistic conformal standard model coupled to gravity, all the way to supergravity. This $(4+2)$ dimensional approach has provided the useful technical tools for the recent advances in $(3+1)$ dimensional cosmology reported in [25–30] and [1–3].

The physics content in the 2T physics formalism in $d+2$ dimensions is the same as the physics content in the conventional one-time (1T) physics formalism in $(d-1)+1$ dimensions except that 2T physics provides a holographic type perspective (as described below) with a much larger set of gauge symmetries, and naturally makes predictions that are not anticipated in 1T physics. Some of the predictions take the forms of hidden symmetries and dualities; in this paper we concentrate mainly on the dualities. The dualities are similar in spirit to dualities encountered in M theory or string theory, in the broader sense of relating theories that look different in conventional 1T formalism, but in reality contain the same physics information once a map is established between them. In fact a lot of the new information from 2T physics, which is not contained systematically in 1T physics, can be expressed in the language of dualities directly in 1T physics. Developing such dualities is our primary objective in this paper.

The idea of using an embedding space X^M in $4+2$ dimensions, which is restricted to the cone, $X\cdot X=0$, in order to realize SO(4,2) conformal symmetry in $3+1$ dimensions, originated with Dirac [31]. This idea was further developed over the years [32–39]. 2T physics connects to this notion of conformal symmetry in one of its duality corners that we discuss in this paper, namely the *conformal shadow*, which is a gauge fixed version of 2T physics in $4+2$ dimensional *flat space-time*. Thus, more recent works, based on the same conformal symmetry notion in flat $4+2$ dimensions, are automatically connected to 2T physics; these include the $4+2$ dimensional formulation of high-spin theory [13,40]–[42], computation of conformal correlators in 3+1 dimensions using $4+2$ dimensions [43,44], conformal bootstrap in the embedding formalism [45], and new mathematical notions related to conformal symmetry [46,47]. We emphasize that these growing sets of connections correspond to only one corner of 2T physics. 2T physics is much more than conformal symmetry in $3+1$ dimensions both conceptually and practically. This is because 2T physics is a gauge theory in phase space (X^M, P_M), generally in $d+2$ curved space-time and, like M theory, has many 1T physics corners with different physical interpretations as illustrated with five specific shadows in this paper. When the idea of *a gauge symmetry in phase space* was introduced in [5] Dirac's idea had faded away; so 2T physics developed as a much richer theory, unaware of Dirac's reasoning or motivation for conformal symmetry. That connection was realized only after the notions of phase space gauge symmetry had taken root and had already revealed new corners of 1T physics well beyond the conformal shadow. We now know that Dirac's idea and modern applications [40–47] are automatically part of 2T physics in the special case when the $\mathrm{Sp}(2,R)$ gauge symmetry generators $Q_{ij}(X,P)$ take their simplest form shown in Eq. (28), and only when the conformal shadow (or gauge) is chosen to connect to 1T physics. This suggests that the broader phase space properties of 2T physics, such as the multishadows and dualities discussed in this paper, that continue to elude the practitioners of the $X\cdot X=0$ constraint even in modern times, can be used to obtain further physical consequences in those settings. Also, 2T physics is a general theory that goes well beyond the *flat* $4+2$ dimensional space-time constraint $X\cdot X=0$: it should be noted that the generalized $\mathrm{Sp}(2,R)$ generators $Q_{ij}(X,P)$ in curved phase space with background fields [11,13,24], [16] and interactions in field theory, including the standard model [18] and gravity [4], lead to far richer applications of 2T physics.

In this paper we will extend previous results on dualities in 1T physics predicted by 2T physics [16]. These take the form of explicit canonical transformations among relativistic and nonrelativistic 1T physics systems in d dimensions, $\tilde{x}^\mu=\mathcal{X}^\mu(x,p)$ and $\tilde{p}_\mu=\mathcal{P}_\mu(x,p)$, that were not obtained before in classical mechanics with one time. These include canonical transformations among some newly constructed solvable 1T systems, such as a relativistic particle in an arbitrary potential, and previously studied simpler systems, such as the relativistic massless particle, relativistic massive particle, nonrelativistic massive particle, H-atom, and several others. All these cases are further generalized in this paper by including arbitrary interactions of a particle with classical background fields (electromagnetic, gravitational, high-spin). It is shown that these more general systems are mapped from one dual system to another by the same duality transformations that are independent of the backgrounds. So the dual systems considered here cover a broad spectrum of interacting 1T physics models. In principle, these classical canonical transformations have counterparts in the quantum version of the same systems and can also be extended to field theory, as has already been demonstrated with simpler examples in the past [48].

One of our aims is to concentrate on the practical aspects of these canonical transformations and to use them for developing new computational methods within the traditional framework of 1T physics. Indeed, our duality methods are useful for performing computations in 1T physics that would be hard or impossible otherwise. The idea is to solve complicated systems by solving much simpler dual systems. As an illustration, we will solve exactly the classical mechanics of a relativistic particle in d dimensions, which is constrained to satisfy $p^2 + V(x^2) = 0$ for any potential $V(x^2)$, such as any power law $V(x^2) = c(x^2)^b$, that we believe has not been solved before, and cannot imagine how to solve without our dualities.

These dualities are predicted in the context of gauge symmetries in phase space that generalize the notion of general coordinate invariance in position space. The examples discussed here are only some representatives of a much larger group of dualities that belong together in a unique symmetric theory in 2T physics as reviewed in Sec. IV. Each one of these 1T systems in d dimensions captures holographically all of the gauge invariant information in the 2T theory in $d+2$ dimensions. We call such 1T systems "shadows" at d dimensional boundaries of the bulk in $d+2$ dimensions. Since each shadow contains all the physical information, the parent theory in the bulk predicts that all shadows must be holographic duals of each other.

Before we discuss specific dualities or the underlying theory, it is useful to outline some concepts that give a sense of direction for why we are interested in examining these dualities. Our canonical transformations in d dimensions, $\tilde{x}^\mu = \mathcal{X}^\mu(x,p)$ and $\tilde{p}_\mu = \mathcal{P}_\mu(x,p)$, include the time coordinate and its canonical conjugate Hamiltonian. Since time and Hamiltonian transform, it is not surprising that we will establish relations among dynamical systems that *a priori* are considered to be different 1T physics dynamical systems with different Hamiltonians. We conceptualize a given phase space (x^μ, p_μ) as the coordinates of a chosen phase space frame for an observer that rides along with a particle on a worldline whose time development $(x^\mu(\tau), p_\mu(\tau))$ is determined by a phase space constraint $Q(x,p) = 0$. An example of such a frame is the massless relativistic particle that satisfies the constraint $p^2 = 0$. This observer is set up to describe all physical phenomena in the Universe (not only the motion of this particle) from the point of view of this frame. A different phase space $(\tilde{x}^\mu, \tilde{p}_\mu)$ with a different constraint $\tilde{Q}(\tilde{x},\tilde{p}) = 0$, such as the constrained relativistic harmonic oscillator, $(\tilde{p}^2 + \omega^2\tilde{x}^2) = 0$, represents the frame of a different observer that also examines all phenomena from this other perspective. The canonical transformation, $\tilde{x}^\mu = \mathcal{X}^\mu(x,p)$ and $\tilde{p}_\mu = \mathcal{P}_\mu(x,p)$, that maps the 1T dynamics $Q(x,p) = 0$ to the 1T dynamics $\tilde{Q}(\tilde{x},\tilde{p}) = 0$ establishes the relations between the frames and therefore all observations made by the two different observers are also related to each other. The reader is invited to think of this setup as the analog of Einstein's observers in different frames that are related to each other by canonical transformations in phase space which generalize Einstein's special or general coordinate transformations. The key in our theory is that the worldlines $(x^\mu(\tau), p_\mu(\tau))$ and $(\tilde{x}^\mu(\tau), \tilde{p}_\mu(\tau))$, that define the frames of the two observers, are actually two shadows of the *same worldline* in the bulk in $d+2$ dimensions $(X^M(\tau), P_M(\tau))$. The two observers see very different 1T physics phenomena from the perspective of their own frames; however in our setup there is already a predicted relationship between the observers since their 1T physics equations are really two gauge choices of the same gauge invariant equations in $d+2$ dimensions. There is a unique set of equations in $d+2$ dimensions supplied by 2T physics that unify the vastly different 1T equations of all such observers in d dimensions. This unification is not at all apparent in the conventional setup of 1T physics. The unification makes predictions of real physical phenomena in 1T physics that can be tested by studying the dualities that capture the hidden correlations of the various 1T observers. Our purpose in this paper is to establish a few examples of such dualities, which are surprising in 1T physics, and in this way show that there is much more physics to be learned from 2T physics predictions that are not supplied systematically in conventional 1T physics.

In this paper we will first present our results for a few specific dualities as canonical transformations, $\tilde{x}^\mu = \mathcal{X}^\mu(x,p)$ and $\tilde{p}_\mu = \mathcal{P}_\mu(x,p)$, purely in the context of conventional 1T physics. Afterwards we will show how they were obtained in the first place as the natural predictions of 2T physics, and also indicate how a vast extension of such dualities can be further obtained from the 2T approach.

The canonical transformations $\tilde{x}^\mu = \mathcal{X}^\mu(x,p)$ and $\tilde{p}_\mu = \mathcal{P}_\mu(x,p)$ discussed in this paper take a special mathematical form. It is shown that they involve a 2×2 matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

of determinant 1, that belongs to the group $\mathrm{Sp}(2,R) = \mathrm{SL}(2,R)$, with entries $(\alpha,\beta,\gamma,\delta)$ that are nonlinear functions of phase space (x^μ, p_μ) including timelike directions. For example, when the origin and target systems are both Lorentz covariant systems, the transformation takes the form,

$$\tilde{x}^\mu = x^\mu \alpha(x,p) + p^\mu \beta(x,p) \equiv \mathcal{X}^\mu(x,p) \quad (1)$$

$$\tilde{p}^\mu = x^\mu \gamma(x,p) + p^\mu \delta(x,p) \equiv \mathcal{P}_\mu(x,p), \quad (2)$$

where α, β, γ, δ are functions of phase space. This means that under the dualities, (x^μ, p^μ) form $\mathrm{Sp}(2,R)$ doublets covariantly in every direction μ of space-time. When one or both systems, (x^μ, p_μ) or $(\tilde{x}^\mu, \tilde{p}_\mu)$, are nonrelativistic, the α, β, γ, δ are not as simple and not Lorentz invariant, but they still belong to the phase space–local $\mathrm{Sp}(2,R)$. It should be emphasized that the set of dualities discussed in this paper [as linear $\mathrm{Sp}(2,R)$ transformations] is just a special case. Our formalism is covariant under the most general nonlinear $\mathrm{Sp}(2,R)$ as the underlying gauge symmetry in phase space. Either the linear or nonlinear $\mathrm{Sp}(2,R)$ transformations are broader than the familiar local gauge transformations or general coordinate transformations since the gauge parameters $(\alpha,\beta,\gamma,\delta)$ are *local in phase space*, not just in position space.[1]

This paper is organized as follows. In Sec. II we review and clarify the gauge symmetries and constraints of the 1T system consisting of a spinless particle in interaction with an arbitrary set of background fields in d dimensions. In Sec. III, we use the notation developed in Sec. II to present our canonical transformations between five different 1T physics systems. These are just examples to illustrate our ideas which apply to a much larger class of 1T physics systems connected to each other by canonical transformations. In Sec. IV we review the idea of general gauge symmetry in phase space, apply it to 2T physics based on the $\mathrm{Sp}(2,R)$ gauge symmetry, and then present five different gauge choices in Sec. IV A in which the gauge fixed forms yield the five different 1T physics systems that appear in Sec. III. In Sec. V we show how to map the five fixed gauge choices to one another by $\mathrm{Sp}(2,R)$ gauge transformations from one fixed gauge to another fixed gauge, thus obtaining the 1T physics canonical transformations described in Sec. III. In Sec. VI we identify the invariant observables under duality transformations and discuss special circumstances when there is a hidden global $\mathrm{SO}(d,2)$ symmetry associated with these invariants. This $\mathrm{SO}(d,2)$ is related to conformal symmetry in one special shadow which we call the conformal shadow, but it is the equivalent of conformal symmetry in all other shadows, including shadows for massive particles. In Sec. VII we illustrate how to use dualities to explicitly solve the dynamics of a relativistic spinless particle with a constraint $p^2 + V(x^2) = 0$ in an arbitrary potential, a problem that could not be solved before. Finally in Sec. VIII we interpret these results from the point of view of $(d+2)$ dimensions, comment on generalizing the concepts of dualities, and discuss what this means for physics and space-time in d dimensions.

II. GAUGE SYMMETRY IN 1T PHYSICS REVISITED

In this section we present all 1T physics systems for a spinless particle in a unified form that will be useful for discussing the dualities and canonical transformations among 1T physics systems that will be the subject of this paper. In the following sections we will use this unified framework in 1T physics to discuss canonical transformations that include spacelike as well as timelike directions (including a change of Hamiltonian) to map various 1T dynamical systems to each other.

To ensure that our ideas are well understood we will begin with a simple familiar example. The worldline action of a freely moving relativistic particle of zero spin and mass m is the familiar expression $S(x) = -m \int_1^2 d\tau \sqrt{-\dot{x}^2}$. Here $\dot{x}^\mu \equiv \partial_\tau x^\mu$ is the velocity of a particle, whose position $x^\mu(\tau)$ as a function of the worldline parameter τ is a covariant vector in $(d-1)$ space and 1 time dimensions. The Euler-Lagrange equations derived from the action are $\partial_\tau p^\mu(\tau) = 0$, where $p^\mu = m\dot{x}^\mu/\sqrt{-\dot{x}^2}$ is the canonical momentum derived from the action. The particle moves freely since the momentum is a constant of motion—indeed this is guaranteed by the fact that this Lagrangian is translationally invariant.

As is well known, this action has a local symmetry under τ-reparametrizations; namely $x^\mu(\tau) \to x^\mu(\tau) + \delta_\varepsilon x^\mu$ with $\delta_\varepsilon x^\mu(\tau) = \varepsilon(\tau)\dot{x}^\mu(\tau)$ is a symmetry of this action $S(x + \delta_\varepsilon x) = S(x)$, as long as the end points are not transformed, $\varepsilon(\tau_1) = \varepsilon(\tau_2) = 0$. This is a transformation that mixes position and momentum locally on the worldline, since we could write $\delta_\varepsilon x^\mu(\tau) = \Lambda(\tau) p^\mu(\tau)$, with another local parameter $\Lambda(\tau) \equiv \varepsilon(\tau)\sqrt{-\dot{x}^2(\tau)}/m$.

This phase space gauge symmetry is crucial to remove the ghost degrees of freedom in the timelike direction of $x^\mu(\tau)$. As usual, any gauge symmetry leads to constraints among the degrees of freedom. A constraint is an equation satisfied by phase space degrees of freedom (x^μ, p_μ) such that no time derivatives occur, and hence it is valid for all times τ. In this case the constraint takes the form $p^2 + m^2 = 0$, which is evidently satisfied by $p^\mu = m\dot{x}^\mu/\sqrt{-\dot{x}^2}$. The physical meaning of the constraint is that this is a massive relativistic particle at all times.

The same physical content is encoded in another form of the action in the first order formalism which treats the phase

[1]An infinitesimal gauge parameter as a function of phase $\varepsilon(x,p)$ packs together the parameters for local gauge transformations $\varepsilon_0(x)$, general coordinate transformations $\varepsilon_1^\mu(x)$, and much more, as seen in an expansion in powers of momentum just as in Eq. (9), $\varepsilon(x,p) = \varepsilon_0(x) + \varepsilon_1^\mu(x)(p_\mu + A_\mu(x)) + \cdots$. As an example, see the familiar transformation on gauge fields, gravitational metric and high-spin fields, organized as phase space transformations, in Eqs. (33)–(37) in [13].

space degrees of freedom $(x^\mu(\tau), p_\mu(\tau))$ as two independent vectors, whose equations of motion are derived by extremizing with respect to all degrees of freedom (x, p, e) in the following Lagrangian:

$$L(x,p,e) = \left[\dot{x}^\mu(\tau)p_\mu(\tau) - \frac{1}{2}e(\tau)(p^2(\tau)+m^2)\right]. \tag{3}$$

Here a new degree of freedom $e(\tau)$ has been added. If first the p_μ, and then the e, degrees of freedom are integrated out in that order, then this action reduces to $S(x) = -m\int_1^2 d\tau\sqrt{-\dot{x}^2}$, and hence the two versions have the same content. However, the first order formalism reveals more clearly the nature of the gauge symmetry, and leads to a full generalization to cover all possible physical systems for a single spinless particle, massive or massless and in interaction with all possible background fields, as seen below.

The phase space gauge symmetry of this first order action is given by

$$\delta_\Lambda x^\mu = \Lambda(\tau)p^\mu(\tau), \quad \delta_\Lambda p_\mu = 0, \quad \delta_\Lambda e = \partial_\tau\Lambda(\tau). \tag{4}$$

Then the action is invariant, $\delta_\Lambda S(x,p,e) = 0$, because the Lagrangian transforms to a total derivative $\delta_\Lambda L(x,p,e) = \partial_\tau(\frac{1}{2}\Lambda(\tau)(p^2(\tau) - m^2))$, while $\Lambda(\tau_1) = \Lambda(\tau_2) = 0$.

This is an example of a more general worldline gauge symmetry formalism that applies to all physical systems as discussed presently. Consider the action $S = \int_1^2 d\tau L$ with the Lagrangian

$$L(x,p,e) = \dot{x}^\mu(\tau)p_\mu(\tau) - e(\tau)Q(x(\tau),p(\tau)). \tag{5}$$

This general $Q(x,p)$ is to be regarded as a generator of *local* canonical transformations for any observable $A(x,p)$ by applying the Poisson bracket, $\delta_\Lambda A = \Lambda(\tau)\{A,Q\}$, where $\Lambda(\tau)$ is the local parameter on the worldline.[2] Furthermore, $e(\tau)$ is to be regarded as a Maxwell-Yang-Mills type Abelian gauge field in 1 dimension (analog of the time component of the gauge field A_0 in Maxwell-Yang-Mills). Note that the gauge field e is coupled to the generator of gauge transformation Q as would be the case in familiar gauge theories. With this point of view, now define a gauge transformation on the phase space degrees of freedom (x^μ, p_μ) by using Poisson brackets to compute $\delta_\Lambda x^\mu$, $\delta_\Lambda p_\mu$, with $Q(x,p)$ as the generator, as follows:

$$\delta_\Lambda x^\mu = \Lambda(\tau)\frac{\partial Q}{\partial p_\mu}, \qquad \delta_\Lambda p_\mu = -\Lambda(\tau)\frac{\partial Q}{\partial x^\mu},$$
$$\delta_\Lambda e = \partial_\tau\Lambda(\tau). \tag{6}$$

Note that $e(\tau)$ does indeed transform like an Abelian gauge field independent of the "matter" content, while the specific choice of $Q(x,p)$ determines the dynamics of the matter degrees of freedom $(x^\mu(\tau), p_\mu(\tau))$ through the equations of motion. It can be checked that the action is invariant because the Lagrangian transforms to a total τ-derivative

$$\delta_\Lambda L = \frac{d}{d\tau}[\Lambda(\tau)(p\cdot\partial_p - 1)Q(x(\tau),p(\tau))]. \tag{7}$$

The equation of motion for the gauge field $\partial L/\partial e(\tau) = 0$ imposes the constraint

$$Q(x,p) = 0. \tag{8}$$

This is the analog of Gauss's law that follows from $\partial L/\partial A_0 = 0$ in Maxwell-Yang-Mills theory. Since $Q(x,p)$ is the generator of gauge transformations, $Q = 0$ identifies the sector of the theory that has zero gauge charge, that is, the gauge invariant sector. So, the meaning of this constraint is that only the gauge invariant subspace of phase space, as identified by the solutions of $Q = 0$, is physical.

In this light, in the simple example where $Q = p^2 + m^2$, the mass-shell constraint $p^2 + m^2 = 0$ implies not only that this is a massive particle for all times, but also that the solutions of the constraint identify the gauge invariant sub–phase space for all times.

The first quantization of the general gauge theory with any $Q(x,p)$ can be performed by using covariant quantization, in which (x^μ, p_μ) are quantized as if they are unconstrained variables. The Hilbert space of this quantum phase space cannot be all physical because it does not take into account the constraint $Q = 0$. However, in this larger Hilbert space, the physical subspace is found by imposing the constraint on the quantum states $\hat{Q}|\Phi\rangle = 0$, where the quantum operator $\hat{Q}$ is defined by an appropriate ordering of the quantum operators $(\hat{x}^\mu, \hat{p}_\mu)$ that appear in $\hat{Q}(\hat{x},\hat{p})$. In particular, in position space $\Phi(x^\mu) \equiv \langle x^\mu|\Phi\rangle$, where the momentum is represented as a derivative on the complete basis for quantum states $\langle x^\mu|$, the constraint takes the form of a differential equation to be satisfied by the physical subset of quantum states $\hat{Q}(x, -i\hbar\partial)\Phi(x^\mu) = 0$. For the example when $\hat{Q} = \hat{p}^2 + m^2$, this becomes the Klein-Gordon equation $(-\hbar^2\partial_x^2 + m^2)\Phi(x^\mu) = 0$. For more complicated cases, the proper definition of the physical sector in the quantum theory is complete only after a quantum ordering of phase space operators is specified for $\hat{Q}(\hat{x},\hat{p})$.

General examples of physical interest that include electromagnetic, gravitational and high-spin relativistic background fields are

[2]It is possible to generalize this first order Lagrangian by including also a Hamiltonian U, $L = \dot{x}\cdot p - eQ(x,p) - U(x,p)$, as long as the Hamiltonian is gauge invariant, meaning a vanishing Poisson bracket $\{Q,U\} = 0$. The inclusion of U does not change our discussion and also does not really provide more physical (gauge invariant) models than those obtainable from all possible expressions for $Q(x,p)$. For this reason we do not find it useful to discuss U any further in this paper.

$$Q(x,p)=\left[\begin{array}{c}\phi(x)+\frac{1}{2}g^{\mu\nu}(x)(p_\mu+A_\mu(x))(p_\nu+A_\nu(x))\\+\sum_{n\geq3}\phi^{\mu_1\cdots\mu_n}(x)(p_{\mu_1}+A_{\mu_1}(x))\cdots(p_{\mu_n}+A_{\mu_n}(x))\end{array}\right].\tag{9}$$

Here we have assumed that $Q(x,p)$ has a Taylor expansion in powers of p_μ, which is a common assumption for many physical systems. If this assumption is not valid for some reason, then we can just as well treat $Q(x,p)$ without an expansion. In any case, when the expansion is valid, $\phi(x)$, $A_\mu(x)$, $h^{\mu\nu}(x)$, $\phi^{\mu_1\cdots\mu_n}(x)$ are the background fields [where $g^{\mu\nu}(x)=\eta^{\mu\nu}+h^{\mu\nu}(x)$, with $\eta^{\mu\nu}$ the flat metric]. Taken as the generator of gauge transformations, the vanishing of this generalized $Q(x,p)$ defines the gauge invariant sector at the classical level. The quantum version (defined after an ordering of quantum operators $\hat{x}$, $\hat{p}$) is a differential operator acting on the gauge invariant physical space, $Q(x,-i\hbar\partial)\Phi(x)=0$, as indicated above. A good first rule for correct quantum ordering is to replace the operators $\hat{p}_\mu$ by generally covariant derivatives, $\hat{p}_\mu\to-i\hbar\nabla_\mu$, which commute with the background metric $g_{\mu\nu}(x)$. Clearly, beyond this, quantum ordering is hard to settle uniquely in the general case without additional guidance from symmetries of the system $Q(x,p)$, or a more complete theory such as field theory. In this paper we do not tackle the quantum issues any further since we will only discuss the purely classical limit here, but instructive examples are treated in [49–50].

It must be noted that not only relativistic mechanics, but also all nonrelativistic mechanics may be presented in this formalism by taking any $Q(t,h,\mathbf{r},\mathbf{p})$, where space and time are considered on the same footing, just as in relativity. Consider the usual nonrelativistic $(d-1)$ dimensional phase space vectors $\mathbf{r}(\tau)$ and $\mathbf{p}(\tau)$, plus the time degree of freedom as a dynamical variable $t(\tau)$ as well as its conjugate variable $h(\tau)$, with their Poisson brackets $\{r^i,p^j\}=\delta^{ij}$ and $\{t,h\}=-1$. Then take the Lagrangian comparable to Eq. (5)

$$L=\dot{\mathbf{r}}(\tau)\cdot\mathbf{p}(\tau)-\dot{t}(\tau)h(\tau)-e(\tau)Q(t(\tau),h(\tau),\mathbf{r}(\tau),\mathbf{p}(\tau)).\tag{10}$$

As already argued above, for any choice of $Q(t,h,\mathbf{r},\mathbf{p})$ there is a gauge symmetry. Now consider the special case of $Q(t,h,\mathbf{r},\mathbf{p})$ given by

$$Q(t,h,\mathbf{r},\mathbf{p})=(H(\mathbf{r},\mathbf{p})-h)\tag{11}$$

which is independent of t and where $H(\mathbf{r},\mathbf{p})$ is any function of the phase space in $(d-1)$ dimensions. To make contact with usual nonrelativistic physics we may choose the gauge $t(\tau)=\tau$ and then solve the constraint in Eq. (11) for the canonical conjugate to t in the form $h=H(\mathbf{r},\mathbf{p})$. Inserting this back in the action, and using $\dot{t}=1$ and $h=H(\mathbf{r},\mathbf{p})$, results in the familiar nonrelativistic formulation of the system for any Hamiltonian $H(\mathbf{r},\mathbf{p})$

$$L=\dot{\mathbf{r}}(\tau)\cdot\mathbf{p}(\tau)-H(\mathbf{r},\mathbf{p}).\tag{12}$$

This shows that, like relativistic systems, nonrelativistic systems, including more complicated versions of $Q(t,h,\mathbf{r},\mathbf{p})$, may also be regarded as gauge symmetric theories, with a dynamical timelike dimension $t(\tau)$ and an appropriate constraint that can be used to determine $h(\tau)$, as described in the unified 1T physics formalism of Eq. (5).

To discuss examples for the nonrelativistic (cases 3, 4 below) and relativistic (cases 1, 2, 5 below) systems in a unified form, we make up the notation, $t=x^0$, $h=p^0=-p_0$, even though Lorentz covariance/invariance is not implied in the rest of the expressions, such as $Q(x,p)$, for the nonrelativistic cases.

III. THE CANONICAL TRANSFORMATIONS

In the remainder of this paper, we will use the approach of the previous section to discuss canonical transformations among a few 1T systems that are the illustrative examples of interest in this paper. These will include the following cases:

Case	Name	$Q(x,p)$ background fields are represented by "..."
1	massless relativistic	$p_1^2+\cdots$
2	massive relativistic	$p_2^2+m_2^2+\cdots$
3	massive non-relativistic	$\mathbf{p}_3^2-2m_3h_3+\cdots$
4	H-atom	$\mathbf{p}_4^2-2m_4\frac{\alpha}{\lvert\mathbf{r}_4\rvert}-2m_4h_4+\cdots$
5	relativistic potential	$p_5^2+V(x_5^2)+\cdots$

(13)

We have labeled the phase space for each case $(x_i^\mu, p_{i\mu})$ with the corresponding case number $i = 1, 2, 3, 4, 5$. Bold characters such as $\mathbf{r}$, $\mathbf{p}$ in cases 3 and 4 imply vectors in $(d-1)$ space dimensions, and in those cases h is the canonical conjugate to t; otherwise x, p imply relativistic vectors as in cases 1, 2, 5, and in those cases p_0 is the canonical conjugate to the timelike coordinate x^0. The choice of $Q(x,p)$, including background fields as in Eq. (9), is what defines the 1T physics dynamics in each case. In the table we indicated the form of $Q(x,p)$ in the limit when all background fields vanish. It is understood that backgrounds represented by $\cdots$ are to be included as follows.

The canonical transformations discussed below are *independent of any set of background fields*. They apply equally well when background fields vanish or when they are included according to the following prescription: first generalize only one of the systems in the table above (say case 1) with any set of background fields as in Eq. (9), and then apply the background-independent canonical transformations below to generate the background fields in all the other dual systems. This is the 1T prescription that emerges from the unified gauge invariant 2T theory including all backgrounds in $d+2$ dimensions, as discussed in Sec. IV.

We found that for each pair i, j the corresponding systems are related by nonlinear canonical transformations $(j\leftarrow i)$ of the form

$$x_j^\mu = \mathcal{X}_j^\mu(x_i, p_i), \qquad p_{j\mu} = \mathcal{P}_{j\mu}(x_i, p_i), \tag{14}$$

that satisfy the Poisson brackets $\{\mathcal{X}_j^\mu(x_i,p_i), \mathcal{X}_j^\nu(x_i,p_i)\} = 0 = \{\mathcal{P}_{j\mu}(x_i,p_i), \mathcal{P}_{j\nu}(x_i,p_i)\}$ and $\{\mathcal{X}_j^\mu(x_i,p_i), \mathcal{P}_{j\nu}(x_i,p_i)\} = \delta_\nu^\mu$, where the brackets are evaluated in the phase space (x_i, p_i) by taking derivatives $\{A,B\} = (\partial_{x_i^\mu}A)(\partial_{p_{i\mu}}B) - (\partial_{x_i^\mu}B)(\partial_{p_{i\mu}}A)$. To illustrate, in this section we exhibit one example, namely the cases $(1\leftarrow 2)$ and $(2\leftarrow 1)$, as the following 2×2 matrix form that gives explicitly the functions $\mathcal{X}_j^\mu(x_i,p_i)$, $\mathcal{P}_j^\mu(x_i,p_i)$ as well as the inverse map

massless relativistic (1) $\leftrightarrow$ massive relativistic (2)

$$\begin{pmatrix} x_1^\mu \\ p_1^\mu \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{2} + \frac{|x_2\cdot p_2|}{2\sqrt{(x_2\cdot p_2)^2 + m_2^2 x_2^2}}\right)^{-1} & 0 \\ \frac{m_2^2\ \mathrm{sign}(x_2\cdot p_2)}{2\sqrt{(x_2\cdot p_2)^2+m_2^2x_2^2}} & \frac{1}{2} + \frac{|x_2\cdot p_2|}{2\sqrt{(x_2\cdot p_2)^2+m_2^2x_2^2}} \end{pmatrix} \begin{pmatrix} x_2^\mu \\ p_2^\mu \end{pmatrix}$$
$$\begin{pmatrix} x_2^\mu \\ p_2^\mu \end{pmatrix} = \begin{pmatrix} \left(1 + \frac{m_2^2 x_1^2}{4(x_1\cdot p_1)^2}\right)^{-1} & 0 \\ -\frac{m_2^2}{2(x_1\cdot p_1)} & \left(1 + \frac{m_2^2 x_1^2}{4(x_1\cdot p_1)^2}\right) \end{pmatrix} \begin{pmatrix} x_1^\mu \\ p_1^\mu \end{pmatrix} \tag{15}$$

For all cases, the explicit $(\mathcal{X}_j^\mu(x_i,p_i), \mathcal{P}_{j\mu}(x_i,p_i))$ are given at the equation numbers specified in the following table.

target\origin	1	2	3	4	5
1		Eq.(69)	Eq.(72)	Eq.(78)	Eq.(82)
2	Eq.(70)		Eq.(75)	$(2\leftarrow 1\leftarrow 4)$	$(2\leftarrow 1\leftarrow 5)$
3	Eq.(73)	Eq.(74)		$(3\leftarrow 1\leftarrow 4)$	$(3\leftarrow 1\leftarrow 5)$
4	Eq.(80)	$(4\leftarrow 1\leftarrow 2)$	$(4\leftarrow 1\leftarrow 3)$		$(4\leftarrow 1\leftarrow 5)$
5	Eq.(85)	$(5\leftarrow 1\leftarrow 2)$	$(5\leftarrow 1\leftarrow 3)$	$(5\leftarrow 1\leftarrow 4)$	

(16)

As an example, the contents of Eq. (15) are indicated at the (12) and (21) entries of this table. The expressions for $\mathcal{X}_j^\mu(x_i,p_i)$, $\mathcal{P}_{j\mu}(x_i,p_i)$ are used directly in 1T physics as canonical transformations, but these results were obtained as predictions from 2T physics. The notation $(j\leftarrow 1\leftarrow i)$ means the composition of two transformations $(1\leftarrow i)$ followed by $(j\leftarrow 1)$, which gives the transformation $(j\leftarrow i)$. We used this notation for cases $(j\leftarrow i)$ in which the direct transformation $(\mathcal{X}_j^\mu(x_i,p_i), \mathcal{P}_{j\mu}(x_i,p_i))$ looks algebraically too involved to be transparent to the reader, and hence we opted for the more transparent notation $(j\leftarrow 1\leftarrow i)$ even though the direct transformation

$(j\leftarrow i)$ is certainly available explicitly. The derivation of these transformations using 2T physics techniques is given in Sec. V B.

In this section we describe some of the general properties of these dualities for all the cases. By definition of momentum as $p_\mu = \partial L/\partial \dot{x}^\mu$, which is in agreement with the only term that contains velocity in the first order Lagrangian (5), $L = \dot{x}\cdot p + \cdots$, the Poisson brackets must be $\{x_i^\mu, p_{i\nu}\} = \delta_\nu^\mu$ for each case i. The claim that we found a canonical map $(i\leftrightarrow j)$ between cases j and i implies that our maps satisfy the following defining property that the first term in the Lagrangian maintains the same form up to a total time derivative

$$\begin{aligned}\dot{x}_j\cdot p_j &= \frac{d}{d\tau}\mathcal{X}_j(x_i,p_i)\cdot\mathcal{P}_j(x_i,p_i)\\ &= \dot{x}_i\cdot p_i + \frac{d}{d\tau}\Lambda_{ji}(x_i,p_i). \qquad (17)\end{aligned}$$

The total derivative may be dropped because it does not contribute to the action or to the equations of motion. This is verified for each duality $(i\leftarrow j)$ and the $\Lambda_{ji}(x_i,p_i)$ is computed in Sec. V B. Consequently our canonical maps have to satisfy the Poisson bracket property (no sum on i or j)

$$\begin{aligned}\{x_j^\mu, p_{j\nu}\} &= \frac{\partial\mathcal{X}_j^\mu(x_i,p_i)}{\partial x_i^\lambda}\frac{\partial\mathcal{P}_{j\nu}(x_i,p_i)}{\partial p_{i\lambda}}\\ &\quad - \frac{\partial\mathcal{P}_{j\nu}(x_i,p_i)}{\partial x_i^\lambda}\frac{\partial\mathcal{X}_j^\mu(x_i,p_i)}{\partial p_{i\lambda}}\\ &= \delta_\nu^\mu. \qquad (18)\end{aligned}$$

We have checked that this is indeed true, but have not included the tedious algebra in this paper. This also guarantees that the Poisson brackets for any observables $\{A,B\}$ give the same result if evaluated in terms of any of the phase spaces listed in table (13).

Our duality maps satisfy the Poisson bracket (18) or the canonical property (17) *off shell*, meaning that they hold for the bigger phase space (including physical and unphysical sectors of phase space) before any equation of motion is used or any constraint $Q(x,p)$ is imposed. That is, they are properties of just the duality transformations among the phase spaces and they are satisfied independently of any specific dynamics or physical model. This means that any set of background fields may be introduced as outlined above without changing the duality transformations. The duality transformations may be thought of as transformations between observers which are set up to describe physics in their own phase space frames, with their own definition of 1T phase space. The canonical transformations connect the frames of such observers to one another in a way that is analogous to general coordinate transformations connecting observers in different frames. In the present case we are considering transformations that connect observers that are local in phase space rather than only in coordinate subspace.

In addition to the model-independent properties (17)–(18), these canonical transformations have the following remarkable property. The five quantities $Q(x,p)$ listed in (13) transform into each other under the dualities. So, up to overall factors these expressions are proportional to each other:

$$\begin{aligned}&p_1^2 \sim (p_2^2+m_2^2) \sim (\mathbf{p}_3^2 - 2m_3h_3)\\ &\quad\sim \left(\mathbf{p}_4^2 - 2m_4\frac{\alpha}{|\mathbf{r}_4|} - 2m_4h_4\right) \sim (p_5^2 + V(x_5^2)). \qquad (19)\end{aligned}$$

The proportionality factors are given precisely by multiplying each constraint $Q(x,p)$ by $(X^{+\prime})^2$ in the same $\mathrm{Sp}(2,R)$ gauge, such as

$$(X_1^{+\prime})^2(p_1^2+\cdots) = (X_2^{+\prime})^2(p_2^2+m_2^2+\cdots),\text{etc.} \qquad (20)$$

where the gauge fixed $X_i^{+\prime}(x_i,p_i)$, $i=1,\cdots,5$, are given for each gauge in Sec. IV A. Thus, when a constraint holds in one of the frames, e.g. $Q_1(x_1,p_1)=0$, it holds automatically also in all the dual frames, including the backgrounds. Although we are considering only five explicit cases in this paper, there are an infinite number of such cases (including their generalizations with background fields represented by the ellipsis $\cdots$). That is, there are an infinite number of observers defined by their own frames in phase space, which are related to each other by canonical transformations, as we will illustrate in Sec. V B. The 1T physics dynamics in each frame is captured by the expression of $Q(x,p)$ as discussed in the previous section. The relations among these $Q(x,p)$ as in (19) allows us to give physical meaning to observations (in the sense of 1T physics) and to the dualities among them.

For example, for simplicity we consider the free massless relativistic particle, with the constraint $Q = p_1^2 = 0$ (no background fields); then via our dualities all the expressions in Eq. (19) must vanish. This means that, while observer 1 interprets this system as the free massless relativistic particle $p_1^2=0$, observer 2 interprets it as the free massive relativistic particle $p_2^2+m_2^2=0$, observer 3 sees it as the free massive nonrelativistic particle with Hamiltonian $h_3 = \frac{\mathbf{p}_3^2}{2m_3}$, observer 4 thinks it is a planetary type or H-atom type interacting system with Hamiltonian $h_4 = \frac{\mathbf{p}_4^2}{2m_4} - \frac{\alpha}{|\mathbf{r}_4|}$, and observer 5 believes it is the relativistic particle in an arbitrary Lorentz invariant potential that satisfies the constraint $p_5^2 + V(x_5^2) = 0$.

In 1T physics, the dynamics of these systems are considered to be independent with no particular relations among them. However, we will show in Sec. VI that there are duality invariant quantities that do not transform, and are exactly equal to each other in all these systems. Hence there are an infinite number of relations among them which are instant predictions that can be verified by experiment or computation. The duality invariants contain all the physical information about the whole collection of these systems. For example, the initial conditions for solving the equations of motion in any one of these systems can be expressed in terms of the duality invariants. If the equations of motion are solved in one system (which is easy for the free cases 1, 2, 3) then they are automatically solved in the difficult systems, such as case 4 and especially 5, by using the duality transformations as well as the duality invariants to relate the initial conditions. Such hidden information is not available in 1T physics, but it is a property of nature which can be verified by physicists in frames related to each other by our transformations. The frame of such observers can in principle be created with proper conditions in a laboratory and the predictions can be verified experimentally.

It is now clear that, based on the model-independent properties of our transformations, we can construct large classes of physical models that are dual to each other by including background fields as in Eq. (9). For each case $i = 1, 2, \cdots$ one may introduce background fields. If the backgrounds are related to each other by the background-independent canonical transformations in Eq. (16) then the models with such backgrounds continue to be duals of each other. For example, if the relativistic massless particle in case 1 is taken with a background electromagnetic field as described by $Q_1(x_1, p_1) = \frac{1}{2}(p_1 + A(x_1))^2$, what are the sets of background fields in the other dual cases? This is computed by applying the canonical transformation $(i \leftarrow 1)$ to obtain the form for $Q_i(x_i, p_i)$ and then expand it in powers of p_i as in Eq. (9) to read off the dual versions of the backgrounds. This is sufficient to see that the type of duality we have been discussing is the norm rather than the exception. There is a huge number of testable physical predictions that can be made in this way by first compiling a list of canonical transformations without background fields, as in the illustrative examples of Eq. (13). This list is in principle infinitely long. The transformations among the members of the list can all be derived from the gauge invariant form of the theory in the framework of 2T physics as will be discussed in Sec. V B. Hence all the corresponding physical predictions are natural consequences of 2T physics.

IV. Sp(2,R) GAUGE SYMMETRY IN 2T PHYSICS

The notion of gauge symmetry in phase space, based on gauging $\mathrm{Sp}(2,R)$ that led to 2T physics, appears at first sight to be generalizable. This generalization is reviewed in [16] where it is shown that the formulation of phase space gauge symmetry for any Lie group would start by constructing a set of Lie algebra generators $Q_a(X,P)$, $a = 1, 2, \cdots, N$, that close under Poisson brackets in phase space. The closure of the Lie algebra is required for the consistency of first class constraints $Q_a(X,P) = 0$ for physical states which follows from gauge invariance.

The case of a single *noncompact* generator $Q(X,P)$ leads to the formulation of all 1T physics as shown in Sec. II. The case of the simplest non-Abelian *noncompact* group $\mathrm{Sp}(2,R) = \mathrm{SL}(2,R)$ with three generators leads uniquely to all 2T physics without any ghosts and is consistent with causality. One may be tempted to speculate that larger noncompact Lie groups may lead to reasonable unitary formulations of physics with more timelike dimensions as formulated in [16]. However, with the phase space degrees of freedom of *a single particle* such attempts have repeatedly failed because we could not find expressions for $Q_a(X,P)$ that yielded nontrivial and ghost-free solutions of the constraints $Q_a(X,P) = 0$, except for the cases of one or three generators. The failure of the attempts may suggest the possibility of a theorem that generalizations with larger noncompact groups [16] must always fail for a *single* spinless particle. A brief review of the $\mathrm{Sp}(2,R)$ case follows.

In the first order formalism, in which position $X^M(\tau)$ and momentum $P_M(\tau)$ are treated on an equal footing, we require our theory to have an $\mathrm{Sp}(2,R)$ gauge symmetry, which is a subset of canonical transformations that mix X and P locally on the worldline. This gauged subset of canonical transformations is generated by three generators written in the form of a symmetric 2×2 tensor Q_{ij}. The indices i, j correspond to doublet indices under $\mathrm{Sp}(2,R)$, i, $j = 1, 2$, while the symmetric tensor Q_{ij} is the triplet that corresponds to the adjoint representation. These generators are constructed from the phase space degrees of freedom $Q_{ij}(X,P)$. The $\mathrm{Sp}(2,R)$ Lie algebra is

$$\{Q_{12}, Q_{11}\} = -2Q_{11}, \qquad \{Q_{12}, Q_{22}\} = 2Q_{22}, \\ \{Q_{11}, Q_{22}\} = 4Q_{12}, \tag{21}$$

where the Poisson brackets $\{Q_{ij}, Q_{kl}\}$ are computed in terms of the (X^M, P_M) phase space. So, to proceed one must find expressions for the $Q_{ij}(X,P)$ that satisfy this Lie algebra. There are an infinite number of such phase space structures for $\mathrm{Sp}(2,R)$, which have been classified in [13]. Assuming some such expression for $Q_{ij}(X,P)$, we proceed as follows.

This $\mathrm{Sp}(2,R)$, which algebraically is the same as $\mathrm{SO}(1,2)$, is equivalent to a *local conformal symmetry* $\mathrm{SO}(1,2)$ on the worldline [i.e. $\mathrm{SO}(d,2)$ with $d = 1$] as seen in a second order formalism where P_M is integrated out [5]. As a guide to readers familiar with string theory, it may be useful to mention that this gauge symmetry may be regarded as being analogous to the local conformal symmetry on the world sheet generated by the Virasoro algebra in string theory. Recall that, like here, the Virasoro algebra is also constructed from the phase space degrees of freedom of the string (harmonic oscillators). As a further guide, it may also be useful to mention that background fields, that are restricted in string theory by equations that come from imposing local conformal symmetry on the world sheet (closure of Virasoro algebra), also appear with analogous restrictions in the $\mathrm{Sp}(2,R)$ gauge theory on the worldline, as seen below.

To implement the gauge symmetry generated by $Q_{ij}(X,P)$, we introduce the gauge field $A^{ij}(\tau)$ in the adjoint representation of $\mathrm{Sp}(2,R)$ and then write the gauge invariant action on the worldline in the first order formalism as follows[3]:

$$L = \dot{X}^M(\tau)P_M(\tau) - \frac{1}{2}A^{ij}(\tau)Q_{ij}(X(\tau),P(\tau)) - \mathcal{H}(X(\tau),P(\tau)),$$
$$\text{where}\quad \mathcal{H}(X,P) \text{ is anything invariant under } \mathrm{Sp}(2,R), \text{i.e.,} \{Q_{ij},\mathcal{H}\}=0. \tag{22}$$

If the gauge generators satisfy the algebra given in Eq. (21), the action is invariant under the following infinitesimal transformations with local parameters $\omega^{ij}(\tau)$:

$$\delta_\omega X^M = \frac{1}{2}\omega^{ij}\{X^M, Q_{ij}\} = \frac{1}{2}\omega^{ij}\frac{\partial Q_{ij}(X,P)}{\partial P_M}, \tag{23}$$

$$\delta_\omega P_M = \frac{1}{2}\omega^{ij}\{P_M, Q_{ij}\} = -\frac{1}{2}\omega^{ij}\frac{\partial Q_{ij}(X,P)}{\partial X^M}, \tag{24}$$

$$\delta_\omega A^{ij} = \frac{d}{d\tau}(\omega^{ij}) + \omega^{ik}\varepsilon_{kl}A^{lj} + \omega^{jk}\varepsilon_{kl}A^{li}, \tag{25}$$

$$\delta_\omega \mathcal{H} = \frac{1}{2}\omega^{ij}\{\mathcal{H}, Q_{ij}\} = 0. \tag{26}$$

These lead to $\delta_\omega Q_{kl} = \frac{1}{2}\omega^{ij}\{Q_{kl}, Q_{ij}\}$, where the right-hand side is given by Eq. (21). Then it is easy to verify that the Lagrangian transforms into a total derivative

$$\delta_\omega L = \frac{d}{d\tau}\left(\frac{1}{2}\omega^{ij}(\tau)P_M\frac{\partial Q_{ij}}{\partial P_M} - \frac{1}{2}\omega^{ij}(\tau)Q_{ij}\right), \tag{27}$$

and therefore the action $S = \int_{\tau_1}^{\tau_2} d\tau L(\tau)$ is invariant, $\delta_\omega S = 0$, provided $\omega^{ij}(\tau)$ vanishes at the end points τ_1, τ_2. This is the $\mathrm{Sp}(2,R)$ gauge symmetry that underlies all 2T physics.

An example of $Q_{ij}(X,P)$ that satisfies the $\mathrm{Sp}(2,R)$ Lie algebra under Poisson brackets is

$$\text{example: } Q_{11} = X\cdot X, \quad Q_{12} = X\cdot P, \quad Q_{22} = P\cdot P, \tag{28}$$

where the dot products are constructed with a flat metric η_{MN} of any signature. But only for $d+2$ dimensions with a signature with two times are there nontrivial solutions to the constraints $Q_{ij}=0$; this means there is a nontrivial gauge invariant physical sub–phase space only when the formalism admits two times or more. Only two times can be admitted because for more timelike dimensions there would be ghosts and the theory would fail to be unitary. Furthermore, with less than two times all solutions of $Q_{ij}=0$ are either identically zero phase space (zero times) or physically trivial phase space (one time, with X and P parallel, so no angular momentum). Hence, only two times, no less and no more, are possible when we demand the $\mathrm{Sp}(2,R)$ gauge symmetry. In the simple case of Eq. (28) the infinitesimal transformations $\delta_\omega X^M$, $\delta_\omega P_M$ above are linear in (X,P), and therefore in that case (X,P) behaves like the doublet of $\mathrm{Sp}(2,R)$ under the local transformation. Hence, if the $Q_{ij}(X,P)$ have the quadratic form (28), then the finite $\mathrm{Sp}(2,R)$ transformation takes the linear form with a matrix of determinant 1 as follows:

$$\begin{pmatrix} X'^M \\ P'^M \end{pmatrix} = \begin{pmatrix} \alpha(\tau) & \beta(\tau) \\ \gamma(\tau) & \frac{1+\beta(\tau)\gamma(\tau)}{\alpha(\tau)} \end{pmatrix}\begin{pmatrix} X^M \\ P^M \end{pmatrix}. \tag{29}$$

More general examples of $Q_{ij}(X,P)$ involve all possible background fields as in Eq. (9). So, when there are background fields, the local infinitesimal transformations $\delta_\omega X^M$, $\delta_\omega P_M$ in (23)–(24) are nonlinear and cannot be written in this linear matrix form. Nevertheless, the transformation of the gauge field A^{ij} is necessarily of the Yang-Mills form, and for finite transformations it can always be written in terms of the matrix with one lower index, $A_i^{\ j} \equiv \varepsilon_{ik}A^{kj}$ where ε_{ij} is the $\mathrm{Sp}(2,R)$ metric, as follows:

$$\begin{pmatrix} A^{12} & A^{22} \\ -A^{11} & -A^{12} \end{pmatrix}' = \begin{pmatrix} \alpha & \beta \\ \gamma & \frac{1+\beta\gamma}{\alpha} \end{pmatrix}\left(\begin{pmatrix} A^{12} & A^{22} \\ -A^{11} & -A^{12} \end{pmatrix} - \partial_\tau\right) \times \begin{pmatrix} \alpha & \beta \\ \gamma & \frac{1+\beta\gamma}{\alpha} \end{pmatrix}^{-1}$$

which gives

$$A'^{11} = \begin{pmatrix} \gamma(1+\beta\gamma)\partial_\tau\alpha^{-1} + \gamma^2\alpha^{-1}\partial_\tau\beta - \alpha^{-1}\partial_\tau\gamma \\ +(\frac{1+\beta\gamma}{\alpha})^2 A^{11} - 2\frac{\gamma}{\alpha}(1+\beta\gamma)A^{12} + \gamma^2 A^{22} \end{pmatrix},$$
$$A'^{12} = \begin{pmatrix} \frac{1}{\alpha}(1+\gamma\beta)\partial_\tau\alpha - \beta\partial_\tau\gamma \\ -\frac{\beta}{\alpha}(1+\beta\gamma)A^{11} + (1+2\beta\gamma)A^{12} - \gamma\alpha A^{22} \end{pmatrix},$$
$$A'^{22} = \begin{pmatrix} \alpha\partial_\tau\beta - \beta\partial_\tau\alpha \\ +\beta^2 A^{11} - 2\beta\alpha A^{12} + \alpha^2 A^{22} \end{pmatrix}. \tag{30}$$

For the more general case with background fields, as in [13] one may argue that, up to canonical transformations of

[3] To continue the analogies to string theory, we mention that string theory, which is usually presented in the second order formulation, could also be reorganized in the first order formalism as here. The second order formulation of the $\mathrm{Sp}(2,R)$ theory could be pursued, but this would be very messy for the general case with all possible background fields, and hence we prefer the first order formalism.

X^M, P_M, the generators $Q_{11}(X,P)$ and $Q_{12}(X,P)$ may be simplified to the following forms,[4]

$$Q_{11}(X,P) = X^M X^N \eta_{MN}, \qquad Q_{12}(X,P) = X^M P_M, \quad (31)$$

while the most general form of $Q_{22}(X,P)$ that satisfies the $\mathrm{Sp}(2,R)$ Lie algebra in Eq. (21) may be parametrized in a power expansion of momentum (when this is permitted) and contains background fields as functions of X as follows [13]:

$$Q_{22}(X,P) = h_0(X) + (\eta^{M_1M_2} + h_2^{M_1M_2}(X))(P_{M_1} + A_{M_1}(X))(P_{M_2} + A_{M_2}(X)) + \sum_{n\geq 3} h_n^{M_1M_2\cdots M_n}(X)(P_{M_1} + A_{M_1}(X))(P_{M_2} + A_{M_2}(X))\cdots(P_{M_n} + A_{M_n}(X)). \quad (32)$$

The background fields are

$$h_0(X), \quad A_M(X) \text{ and } h_n^{M_1M_2\cdots M_n}(X), \quad \text{with } n = 2,3,\cdots. \quad (33)$$

When all of these vanish we obtain the simple case $Q_{22}(X,P) = P^2$ in Eq. (28). The vector $A_M(X)$ is a $U(1)$ gauge field coupled covariantly to momentum $(P_M + A_M(X))$. The 2-tensor $g^{M_1M_2}(X) = \eta^{M_1M_2} + h_2^{M_1M_2}(X)$ is a general metric in curved space. $h_0(X)$ is a scalar field, while the $h_n^{M_1M_2\cdots M_n}(X)$, which are symmetric traceless tensors with $n \geq 3$ indices, are higher spin fields with spin n. There is no independent vector $h_1^M(X)$ associated with the first power of P_M, because in a rearrangement in powers of P rather than $(P + A)$, the vector $h_1^M(X)$ emerges as a combination of the vector $A_M(X)$ and the other $h_n^{M_1M_2\cdots M_n}$, i.e. $h_1^M = 2(\eta^{M_1M_2} + h_2^{M_1M_2})A_{M_2} + \cdots$.

For the $\mathrm{Sp}(2,R)$ Lie algebra to close properly as in Eq. (21) it is necessary to impose restrictions on the background fields. The closure requires that the two-form, $F_{MN} \equiv \partial_M A_N - \partial_N A_M$, and all the high-spin fields be transverse to the vector X^M [13]

$$X^M F_{MN} = 0, \text{ and } \eta_{MM_1} X^M h_n^{M_1M_2\cdots M_n} = 0, \quad n = 2,3,\cdots \quad (34)$$

and that all other backgrounds be homogeneous fields with definite scaling dimensions for $n = 0,2,3,\cdots$ [13]

$$(X^M\partial_M - (n-2))h_n^{M_1M_2\cdots M_n}(X) = 0, \quad \text{or} \quad h_n^{M_1M_2\cdots M_n}(\lambda X) = \lambda^{n-2} h_n^{M_1M_2\cdots M_n}(X). \quad (35)$$

The $\mathrm{Sp}(2,R)$ algebra among the $Q_{ij}(X,P)$closes only if the background fields satisfy the transversality and homogeneity conditions in Eqs. (34)–(35). Hence, to define the model with an $\mathrm{Sp}(2,R)$ gauge symmetry, it is necessary to impose these as *a priori* conditions on the background fields.

For the reader familiar with string theory, these $\mathrm{Sp}(2,R)$ conditions on the backgrounds in the worldline formalism are analogous to the conditions on backgrounds that emerge from conformal symmetry on the world sheet (closure of the Virasoro algebra).

It is useful to work in a fixed axial type gauge for the U(1) background gauge field, $X \cdot A = 0$, which makes it a transverse vector, just like all other tensors as in Eq. (34). In that case the constraint $X^M F_{MN} = 0$ simplifies to the following homogeneity condition on A_M [13], which is also similar to all other tensors as in Eq. (35):

$$X \cdot A = 0, \qquad (X^M\partial_M + 1)A_M = 0, \quad \text{or} \quad A_M(\lambda X) = \lambda^{-1} A_M(X). \quad (36)$$

The generalization of these equations to spinning systems was given in [9–11] but we will not discuss this here since in this paper we are concentrating only on spinless particles.

It may be of interest to emphasize that the constraints on 6 dimensional fields found by trial and error by Weinberg [44] in order to have 6 dimensional correlators consistent with conformal symmetry in $3+1$ dimensions are identical to the $\mathrm{Sp}(2,R)$ gauge symmetry conditions on fields that were already derived in [9–11,13,24] as given above. So these constraints on fields, which were also naturally incorporated in the 2T standard model [18] and 2T gravity [4], including fermions and gauge bosons, follow directly from a fundamental gauge symmetry $\mathrm{Sp}(2,R)$ in phase space, and their underlying role is to ensure a unitary and causal theory with two times in $d+2$ dimensions.

[4]A generally covariant form that avoids the appearance of explicit X^M is given in [4,11,13] as follows: $Q_{11} = W(X)$ and $Q_{12} = V^M(X)P_M$, where $W(X)$, $V^M(X)$ are background fields like the others, and instead of $h_2^{MN}(X) + \eta^{MN}$ in Q_{22} we simply write the general metric $g^{MN}(X)$. Then closure for $\mathrm{Sp}(2,R)$ restricts these background fields to obey some homothety conditions as given in [4,11,13]. The simplified version, with the explicit X^M used in this paper, is a choice of coordinates under general coordinate transformations that is equivalent to the general version, while maintaining covariance with respect to the $\mathrm{SO}(d,2)$ global transformations as a subset of general coordinate transformations. The simplified version satisfies the homothety conditions automatically.

As explained in footnote (4), we made a special choice of basis of phase space (X^M, P_M) such that the expression for $Q_{11} = X^M X^N \eta_{MN}$ introduced the flat metric η_{MN} which is invariant under $\mathrm{SO}(d,2)$. Using this flat metric we may raise or lower indices, such as $P^M \equiv \eta^{MN} P_N$ or $X_M \equiv \eta_{MN} X^N$, which should not be confused with raising or lowering indices with the full metric $g^{M_1M_2}(X) = \eta^{M_1M_2} + h_2^{M_1M_2}(X)$. With this definition of P^M we define the generators of $\mathrm{SO}(d,2)$ transformations

$$L^{MN} = X^M P^N - X^N P^M. \tag{37}$$

Under Poisson brackets these commute with all dot products $(X^M X^N \eta_{MN})$, $(X^M P_N)$, $(P_M P_N \eta^{MN})$. In particular they commute with the two $\mathrm{Sp}(2,R)$ generators $Q_{11} = X^M X^N \eta_{MN}$ and $Q_{12} = X^M P_M$

$$\{Q_{11}, L^{MN}\} = 0, \qquad \{Q_{12}, L^{MN}\} = 0. \tag{38}$$

This means that Q_{11}, Q_{12} are invariant under global $\mathrm{SO}(d,2)$ transformations, but it also means that the L^{MN} are *gauge invariant* under the subgroup of $\mathrm{Sp}(2,R)$ transformations generated by Q_{11}, Q_{12}. Since these two generators are quadratic, the two-parameter gauge transformation they induce on (X^M, P^M) is linear just as Eq. (29), with the parameter $\beta = 0$. This subgroup of gauge transformations will play an important role in the dualities we will discuss in this paper. The fact that L^{MN} are gauge invariant under this subgroup of $\mathrm{Sp}(2,R)$ predicts that these L^{MN} *are invariants under the dualities* as discussed in Sec. VI.

In the presence of background fields denoted by "$\cdots$" the third $\mathrm{Sp}(2,R)$ generator, $Q_{22} = (P^2 + \cdots)$, does not commute with L^{MN} except for its first term $\{P^2, L^{MN}\} = 0$, but when the background fields vanish then Q_{22} becomes $\mathrm{SO}(d,2)$ invariant while L^{MN} becomes gauge invariant under the full $\mathrm{Sp}(2,R)$.

A. Five gauges and five shadows

In this section, we give five different gauge fixed configurations of (X^M, P_M) such that, when inserted in the 2T action (22), they result in five shadows in two fewer dimensions and can be interpreted as five different 1T physics systems. Each 1T shadow is expressed by 1T Lagrangians L_i, $i = 1, \ldots, 5$, as in Eq. (5), but with five different constraints $Q_i(x_i, p_i)$ as in (9), and parametrized in terms of five canonical sets of degrees of freedom $(x_i(\tau), p_i(\tau))$, as listed in Eq. (13). It should be mentioned that the emerging 1T Lagrangians L_i are defined up to a total derivative $L_i \to L_i + \frac{d\Lambda_i}{d\tau}$. The total derivative could be dropped since it does not contribute to the action or the equations of motion, but here we will give the $\Lambda_i(x,p)$ that emerge directly from the gauge fixing, so that the interested reader can verify the result.

It should be emphasized that for these five shadows the parent 2T theory in general contains any set of background fields, since $Q_{22}(X,P) = P^2 + (\text{backgrounds})$, but for simplicity we will not explicitly write down specific backgrounds. Also, we will discuss only the case of the 2T system in Eq. (22) in which $\mathcal{H} = 0$ because this is sufficient to illustrate our methods, while the addition of a nontrivial $\mathcal{H}$ does not change the essential part of the discussion.

We now give a list of five configurations for X^M, P^M [where $P^M = \eta^{MN} P_N$, using the η^{MN} already introduced in (31)–(32)], for which two gauges have been fixed and the two constraints $X^2 = 0$ and $X \cdot P$ have been solved explicitly. So each configuration is parametrized in terms of the remaining 1T degrees of freedom (x_i^μ, p_{i_μ}) in two fewer dimensions. In each gauge the resulting $Q_i(x_i, p_i)$ and $\Lambda_i(x_i, p_i)$ are computed. The algebra to get these results is straightforward. We will illustrate this in detail for the simplest case 1 and most complicated case 5, while cases 2, 3, 4 are sketched with sufficient detail but leaving a small exercise for the reader.

1. Shadow 1, massless relativistic

The light cone basis in the extra dimensions $\pm'$ is defined as $X^{\pm'} = \frac{1}{\sqrt{2}}(X^{0'} \pm X^{1'})$, and similarly for the momenta. The two gauge choices are $X_1^{+'}(\tau) = 1$ and $P_1^{+'}(\tau) = 0$ for all τ. The components $X_1^{-'}(\tau) = \frac{1}{2}x_1^2$ and $P_1^{-'}(\tau) = x_1 \cdot p_1$ are computed to satisfy the constraints, $X \cdot X = 0 = -2X^{+'}X^{-'} + X^\mu X_\mu$, and similarly for $0 = X \cdot P$. The gauge fixed configuration of (X^M, P^M) is then

$M =$	$+'$	$-'$	μ
$X_1^M =$	1	$\frac{1}{2}x_1^2$	x_1^μ
$P_1^M =$	0	$x_1 \cdot p_1$	p_1^μ

,

$P_1^2 = p_1^2$
$\Lambda_1 = 0$

(39)

Now, to obtain the gauge fixed form of the action (22) up to a total τ derivative, we compute $\dot{X}_1^M = (0, \dot{x}_1 \cdot x_1, \dot{x}_1^\mu)$ which gives $\dot{X}_1 \cdot P_1 = \dot{x}_1 \cdot p_1 + d\Lambda_1/d\tau$. We see that $\Lambda_1 = 0$ since we find no extra total time derivative. We also compute the third constraint given in (32), $Q_{22} = P_1^2 + \cdots$, which becomes $Q_{22} = p_1^2 + \cdots$, where $\cdots$ stands for background fields consistent with the constraints (32)–(35). Inserting these in the 2T Lagrangian (22) we obtain the 1T shadow Lagrangian

$$L_1 = \dot{x}_1 \cdot p_1 - \frac{1}{2}A_1^{22}(\tau)(p_1^2 + \cdots). \tag{40}$$

After imposing the transversality and homogeneity constraints in (34)–(35) on the background fields in $d+2$ dimensions, we find that the surviving background fields denoted by $\cdots$ are precisely the background fields in d dimensions displayed in Eq. (9) and [13]. The emergent shadow in d dimensions is evidently the Lagrangian for the interacting 1T massless relativistic particle as discussed in Eqs. (5), (9).

We call this gauge the conformal shadow. This is the shadow in which linear $\mathrm{SO}(d,2)$ transformations on (X^M, P_M), that leave the flat metric η_{MN} invariant, become the familiar nonlinear conformal transformations in phase space in $(d-1)+1$ dimensions. To see this, the reader is invited to evaluate the $\mathrm{SO}(d,2)$ generators $L^{MN} = X^M P^N - X^N P^M$ for the gauged fixed configuration (X_1^M, P_1^N) of Eq. (39) and verify that these L^{MN} take the form of the familiar $\mathrm{SO}(d,2)$ conformal transformations in d dimensions. That we should expect such a hidden symmetry in Eq. (40) when all background fields vanish is predicted from the fully covariant parent 2T theory (22) before gauges are fixed.

2. Shadow 2, massive relativistic

We will be brief because the procedure is the same and the result was given before (see references in [16]). The gauge fixed configuration that also satisfies $X_2^2 = 0 = X_2 \cdot P_2$ is

$M =$	$+'$	$-'$	μ
$X_2^M =$	$\frac{1+a}{2a}$	$\frac{x_2^2 a}{1+a}$	x_2^μ
$P_2^M =$	$\frac{-m_2^2}{2(x_2\cdot p_2)a}$	$(x_2 \cdot p_2)\, a$	p_2^μ

,

$a \equiv \sqrt{1 + \frac{m_2^2 x_2^2}{(x_2\cdot p_2)^2}}$
$P_2^2 = p_2^2 + m_2^2$
$\Lambda_2 = (x_2 \cdot p_2)\,(a-1)$

(41)

The steps leading from the 2T Lagrangian to the 1T shadow are parallel to those in case 1. We find $\dot{X}_2 \cdot P_2 = \dot{x}_2 \cdot p_2 + d\Lambda_2/d\tau$ with the Λ_2 given in (41), and $P_2^2 = p_2^2 + m_2^2$. Inserting these in the 2T Lagrangian (22) we obtain the 1T shadow action

$$L_2 = \dot{x}_2 \cdot p_2 - \frac{1}{2} A_2^{22}(\tau)(p_2^2 + m_2^2 + \cdots), \qquad (42)$$

in which we have dropped the total derivative $d\Lambda_2/d\tau$. Here the remaining constraint is the same Q_{22} in (32), but now written in gauge 2, $0 = Q_{22} = P_2^2 + \cdots = (p_2^2 + m_2^2 + \cdots) \equiv Q_2(x_2, p_2)$, as listed in (13). The background fields in L_2 are inherited from those in $d+2$ dimensions by specializing to the gauge 2. This is evidently the Lagrangian for the 1T massive relativistic particle, with mass m_2, and generally interacting with background fields. The mass can now be viewed as a modulus in the embedding of the d dimensional phase space $(x_2^\mu, p_{2\mu})$ in the $(d+2)$ dimensional phase space (X^M, P_M). So, it is a property of the 1T observer as he/she parametrizes from this perspective the phenomena that occur in $(d+2)$ dimensional phase space.

We should expect a relationship between the background fields in shadow 1 and shadow 2 since they are both derived from those in $d+2$ dimensions. As we have summarized in the paragraph just before Eq. (14), this relationship is given by the background-independent duality transformation between shadows 1 and 2 which takes the form of canonical transformations displayed in Eq. (15).

When all backgrounds vanish, the massive particle system described by (42) has a hidden $\mathrm{SO}(d,2)$ symmetry given by the conserved generators, $L^{MN} = X_2^M P_2^N - X_2^N P_2^M$, as demonstrated in [16]. That we should expect such a hidden symmetry in Eq. (42) when all backgrounds vanish is evident from the fully covariant parent theory (22) before gauges are fixed.

3. Shadow 3, massive nonrelativistic

The gauge fixed configuration that also satisfies $X_3^2 = 0 = X_3 \cdot P_3$ is (here we use the parameters t_3 for the timelike coordinate and h_3 for its canonical conjugate since these are more intuitive symbols in nonrelativistic physics)

$M =$	$+'$	$-'$	0	i
$X_3^M =$	t_3	u	s	$\mathbf{x}_3^i$
$P_3^M =$	m_3	h_3	0	$\mathbf{p}_3^i$

,

$u \equiv \frac{1}{m_3}(\mathbf{x}_3 \cdot \mathbf{p}_3 - t_3 h_3)$
$s^2 \equiv \mathbf{x}_3^2 - \frac{2t_3}{m_3}\mathbf{x}_3 \cdot \mathbf{p}_3 + \frac{2t_3^2}{m_3} h_3$
$P_3^2 = \mathbf{p}_3^2 - 2m_3 h_3$
$\Lambda_3 = -m_3 u$

(43)

The steps leading from the 2T Lagrangian to the 1T shadow are parallel to those in cases 1 and 2. We find $\dot{X}_3 \cdot P_3 = \dot{\mathbf{x}}_3 \cdot \mathbf{p}_3 - \dot{t}_3 h_3 + d\Lambda_3/d\tau$, and $P_3^2 = -2m_3 h_3 + \mathbf{p}_3^2$. Inserting these in the 2T Lagrangian (22) we obtain the 1T shadow 3 action

$$L_3 = \dot{\mathbf{x}}_3 \cdot \mathbf{p}_3 - \dot{t}_3 h_3 - \frac{1}{2} A_3^{22}(\tau)(\mathbf{p}_3^2 - 2m_3 h_3 + \cdots), \quad (44)$$

in which we dropped the total derivative $d\Lambda_3/d\tau$. The remaining constraint is the same Q_{22} now written in gauge 3, $0 = Q_{22} = P_3^2 + \cdots = (\mathbf{p}_3^2 - 2m_3 h_3 + \cdots) \equiv Q_3(x_3, p_3)$ as listed in (13). This is evidently the Lagrangian for the massive nonrelativistic particle, with mass m_3, as discussed in Eqs. (10)–(12). The mass m_3 can be viewed as a modulus in the embedding of the d dimensional nonrelativistic phase space $(\mathbf{x}_3, \mathbf{p}_3, t_3, h_3)$ in the $(d+2)$ dimensional phase space (X^M, P_M). So, it is a property of the 1T nonrelativistic observer as he/she parametrizes from this perspective the phenomena that occur in $(d+2)$ dimensional phase space. The background fields represented by $\cdots$ are again inherited from those in $d+2$ dimensions, and therefore are related to the background fields in shadows 1 and 2 by the background-independent canonical transformations given in Eqs. (72)–(73) and Eqs. (74)–(75).

When all backgrounds vanish, the massive particle system described by (44) has a hidden $\mathrm{SO}(d,2)$ symmetry given by the conserved generators, $L^{MN} = X_3^M P_3^N - X_3^N P_3^M$, as demonstrated in [16]. That we should expect such a hidden symmetry in Eq. (44) when all backgrounds vanish is evident from the fully covariant parent theory (22) before gauges are fixed.

4. Shadow 4, H-atom

The gauge fixed configuration that already satisfies $X_4^2 = 0 = X_4 \cdot P_4$ is

$$\begin{array}{|c|c|c|c|}
\hline
M = & \pm' & 0 & i \\
\hline
X_4^M = & \frac{1}{\sqrt{-4m_4h_4}} \left\{ \begin{array}{c} |\mathbf{x}_4| \sqrt{-2m_4h_4} \sin u \\ + (\mathbf{x}_4 \cdot \mathbf{p}_4)(\cos u \pm 1) \end{array} \right\}, & \left\{ \begin{array}{c} |\mathbf{x}_4| \cos u \\ - \frac{\mathbf{x}_4 \cdot \mathbf{p}_4}{\sqrt{-2m_4h_4}} \sin u \end{array} \right\}, & \mathbf{x}_4^i \\
\hline
P_4^M = & \frac{1}{\sqrt{-4m_4h_4}} \left\{ \begin{array}{c} \pm 2m_4h_4 \\ + \frac{m_4 e_4^2}{|\mathbf{x}_4|}(\cos u \pm 1) \end{array} \right\}, & -\frac{m_4 e_4^2}{|\mathbf{x}_4|\sqrt{-2m_4h_4}} \sin u\ , & \mathbf{p}_4^i \\
\hline
\multicolumn{4}{|c|}{\text{in this paper it is understood that } \sqrt{-2m_4h_4} \text{ means } \pm\sqrt{-2m_4h_4}.} \\
\hline
\end{array} \quad (45)$$

with

$$\begin{array}{|l|}
\hline
u \equiv \frac{\sqrt{-2m_4h_4}}{m_4 e_4^2}(\mathbf{x}_4 \cdot \mathbf{p}_4 - 2h_4 t_4) \\
\hline
P_4^2 = \left(\mathbf{p}_4^2 - 2m_4 \frac{e_4^2}{|\mathbf{x}_4|} - 2m_4 h_4\right) \\
\hline
\Lambda_4 = 3h_4 t_4 - 2(\mathbf{x}_4 \cdot \mathbf{p}_4) \\
\hline
\end{array} \quad (46)$$

Here we use the parameters t_4 for the timelike coordinate and h_4 for its canonical conjugate, and assume $h_4 < 0$ for bound states.[5] The remaining constraint is the same Q_{22} now written in gauge 4, $0 = Q_{22} = P_4^2 + \cdots = (\mathbf{p}_4^2 - 2\frac{m_4 e_4^2}{|\mathbf{x}_4|} - 2m_4 h_4 + \cdots) \equiv Q_4(x_4, p_4)$ as listed in (13). The steps leading from the 2T Lagrangian to the 1T shadow are parallel to those in case 1. We find $\dot{X}_4 \cdot P_4 = \dot{\mathbf{x}}_4 \cdot \mathbf{p}_4 - \dot{t}_4 h_4 + d\Lambda_4/d\tau$, and $P_4^2 = (\mathbf{p}_4^2 - 2\frac{m_4 e_4^2}{|\mathbf{x}_4|} - 2m_4 h_4)$. Inserting these in the 2T Lagrangian (22) we obtain the 1T shadow action (in which we drop the total derivative $d\Lambda_4/d\tau$)

$$L_4 = \dot{\mathbf{x}}_4 \cdot \mathbf{p}_4 - \dot{t}_4 h_4 - \frac{1}{2} A_4^{22}(\tau)\left(\mathbf{p}_4^2 - \frac{2m_4 e_4^2}{|\mathbf{x}_4|} - 2m_4 h_4 + \cdots\right). \quad (47)$$

To see that this is equivalent to the Lagrangian for a particle in the $1/r$ potential (like the H-atom or planetary motion), we use the remaining gauge freedom to choose the gauge $t_4(\tau) = \tau$ and solve the constraint $Q_4 = 0$ for the canonical conjugate h_4. In the case of no background fields we get $h_4 = \frac{\mathbf{p}_4^2}{2m_4} - \frac{e_4^2}{|\mathbf{x}_4|}$. Then, using $\dot{t}_4(\tau) = 1$ and the solved form for h_4, the Lagrangian L_4 reduces to the familiar form for the particle in the $1/r$ potential

$$L_4 \to \dot{\mathbf{x}}_4 \cdot \mathbf{p}_4 - \left(\frac{\mathbf{p}_4^2}{2m_4} - \frac{e_4^2}{|\mathbf{x}_4|}\right), \quad \text{if no backgrounds.} \quad (48)$$

The mass m_4 and coupling strength e_4^2 can be viewed as moduli in the embedding of the d dimensional

[5] The analytic continuation to the phase space region $h_4 > 0$ for scattering states looks similar, but to ensure a real parametrization we also swap a timelike coordinate with a spacelike coordinate. See Table II in [Phys. Rev. D 76, 065016 (2007)] for details.

nonrelativistic phase space $(\mathbf{x}_4, \mathbf{p}_4, t_4, h_4)$ in the $(d+2)$ dimensional phase space (X^M, P_M). So, these are properties of the 1T nonrelativistic observer as he/she parametrizes from this perspective the phenomena that occur in $(d+2)$ dimensional phase space. If there are background fields then they are inherited from those in $d+2$ dimensions specialized to gauge 4 and related to those in shadows 1, 2, 3 by the duality transformations indicated in Table II.

When all backgrounds vanish, the massive particle system described by (48) has a hidden SO$(d,2)$ symmetry given by the conserved generators, $L^{MN} = X_4^M P_4^N - X_4^N P_4^M$, as demonstrated in [16]. That we should expect such a hidden symmetry in Eq. (48) when all backgrounds vanish is evident from the fully covariant parent theory (22) before gauges are fixed.

5. Shadow 5, relativistic potential $V(x^2)$

The discussion of this case will also shed some light on how to proceed to construct more general shadows. The gauge fixed configuration is

$M=$	$+'$	$-'$	μ
$X_5^M=$	A	$\frac{1}{2A}x_5^2$	x_5^μ
$P_5^M=$	$\frac{A}{x_5^2}(x_5\cdot p_5-\phi)$	$\frac{1}{2A}(x_5\cdot p_5+\phi)$	p_5^μ

(49)

The constraints $X_5^2 = 0 = X_5 \cdot P_5$ are already satisfied for any $A(x_5, p_5)$ and $\phi(x_5, p_5)$. The A, ϕ are constructed to obtain the dynamics described by the following constraint:

$$Q_5(x_5, p_5) = (P_5^2 + \cdots) = [p_5^2 + V(x_5^2) + \cdots] = 0, \quad (50)$$

where $\cdots$ represents the contribution of background fields, and $V(x^2)$ is any function of the Lorentz invariant x_5^2. Hence, to obtain

$$P_5^2 = p_5^2 - \frac{1}{x_5^2}((x_5\cdot p_5)^2 - \phi^2) = p_5^2 + V(x_5^2), \quad (51)$$

we chose $\phi(x_5, p_5)$ such that

$$\phi(x_5, p_5) = (x_5 \cdot p_5)\sqrt{1 + \frac{x_5^2 V(x_5^2)}{(x_5\cdot p_5)^2}}. \quad (52)$$

More general relativistic shadows follow from more general choices of $\phi(x_5, p_5)$.

To ensure that $(x_5^\mu, p_{5\mu})$ are canonical conjugates in the emergent fifth shadow, rather than being some random symbols, we must also require

$$\dot{X}_5^M P_{5M} = \dot{x}_5^\mu p_{5\mu} + \frac{d\Lambda_5}{d\tau}, \quad (53)$$

where the second term is a total time derivative. Then $\Lambda_5(x_5(\tau), p_5(\tau))$ may be dropped from the action since it does not contribute to the equations of motion of $(x_5^\mu, p_{5\mu})$. Inserting the (X_5^M, P_5^M) of Eq. (49) into this requirement results in a nontrivial restriction on $X_5^{+'} \equiv A$ as follows:

$$\frac{1}{2A}(x_5\cdot p_5 + \phi)\frac{dA}{d\tau} + \frac{A}{x_5^2}(x_5\cdot p_5 - \phi)\frac{d}{d\tau}\left(\frac{x_5^2}{2A}\right) = \frac{d\Lambda_5}{d\tau}. \quad (54)$$

The general solution of this equation for the special $\phi(x_5, p_5)$ in Eq. (52) is given by

$$A(x_5, p_5) = F(\phi)\sqrt{x_5^2}\exp\left[-\frac{1}{2}\int^{x_5^2}\frac{du}{u}\left(1 - \frac{uV(u)}{\phi^2}\right)^{-1/2}\right], \quad (55)$$

and

$$\Lambda_5(x_5, p_5) = -\int^{x_5^2} du V(u)(\phi^2 - uV(u))^{-1/2} + 2\int^{\phi} dz z\frac{d}{dz}\ln F(z), \quad (56)$$

where $F(\phi)$ is a general function of its argument $\phi(x_5, p_5)$ given in (52). Then we see that the emerging fifth shadow Lagrangian which determines the dynamics of the remaining degrees of freedom $(x_5^\mu, p_{5\mu})$ as derived from Eq. (22) is given by

$$L_5 = \dot{x}_5^\mu p_{5\mu} - \frac{1}{2}A_5^{22}(p_5^2 + V(x_5^2) + \cdots). \quad (57)$$

Note that the dynamics of (x_5, p_5) is independent of the solution $X_5^{+'} = A(x_5, p_5)$ given in Eq. (55), but the expression for $A(x_5, p_5)$ in (55) is needed to fully determine the embedding of the fifth shadow in $d+2$ dimensional phase space as given in Eq. (49). Also, $X_5^{+'} = A(x_5, p_5)$ is needed to obtain the duality transformation to the other shadows. Furthermore, $A(x_5, p_5)$ determines also the components $L^{\pm'\mu}$ of the SO$(d,2)$ generators.

When all background fields $\cdots$ vanish, the action in Eq. (57) has a hidden $SO(d,2)$ symmetry just as in all previous cases discussed above. The generators of this symmetry are again, $L^{MN} = X_5^M P_5^N - X_5^N P_5^M$, where we insert the gauge fixed (X_5^M, P_5^M) given in Eq. (49). These symmetry generators are the Noether charges that are conserved using the equations of motion derived from (57). In particular the conserved generator $L^{+'-'}$ coincides with $\phi(x_5, p_5)$ given in Eq. (52), namely $L^{+'-'} = X_5^{+'} P_5^{-'} - X_5^{-'} P_5^{+'} = \phi$, as derived from (49). That we should expect a hidden $SO(d,2)$ symmetry for the action in Eq. (57) when all backgrounds vanish is evident from the fully covariant parent theory (22) before the gauge is fixed.

As an example, consider $V(x^2) = c(x^2)^b$ where c, b are arbitrary constants. For this case the integrals in Eq. (55) can be done explicitly, yielding

$$A^{1+b} = (F(\phi))^{1+b} \frac{|x_5 \cdot p_5|}{\sqrt{c}} \left[1 + \left(1 + \frac{c(x_5^2)^{1+b}}{(x_5 \cdot p_5)^2} \right)^{1/2} \right]. \tag{58}$$

As a check, we compare this result to shadow 2 given in Eq. (41). We find agreement when we specialize by taking $V(x^2) = m_2^2$, or $c = m_2^2$, $b = 0$, with $F(\phi) = m_2/(2|\phi|)$.

Note that solving for $A(x_5, p_5)$ from the expression (58) involves branch cuts. Therefore A may need to be redefined up to various signs in neighboring patches of phase space (x_5^μ, p_5^μ) so as to be able to cover continuously the phase space in $d+2$ dimensions (X^M, P^M). We leave this issue open here, but we return to make comments about it in Sec. VII in the context of obtaining solutions of the constrained system (57) by using dualities.

This example yields another interesting shadow, namely the relativistic harmonic oscillator, with the constraint $Q(x,p) = (p^2 + \omega^2 x^2 + \cdots) = 0$, when $V(x^2) = c(x^2)^b$ is taken with the special constants $c = \omega^2$, $b = 1$. When the background fields $\cdots$ vanish, the physical sector of the constrained relativistic harmonic oscillator in $(d-1)+1$ dimensions is the same as the unconstrained nonrelativistic harmonic oscillator in $(d-1)$ space dimensions. This was demonstrated in [51] by the following canonical transformation for the timelike phase space:

$$\begin{aligned} \omega x^0(\tau) &= \pm\sqrt{2(h(\tau) - E_0)} \sin(t(\tau)), \\ p^0(\tau) &= \pm\sqrt{2(h(\tau) - E_0)} \cos(t(\tau)), \end{aligned} \tag{59}$$

where E_0 is a constant. Choosing the gauge $t(\tau) = \tau$, and solving the relativistic constraint for h from $(p^2 + \omega^2 x^2) = (\mathbf{p}^2 + \omega^2 \mathbf{r}^2) - 2(h(\tau) - E_0) = 0$, reduces the problem to only the physical phase space degrees of freedom $(\mathbf{r}, \mathbf{p})$ with the Hamiltonian $h = \frac{1}{2}(\mathbf{p}^2 + \mathbf{r}^2) + E_0$, which describes the nonrelativistic oscillator.

The more general case of the constrained system, $p^2 + c(x^2)^b = 0$ with general b, c, is a rather complicated problem whose solution was not known until now. But we will show in Sec. VII that the duality methods discussed in this paper will provide the means to solve it analytically. The same methods apply also to the even more general case $p^2 + V(x^2) = 0$, with any $V(x^2)$, thus demonstrating the power of our duality methods derived from 2T physics.

V. SP(2,R) GAUGE TRANSFORMATIONS AND DUALITIES

As was already discussed in the previous section, in the case of 2T physics, the five gauges that were studied above correspond to different physical systems in 1T physics. However, all of them are holographic shadows of the same theory in 2T physics, meaning that each shadow contains all the gauge invariant 2T physics by virtue of being just a gauge choice. So, there must exist duality relations that map the 1T shadows into each other. The 1T physics observed in the respective shadows, although they have different 1T physics interpretations, must be related to each other by dualities, and must describe the same gauge invariant content of the 2T physics theory from which the shadows are derived. This is hidden information among 1T physics systems that 1T physics does not provide systematically, but is a prediction of the 2T physics formulation which can be tested and verified directly in 1T physics by using our dualities.

These dualities have to be $Sp(2,R)$ gauge transformations acting on the 2T phase space (X^M, P_M). The parameters of these transformations are local on the worldline parametrized by τ, but since the interest is in transforming one fixed gauge (X_i^M, P_{iM}) to another (X_j^M, P_{jM}), the parameters of the gauge transformation would be written in terms of the τ-dependent phase space coordinates of the corresponding 1T shadows themselves. So, these $Sp(2,R)$ gauge transformations must take the form of canonical transformations among the 1T shadows. In this section we will illustrate these ideas by considering a special subset of canonical transformations that connect the five shadows to each other.

We can compute algebraically the gauge transformations that relate the phase space degrees of freedom of any two shadows to each other. To do this we consider the gauge transformations generated by the $Sp(2,R)$ charges of the form given in Eqs. (31)–(35)

$$Q_{11} = X \cdot X, \quad Q_{12} = X \cdot P, \quad Q_{22} = P \cdot P + \cdots. \tag{60}$$

The gauge transformations (23)–(26) generated by the first two charges Q_{11}, Q_{12} are linear $Sp(2,R)$ transformations that are written as a 2×2 matrix of the general form (29), but with $\beta = 0$, namely $\begin{pmatrix} \alpha & 0 \\ \gamma & 1/\alpha \end{pmatrix}$ because the transformation generated by Q_{22} is not included in the present duality discussion. In fact, the $\mathrm{Sp}(2,R)$ transformations generated by Q_{22} are nonlinear in the capital (X,P), because Q_{22} generally contains background fields denoted by $\cdots$ that we wish to keep as general as possible in our discussion. By contrast, since Q_{11}, Q_{12} are purely quadratic in phase space these charges induce only linear transformations via (23)–(26). Recall that $X\cdot X = X\cdot P = 0$ are already satisfied explicitly in each shadow. The gauge transformations generated by $(X\cdot X)$, $(X\cdot P)$ close into a subgroup of $\mathrm{Sp}(2,R)$ and hence they act within the physical space that already satisfies these constraints in each shadow, namely $X\cdot X = X\cdot P = 0$. Furthermore, within this restricted phase space, the subgroup of transformations generated by (Q_{11}, Q_{12}) transform Q_{22}, and the corresponding gauge field A^{22}, only by an overall scaling $Q_{22} \to \alpha^{-2} Q_{22}$ and $A^{22} \to \alpha^2 A^{22}$, as seen from Eqs. (30) at $\beta = 0$. So the remaining term in the gauge fixed action $A^{22}Q_{22}$ is invariant while compatible with being in the subspace $X\cdot X = X\cdot P = 0$. Hence these duality transformations change one shadow into another without changing the remaining constraint Q_{22} (which is eventually applied as $Q_{22} = 0$ in each shadow). This is the reason that the duality transformations we discuss take the form of 2×2 matrices as in (29), with $\alpha(x,p)$, $\gamma(x,p)$ taken as functions of 1T phase space $(x(\tau), p(\tau))$, and with $\beta = 0$.

We can, therefore, use the matrix method to find explicitly the duality transformation constructed from phase space, with $\alpha(x,p)$, $\gamma(x,p)$, and $\beta = 0$, given that the gauge fixed forms of the shadows that we want to relate to each other by 2×2 matrices are already specified in Eqs. (39), (41), (43), (45), (49) as $\begin{pmatrix} X \\ P \end{pmatrix}$ doublets in each direction M. These dualities must also be canonical transformations since they are written only in terms of phase space degrees of freedom and map one canonical phase space to another canonical phase space.

A. The general duality transformation

In order to find the duality transformation we consider the general linear form of an element of $\mathrm{Sp}(2,R)$ given by Eq. (29) with $\beta = 0$, as explained above. We recall that the gauge fixed X^M and P^M for each shadow are given explicitly as doublets in Sec. IV A. First we set up a 2×2 matrix transformation between two shadows i and j for every direction M

$$\begin{pmatrix} X_j^M(x_j^\mu, p_j^\mu) \\ P_j^M(x_j^\mu, p_j^\mu) \end{pmatrix} = \begin{pmatrix} \alpha(\tau) & 0 \\ \gamma(\tau) & \alpha^{-1}(\tau) \end{pmatrix} \begin{pmatrix} X_i^M(x_i^\mu, p_i^\mu) \\ P_i^M(x_i^\mu, p_i^\mu) \end{pmatrix}. \tag{61}$$

Note that shadow i is parametrized in terms of phase space (x_i^μ, p_i^μ) while shadow j is parametrized in terms of phase space (x_j^μ, p_j^μ). Next, we solve for $\alpha(\tau)$ and $\gamma(\tau)$ such that the 2×2 matrix corresponds to the gauge transformation from one fixed gauge to another. This is done by using some doublets in convenient directions M that contain information on how the gauge was fixed in shadows i and j. For example, in the case of shadow 1 in Eq. (39) the doublet $M = +'$ is convenient since it is fully fixed to $X_1^{+'} = 1$ and $P_1^{+'} = 0$. Finally, we express the transformation parameters $\alpha(x_i^\mu, p_i^\mu)$, $\gamma(x_i^\mu, p_i^\mu)$ in terms of the degrees of freedom (x_i^μ, p_i^μ) of the shadow of origin. Having fixed the matrix, the canonical transformation $(x_j^\mu, p_j^\mu) \leftarrow (x_i^\mu, p_i^\mu)$ is now obtained by taking the $M = \mu$ direction in Eq. (61) as shown in the example (15). Using this procedure, we found the explicit canonical transformations in Sec. V B, and listed the results of Sec. V B in the table of Eq. (16).

To show that these gauge transformations are canonical transformations (including time and Hamiltonian), we must also show that the canonical structure holds up to a total derivative

$$\dot{x}_j^\mu p_{j\mu} = \dot{x}_i^\mu p_{i\mu} + \frac{d}{d\tau}\Lambda_{ji}(\tau). \tag{62}$$

When this is true, the invariance of Poisson brackets for any two quantities $\{A, B\}$ is also guaranteed when they are evaluated as derivatives in terms of either shadow. The validity of Eq. (62) can be checked by using our explicit transformations in Sec. V B. However, the result (62) is already guaranteed by the canonical structures that descended from $d+2$ dimensions. The essential observation is that we can equate two gauge fixed forms of the same gauge invariant. That is, the gauge invariant Lagrangian of the 2T theory (22) can be equated to its five gauge fixed versions in the five shadows of Sec. IV A. Consider two shadows i and j which satisfy the following relations due to the gauge invariance of the Lagrangian:

$$L = \dot{X}^M P_M + \frac{1}{2} A^{kl} Q_{kl} \tag{63}$$

$$= \dot{x}_i^\mu p_{i\mu} + \frac{1}{2} A_i^{22}(\tau) Q_{22}(x_i^\mu, p_i^\mu) + \frac{d\Lambda_i}{d\tau}, \tag{64}$$

$$= \dot{x}_j^\mu p_{j\mu} + \frac{1}{2} A_j^{22}(\tau) Q_{22}(x_j^\mu, p_j^\mu) + \frac{d\Lambda_j}{d\tau}. \tag{65}$$

We have shown in (30) that under the gauge transformation (61), A_i^{22} is related to A_j^{22}, by $A_j^{22} = \alpha^2 A_i^{22}$, since $\beta = 0$. Similarly $Q_{22}(x_j^\mu, p_j^\mu)$ must be related to $Q_{22}(x_i^\mu, p_i^\mu)$ by the inverse transformation, $Q_{22}(x_j^\mu, p_j^\mu) = \alpha^{-2} Q_{22}(x_i^\mu, p_i^\mu)$, so that the combination $A^{22}Q_{22}$ is invariant under the gauge transformation (61)

$$A_j^{22}(\tau) Q_{22}(x_j^\mu, p_j^\mu) = A_i^{22}(\tau) Q_{22}(x_i^\mu, p_i^\mu). \tag{66}$$

This is because A^{22} and Q_{22} are both members of the adjoint representation whose dot product $A^{kl}Q_{kl}$ remains invariant under the nonderivative parts of any gauge transformation. Since Q_{11} and Q_{12} are already identically zero, then the above relation must hold under the gauge transformation (61). After taking into account this identity, equating the expressions in Eqs. (64)–(65) establishes that the canonical transformation described in Eq. (62) is guaranteed and predicts that the total derivative $d\Lambda_{ij}/d\tau$ must be given by

$$\Lambda_{ji}(\tau) = \Lambda_j(x_j^\mu(\tau), p_j^\mu(\tau)) - \Lambda_i(x_i^\mu(\tau), p_i^\mu(\tau)) \qquad (67)$$

where the $\Lambda_i(x_i^\mu(\tau), p_i^\mu(\tau))$ have been computed and given explicitly for each i in Eqs. (39), (41), (43), (46), (56). The reader may verify this expression also directly from the five explicit canonical transformations $x_j^\mu = \mathcal{X}_j^\mu(x_i, p_i)$, $p_{j\mu} = \mathcal{P}_{j\mu}(x_i, p_i)$ given in the next section.

Finally, we should remark that a further consistency check for the dualities is to verify that the different constraints of the gauge fixed shadows transform into each other up to overall factors as indicated in Eqs. (19)–(20). This is evident from the remarks made above on how the generator Q_{22} transforms with an overall factor α^{-2} under the gauge transformation (61), and noting that this factor can be written as $\alpha^{-2} = (X_i^{+'}/X_j^{+'})^2$.

B. Explicit canonical transformations

Now we give explicitly each one of the duality relations between the five shadows under study. They were obtained through the methods described above.

1. Dualities (1↔2)

For the duality (1←2) between shadows 1 and 2 we first consider the transformation (61) by using Eqs. (39), (41) in the direction $M = 0$ and obtain the relation

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha(\tau) & 0 \\ \gamma(\tau) & \alpha^{-1}(\tau) \end{pmatrix} \begin{pmatrix} \frac{1+a}{2a} \\ \frac{-m_2^2}{2(x_2\cdot p_2)a} \end{pmatrix}. \qquad (68)$$

From this equation we determine both $\alpha = 2a/(1+a)$ and $\gamma = \frac{m_2^2}{2(x_2\cdot p_2)a}$ as functions of (x_2, p_2). We insert them back in Eq. (61) to obtain the canonical transformation (1←2) as follows:

massless relativistic (1) ← massive relativistic (2)

$$\begin{pmatrix} x_1^\mu \\ p_1^\mu \end{pmatrix} = \begin{pmatrix} \frac{2a}{1+a} & 0 \\ \frac{m_2^2}{2(x_2\cdot p_2)a} & \frac{1+a}{2a} \end{pmatrix} \begin{pmatrix} x_2^\mu \\ p_2^\mu \end{pmatrix}$$
$$a \equiv \sqrt{1 + \frac{m_2^2 x_2^2}{(x_2\cdot p_2)^2}}\ , \quad \Lambda_{12} = (a-1)\,(x_2\cdot p_2) \qquad (69)$$

The inverse transformation (2←1) is given by the inverse matrix, but α and γ must be rewritten in terms of (x_1^μ, p_1^μ). After some algebra one gets $\frac{2a}{1+a} = 1 + \frac{m_2^2 x_1^2}{4(x_1\cdot p_1)^2}$, which yields the inverse matrix as follows:

massive relativistic (2) ← massless relativistic (1)

$$\begin{pmatrix} x_2^\mu \\ p_2^\mu \end{pmatrix} = \begin{pmatrix} \left(1 + \frac{m_2^2 x_1^2}{4(x_1\cdot p_1)^2}\right)^{-1} & 0 \\ -\frac{m_2^2}{2(x_1\cdot p_1)} & \left(1 + \frac{m_2^2 x_1^2}{4(x_1\cdot p_1)^2}\right) \end{pmatrix} \begin{pmatrix} x_1^\mu \\ p_1^\mu \end{pmatrix} \qquad (70)$$
$$\Lambda_{21} = -\frac{2m_2^2 x_1^2 (x_1\cdot p_1)}{4(x_1\cdot p_1)^2 + m_2^2 x_1^2}$$

2. Dualities (1↔3)

To determine the duality transformation (1←3) between shadows 1 and 3 we first consider the transformation (61) by using Eqs. (39), (43) in the direction $M = 0$ and obtain the relation

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha(\tau) & 0 \\ \gamma(\tau) & \alpha^{-1}(\tau) \end{pmatrix} \begin{pmatrix} t_3 \\ m_3 \end{pmatrix} \qquad (71)$$

which determines $\alpha = (t_3)^{-1}$ and $\gamma = -m_3$. We insert this back in Eq. (61) to obtain the canonical transformation (1←3) as follows:

$$\boxed{\begin{array}{c}\text{massless relativistic (1) } \leftarrow \text{massive nonrelativistic (3)}\\ \hline \begin{pmatrix}\mathbf{x}_1\\ \mathbf{p}_1\end{pmatrix}=\begin{pmatrix}(t_3)^{-1} & 0\\ -m_3 & t_3\end{pmatrix}\begin{pmatrix}\mathbf{x}_3\\ \mathbf{p}_3\end{pmatrix}\\ \begin{pmatrix}x_1^0\\ p_1^0\end{pmatrix}=\begin{pmatrix}(t_3)^{-1} & 0\\ -m_3 & t_3\end{pmatrix}\begin{pmatrix}s\\ 0\end{pmatrix}\\ s^2\equiv(\mathbf{x}_3)^2-\frac{2t_3}{m_3}\mathbf{x}_3\cdot\mathbf{p}_3+\frac{2(t_3)^2}{m_3}h_3\\ \Lambda_{13}=t_3h_3-\mathbf{x}_3\cdot\mathbf{p}_3\end{array}} \tag{72}$$

The inverse transformation (3←1) is given by the inverse matrix, but α must be rewritten in terms of (x_1^μ, p_1^μ) as $\alpha=-m_3x_1^0/p_1^0$, so that the inverse transformation takes the form

$$\boxed{\begin{array}{c}\text{massive nonrelativistic (3) } \leftarrow \text{massless relativistic (1)}\\ \hline \begin{pmatrix}\mathbf{x}_3\\ \mathbf{p}_3\end{pmatrix}=\begin{pmatrix}-\frac{p_1^0}{m_3x_1^0} & 0\\ m_3 & -\frac{m_3x_1^0}{p_1^0}\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{p}_1\end{pmatrix}\\ t_3=-\frac{p_1^0}{m_3x_1^0}\\ h_3=\frac{m_3}{2}\left(\mathbf{x}_1^2+(x_1^0)^2\right)-\frac{m_3x_1^0}{p_1^0}\mathbf{x}_1\cdot\mathbf{p}_1\\ \Lambda_{31}=-\frac{p_1^0}{2x_1^0}x_1^\mu x_{1\mu}\end{array}} \tag{73}$$

3. Dualities (2↔3)

To determine the duality transformation (3←2) between shadows 2 and 3 we can use shadow 1 as an intermediate step since we already know the transformations back and forth (1↔2) and (1↔3). Hence we construct (3←2) via the steps (1←2) followed by (3←1). This gives the following explicit transformation for (3←2):

$$\boxed{\begin{array}{c}\text{massive nonrelativistic (3) } \leftarrow \text{massive relativistic (2)}\\ \hline \begin{pmatrix}\mathbf{x}_3\\ \mathbf{p}_3\end{pmatrix}=\begin{pmatrix}-\frac{1}{m_3}w(x_2,p_2) & 0\\ \frac{p_2^0}{x_2^0}\frac{m_3}{w(x_2,p_2)} & -\frac{m_3}{w(x_2,p_2)}\end{pmatrix}\begin{pmatrix}\mathbf{x}_2\\ \mathbf{p}_2\end{pmatrix}\\ w(x_2,p_2)\equiv\left(\frac{m_2^2}{2(x_2\cdot p_2)a}+\frac{1+a}{2a}\frac{p_2^0}{x_2^0}\right),\ a\equiv\sqrt{1+\frac{m_2^2x_2^2}{(x_2\cdot p_2)^2}}\\ t_3=-\frac{1}{m_3}w(x_2,p_2)\frac{1+a}{2a}\\ h_3=\frac{m_3}{w(x_2,p_2)}\left(\frac{p_2^0}{x_2^0}\frac{a}{1+a}\frac{x_2^2}{}-a(x_2\cdot p_2)\right)\\ \Lambda_{32}=\frac{a-1}{2}(x_2\cdot p_2)-\frac{p_2^0}{2x_2^0}x_2^2\end{array}} \tag{74}$$

The inverse transformation (2←3) is built in the same manner, with the result massive relativistic (2) ← massive nonrelativistic (3)

$$\begin{pmatrix}\mathbf{x}_2\\ \mathbf{p}_2\end{pmatrix}=\begin{pmatrix}\left(t_3+\frac{m_2^2}{2m_3}\frac{\mathbf{x}_3\cdot\mathbf{p}_3-t_3h_3}{(2t_3h_3-\mathbf{x}_3\cdot\mathbf{p}_3)^2}\right)^{-1} & 0\\ -\left(m_3+\frac{m_2^2h_3}{2(2t_3h_3-\mathbf{x}_3\cdot\mathbf{p}_3)^2}\right) & \left(t_3+\frac{m_2^2}{2m_3}\frac{\mathbf{x}_3\cdot\mathbf{p}_3-t_3h_3}{(2t_3h_3-\mathbf{x}_3\cdot\mathbf{p}_3)^2}\right)\end{pmatrix}\begin{pmatrix}\mathbf{x}_3\\ \mathbf{p}_3\end{pmatrix}$$

$$x_2^0=\left(t_3+\frac{m_2^2}{2m_3}\frac{\mathbf{x}_3\cdot\mathbf{p}_3-t_3h_3}{(2t_3h_3-\mathbf{x}_3\cdot\mathbf{p}_3)^2}\right)^{-1}s\left(x_3,p_3\right)$$

$$p_2^0=-\left(m_3+\frac{m_2^2h_3}{2(2t_3h_3-\mathbf{x}_3\cdot\mathbf{p}_3)^2}\right)s\left(x_3,p_3\right)\tag{75}$$

$$s\left(x_3,p_3\right)\equiv\pm\sqrt{\mathbf{x}_3^2-\frac{2t_3}{m_3}\mathbf{x}_3\cdot\mathbf{p}_3+\frac{2t_3^2}{m_3}h_3}$$

$$\Lambda_{23}=\left(x_2\cdot p_2\right)\left(a\left(x_2,p_2\right)-1\right)-\left(\mathbf{x}_3\cdot\mathbf{p}_3-t_3h_3\right)$$

4. Dualities (1↔4)

For the duality (1←4) between shadows 1 and 4 we first consider the transformation (61) by using Eqs. (39), (45) in the direction $M=0$ and obtain the relation

$$\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\alpha & 0\\ \gamma & \alpha^{-1}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{-4m_4h_4}}[|\mathbf{x}_4|\sqrt{-2m_4h_4}\ \sin\ u+\mathbf{x}_4\cdot\mathbf{p}_4(\cos\ u+1)]\\ \frac{1}{\sqrt{-4m_4h_4}}[2m_4h_4|\mathbf{x}_4|+m_4e_4^2(\cos\ u+1)]\end{pmatrix}.\tag{76}$$

This determines both α and γ as functions of $(\mathbf{x}_4,\mathbf{p}_4,t_4,h_4)$:

$$\alpha=\frac{\sqrt{-4m_4h_4}}{|\mathbf{x}_4|\sqrt{-2m_4h_4}\sin u+\mathbf{x}_4\cdot\mathbf{p}_4(\cos u+1)},\qquad \gamma=-\frac{2m_4h_4|\mathbf{x}_4|+m_4e_4^2(\cos u+1)}{\sqrt{-4m_4h_4}}.\tag{77}$$

We insert them back in Eq. (61) to obtain the canonical transformation (1←4) as follows:

massless relativistic (1) ← H-atom (4)

$$\begin{pmatrix}\mathbf{x}_1\\ \mathbf{p}_1\end{pmatrix}=\begin{pmatrix}\alpha & 0\\ \gamma & \alpha^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{x}_4\\ \mathbf{p}_4\end{pmatrix}$$

$$\begin{pmatrix}x_1^0\\ p_1^0\end{pmatrix}=\begin{pmatrix}\alpha & 0\\ \gamma & \alpha^{-1}\end{pmatrix}\begin{pmatrix}|\mathbf{x}_4|\cos u-\frac{\mathbf{x}_4\cdot\mathbf{p}_4\ \sin u}{\sqrt{-2m_4h_4}}\\ -\frac{m_4e_4^2\ \sin u}{|\mathbf{x}_4|\sqrt{-2m_4h_4}}\end{pmatrix}\tag{78}$$

$$u\equiv\frac{\sqrt{-2m_4h_4}}{m_4e_4^2}(\mathbf{x}_4\cdot\mathbf{p}_4-2h_4t_4).$$

$$\Lambda_{14}=3h_4t_4-2\left(\mathbf{x}_4\cdot\mathbf{p}_4\right)$$

To construct the inverse transformation we must rewrite the α, γ of Eqs. (77) as functions of (x_1^μ,p_1^μ) by using (78). This gives

$$\alpha(x_1,p_1)=\frac{m_4e_4^2(1+\frac{1}{2}(x_1^0-|\mathbf{x}_1|)^2)}{(|\mathbf{x}_1|p_1^0-\mathbf{x}_1\cdot\mathbf{p}_1)\sqrt{2}L^{\prime0}},\qquad \gamma(x_1,p_1)=m_4e_4^2\left[\frac{1}{\sqrt{2}L^{\prime0}}-\frac{1}{|\mathbf{x}_1|(|\mathbf{x}_1|p_1^0-\mathbf{x}_1\cdot\mathbf{p}_1)}\right]$$

$$\text{with}\quad \sqrt{2}L^{\prime0}\equiv p_1^0(1+\frac{1}{2}(x_1^0-|\mathbf{x}_1|)^2)+x_1^0(|\mathbf{x}_1|p_1^0-\mathbf{x}_1\cdot\mathbf{p}_1).\tag{79}$$

The inverse transformation (4←1) is then

$$\boxed{\begin{array}{c}\text{H-atom (4)} \leftarrow \text{ massless relativistic (1)}\\ \hline \begin{pmatrix}\mathbf{x}_4\\ \mathbf{p}_4\end{pmatrix}=\begin{pmatrix}\alpha^{-1}(x_1,p_1) & 0\\ -\gamma(x_1,p_1) & \alpha(x_1,p_1)\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{p}_1\end{pmatrix}\\ t_4=\alpha^{-1}(x_1,p_1)\\ h_4=-\frac{1}{2}x_1^2\gamma(x_1,p_1)+(x_1\cdot p_1)\,\alpha(x_1,p_1)\\ \Lambda_{41}=-\Lambda_{14}\end{array}} \tag{80}$$

5. Dualities (1↔5) for general $V(x^2)$

For the duality (1←5) between shadows 1 and 5 we first consider the transformation (61) by using Eqs. (39), (49) in the direction $M=0$ and obtain the relation

$$\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\alpha & 0\\ \gamma & \alpha^{-1}\end{pmatrix}\begin{pmatrix}A\\ \frac{A}{x^2}(x_5\cdot p_5-\phi)\end{pmatrix}\text{with}\quad\begin{cases}\phi(x_5,p_5)\equiv(x_5\cdot p_5)(1+x_5^2V(x_5^2)(x_5\cdot p_5)^{-2})^{1/2}\\ A(x_5,p_5)\equiv F(\phi)\sqrt{x_5^2}\exp\left[-\frac{1}{2}\int^{x_5^2}\frac{du}{u}(1-\phi^{-2}uV(u))^{-1/2}\right]\end{cases}. \tag{81}$$

This determines $\alpha(x_5,p_5)=A^{-1}$ and $\gamma(x_5,p_5)=-\frac{A}{x^2}(x_5\cdot p_5-\phi)$ as functions of (x_5,p_5). We insert them back in Eq. (61) to obtain the canonical transformation (1←5) as follows:

$$\boxed{\begin{array}{c}\text{massless relativistic (1)} \leftarrow \text{ relativistic potential (5), general } V(x^2)\\ \hline \begin{pmatrix}x_1^\mu\\ p_1^\mu\end{pmatrix}=\begin{pmatrix}A^{-1} & 0\\ -\frac{A}{x_5^2}(x_5\cdot p_5-\phi) & A\end{pmatrix}\begin{pmatrix}x_5^\mu\\ p_5^\mu\end{pmatrix}\\ \Lambda_{15}=-\frac{1}{\phi}\int^{x_5^2}du V(u)\left(1-\phi^{-2}uV(u)\right)^{-1/2}+2\int^{\phi}dz\; z\frac{d}{dz}\ln F(z)\end{array}} \tag{82}$$

To construct the inverse transformation we must rewrite α, γ, or equivalently A, ϕ, in terms of (x_1,p_1). To construct this, it is useful to remember that the SO$(d,2)$ generator $L^{+'-'}$ is invariant under the Sp$(2,R)$ gauge transformations. It takes the form $L^{+'-'}=x_1\cdot p_1$ in shadow 1 while it is given by $L^{+'-'}=\phi(x_5,p_5)$ in shadow 5, but due to the gauge invariance of $L^{+'-'}$ we have

$$L^{+'-'}=\phi=(x_5\cdot p_5)(1+x_5^2V(x_5^2)(x_5\cdot p_5)^{-2})^{1/2}=x_1\cdot p_1. \tag{83}$$

Thus, we obtain $\phi=x_1\cdot p_1$, and $x_5^2=Ax_1^2$, while $A(x_1,p_1)$ is determined implicitly by solving the following algebraic equation [which is a rewriting of (55) after inserting $x_5^\mu=Ax_1^\mu$ from (49)]:

$$\int^{x_1^2A^2}\frac{du}{u}(1-(x_1\cdot p_1)^{-2}uV(u))^{-1/2}=\ln[x_1^2F^2(x_1\cdot p_1)]. \tag{84}$$

Inserting these results, we obtain the transformation (5←1) as follows:

$$\boxed{\begin{array}{c}\text{relativistic potential (5), general } V\left(x^2\right) \leftarrow \text{massless relativistic (1)}\\ \hline \begin{pmatrix} x_5^\mu \\ p_5^\mu \end{pmatrix} = \begin{pmatrix} A\left(x_1,p_1\right) & 0 \\ \frac{x_1\cdot p_1}{Ax_1^2}\left(\sqrt{1-A^2x_1^2V\left(A^2x_1^2\right)\left(x_1\cdot p_1\right)^{-2}}-1\right) & A^{-1}\left(x_1,p_1\right)\end{pmatrix}\begin{pmatrix} x_1^\mu \\ p_1^\mu \end{pmatrix}\\ \Lambda_{51}=\frac{1}{\phi}\int^{x_5^2}duV\left(u\right)\left(1-\phi^{-2}uV\left(u\right)\right)^{-1/2}+2\int^{\phi}dz\ z\frac{d}{dz}\ln F\left(z\right)\end{array}} \tag{85}$$

6. Dualities (1↔5) for $V(x^2)=c(x_5^2)^b$

To be completely explicit, we specialize to $V(x^2)=c(x^2)^b$. Then from Eqs. (81), (83), (84) we compute the explicit forms for ϕ, A written in terms of the phase spaces of either shadow:

$V=c\left(x_5^2\right)^b$	as function of $\left(x_5,p_5\right)$	as function of $\left(x_1,p_1\right)$
$\phi=$	$\left(x_5\cdot p_5\right)\sqrt{1+c\left(x_5^2\right)^{1+b}\left(x_5\cdot p_5\right)^{-2}}$	$=\left(x_1\cdot p_1\right)$
$A=$	$\left(\lvert x_5\cdot p_5\rvert+\sqrt{\left(x_5\cdot p_5\right)^2+c\left(x_5^2\right)^{1+b}}\right)^{\frac{1}{1+b}}\frac{F(\phi)}{c^{1/(2+2b)}}$	$=\left[\frac{4(x_1\cdot p_1)^2}{\left(1+\left(F^2(\phi)x_1^2\right)^{1+b}\right)^2}\right]^{\frac{1}{2+2b}}\frac{F(\phi)}{c^{1/(2+2b)}}$

(86)

A quick way of proving the equality of the two forms of ϕ is to use the gauge invariance of the L^{MN}; then note that $L^{+'-'}=\phi$ when computed in shadow 5 of Eq. (49), and $L^{+'-'}=(x_1\cdot p_1)$ when evaluated in shadow 1 of Eq. (39). Hence the result above for ϕ is obtained. Inserting this in Eqs. (82), (85) gives the duality transformations (5↔1) for the potential $V=c(x^2)^b$ as follows:

$$\boxed{\begin{array}{c}\text{massless relativistic (1)} \leftarrow \text{ relativistic potential (5), } V=c\left(x^2\right)^b\\ \hline \begin{pmatrix} x_1^\mu \\ p_1^\mu \end{pmatrix} = \begin{pmatrix} \left(A\left(x_5,p_5\right)\right)^{-1} & 0 \\ \frac{c\left(x_5^2\right)^b\ A(x_5,p_5)}{\phi(x_5,p_5)} & A\left(x_5,p_5\right)\end{pmatrix}\begin{pmatrix} x_5^\mu \\ p_5^\mu \end{pmatrix}\\ \hline \phi\equiv\left(x_5\cdot p_5\right)\sqrt{1+c\left(x_5^2\right)^{1+b}\left(x_5\cdot p_5\right)^{-2}}\end{array}} \tag{87}$$

where the $F(\phi)$ that appears in $A(x_5,p_5)$ is an arbitrary function of its argument. Similarly, the inverse transformation is

$$\boxed{\begin{array}{c}\text{relativistic potential (5), } V=c\left(x^2\right)^b \leftarrow \text{massless relativistic (1)}\\ \hline \begin{pmatrix} x_5^\mu \\ p_5^\mu \end{pmatrix} = \begin{pmatrix} A\left(x_1,p_1\right) & 0 \\ -\frac{2}{A(x_1,p_1)}\frac{x_1\cdot p_1}{x_1^2}\frac{\left(x_1^2F^2(\phi)\right)^{1+b}}{1+\left(x_1^2F^2(\phi)\right)^{1+b}} & \frac{1}{A(x_1,p_1)}\end{pmatrix}\begin{pmatrix} x_1^\mu \\ p_1^\mu \end{pmatrix},\\ F\left(\phi\right)\ \text{arbitrary function of its argument } \phi=x_1\cdot p_1,\\ A\left(x_1,p_1\right)\ \text{given in Eq.(86).}\end{array}} \tag{88}$$

One can verify that these expressions satisfy

$$(p_5^2+c(x_5^2)^b+\cdots)=A^{-2}(p_1^2+\cdots), \tag{89}$$

where $\cdots$ represent the background fields in the respective shadows. Then, by using the constraint in shadow 1, $p_1^2 + \cdots = 0$, the constraint in shadow 5 is automatically satisfied and vice versa.

For a consistency check one may specialize to $b = 0$, $c = m_2^2$ and $F(\phi) = m_2/(2|\phi|)$, to see that the $(1\leftrightarrow5)$ duality expressions in this subsection agree with the duality expressions $(1\leftrightarrow2)$ given above.

VI. DUALITY INVARIANTS AND HIDDEN SO(d,2)

In Eqs. (37)–(38) we argued that the SO$(d,2)$ generators $L^{MN} = X^M P^N - X^N P^M$ are gauge invariant under the subgroup of Sp$(2,R)$ gauge transformations that correspond to the duality transformations. Therefore, it is predicted that any function of the L^{MN} must be invariant under the duality transformations. Namely, if one inserts the gauge fixed versions of (X_i^M, P_i^M) for $i = 1, 2, 3, 4, 5$, given in Eqs. (39), (41), (43), (45), (49), into $L^{MN} = X^M P^N - X^N P^M$, they must equal each other:

$$L^{MN} = X_1^{[M} P_1^{N]} = X_2^{[M} P_2^{N]} = X_3^{[M} P_3^{N]} = X_4^{[M} P_4^{N]} = X_5^{[M} P_5^{N]}. \tag{90}$$

We list below the different shadow forms of the $L^{MN} = (L^{+'-'}, L^{+'\mu}, L^{-'\mu}, L^{\mu\nu})$ for each of the five shadows. These L^{MN} in the shadows satisfy the Lie algebra of SO$(d,2)$ under Poisson brackets computed in terms of $\{x_i^\mu, p_{i\mu}\}$ in each shadow i. The closure of the SO$(d,2)$ algebra in each shadow holds whether or not these L^{MN} are conserved, that is whether background fields are present or not.

Hence each shadow provides a new phase space representation of SO$(d,2)$. One of these, shadow 1, which we sometimes call the conformal shadow, yields the familiar form of conformal transformations SO$(d,2)$ in d dimensions. Namely $\delta_\omega x_1^\mu = \omega_{MN}\{L^{MN}, x_1^\mu\}$, computed with the Poisson brackets of shadow 1, gives precisely the infinitesimal SO$(d,2)$ conformal transformations of x_1^μ. However, we claim that all shadows, including the shadows with mass, also provide a representation space for SO$(d,2)$, with an action of L^{MN} on that phase space that is the dual of a conformal transformation in shadow 1.

The L^{MN} are not necessarily symmetry generators of the full theory, but there are specialized forms of the theory in which they do generate the natural SO$(d,2)$ rotation type symmetry of flat $d+2$ dimensions. First, it is important to note that the L^{MN} are generators of an SO$(d,2)$ *symmetry* of the first two constraints, since they do commute with each other under Poisson brackets in the bulk, $\{L^{MN}, X\cdot X\} = 0$ and $\{L^{MN}, X\cdot P\} = 0$. The third constraint, $(P^2 + \cdots) = 0$, does not commute with L^{MN} if the background fields $\cdots$ are present in general. But, in the case when all background fields vanish, $\cdots = 0$, the third constraint, and indeed the full 2T Lagrangian, is invariant under SO$(d,2)$ transformations. Therefore, by Noether's theorem, the L^{MN} must be conserved generators $dL^{MN}/d\tau = 0$ of the SO$(d,2)$ symmetry in that case. Since the L^{MN} are gauge (or duality) invariants, then their shadows listed below, $L^{MN} = (L^{+'-'}, L^{+'\mu}, L^{-'\mu}, L^{\mu\nu})$, must also be conserved in each shadow by virtue of being the generators of a hidden SO$(d,2)$ symmetry in the case of no background fields. Prior to the introduction of 2T physics in 1998, the presence of a hidden SO$(d,2)$ symmetry in shadows 2, 3, 4, 5 without backgrounds had not been noticed in 1T physics. This hidden symmetry in the other shadows, which is a close cousin of the familiar conformal symmetry in shadow 1, is just as powerful and just as fundamental as conformal symmetry. Indeed all forms of this hidden symmetry in the shadows are the same symmetry of the bulk, which turns out to be realized *in the same irreducible unitary representation of* SO$(d,2)$ in each shadow, as further discussed below at the classical or quantum levels.

In the following we use definitions for symbols given earlier in the paper, which include

Shadow #	Definitions
2	$a \equiv \left[1 + m_2^2 x_2^2 (x_2\cdot p_2)^{-2}\right]^{1/2}$
3	$s^2 \equiv \mathbf{x}_3^2 - \frac{2t_3}{m_3}\mathbf{x}_3\cdot\mathbf{p}_3 + \frac{2t_3^2}{m_3}h_3$
4	$u \equiv \frac{\sqrt{-2m_4h_4}}{m_4e_4^2}(\mathbf{x}_4\cdot\mathbf{p}_4 - 2h_4t_4)$
5	$\phi \equiv (x_5\cdot p_5)\left[1 + x_5^2 V(x_5^2)(x_5\cdot p_5)^{-2}\right]^{1/2}$ $A \equiv F(\phi)\sqrt{x_5^2}\exp\left[-\frac{1}{2}\int^{x_5^2}\frac{du}{u}\left(1 - \frac{uV(u)}{\phi^2}\right)^{-1/2}\right]$

(91)

The dual forms of the gauge invariant $L^{+'-'}$ and $L^{\mu\nu}$ in the five shadows are (these are conserved if the backgrounds vanish)

(92)

Shadow #	$L^{+'-'}$	$L^{\mu\nu}$
1	$x_1 \cdot p_1$	$x_1^\mu p_1^\nu - x_1^\nu p_1^\mu$
2	$(x_2 \cdot p_2)\, a$	$x_2^\mu p_2^\nu - x_2^\nu p_2^\mu$
3	$2t_3 h_3 - \mathbf{x}_3 \cdot \mathbf{p}_3$	$L^{0i} = s\mathbf{p}_3^i$ $L^{ij} = \mathbf{x}_3^i \mathbf{p}_3^j - \mathbf{x}_3^j \mathbf{p}_3^i$
4	$-(\mathbf{x}_4 \cdot \mathbf{p}_4)\cos u + \frac{2m_4h_4\lvert\mathbf{x}_4\rvert + m_4e_4^2}{\sqrt{-2m_4h_4}}\sin u$	$L^{0i} = \left[\begin{pmatrix} \lvert\mathbf{x}_4\rvert\cos u \\ -\frac{\mathbf{x}_4\cdot\mathbf{p}_4}{\sqrt{-2m_4h_4}}\sin u \end{pmatrix}\mathbf{p}_4^i + \frac{m_4e_4^2}{\lvert\mathbf{x}_4\rvert\sqrt{-2m_4h_4}}\sin u\ \mathbf{x}_4^i\right]$ $L^{ij} = \mathbf{x}_4^i \mathbf{p}_4^j - \mathbf{x}_4^j \mathbf{p}_4^i$
5	$(x_5 \cdot p_5)\sqrt{1 + x_5^2 V(x_5^2)(x_5\cdot p_5)^{-2}}$	$x_5^\mu p_5^\nu - x_5^\nu p_5^\mu$

Similarly, the dual forms of the gauge invariant $L^{\pm'\mu}$ in the five shadows are (these are conserved if the backgrounds vanish)

(93)

Shadow #	$L^{+'\mu}$	$L^{-'\mu}$
1	p_1^μ	$\frac{1}{2}x_1^2 p_1^\mu - (x_1 \cdot p_1)\, x_1^\mu$
2	$\frac{1+a}{2a}p_2^\mu + \frac{m_2^2}{2(x_2\cdot p_2)a}x_2^\mu$	$\frac{a}{1+a}\left[x_2^2 p_2^\mu - (1+a)(x_2\cdot p_2)\, x_2^\mu\right]$
3	$L^{+'0} = -m_3 s$ $L^{+'i} = t_3\mathbf{p}_3^i - m_3\mathbf{x}_3^i$	$L^{-'0} = -h_3 s$ $L^{-'i} = \frac{1}{m_3}(\mathbf{x}_3\cdot\mathbf{p}_3 - t_3h_3)\,\mathbf{p}_3^i - h_3\mathbf{x}_3^i$
4	$\frac{1}{\sqrt{2}}\left(L^{0'\mu} + L^{1'\mu}\right)$, see (94)	$\frac{1}{\sqrt{2}}\left(L^{0'\mu} - L^{1'\mu}\right)$, see (94)
5	$Ap_5^\mu - \frac{A}{x_5^2}(x_5\cdot p_5 - \phi)\, x_5^\mu$	$\frac{1}{2A}\left[x_5^2 p_5^\mu - (x_5\cdot p_5 + \phi)\, x_5^\mu\right]$

where for shadow 4 it is more convenient to give $L^{0'\mu}$, $L^{1'\mu}$ instead of $L^{\pm'\mu}$, as follows:

Shadow #4, $L^{\pm'\mu} \equiv \frac{1}{\sqrt{2}}\left(L^{0'\mu} \pm L^{1'\mu}\right)$,

$$
\begin{aligned}
L^{0'0} &= -\frac{m_4e_4^2}{\sqrt{-2m_4h_4}} \\
L^{1'0} &= \sin u\ \mathbf{x}_4\cdot\mathbf{p}_4 - \frac{\lvert\mathbf{x}_4\rvert\cos u}{\sqrt{-2m_4h_4}}\left(\frac{m_4e_4^2}{\lvert\mathbf{x}_4\rvert} + 2m_4h_4\right) \\
L^{0'i} &= \left(\lvert\mathbf{x}_4\rvert\sin u + \frac{\mathbf{x}_4\cdot\mathbf{p}_4}{\sqrt{-2m_4h_4}}\cos u\right)\mathbf{p}_4^i - \frac{m_4e_4^2}{\lvert\mathbf{x}_4\rvert\sqrt{-2m_4h_4}}\cos u\ \mathbf{x}_4^i \\
L^{1'i} &= \frac{1}{\sqrt{-2m_4h_4}}\left[-(\mathbf{x}_4\cdot\mathbf{p}_4)\,\mathbf{p}_4^i + \left(\frac{m_4e_4^2}{\lvert\mathbf{x}_4\rvert} + 2m_4h_4\right)\mathbf{x}_4^i\right]
\end{aligned}
\tag{94}
$$

It is interesting to point out that, in shadow 4, the last listed $L^{1'i}$ is the famous Runge-Lenz vector, which is recognized as follows. After the τ-reparametrization gauge is chosen, $t_4(\tau) = \tau$ and the constraint of Eq. (47) is solved when backgrounds are absent as in Eq. (48), yielding $h_4 = \mathbf{p}_4^2/2m_4 - e_4^2/|\mathbf{x}_4|$, the $L^{1'i}$ takes the familiar form proportional to the Runge-Lenz vector, $L^{1'i} = [-(\mathbf{x}_4\cdot\mathbf{p}_4)\mathbf{p}_4^i + \mathbf{p}_4^2\mathbf{x}_4^i - \frac{m_4 e_4^2}{|\mathbf{x}_4|}\mathbf{x}_4^i]/\sqrt{-2m_4h_4}$. This conserved vector in the H-atom or in a planetary system is responsible for explaining the seemingly accidental systematic degeneracies of the H-atom levels, or why planetary ellipses do not precess. In the 2T physics approach the explanation is because the dimension labeled by $1'$ is an extra hidden space dimension that, in the bulk, is at the same footing as the other usual three space dimensions. There is a natural rotation symmetry SO(4) which is part of the hidden SO(4, 2) symmetry in the H-atom shadow. The conservation of angular momentum in all four dimensions, not only in the first three dimensions, is the real explanation of the interesting observations in the H-atom or in planetary systems. Another familiar case of hidden symmetry is the more familiar conformal symmetry SO(4, 2) of the conformal shadow. These are examples of the more general hidden symmetries and conservation rules that we are advocating in this section. For more examples of previously unknown hidden $SO(d,2)$ symmetries see [49,50].

Using the duality invariants L^{MN} (whether the theory has background fields or not) we make an infinite number of predictions that may be checked both experimentally and theoretically by comparing any function of the L^{MN} in various shadows. Namely, at the classical level we predict

$$f(L^{\mathrm{MN}}_{\mathrm{shadow}\,i}) = f(L^{\mathrm{MN}}_{\mathrm{shadow}\,j}), \quad \text{any function } f, \quad \text{any dual pair } (i\leftrightarrow j).$$

The functions f need not be $SO(d,2)$ invariants. For example, we may take any function of just $L^{+'-'}$ and use the corresponding expressions that are listed in the table above to make predictions that follow from our dualities. Some such functions are the Casimir invariants of $SO(d,2)$; for example the quadratic Casimir is $C_2 = \frac{1}{2}L^{MN}L_{MN}$. At the classical level we find $C_2 = X^2P^2 - (X\cdot P)^2 = 0$, since $X^2 = 0$ and $X\cdot P = 0$ is satisfied in every shadow *for any set of background fields* (namely without constraining P^2, leaving it off shell). The same vanishing result, with off-shell P^2, is found for all higher Casimirs at the classical level, $C_n = \frac{1}{n!}\mathrm{Tr}((iL)^n) = 0$, $n = 2,4,6,\ldots$, where L^M_N is treated like a matrix to evaluate the trace.

However, at the quantum level there are quantum ordering issues that must be resolved in order to satisfy Hermiticity of the L^{MN} and the $SO(d,2)$ Lie algebra by using the quantum commutators for the operators (X^M, P^M). Hermiticity implies that at the quantum level we deal with unitary representations of $SO(d,2)$. These requirements lead to nonzero but definite eigenvalues for all the Casimir operators in the physical subspace. The gauge invariant physical quantum states are those that satisfy the vanishing of the $Sp(2,R)$ generators, $X^2|\mathrm{phys}\rangle = 0$ and $(X\cdot P + P\cdot X)|\mathrm{phys}\rangle = 0$, in $SO(d,2)$ covariant quantization. The third $Sp(2,R)$ generator $Q_{22} = (P^2 + \cdots)$ is to be imposed as well, but since the background fields $\cdots$ are not yet specified, we consider P^2 to be off shell, namely so far unconstrained. This definition of physical states is compatible with the duality transformations which do not alter the Q_{22} constraint. In the physical sector as defined, we obtain a definite numerical eigenvalue for the $SO(d,2)$ quadratic Casimir operator

$$C_2|\mathrm{phys}\rangle = \left(1 - \frac{d^2}{4}\right)|\mathrm{phys}\rangle, \quad P^2 \text{off shell}.$$

To see how this result is obtained we construct the Hermitian quantum quadratic Casimir operator and reorder operator factors as follows:

$$C_2 = \frac{1}{2}L^{MN}L_{MN} = \begin{bmatrix} P^2X^2 + i(X\cdot P + P\cdot X) \\ -\frac{1}{4}(X\cdot P + P\cdot X)^2 + (1 - d^2/4) \end{bmatrix},$$

where X^2 has been pulled to the right and $X\cdot P$ has been written in Hermitian form. Applying this on physical states, we see that all operator parts vanish, leaving behind a constant eigenvalue. Note that no constraint has been imposed on P^2; hence the result $C_2 \to (1 - \frac{d^2}{4})$ for physical states works for any set of background fields. Similarly, for physical states we get nonzero *numerical eigenvalues* for all higher Casimirs C_n, for any set of background fields, since P^2 is off shell.

Due to the gauge invariance of the L^{MN}, the quantum theory in each shadow must agree with the $SO(d,2)$ covariant quantization just described. This requires that in each shadow the quantum ordering of the L^{MN} must be performed so that the same gauge invariant physical result is obtained for the Casimir eigenvalues independent of the shadow and independent of the background fields. Examples of how this quantum ordering is done in a few shadows were given in [49,50]. The numerical eigenvalues of the Casimirs obtained in covariant quantization already identify the specific unitary representation of $SO(d,2)$, which turns out to be the unitary *singleton representation* of $SO(d,2)$. This result was known before in the absence of background fields [49,50], and now we have established it for *any set of background fields and any shadow* since we have shown it holds for P^2 off shell.

We see now that the duality invariants must also hold at the quantum level in every shadow; namely, once the L^{MN} listed above are quantum ordered properly in two dual shadows $(i\leftrightarrow j)$, they are equal to each other as operators acting on a complete set of states in the physical Hilbert space

$$L^{MN}_{\mathrm{quantum}}(x_i, p_i) = L^{MN}_{\mathrm{quantum}}(x_j, p_j).$$

A subset of these identities is the numerical values of the $SO(d,2)$ Casimir operators being the same in every shadow, which is already guaranteed by the correct quantum ordering. But, well beyond this, all matrix elements between any set of quantum states for any function $f(L^{MN}_{\text{quantum}})$ must also yield identical results for either the left or the right side of this operator equation for every set of dual shadows $(i \leftrightarrow j)$. This is a huge set of *quantum relations* between 1T physics systems that can be tested as *predictions of our dualities derived from 2T physics*.

VII. SOLVING PROBLEMS USING DUALITIES

To illustrate the usefulness of our dualities we will solve the classical equations of motion of the constrained system in shadow 5 at zero background fields. The equations of motion and constraint are given by the 1T Lagrangian

$$L_5 = \dot{x}^\mu_5 p_{5\mu} - \frac{1}{2} A^{22}_5 (p^2_5 + V(x^2_5)). \tag{95}$$

As far as we know, the solution to the classical equations of motion of this constrained system is not available in the literature for general $V(x^2)$, or even for the specialized case $V(x^2) = c(x^2)^b$, except for $b = 0$ (massive particle) or $b = 1$ (constrained relativistic harmonic oscillator [51]). Furthermore, attempting to solve it with standard methods, such as choosing a gauge, and solving the constraint, leads to a time-dependent potential that is difficult or impossible to solve in closed form. However, by using our dualities, we obtain the desired analytic solutions easily as follows.

The equations that determine $(x^\mu_5(\tau), p_{5\mu}(\tau))$ are

$$\dot{x}^\mu_5 = A^{22}_5 p^\mu_5, \qquad \dot{p}^\mu_5 = -A^{22}_5 V'(x^2) x^\mu_5, \qquad p^2_5 + V(x^2_5) = 0. \tag{96}$$

We can make a gauge choice for τ-reparametrizations by making some convenient choice for $A^{22}_5(\tau)$ as an explicit function of τ. The solution we display below corresponds to an insightful gauge choice for the gauge field $A^{22}_5(\tau)$ that yields the analytic solution for any $V(x^2)$. It would be impossible to foresee such a gauge choice without our dualities. We proceed as follows.

First we transform the equations derived from (95) to shadow 1, where the equations of motion and constraints are easily solved for $x^\mu_1(\tau)$, $p_{1\mu}(\tau)$ in the gauge $A^{22}_1(\tau) = 1$ as follows:

$$x^\mu_1(\tau) = q^\mu_1 + \tau p^\mu_1, \text{ with constant } p^\mu_1, \text{ and constraint } p^2_1 = 0. \tag{97}$$

Then transforming this solution in shadow 1 back to shadow 5 we obtain the desired analytic solution. Thus, we use the duality $(5 \leftarrow 1)$ given in Eq. (85) to write $(x^\mu_5, p_{5\mu})$ in terms of $(x^\mu_1, p_{1\mu})$ and insert the solution (97) to obtain the time dependence of the classical trajectories $x^\mu_5(\tau)$ and $p_{5\mu}(\tau)$ that solve the equations of motion as well as the constraint, $p^2_5 + V(x^2_5) = 0$, derived from the Lagrangian L_5.

To be completely explicit, we specialize to $V(x^2) = c(x^2)^b$ and use the duality in Eq. (88) in which we insert the explicit time dependence for $x^\mu_1(\tau)$, $p_{1\mu}(\tau)$ given in (97). To make all τ dependence evident, we note that p^μ_1 is a constant that also satisfies $p^2_1 = 0$, while the other dot products have the following explicit τ dependence:

$$x^2_1(\tau) = q^2_1 + 2(q_1 \cdot p_1)\tau, \qquad p_1 \cdot x_1(\tau) = q_1 \cdot p_1. \tag{98}$$

The $F(\phi)$ that appears in Eq. (88) is evaluated as $F(q_1 \cdot p_1)$, so it is another τ-independent constant that we will denote simply as a constant F. Then the explicit τ dependence of the solution $(x^\mu_5(\tau), p_{5\mu}(\tau))$ follows from Eq. (88). After some simplifications it takes the form

$$x^\mu_5(\tau) = \left(\frac{4(q_1 \cdot p_1)^2}{cF^{2+2b}}\right)^{\frac{1}{2+2b}} \times \frac{q^\mu_1 + \tau p^\mu_1}{(F^{-2-2b} + (q^2_1 + 2\tau q_1 \cdot p_1)^{1+b})^{\frac{1}{1+b}}}, \tag{99}$$

and

$$p^\mu_5(\tau) = \left(\frac{cF^{2+2b}}{4(q_1 \cdot p_1)^2}\right)^{\frac{1}{2+2b}} \times \frac{p^\mu_1 + [q^2_1 p^\mu_1 - 2(q_1 \cdot p_1) q^\mu_1](q^2_1 + 2\tau q_1 \cdot p_1)^b}{(F^{-2-2b} + (q^2_1 + 2\tau q_1 \cdot p_1)^{1+b})^{\frac{b}{1+b}}}. \tag{100}$$

The τ in these expressions is the τ parameter conveniently gauge fixed for shadow 1 which *a priori* would not occur naturally as a gauge choice for shadow 5, although these are related to each other by τ reparametrizations. The τ-gauge in each shadow amounts to making a choice for $A^{22}(\tau)$. The gauge choice, $A^{22}_1(\tau) = 1$, was already made in shadow 1 when writing the solution for $(x^\mu_1(\tau), p_{1\mu}(\tau))$ in the form (97). Hence, using the solution as it stands, without further reparametrizing τ, amounts to making a definite choice for $A^{22}_5(\tau)$ which is given by the $\mathrm{Sp}(2,R)$ transformation in Eq. (30) with $\beta = 0$. Therefore, we must take $A^{22}_5(\tau) = \alpha^2(\tau) A^{22}_1(\tau) = (A(x_1(\tau), p_1(\tau)))^2 \times 1$, where $A(x_1, p_1)$ is given in Eq. (86),

$$A^{22}_5(\tau) = F^{-2b} \left[\frac{4(q_1 \cdot p_1)^2}{c(F^{-2-2b} + (q^2_1 + 2\tau q_1 \cdot p_1)^{1+b})^2}\right]^{\frac{1}{1+b}}. \tag{101}$$

So, the solution for $(x^\mu_5(\tau), p_{5\mu}(\tau))$ given in (99), (100), (101) is expressed in terms of this choice of τ-gauge applied to Eqs. (96) with $V(x^2) = c(x^2)^b$. This is a highly nontrivial gauge choice for $A^{22}_5(\tau)$ that would be hard to imagine without the guidance of the duality transformation. With this understanding of the τ-gauge, one may now check explicitly that the equations of motion and the constraints in Eq. (96) are indeed completely solved by the $(x_5^\mu(\tau), p_{5\mu}(\tau), A_5^{22}(\tau))$ given above.

Note that this solution is defined up to $\pm$ signs in different regions of τ since it contains branch cuts in the complex τ plane (see below for an example). This means that as τ changes, the corresponding expressions must be continued across the branch cuts in order to get continuously all the patches of the solution. An example of this for the case of $b = 1$ is a square-root branch cut as discussed below.

As a check, we may specialize to two cases, namely $b = 0, 1$, for which we do have a direct means of obtaining analytic solutions without using dualities, that we may compare to the general case given above.

(i) When $b = 0$, or $V(x_5^2) = c$, the Lagrangian L_5 reduces to the free massive relativistic particle with mass $c \equiv m^2$, satisfying the equations of motion, $\dot{x}_5^\mu = A_5^{22} p_5^\mu$, $\dot{p}_5^\mu = 0$, $p_5^2 + c = 0$, for which a direct solution is obtained as

$$b = 0{:}\ \begin{cases} x_5^\mu(\tau) = q_5^\mu + p_5^\mu \int^\tau A_5^{22}(\tau')d\tau', \\ p_5^\mu(\tau) = p_5^\mu, \\ p_5^2 + c = 0, \quad \text{with} \quad q_5^\mu, p_5^\mu \text{ constants.} \end{cases} \tag{102}$$

The solution in Eqs. (99), (100), (101) is indeed of this form when $b = 0$. This is verified by noting that in the gauge (101) we have

$$b=0{:}\ \int^\tau A_5^{22}(\tau')d\tau' = \frac{-2(q_1\cdot p_1)F^{-2}}{c[F^{-2} + (q_1 + p_1\tau)^2]}, \tag{103}$$

and that the constrained constants p_5^μ are parametrized in terms of the constants q_1^μ, p_1^μ with $p_1^2 = 0$. The choice of $A_5^{22}(\tau)$ in Eq. (101) as a function of τ to express the solution (99), (100) is clearly τ-gauge dependent, but this does not affect the gauge invariant physics. To see this in the case $b = 0$, we can write the solution *for any* $A_5^{22}(\tau)$ given in (102) in terms of the gauge invariant variables $(x_5^0, \mathbf{x}_5^i)$ and $(p_5^0, \mathbf{p}_5^i)$ as follows. First we write $x_5^0(\tau) = q_5^0 + p_5^0 \int^\tau A_5^{22}(\tau')d\tau'$, from which we solve for $\int^\tau A_5^{22}(\tau')d\tau' = (x_5^0(\tau) - q_5^0)/p^0$, and replace it in the solution for $\mathbf{x}_5^i(\tau)$ in (102) to obtain

$$\mathbf{x}_5^i = \mathbf{q}^i + \frac{\mathbf{p}_5^i}{p_5^0}(x_5^0 - q_5^0), \quad \text{with} \quad p_5^0 = \sqrt{\mathbf{p}_5^2 + c}.$$

This expression, written in terms of x_5^0, is gauge independent because it has the same form in terms of $(x_5^0(\tau), \mathbf{x}_5^i(\tau))$ for any choice of the function $A_5^{22}(\tau)$. Hence all the physical information about the solution is encoded in any gauge choice for $A_5^{22}(\tau)$, including the choice of gauge in (101) that made the solution (99), (100) possible for any b.

(ii) When $b = 1$, or $V(x_5^2) = cx_5^2$, the Lagrangian L_5 reduces to the constrained relativistic harmonic oscillator with frequency $\sqrt{c}$, satisfying the equations of motion, $\dot{x}_5^\mu = A_5^{22} p_5^\mu$, $\dot{p}_5^\mu = -A_5^{22} c x_5^\mu$, $p_5^2 + cx_5^2 = 0$, for which a direct solution is obtained as

$$b = 1{:}\ \begin{cases} x_5^\mu(\tau) = x_0^\mu \cos\left(\sqrt{c}\int^\tau A_5^{22}(\tau')d\tau'\right) + \frac{1}{\sqrt{c}} p_0^\mu \sin\left(\sqrt{c}\int^\tau A_5^{22}(\tau')d\tau'\right), \\ p_5^\mu(\tau) = p_0^\mu \cos\left(\sqrt{c}\int^\tau A_5^{22}(\tau')d\tau'\right) - \sqrt{c}x_0^\mu \sin\left(\sqrt{c}\int^\tau A_5^{22}(\tau')d\tau'\right) \\ p_0^2 + cx_0^2 = 0, \quad \text{with} \quad (x_0^\mu, p_0^\mu) \text{ constants.} \end{cases} \tag{104}$$

The solution in Eqs. (99), (100), (101) is indeed of this form when $b = 1$. This is verified by noting that the constrained constants (x_0^μ, p_0^μ) are parametrized in terms of the constants (q_1^μ, p_1^μ) with $p_1^2 = 0$, and that the τ dependence in the gauge (101) becomes

$$b = 1{:}\ \sqrt{c}\int^\tau A_5^{22}(\tau')d\tau' = \arctan\,[F^2 q_1^2 + 2F^2(q_1\cdot p_1)\tau].$$

Then

$$\cos\left(\sqrt{c}\int^\tau A_5^{22}(\tau')d\tau'\right) = \pm(1 + F^4(q_1^2 + 2\tau q_1\cdot p_1)^2)^{-1/2}$$
$$\sin\left(\sqrt{c}\int^\tau A_5^{22}(\tau')d\tau'\right) = F^2(q_1^2 + 2\tau q_1\cdot p_1)(1 + F^4(q_1^2 + 2\tau q_1\cdot p_1)^2)^{-1/2} \tag{105}$$

reproduces the expressions in (99), (100) with $b = 1$. Note that the $\pm$ in the cosine expression is related to recovering all the patches of the solution; this set of signs corresponds to the continuation of the expression across the square-root cut in the complex τ plane, as mentioned above more generally for any b. Again we may argue that the gauge choice for $A_5^{22}(\tau)$ is immaterial because the same gauge invariant physics is reproduced when written in terms of a gauge invariant choice of the time coordinate. For $b = 1$, a natural gauge invariant time coordinate $t(\tau)$ was given in Eq. (59) which amounts to $t(\tau) = \arctan(\sqrt{c}x^0(\tau)/p^0(\tau))$; in that case the solution (99), (100) is seen to capture all the motions of the nonrelativistic harmonic oscillator when rewritten in terms of t. To obtain the gauge invariant motion it is not necessary to explicitly solve for $t(\tau)$ or for the inverse $\tau(t)$; instead one may simply do a parametric plot of $(\mathbf{x}_5(\tau), t(\tau))$ and $(\mathbf{p}_5(\tau), t(\tau))$ which is gauge invariant. Hence, again, all the physical information about the solution is encoded in any gauge choice for $A_5^{22}(\tau)$, including the choice of gauge in (101) that made the solution (99), (100) possible for any b.

We do not know of another approach to solve the equations (96) analytically for $V(x^2) = c(x^2)^b$ with any b, c, except for the duality methods discussed in this section. Even more impressive is that we have obtained the analytic expressions for any $V(x^2)$. This demonstrates the utility and power of our system of dualities.

VIII. OUTLOOK

The study of dualities in 1T physics in d dimensions is equivalent to probing the properties of the underlying $d + 2$ dimensions including the extra $1 + 1$ dimensions. In this sense the extra dimensions are not hidden and can be investigated both experimentally and theoretically via the dualities directly in $3 + 1$ dimensions, with the guidance of 2T physics. It is apparent that the discussion given in this paper is just the tip of an iceberg of dualities that will take a long time to mine.

We have seen that the underlying meaning of the dualities $(i \leftrightarrow j)$, which were realized here as canonical transformations, is really gauge transformations from one fixed gauge to another fixed gauge for the gauge group $\mathrm{Sp}(2, R)$ acting in phase space in $d + 2$ dimensions. There definitely are gauge invariants (equivalently duality invariants). Specifically, any function of the L^{MN} is an invariant, as explained in Sec. VI, but time, Hamiltonian or more generally space-time, as interpreted by observers in any 1T shadow, is not among the gauge invariants. This is why 1T physics is different in different shadows, but yet there are deep relations and corresponding physical predictions among observers because of the underlying gauge symmetry. This concept of gauge symmetry in phase space is more general than the more familiar gauge symmetries, such as Yang-Mills or general coordinate transformations, that act locally only in space-time, rather than in phase space.

The reader may better grasp the significance of these statements by considering the concept of observers outlined in the introduction, i.e. that a given phase space $(x_i^\mu, p_{i\mu})$ in shadow i defines the frame of an observer that rides along with a particle on a worldline $(x_i^\mu(\tau), p_{i\mu}(\tau))$ which is embedded in the bulk in $d + 2$ dimensions. Such an observer, which may be said to live on a "screen i" or "boundary i" or "shadow i" in $1 + 1$ fewer dimensions, interprets all the *gauge invariant phenomena* occurring in the bulk in $d + 2$ dimensions from his/her perspective i, which is totally different than perspective j defined by another observer riding along worldline $(x_j^\mu(\tau), p_{j\mu}(\tau))$ that defines shadow j. For the five different shadows discussed in this paper we have seen that these five perspectives are indeed very different forms of 1T physics. Nevertheless each shadow, being just a gauge choice, captures all the gauge invariant information in the bulk. Therefore each shadow is holographic and hence must be dual to all other shadows. Indeed, we have shown that all shadows are closely related to one another by explicit dualities, and even more strongly, that all of the different 1T physics equations in various shadows are united and captured in a unified form of gauge invariant equations for the phase space (X^M, P_M) in the bulk in $d + 2$ dimensions, namely just $X^2 = 0$, $X \cdot P = 0$, and $P^2 + \cdots = 0$.

These ideas resonate with Einstein's concepts of observers in his thought experiments in various frames in special or general relativity in 1T physics. In our case, the analogous infinite set of frames are connected to each other by phase space transformations. This is a much larger set as compared to the set of frames connected to each other by only position space transformations. Hence the unification of 1T observers is much larger in the framework of 2T physics, while the corresponding unification of their diverse 1T equations is a unique set of equations in $d + 2$ dimensions, whose form is dictated by gauge symmetry in phase space.

More generally, the dualities generated by 2T physics go well beyond the realm of canonical transformations in 1T phase space because they include additional degrees of freedom besides (x, p). We remind the reader that the 2T formalism includes also the degrees of freedom of spinning systems [9–12], supersymmetric systems [14], twistors [15], fields in local field theory [18], [4], [19–21], and fields in phase space [24]. Hence 2T physics provides a new path to unification of 1T systems that is not available among the familiar concepts in 1T physics.

Extrapolating from a particle's phase space to the corresponding situation in field theory, the shadow i in field theory, derived from the field theory in the bulk in $d + 2$ dimensions (such as the standard model in $4 + 2$ dimensions [18]) describes all the physics as seen from the perspective of observer i, and similarly the shadow field theory j describes all the physical phenomena as seen from the perspective of observer j. These are the dual field theories that correspond to the duality $(i \leftrightarrow j)$ in phase space. These dual field theories predict the relationships between observers i and j and capture all the gauge invariant phenomena that the observers could measure, so the duality between 1T field theories predicted by 2T field theory leads to much broader verifiable tests of the entire approach described here. For some simple cases of $(i \leftrightarrow j)$ dualities discussed in the past (simpler than the five cases in this paper), examples of such dual field theories are developed in [48]. For the harder cases $(i \leftrightarrow j)$ discussed in this paper it is also possible in principle to construct the corresponding dual field theories. Our future goals include the construction of dual versions of the standard model and their use as new tools of investigation. The recent successful application in cosmology (involving transformations between different fixed Weyl gauges to solve and interpret cosmological equations) [1,2] is a simple example of this idea involving "Weyl dualities" in $3+1$ dimensions which originated from 2T physics gauge symmetries.

We have argued that phase space gauge symmetry in 2T physics offers superior unifying power compared to gauge symmetry in 1T physics. Having seen that even in simple classical mechanics systems there does exist a deeper unification, as shown in this paper, it is natural to expect that the same must also be true at the deepest level of physics principles. Hence 2T physics is likely to show the right path to the ultimate theory. Therefore, we posit that there is much benefit in developing further this formalism and in studying its consequences, such as the types of dualities discussed in this paper, and much more, in order to better understand the meaning of space-time and true unification. Along this path we should also benefit from new computational techniques in 1T physics that emerge from 2T physics. There is still much to be accomplished in 2T physics even in classical and quantum mechanics, not to mention field theory, string theory and M theory.

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