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AdS_5 with two boundaries and holography of $\mathcal{N}=4$ SYM theory

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According to the AdS/CFT correspondence, the $\mathcal{N}=4$ supersymmetric Yang-Mills theory is studied through its gravity dual, whose configuration has two boundaries at the opposite sides of the fifth coordinate. At these boundaries, in general, the four-dimensional (4D) metrics are different; then we expect different properties for the theory living in two boundaries. It is studied how these two different properties of the theory are obtained from a common 5D bulk manifold in terms of the holographic method. We could show in our case that the two theories on the different boundaries are described by the AdS_5, which is separated into two regions by a domain wall. This domain wall is given by a special point of the fifth coordinate. Some issues of the entanglement entropy related to this bulk configuration are also discussed.

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I. INTRODUCTION

Up to now, many holographic approaches to the $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory have been performed in terms of the dual supergravity [1–10]. These approaches are based on a conjectured correspondence between a conformal field theory on the boundary M_d of an asymptotic anti–de Sitter space (AdS_{d+1}) and string theories on the product of AdS_{d+1} with a compact manifold. In many cases, the boundary M_d is set as a Minkowski space-time, and the bulk manifolds have a structure that they have a boundary at the ultraviolet (UV) side of the dual d-dimensional conformal field theory (CFT). On the other hand, in the infrared side, they have a horizon. Then the holographic analyses for CFT in M_d are performed in the region between the horizon and the boundary for the gravity side.

In these approaches, the research has been extended to the SYM theory in the background of $M_4 = dS_4(\mathrm{AdS}_4)$ by introducing a 4D cosmological constant $[\Lambda_4 > 0(\Lambda_4 < 0)]$ [11–16] in the supergravity solutions. In the bulk supergravity solutions, Λ_4 appears as a free parameter in the step of solving the equations of motion. However, this parameter plays an important role, since the 4D geometry of the boundary is controlled by this parameter. Another important point is that the form of the bulk metric is also deformed by this parameter. As a result, we could see how the dynamical properties of the SYM theory are changed by the 4D geometry which is changed from the Minkowski space-time to $dS_4(\mathrm{AdS}_4)$.

Actually, in the cases of the dS_4 [11,12] and AdS_4 [13], we find quite different properties of the SYM theory from the one observed in the Minkowski space-time. For the dS_4 background, we observe a horizon in the infrared side of the fifth coordinate, and we find the phenomena similar to the one of the finite temperature SYM theory in the deconfinement phase. On the other hand, in the case of $\Lambda_4 < 0$ (for the AdS_4 boundary), we could find that the theory is in the confining phase [13]. Furthermore, we found that the meson spectrum obtained in our analysis is consistent with the one obtained in the usual field theory in AdS_4 [17].

Furthermore, we should notice that it is possible to introduce another free parameter in the bulk AdS_5 solution. This parameter is called dark radiation, which corresponds to the thermal excitation of SYM fields, and plays an important role in determining the (de)confining phase of the theory [14–16].

Our purpose in this article is to point out and discuss a characteristic holographic feature of the AdS_5 bulk solution. The point is that a second boundary appears in the solution with the AdS_4 boundary without horizons. It is found at the "infrared limit" ($r=0$),[1] which is opposite to the boundary at the "ultraviolet limit" ($r=\infty$). Here r denotes the fifth coordinate of the AdS_5. Then the gravity of the bulk AdS_5 is dual to the two theories living on the two boundaries separately.

We should notice that two boundaries are also seen in the case of the solution with dS_4 boundary. However, in this case, a horizon appears between the boundaries, and then we can restrict the region of the holographic dual to the one between the horizon and a boundary to study the dynamical properties of the theory living on the boundary. In the other

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[1]As shown below, the infrared limit does not mean the infrared limit of the SYM theory on the boundary.

half region, the same things are considered; however, the physics of the two boundaries might be independent of each other.[2] In the case of the AdS_4 boundary, on the other hand, there is no horizon as such a border between the two boundaries.

The problem in this case (AdS_4 boundary) is how the bulk manifold could provide dynamical properties of the two field theories, in other words, how we could get information of two theories separately from the common bulk geometry. This problem is resolved due to the presence of a sharp domain wall in the bulk. The dynamical properties of the boundary theory are found through various stringy objects embedded in the bulk as probes, since the probes are controlled by the bulk configuration which reflects the vacuum structure of the dual theory. In the present case, we could find that the embedded objects are confined in one side and never cross the wall to penetrate to the other side. Then, in this sense, the gravity duals for the two boundaries are separated clearly by this wall. Therefore, this wall separates the manifold to two regions which are surrounded by the boundaries at $r = 0$ and $r = \infty$, respectively. They correspond to two dual field theories of the two boundaries. The situation is shown in Fig. 1, where the two regions are shown by IR and UV, respectively. We address the holographic problem from this viewpoint and examine the robustness of the wall.

This statement would be correct at the level of classical in the gravity side of the bulk. When we consider the quantum fluctuations of the bulk, they could cross the wall, since there are no obstacles to prevent their propagation like a singularity at this wall point. This problem, we will discuss in a future article.

When we add the dark radiation, our solutions of Friedmann-Robertson-Walker (FRW) type are modified. We find that the role of the dark radiation is to shift the position of the horizon and the domain wall for the dS_4 and AdS_4 cases, respectively. In the latter case, a phase transition from confinement to the deconfinement phase is seen when the magnitude of this term exceeds a critical value as shown in Ref. [14]. Another observation is that this term deforms the geometry of the IR boundary. On the other hand, the metric at the UV boundary is not affected by the dark radiation. This fact seems to be curious but interesting. We will give more details on this point in the future publication.

In the next section, our model to be examined is given and two boundaries of the gravity dual are shown. Then, in Sec. III, we show the existence of the domain wall which divides the bulk region of two boundary theories through the Wilson loop, D7 and D5 embeddings. From these, we can say quarks, flavored mesons, and baryons are all separately examined in each bulk region corresponding to the dual theory in each boundary. In Sec. IV, the entanglement entropy is examined. In this case also, the minimal surface giving the entanglement entropy of a theory in one boundary cannot penetrate into the region which is dual to the other boundary theory, since the penetration is protected by the domain wall. This fact implies that there is no entanglement of the two theories of each boundary. A summary and discussions are given in the final section.

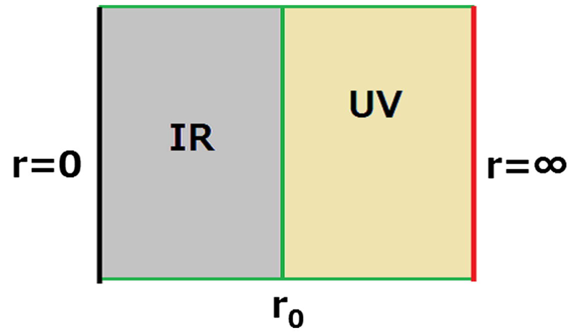

FIG. 1 (color online). A schematic picture which represents the bulk AdS_5 with two boundaries (shown by $r = 0$ and $r = \infty$). The middle line ($r = r_0$) shows the domain wall (horizon) for the AdS_4 (dS_4) boundary. This line separates the bulk into two regions shown by "IR" and "UV," which are dual to the theory on the boundaries at $r = 0$ and $r = \infty$.

II. SETUP OF THE MODEL

First, we briefly review our model [14–16]. We start from the 10D type IIB supergravity retaining the dilaton Φ, axion χ, and self-dual five-form field strength $F_{(5)}$:

$$S=\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(R-\frac{1}{2}(\partial\Phi)^2+\frac{1}{2}e^{2\Phi}(\partial\chi)^2-\frac{1}{4\cdot 5!}F_{(5)}^2\right), \tag{1}$$

where other fields are neglected, since we do not need them, and χ is Wick rotated [18]. Under the Freund-Rubin ansatz for $F_{(5)}$, $F_{\mu_1\ldots\mu_5} = -\sqrt{\Lambda}/2\epsilon_{\mu_1\ldots\mu_5}$ [19,20], and for the 10D metric as $M_5 \times S^5$,

$$\begin{aligned} ds_{10}^2 &= g_{MN}dx^M dx^N + g_{ij}dx^i dx^j \\ &= g_{MN}dx^M dx^N + R^2 d\Omega_5^2, \end{aligned}$$

we consider the solution. Here, the parameter is set as $(\mu =)1/R = \sqrt{\Lambda}/2$.

While the dilaton Φ and the axion χ play an important role when the boundary of M_5 is given by Minkowski space-time [19,20], we neglect them here, since we study the case of (A)dS_4 boundary. Then the equations

[2]We could see a similar situation in the case of the topological black hole solution in terms of the global coordinate r.

of motion of noncompact five-dimensional part M_5 are written as[3]

$$R_{MN} = -\Lambda g_{MN}. \tag{3}$$

While this equation leads to the solution of AdS$_5$, there are various AdS$_5$ forms of the solutions which are discriminated by the geometry of their 4D boundary as shown below.

A. Solution

A class of solutions of the above equation (3) are obtained in the following form of the metric [16]:

$$ds_{10}^2 = \frac{r^2}{R^2}(-\bar{n}^2 dt^2 + \bar{A}^2 a_0^2(t)\gamma_{ij}(x)dx^i dx^j) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2. \tag{4}$$

where

$$\gamma_{ij}(x) = \delta_{ij}\left(1 + k\frac{\bar{r}^2}{4\bar{r_0}^2}\right)^{-2}, \qquad \bar{r}^2 = \sum_{i=1}^{3}(x^i)^2, \tag{5}$$

and $k = \pm 1$ or 0. The arbitrary scale parameter $\bar{r_0}$ is set hereafter as $\bar{r_0} = 1$. For the undetermined noncompact five-dimensional part, the following equation is obtained from the tt and rr components of (3) [21,22]:

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 + \frac{k}{a_0^2} = -\frac{\Lambda}{4}A^2 + \left(\frac{r}{R}A'\right)^2 + \frac{C}{a_0^4 A^2}, \tag{6}$$

where $\dot{a_0} = \partial a_0/\partial t$, $A' = \partial A/\partial r$, and

$$A = \frac{r}{R}\bar{A}, \qquad \frac{\partial_t(a_0(t)A)}{\dot{a}_0(t)} = \frac{r}{R}\bar{n}. \tag{7}$$

The constant C is given as an integral constant in obtaining (6), we could understand that it corresponds to the thermal excitation of the $\mathcal{N} = 4$ SYM theory for $a_0(t) = 1$, and it is called dark radiation [21,22].

At this stage, two undetermined functions, $\bar{A}(r,t)$ and $a_0(t)$, remain. However, the equation to solve them is Eq. (6) only. Therefore, we could determine $a_0(t)$ by introducing the 4D Friedmann equation, which is independent of (3). However, it should be realized on the boundary where various kinds of matter could be added in order to form the presumed FRW universe as in [16]

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 + \frac{k}{a_0^2} = \frac{\Lambda_4}{3} + \frac{\kappa_4^2}{3}\left(\frac{\rho_m}{a_0^3} + \frac{\rho_r}{a_0^4} + \frac{\rho_u}{a_0^{3(1+u)}}\right) \equiv \lambda(t), \tag{8}$$

where κ_4 (Λ_4) denotes the 4D gravitational constant (cosmological constant). The quantities ρ_m and ρ_r denote the energy density of the nonrelativistic matter and the radiation of the 4D theory, respectively. The most right-hand side expression $\lambda(t)$ in (8) is given as a simple form of the most left-hand side of (8) given by using $a_0(t)$. Then the remaining function $A(t,r)$ is obtained from (6) in terms of $\lambda(t)$. The last term ρ_u in the middle of (8) represents an unknown matter with the equation of state, $p_u = u\rho_u$, where p_u and ρ_u denote its pressure and energy density, respectively. It is important to be able to solve the bulk equation (6) in this way by relating its left-hand side to the Friedmann equation defined on the boundary [16], since we could have a clear image for the solution.

Finally, the solution is obtained as

$$\bar{A} = \left(\left(1 - \frac{\lambda}{4\mu^2}\left(\frac{R}{r}\right)^2\right)^2 + \tilde{c}_0\left(\frac{R}{r}\right)^4\right)^{1/2}, \tag{9}$$

$$\bar{n} = \frac{\left(1 - \frac{\lambda}{4\mu^2}\left(\frac{R}{r}\right)^2\right)\left(1 - \frac{\lambda + \frac{a_0}{\dot{a}_0}\dot{\lambda}}{4\mu^2}\left(\frac{R}{r}\right)^2\right) - \tilde{c}_0\left(\frac{R}{r}\right)^4}{\sqrt{\left(1 - \frac{\lambda}{4\mu^2}\left(\frac{R}{r}\right)^2\right)^2 + \tilde{c}_0\left(\frac{R}{r}\right)^4}}, \tag{10}$$

where

$$\tilde{c}_0 = C/(4\mu^2 a_0^4). \tag{11}$$

B. Two boundaries

1. Ultraviolet boundary $r \to \infty$

In the case of the above solution, there is a boundary at $r \to \infty$, where the energy scale of the dual field theory is at the ultraviolet limit. The boundary should be set at the position where the metric has a second-order pole [23], since the manifold is not well defined there. At this boundary $r \to \infty$, the 4D metric is given as

$$ds_{\mathrm{FRW}}^2 = -dt^2 + a_0(t)^2\gamma_{ij}dx^i dx^j, \tag{12}$$

since the above solution behaves as $\bar{n} \to 1$ and $\bar{A}(r,t) \to 1$ for $r \to \infty$. This is the well-known FRW metric, which is usually used in cosmology to study the time development of our Universe. In the present case, therefore, we can study the SYM theory in this FRW universe from the bulk metric (4) which is the holographic dual as shown in [16].

[3] The five-dimensional M_5 part of the solution is obtained by solving the following reduced Einstein frame 5D action:

$$S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}(R + 3\Lambda), \tag{2}$$

which is written in the string frame and taking $\alpha' = g_s = 1$ and the opposite sign of the kinetic term of χ is due to the fact that the Euclidean version is considered here [18].

2. Infrared boundary $r \to 0$

Next, from (9) and (10), we find that there is another boundary at $r \to 0$, in the infrared limit, for $\Lambda_4 < 0$, small C, and a tiny time dependence of $\lambda(t)$. However, the appearance of this boundary depends on the time when the effect of C and the time dependence of $\lambda(t)$ are considered. The situation is therefore a little complicated. For example, consider the case of $\dot{\lambda} = 0$ for simplicity; then the second boundary is found for $\lambda < 0$ and $|\frac{\lambda}{4\mu^2}| > \sqrt{\tilde{c}_0}$. However, the last inequality depends on time, and it is satisfied for a restricted time interval. So a more simple case is considered below.

(i) *For the case of $C = 0$ and negative constant $\lambda(= -\lambda_0)$*.—In order to make clear the two boundaries, the situation is simplified by considering the case of $\lambda = -\lambda_0$ and $C = 0$, where λ_0 is a positive constant. This corresponds to the case of negative Λ_4 and $\lambda_0 = -\Lambda_4/3$. In this case, the scale factor is given by solving Eq. (8) for $k = -1$ as follows:

$$a_0(t) = \sin(\sqrt{\lambda_0}t)/\sqrt{\lambda_0}, \tag{13}$$

and then the metric is written as

$$ds_{10}^2 = ds_5^2 + R^2 d\Omega_5^2, \tag{14}$$

$$ds_5^2 = \frac{r^2}{R^2}\left(1 + \frac{r_0^2}{r^2}\right)^2 (-dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j) + \frac{R^2}{r^2}dr^2, \tag{15}$$

where $r_0^2 = \lambda_0 R^4/4$ and

$$\gamma_{ij}(x) = \delta_{ij}\left(1 - \frac{\bar{r}^2}{4}\right)^{-2}. \tag{16}$$

In this case, the boundary represents a typical AdS_4 manifold, and the SYM theory on this manifold has been holographically examined well previously [13]. In this case, the analysis has been performed by supposing that the bulk is dual to the theory on the boundary $r = \infty$. And we have paid no attention to the other possible boundary at $r = 0$.

However, in order to have correct results of the analysis, we must notice the fact that there is actually another boundary at $r \to 0$ in the bulk of (19). In order to see this point clearly, we rewrite the above metric by changing the coordinate r as $r = r_0^2/z$; then we have

$$ds_{10}^2 = \frac{z^2}{R^2}\left(1 + \frac{r_0^2}{z^2}\right)^2 (-dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j) + \frac{R^2}{z^2}dz^2 + R^2 d\Omega_5^2. \tag{17}$$

Then we find again the same form of metric with (19), but r is replaced by z. This implies the following two points. (i) There must be another bulk region near $z = \infty$ which is dual to SYM theory living on AdS_4 at $z = \infty$. (ii) Second, the limit of $z = \infty$ is also the ultraviolet region of the SYM theory as understood from the form of (17). Then we could obtain the same dynamical information, from the metric (17), of the theory with the one given for the theory at $r = \infty$. In other words, in the bulk manifold, the same two dual theories should be expressed by two regions which are separated at some point of the coordinate r. This point is called a domain wall, and we could find it at $r = r_0$ as shown below.

(ii) *For the case of $C = 0$ and positive constant $\lambda = \lambda_0$*.—In the case of $\lambda = \lambda_0 > 0$, Λ_4 is positive and the scale factor is given by solving Eq. (8) for $k = 0$ as follows:

$$a_0(t) = a(0)e^{\sqrt{\lambda_0}t}, \tag{18}$$

and then the metric is written as

$$ds_{10}^2 = d\tilde{s}_5^2 + R^2 d\Omega_5^2, \tag{19}$$

$$d\tilde{s}_5^2 = \frac{r^2}{R^2}\left(1 - \frac{r_0^2}{r^2}\right)^2 (-dt^2 + a_0^2(t)\delta_{ij}(x)dx^i dx^j) + \frac{R^2}{r^2}dr^2. \tag{20}$$

In this case, the boundary represents the dS_4 manifold, and the SYM theory on this manifold has been holographically examined well previously [12] for the theory on the boundary $r = \infty$. And we have considered only for the half region of $r_0 < r < \infty$; then no attention is paid to the other possible boundary at $r = 0$. In this case, however, the situation is different from the above case, and it would be reasonable to restrict to the region $r_0 < r < \infty$, since the point $r = r_0$ represents the horizon. The situation is similar to the case of the Schwartzschild-AdS background, where the holographic region is restricted to the region from the horizon to $r = \infty$.

It would be an interesting problem to study the theory at the $r = 0$ boundary by considering the region of $0 < r < r_0$. We expect similar behavior to the theory on $r = \infty$. However, here, we give such study in the future article.

(iii) *General case of $C \neq 0$*.—Here we consider the metric of the general case of $C \neq 0$, namely, (4) with (9)–(11). In this case, we find $\bar{A} \neq \bar{n}$, since $\bar{A}$ and $\bar{n}$ are modified by the term $\tilde{c}$, and $\bar{n}$ could have a zero point as in the case of the black hole configuration when $\tilde{c}$ exceeds a critical value. Then we could find a phase transition by adding the term C to the solution with the AdS_4 boundary [14].

In the confinement phase, we find the position of the domain wall is pushed to $r=0$ by increasing $\tilde{c}$. In the case of the dS_4 boundary, the horizon is pushed toward larger r. In any case, the boundary metric at $r=0$ is deformed. This point is seen as follows. The five-dimensional part of the metric is rewritten in terms of $z^* = 1/z^2$ as follows:

$$ds^2_{(5)} = R^2\left(\frac{1}{z^*}\hat{g}_{\mu\nu}dx^\mu dx^\nu + \frac{dz^{*2}}{4z^{*2}}\right), \tag{21}$$

and the 4D part is expanded by the powers of z for $R=1$, as follows:

$$\hat{g}_{\mu\nu} = \hat{g}_{(0)\mu\nu} + \hat{g}_{(2)\mu\nu}z^* + z^{*2}(\hat{g}_{(4)\mu\nu} + \hat{h}_{1(4)\mu\nu}\log z^* + \hat{h}_{2(4)\mu\nu}(\log z^*)^2) + \cdots. \tag{22}$$

The first term is given as

$$\hat{g}_{(0)\mu\nu} = (\hat{g}_{(0)00}, \hat{g}_{(0)ij}) = \left(-\frac{((bb_1)^2 - \tilde{c}_0)^2}{(b^4+\tilde{c}_0)r_0^4}, \frac{b^4+\tilde{c}_0}{r_0^4}a_0(t)^2\gamma_{i,j}\right), \tag{23}$$

where

$$b^2 = -\frac{\lambda}{4}R^4, \qquad b_1^2 = -\frac{\lambda + \dot{\lambda}a_0/\dot{a}_0}{4}R^4. \tag{24}$$

This implies that the boundary metric depends on the dark radiation, namely, the SYM fields. Then this fact seems to contradict the expectation of the decoupling of the SYM theory and the gravity on the boundary. On the other hand, at $r=\infty$, the boundary metric is not affected by this dark radiation C as expected. Then the holographic situation is modified in the side of $r=0$ for $C \neq 0$, where the dual 4D theory couples with gravity through the energy momentum tensor[4] generated in the side of the SYM theory. It is an interesting problem to make clear this point and to investigate the holography of a SYM theory coupled with gravity. We, however, postpone to investigate this problem to the future, and we restrict to the case of $C=0$ hereafter.

III. GRAVITY DUAL AND DOMAIN WALL

We consider the gravity dual of the two theories living on different boundaries. It is represented by a common bulk manifold. We study how we can see the holographic properties of two field theories on the same bulk manifold through various objects which are responsible for field theories.

A. Wilson loop and quark confinement

The potential between a quark and an antiquark is studied by the Wilson loop. It is obtained holographically from the U-shaped (in the $r-x$ plane) string which is embedded in the bulk, and its two end points are on the boundary. By supposing a string whose world volume is set in the (t,x) plane,[5] the energy E of this state is obtained as a function of the distance (L) between the quark and antiquark according to Ref. [12].

Taking the gauge as $X^0 = t = \tau$ and $X^1 = x^1 = \sigma$ for the coordinates (τ,σ) of string world volume, the Nambu-Goto Lagrangian in the present background (4) becomes

$$L_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'}\int d\sigma\bar{n}(r)\sqrt{r'^2 + \left(\frac{r}{R}\right)^4(\bar{A}(r)a_0(t)\gamma(x))^2}, \tag{25}$$

where

$$\gamma(x) = \frac{1}{1-x^2/4} \tag{26}$$

and we notice $r' = \partial r/\partial x = \partial r/\partial\sigma$ and $\bar{n}, \bar{A}$ have no time dependence since we here use the metric (20). The general forms of the solution for $\bar{n}$ and $\bar{A}$ are given in (9)–(11), and they depend on the time through the scale factor $a_0(t)$. Then, in order to see the static energy as $E = -L_{\mathrm{NG}}$, we should restrict the solutions to the case of $C=0$ and $\lambda = -\lambda_0$. In this case, the two boundaries have the same form of metric as given in (13)–(17) with different notation of the radial coordinate for each boundary, r and z, respectively.

In the case of (13)–(17), the energy is rewritten to a more convenient form by introducing the factor n_s (given below) [24] as

$$E = -L_{\mathrm{NG}} = \frac{1}{2\pi\alpha'}\int d\tilde{\sigma}n_s\sqrt{1 + \left(\frac{R^2}{r^2\bar{A}}\partial_{\tilde{\sigma}}r\right)^2}, \tag{27}$$

$$\tilde{\sigma} = a_0(t)\int d\sigma\gamma(\sigma) = a_0(t)\int d\sigma\frac{1}{1-\sigma^2/4}, \tag{28}$$

$$n_s = \left(\frac{r}{R}\right)^2\bar{A}\,\bar{n} = \left(\frac{r}{R}\right)^2\left(1 + \frac{r_0^2}{r^2}\right)^2. \tag{29}$$

[4] We could show that the vacuum expectation value of the energy momentum tenser in the IR side boundary is also derived according to the renormalization group method used in the UV side. The result at the IR side is given by the same form of the one at the UV side by replacing the curvatures written by the metric (II B). For example, the trace anomaly is given by $\langle T^\mu_\mu\rangle = \frac{N^2}{32\pi^2}(R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2)$. See Appendix B.

[5] Here x denotes one of the three coordinate x^i, and we take x^1 in the present case.

Here, we use the proper coordinate $\tilde{\sigma}$ instead of the comoving coordinate σ to measure the distance between the quark and antiquark.

In this form, the criterion of the confinement is stated such that n_s has a finite minimum value at some appropriate $r(=r^*)$. In the present case, we find $r^* = r_0$. Actually, in such a case, E is approximated as [12]

$$E \sim \frac{n_s(r^*)}{2\pi\alpha'} L, \tag{30}$$

where

$$L = 2\int_{\tilde{\sigma}_{\min}}^{\tilde{\sigma}_{\max}} d\tilde{\sigma} \tag{31}$$

and $\tilde{\sigma}_{\min}$ ($\tilde{\sigma}_{\max}$) is the value at $r_{\min}$ ($r = \infty$) of the string configuration [13]. The tension of the linear potential between the quark and antiquark is therefore given as

$$\tau_{q\bar{q}} = \frac{n_s(r_0)}{2\pi\alpha'}. \tag{32}$$

We notice that the U-shaped string configuration whose bottom point is near r_0 and the string on both sides goes up toward the boundary $r = \infty$. When the bottom approaches r_0, the length L goes to ∞. In other words, the string configuration is bounded at $r = r_0$ and cannot exceed this point to smaller r.

In the case of (17), the procedure of the calculation of the Wilson loop is completely parallel to the above case only by replacing r by z. Then we find the same tension of the linear potential between the quark and antiquark, which are living on the boundary $r = 0$ or $z = \infty$, is obtained as

$$\tau_{q\bar{q}} = \frac{n_s(r_0)}{2\pi\alpha'}. \tag{33}$$

In the present case, the string on both sides goes up toward the boundary, namely, to $r = 0$. So the U-shaped configuration of the string has a form which has been upside down to the one obtained above. Then we will find two types of string configurations which are responsible to the Wilson loop calculation. The end points of the one type of string go towards $r = \infty$, and the one of the other type goes to $r = 0$. This equation is very complicated, so we show its numerical result in Fig. 2.

1. *String configurations and domain wall*

Here we show the string configurations mentioned above to make clear the situation. They are obtained by solving the equation of motion for the profile of the string. Both the solutions belonging to the boundary $r = \infty$ and $r = 0$ are obtained by solving the same equation, which is given from (25) as follows:

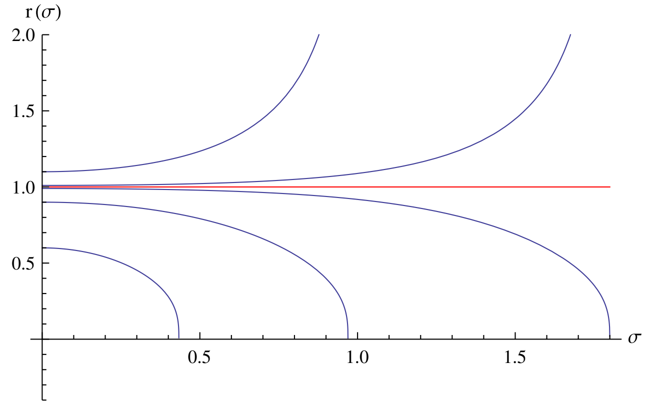

FIG. 2 (color online). Solutions for $r(\sigma)$, where $\tilde{\sigma}$ is denoted by σ. The curves denote for $r(0) = 1.1, 1.01, 0.99, 0.9, 0.6$ from the above one. The horizontal line $r = 1.0$ shows the domain wall.

$$r'' - r'\left(\log\left(\frac{r^4}{R^4} + \left(\frac{r'}{(1+(r_0/r)^2)}\right)^2\right)\right)' + 2\frac{r_0^3}{r^3}\frac{r'^2}{1+(r_0/r)^2} - 2\frac{r^3}{R^4}\left(1-\left(\frac{r_0}{r}\right)^4\right) = 0, \tag{34}$$

where a prime denotes the differentiation with respect to $\tilde{\sigma}$ as $r' = \partial_{\tilde{\sigma}} r$. This equation is complicated, so we solve it numerically.

Several configurations are shown in Fig. 2, from which we can see the solutions are separated to two groups by the boundary condition at $\sigma = x = 0$, namely, the value of $r(0)$. The wall which separates two classes of the solutions is found at $r(0) = r_0$.

B. D7 brane embedding and domain wall

Here, we study the D7 brane embedding, which is responsible for studying the meson spectrum and the chiral condensate of the boundary theory. The D7 brane action is given by the Dirac-Born-Infeld (DBI) and the Chern-Simons terms as follows:

$$S_{D7} = -T_7\int d^8\xi e^{-\Phi}\sqrt{-\det(g_{ab} + 2\pi\alpha' F_{ab})} + T_7\int\sum_i (e^{2\pi\alpha' F_{(2)}} \wedge c_{(a_1\ldots a_i)})_{0\ldots 7},$$
$$g_{ab} \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}, \qquad c_{a_1\ldots a_i} \equiv \partial_{a_1} X^{\mu_1}\ldots\partial_{a_i} X^{\mu_i} C_{\mu_1\ldots\mu_i}, \tag{35}$$

where T_7 is the brane tension. The DBI action involves the induced metric g_{ab} and the $U(1)$ world volume field strength $F_{(2)} = dA_{(1)}$.

1. *Near $r = \infty$*

For simplicity, we consider the background (19)–(20) near the boundary $r = \infty$. The metric of the extra six-dimensional part of this metric is rewritten as follows:

$$\frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 = \frac{R^2}{r^2}\left(d\rho^2 + \rho^2 d\Omega_3^2 + \sum_{i=8}^{9} dX^{i2}\right), \tag{36}$$

where the new coordinate ρ is introduced instead of r with the relation

$$r^2 = \rho^2 + (X^8)^2 + (X^9)^2. \tag{37}$$

Thus, the induced metric of the D7 brane is obtained as

$$ds_8^2 = \frac{r^2}{R^2}\left(1+\frac{r_0^2}{r^2}\right)^2(-dt^2 + a_0^2(t)\gamma^2(x)(dx^i)^2) + \frac{R^2}{r^2}((1+w'^2)d\rho^2 + \rho^2 d\Omega_3^2), \tag{38}$$

where the profile of the D7 brane is taken as $(X^8, X^9) = (w(\rho), 0)$ and $w' = \partial_\rho w$; then

$$r^2 = \rho^2 + w^2. \tag{39}$$

In the present case, there is no Ramond-Ramond field, so the action is given only by the one of DBI as

$$S_{D7} = -T_7\Omega_3 \int d^4x a_0^3(t)\gamma^3(x) \int d\rho \rho^3 \bar{A}^4 \sqrt{1+w'^2(\rho)}, \tag{40}$$

where

$$\bar{A} = \left(1+\frac{r_0^2}{r^2}\right)^2 \tag{41}$$

and Ω_3 denotes the volume of S^3 of the D7's world volume.

From this action, the equation of motion for w is obtained as

$$w'' + \left(\frac{3}{\rho} + \frac{\rho + ww'}{r}\partial_r(\log(\bar{A}^4))\right)w'(1+w'^2) - \frac{w}{r}(1+w'^2)^2\partial_r(\log(\bar{A}^4)) = 0. \tag{42}$$

The constant w is not the solution of this equation, so the supersymmetry is broken. The numerical solutions of (42) for $w(\rho)$ are shown in Fig. 3. In general, in this case, we find finite chiral condensate $\langle\bar{\Psi}\Psi\rangle = c$ for any $m_q \geq 0$, since the curves decrease from above with increasing ρ. For all curves, we find the behavior given by the following asymptotic form:

$$w = m_q + \frac{c + 4m_q^2 r_0^2 \log(\rho)}{\rho^2} + \cdots, \tag{43}$$

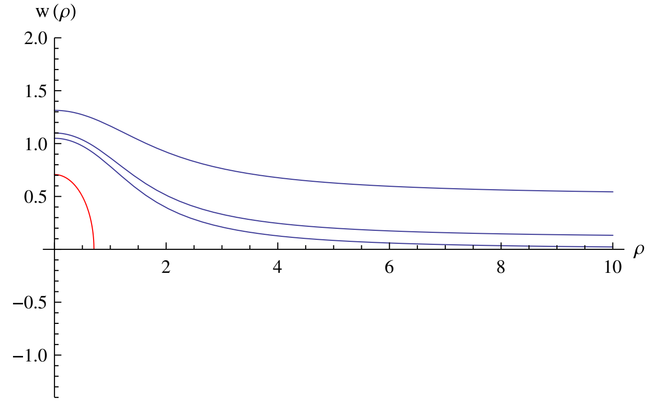

FIG. 3 (color online). Typical solutions of $w(\rho)$ for $\lambda_0 = 2, \mu = 1/R = 1.0$, and $r_0 = 1/\sqrt{2}$. The curves are given for $w(0) = 1.3, 1.1, 1.05$ from above to below. The circle represents $r = \sqrt{r_0^2 - \rho^2}$, which corresponds to the domain wall of the dual bulk manifold for two boundaries.

at large ρ with $c > 0$. Here, the term proportional to $\log(\rho)$ comes from the breaking of the conformal invariance due to the cosmological constant in the theory [11–13]. We can observe spontaneous chiral symmetry breaking from the third curve, which corresponds to $m_q = 0$. It shows the mass generation of a massless quark due to the chiral condensate $\langle\bar{\Psi}\Psi\rangle$.

As a result, we could say that the spontaneous mass generation of massless quarks is realized in the theory on the boundary at $r = \infty$. This point was already found previously [16]. We notice here that the embedded region of the D7 brane with $m_q \geq 0$ is restricted to the region $r > r_0$. Furthermore, there is no D7 brane configuration which crosses the domain wall $r = r_0$ in the $w - \rho$ plane. Then the quarks introduced in the dual SYM theory on the boundary $r = \infty$ can be represented by the D7 brane embedded in the region of $r > r_0$.

2. Near $r = 0$

For the flavor brane near $r = 0$, its embedding is performed as follows. First, by adopting the bulk metric (17), the procedure is completely parallel to the above case by replacing r by z. Then the embedded D7 branes of $m_q \geq 0$ are all obtained in the region of $z > r_0$, and we find the dual theory with the chiral symmetry breaking phase at the boundary $r = 0$. In the present case, the region $z > r_0$ means $r < r_0$ since $z = r_0^2/r$. Then we find the fact that each theory in two boundaries of the bulk is separated by the wall at $r = r_0$. Namely, we can study each dual theories given by considering the gravity within each region.

C. D5 Branes and baryon

Next, we consider the baryon. It is constructed from a vertex and N_c quarks, and the latter are expressed by fundamental strings. The vertex is identified with the D5 brane, which is embedded in the bulk as a probe with a nontrivial $U(1)$ flux in it. Then a baryon is discussed through the D5 brane embedding given as follows.

First, we briefly review the model based on type IIB superstring theory [25–29]. In the type IIB model, the vertex is described by the D5 brane which wraps S^5 of the 10D manifold $M_5 \times S^5$. In this case, in the bulk, there exists the following form of self-dual Ramond-Ramond field strength:

$$G_{(5)} \equiv dC_{(4)} = \frac{4}{R}(\epsilon_{S^5} + {}^*\epsilon_{S^5}), \tag{44}$$

$$\epsilon_{S^5} = R^5 \mathrm{vol}(S^5) d\theta_1 \wedge \cdots \wedge d\theta_5, \tag{45}$$

where $\mathrm{vol}(S^5) \equiv \sin^4\theta_1 \mathrm{vol}(S^4) \equiv \sin^4\theta_1 \sin^3\theta_2 \sin^2\theta_3 \sin\theta_4$ and ϵ_{S^5} denotes the volume form of the S^5 part. The flux from the stacked D3 branes flows into the D5 brane as a $U(1)$ field which is living in the D5 brane.

The effective action of the D5 brane is given by using the Born-Infeld and Chern-Simons term as follows:

$$\begin{aligned} S_{D5} &= -T_5 \int d^6\xi e^{-\Phi} \sqrt{-\det(g_{ab} + 2\pi\alpha' F_{ab})} \\ &\quad + T_5 \int (2\pi\alpha' F_{(2)} \wedge c_{(4)})_{0\ldots 5}, \\ g_{ab} &\equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}, \\ c_{a_1\ldots a_4} &\equiv \partial_{a_1} X^{\mu_1} \ldots \partial_{a_4} X^{\mu_4} C_{\mu_1\ldots\mu_4}, \end{aligned} \tag{46}$$

where $T_5 = 1/(g_s(2\pi)^5 l_s{}^6)$ and $F_{(2)} = dA_{(1)}$, which represents the $U(1)$ world volume field strength. In terms of (the pullback of) the background five-form field strength $G_{(5)}$, the above action can be rewritten as

$$S_{D5} = -T_5 \int d^6\xi e^{-\Phi} \sqrt{-\det(g+F)} + T_5 \int A_{(1)} \wedge G_{(5)}.$$

The embedding of the D5 brane is performed by solving the $r(\theta)$, $x(\theta)$, and $A_{(1)}(\theta)$ [29]. They are retained as dynamical fields in the D5 brane action as the function of $\theta \equiv \theta_1$ only. The equation of motion for the gauge field $A_{(1)}$ is written as

$$\partial_\theta D = -4\sin^4\theta,$$

where the dimensionless displacement is defined as the variation of the action with respect to $E = F_{t\theta}$, namely, $D = \delta\tilde{S}/\delta F_{t\theta}$ and $\tilde{S} = S/T_5\Omega_4 R^4$. The solution to this equation is

$$D \equiv D(\nu,\theta) = \left[\frac{3}{2}(\nu\pi - \theta) + \frac{3}{2}\sin\theta\cos\theta + \sin^3\theta\cos\theta\right]. \tag{47}$$

Here, the integration constant ν is expressed as $0 \le \nu = k/N_c \le 1$, where k denotes the number of Born-Infeld strings emerging from one of the pole of the $\mathbf{S}^5$.

Next, it is convenient to eliminate the gauge field in favor of D; then the Legendre transformation is performed for the original Lagrangian to obtain an energy functional as [27–29]

$$U = \frac{N}{3\pi^2\alpha'} \int d\theta \bar{n} \sqrt{r^2 + r'^2 + (r/R)^4 x'^2 (\bar{A} a_0 \gamma)^2} \sqrt{V_\nu(\theta)}, \tag{48}$$

$$V_\nu(\theta) = D(\nu,\theta)^2 + \sin^8\theta, \tag{49}$$

where we used $T_5\Omega_4 R^4 = N/(3\pi^2\alpha')$ and we use the metric form (4). Then, in this expression, (48), $r(\theta)$ and $x(\theta)$ remain, and they are solved by minimizing U. As a result, the D5 brane configuration is determined.

For simplicity, here, we restrict to the pointlike configuration; namely, r and x are constants. Furthermore, the simple metric (20) is adopted. In this case, we have for the matter considered here

$$U = r\bar{n}(r) U_0 = r\left(1 + \frac{r_0^2}{r^2}\right) U_0, \tag{50}$$

where U_0 is a constant given as

$$U_0 = \frac{N}{3\pi^2\alpha'} \int d\theta \sqrt{V_\nu(\theta)}. \tag{51}$$

From (50), we find that U has a minimum at $r_m = r_0$. Then the vertex is trapped at the domain wall.

We notice, however, that the embedded regions of the fundamental strings, quarks, are separated to two regions by the domain wall. Namely, the strings cannot cross the domain wall. In this sense, the baryons are also separated to two theories by the wall in the gravity side.

IV. ENTANGLEMENT ENTROPY AND DOMAIN WALL

Next, we consider the entanglement entropy for the theory of one boundary. It is given by calculating the minimum area of the surface A whose boundary ∂A is set at the boundary of the bulk and the surface could be extended in the bulk. In this calculation, there is a possibility that the minimal surface could penetrate into the bulk region corresponding to the theory living in the other boundary. When this situation is realized, we could see a new entanglement of two theories which are living in the separated boundaries.

In order to see such a phenomenon, we estimate the entanglement entropy of a theory in one boundary according to formula (3.3) in [30]:

$$S_{\rm EE} = \frac{{\rm Area}(\gamma)}{4G_N^{(5)}}, \tag{52}$$

where γ_A denotes the minimal surface, whose boundary is defined by ∂A and the surface is extended into the bulk. And $G_N^{(5)} = G_N^{(10)}/(\pi^3 R^5)$ denotes the 5D Newton constant reduced from the 10D one $G_N^{(10)}$. If, in this calculation, the minimal surface crosses the domain wall, then we can say that a new kind of an entanglement between two theories on each boundary may exist. This is because the surface or equivalently the entanglement entropy is controlled by the dynamics of the other theory.

We adopt (17) as the bulk metric, which is given as

$$ds_{10}^2 = \frac{z^2}{R^2}\left(1+\frac{r_0^2}{z^2}\right)^2 ds_{{\rm FRW}_4}^2 + \frac{R^2}{z^2}dz^2 + R^2 d\Omega_5^2, \tag{53}$$

where

$$ds_{{\rm FRW}_4}^2 = -dt^2 + a_0^2(t)\gamma^2(dp^2 + p^2 d\Omega_2^2), \tag{54}$$

$$p = \frac{\bar{r}}{\bar{r}_0}, \qquad \gamma = 1/(1-p^2/4). \tag{55}$$

We notice here that the mass dimension is -1 for a_0, but p has no mass dimension since it is scaled by $\bar{r}_0$. In order to study the entanglement entropy (EE), we separate the 3D space of the boundary at a fixed time by a constant value for $p = p_0$. Then the EE for the restricted space $p < p_0$ is obtained holographically by finding the minimum value of the following quantity:

$$\frac{S_{\rm area}}{4\pi} = \left(\frac{r_0}{R}a_0\right)^3 \int_{x_0}^{\epsilon} dx f_1(x), \tag{56}$$

$$f_1(x) = p^2\left(\frac{1+x^2}{x}\right)^3 \gamma^3 \sqrt{p'^2 + \frac{b}{\gamma^2(1+x^2)^2}}, \tag{57}$$

$$b = \frac{R^4}{a_0^2 r_0^4}, \qquad p' = \frac{\partial p}{\partial x}, \tag{58}$$

where $x = z/r_0$ and x_0 denotes the end point of the embedded surface. Since this integral diverges at the UV limit, the UV cutoff ϵ is introduced. This represents the minimal surface of the ball embedded in the bulk.

In order to obtain the minimum of $S_{\rm area}$, we must solve the variational equation for $p(x)$ which is extended in the region $0 < x < x_0$ of the bulk space. On the other hand, the information of the two boundary theories is divided by the domain wall as mentioned above. Then we will see the upper bound of x_0 at $x_0 = 1$. This point corresponds to

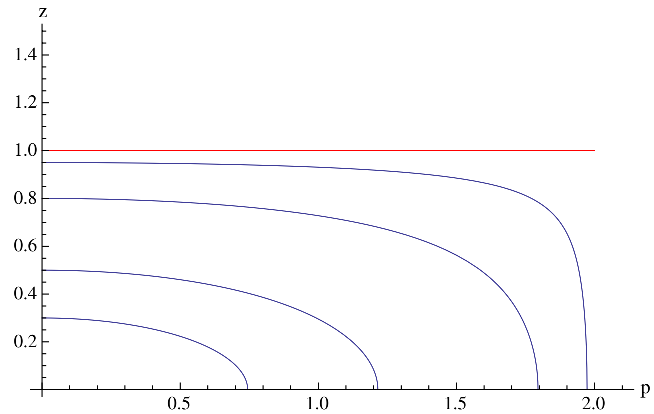

FIG. 4 (color online). Embedded solutions of $p(z)$ for $p_0 = 0.74$, 1.21, 1.79, and 1.97 from below. Other parameters are set as $r_0 = 1$, $R = 1$, and $a_0 = 0.4$. The horizontal line represents the domain wall.

$r = r_0$. This is actually assured by rewriting the above $S_{\rm area}$ as follows[6]:

$$\frac{S_{\rm area}}{4\pi} = \left(\frac{r_0}{R}a_0\right)^3 \int dy f_2(y), \tag{59}$$

$$f_2(y) = \left(\frac{1+x^2}{x}\right)^3 \sqrt{1 + \frac{b(x'p^2\gamma^2)^2}{(1+x^2)^2}}, \tag{60}$$

$$y = y(p) = \int^p dp \frac{p^2}{(1-p^2/4)^3}, \qquad x' = \partial_y x. \tag{61}$$

These formulas are very similar to the case of the Wilson loop calculation, where the embedded string configuration is obtained as a U-shaped one, and its bottom point is bounded at the minimum of the prefactor of the integrand. It corresponds here to

$$n_{\rm Sphere} = \left(\frac{1+x^2}{x}\right)^3. \tag{62}$$

In fact, we can see that $n_{\rm Sphere}$ has a minimum at $x = 1$. Then the embedded solution of the ball would be bounded in the region $0 < x < 1$, and this is also assured from the numerical calculation as shown in Fig. 4.

The analysis given above is obtained for $p(z)$ with various p_0, which is the value of p at the UV limit $z = 0$. The bottom point of $p(z)$ approaches to the value for $z = r_0$ (the horizontal line $x = 1$). However, it does not ever exceed this line. In other words, the quantum information of the theory on the other boundary does not affect the EE calculation of the theory at $z = 0$. This implies that there is no entanglement between the two theories on the opposite boundaries at $r = 0 (z = \infty)$ and $r = \infty (z = 0)$.

[6]Notice that p in Eq. (60) is a function of y as solved from Eq. (61).

A. Divergent term and central charge of the theory

While it is difficult to find an analytic solution of $p(z)$ in the present case, it is possible to see the divergent form of $\frac{S_{\rm area}}{4\pi}$ near the UV limit by using the approximate solution near the boundary. Before solving our present case, this point is shown first for the bulk AdS$_5$ case with a Minkowski boundary metric. Through this analysis, we could obtain knowledge related to the central charge of the theory. We write the AdS$_5$ metric as

$$ds^2_{\rm AdS_5} = \left(\frac{R}{z}\right)^2 (\eta_{\mu\nu}dx^\mu dx^\nu + dz^2). \tag{63}$$

In this case also, we use the same notation p for the three space radial coordinate as

$$ds^2_{(4)} = \eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dp^2 + p^2 d\Omega^2_{(2)}. \tag{64}$$

Then the 3D area of the embedded ball with radius p_0 is given as

$$\frac{S_{\rm area}}{4\pi} = R^3 \int_{z_{\rm min}}^{z_{\rm max}} \frac{dz}{z^3} p^2 \sqrt{1+p'^2}, \tag{65}$$

where $p' = \partial p/\partial z$ and ϵ is the cutoff, namely, $z > \epsilon$. In order to get the area of minimal surface, we should minimize $S_{\rm area}$. This requirement is achieved by the variational principle. The variational equation for $p(z)$ is solved in this case as

$$p = p_0 + p_2 z^2 + p_4 z^4 + \cdots. \tag{66}$$

In the above, p_0 and p_4 are two arbitrary constants. This is the general solution. Simple power counting assures that only p_0 and p_2 are necessary to get divergent terms of $S_{\rm area}$. The value of p_2 is determined as

$$p_2 = -\frac{1}{2p_0}. \tag{67}$$

Using this solution, we can estimate the leading UV ($\epsilon \to 0$) divergent terms as

$$\frac{S_{\rm area}}{4\pi} = \frac{1}{2}R^3 \left[\left(\frac{p_0^2}{\epsilon^2}\right) + \log\left(\frac{\epsilon}{p_0}\right)\right] + \text{finite terms}, \tag{68}$$

where the parameter p_0, which characterizes the present physical system, is introduced according to Refs. [30,31].

The entanglement entropy is then expressed in the form used in Ref. [31] as follows:

$$S_{EE} = \frac{\gamma_1}{2}\cdot\frac{\text{Area}(\partial A)}{4\pi\epsilon^2} + \gamma_2 \log\left(\frac{p_0}{\epsilon}\right) + \text{finite terms}, \tag{69}$$

where Area(∂A) denotes the area of the surface A, and γ_1 and γ_2 are numerical constants. In the present case, Area$(\partial A) = 4\pi p_0^2$; then the coefficients γ_1 and γ_2 are obtained as

$$\frac{\gamma_1}{2} = \frac{2\pi R^3}{4G_N^{(5)}} = N^2, \qquad \gamma_2 = \frac{2\pi R^3}{4G_N^{(5)}} = N^2, \tag{70}$$

with the use of (52) and the relation $R^4 = 4\pi g_s \alpha'^2 N$. The result is compared to the corresponding divergent terms of our AdS$_4$ space model in the following.

Now we return to (56). In this case, the variational equation is solved by using the following expansion:

$$p = p_0 + p_2 x^2 + p_{4L} x^4 \log x + p_4 x^4 + \cdots. \tag{71}$$

The coefficients of this series expansion are determined by the two arbitrary constants p_0 and p_4. The values of p_2 and p_{4L} are determined, respectively, as

$$p_2 = -\frac{(1-(p_0^2/4)^2)R^4}{2a_0^2 p_0 r_0^2} \tag{72}$$

and

$$p_{4L} = -\frac{(1-(\frac{p_0^2}{4})^2)R^8 \cos^2(\sqrt{\lambda_0}t)}{4a_0^4 p_0 r_0^8}. \tag{73}$$

This solution is not analytical in contrast to (66) due to the term $\log x$. However, it is not important, since only p_0 and p_2 contribute to the divergent terms of $S_{\rm EE}$ as is mentioned above. The value of p_2 is determined as

$$p_2 = -\frac{(1-(p_0^2/4)^2)R^4}{2a_0^2 p_0 r_0^2}. \tag{74}$$

With this solution, we obtain

$$\frac{S_{\rm area}}{4\pi} = \frac{1}{2}R^3 \left[\left(\frac{k_0^2}{4\epsilon^2}\sin^2(\sqrt{\lambda_0}t)\right) + (1 + k_0^2\cos^2(\sqrt{\lambda_0}t))\log\left(\frac{\epsilon}{p_0}\right)\right] + \text{finite terms}, \tag{75}$$

where $k_0 = p_0/(1 - p_0^2/4)$. This result is also written in the form of (69) with the following coefficients γ_i:

$$\frac{\gamma_1}{2} = \frac{\lambda_0}{4}N^2, \tag{76}$$

$$\gamma_2 = N^2(1 + k_0^2\cos^2(\sqrt{\lambda_0}t)), \tag{77}$$

where we used the following proper area in this case:

$$\mathrm{Area}(\partial A) = 4\pi k_0^2 a_0(t)^2. \tag{78}$$

In this case, $\gamma_1/2$ is slightly different by the factor $\lambda_0/4$ from the one of the AdS_5 case in (70). However, this difference can be removed by redefinition of the cutoff parameter ϵ. Then the remaining coefficient N^2 represents the freedom of the dual theory, and this is consistent with the previous result that the central charge of the dual theory has been given by N^2 through the calculation of the energy momentum tensor holographically (see Appendix A) [16].

On the other hand, we find a definite difference in γ_2. This is understood from the fact that γ_2 depends on the curvatures in the 4D boundary and the extrinsic curvatures of the boundary A in general [31]. When $t = (n + 1/2)\pi/\sqrt{\lambda_0}$ (n is an integer), $p_{4L} = 0$ as seen from (73). In this case, γ_2 becomes the same with (70) and independent of p_0. Thus, there is a relation between γ_2 and p_{4L}. More precisely, p_{4L} and the extra term in γ_2 both contain the factor $\cos^2(\sqrt{\lambda_0}t)$. While we are still considering the physical interpretation of this relation, this remains as an open question.

It would be an interesting problem to assure that our result could coincide with the one given from the side of the dual field theory, the SYM theory in the AdS_4 background, in order to see the validity of the gauge or gravity corresponding to our present model. This remains here as an open question.

V. SUMMARY AND DISCUSSIONS

In this paper, we have put forward an extended form of $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality proposed in the previous paper [16], where AdS_5 is replaced by $\widetilde{\mathrm{AdS}}_5$ whose boundary (in the ultraviolet side) is expressed by the FRW_4 space-time with finite 4D curvature. However, the notation $\widetilde{\mathrm{AdS}}_5$ might be misleading. It is because one might consider that, in order to get the solution $\widetilde{\mathrm{AdS}}_5$, the equation of motion would be different from the one which leads to the AdS_5. Contrary to this expectation, $\widetilde{\mathrm{AdS}}_5$ is a solution of the same 5D Einstein equation which leads to the typical AdS_5. Then the two solutions are locally the same with each other. On this point, we will discuss more in the future.

Two cases of the boundary geometry, AdS_4 and dS_4, are possible for this FRW_4 depending on the sign of the 4D cosmological constant Λ_4. The parameter Λ_4 can be introduced as an arbitrary constant in the process of solving the 5D Einstein equation with negative 5D cosmological constant Λ, which comes from five-form field strength and is independent of Λ_4.

Here we point out a new holographic feature of the $\widetilde{\mathrm{AdS}}_5$ with the AdS_4 boundary. In this case, we observe a second boundary in $\widetilde{\mathrm{AdS}}_5$ at the opposite side of the fifth coordinate, namely, at $r = 0$ in addition to the one at $r = \infty$. This fact is in sharp contrast to the usual asymptotic AdS_5 case, in which the boundary appears only at $r = \infty$ and the point $r = 0$ is usually set as a horizon.

This situation depends also on the other parameter C in the general form of $\widetilde{\mathrm{AdS}}_5$ given in (9)–(11), where C denotes the dark radiation. While this term pushes the domain wall to smaller r, we could find the boundary at $r = 0$ for a small value of $\tilde{c}_0$. However, the geometry of the boundary at $r = 0$ is generally different from a simple AdS_4 which is realized at $r = \infty$. In Sec. 2.3, a short discussion is given in the case of $C \neq 0$, in which it is pointed out that the metric of the IR boundary depends on the dark radiation or the SYM fields. On the other hand, at the UV boundary the situation is different. Its geometry is not affected by the dark radiation. This point is interesting, but we postpone to resolve this problem in the future.

Thus we restricted here to the case of $C = 0$ and constant λ in order to simplify the problem of two boundaries discussed in this article. In this case, we find that the metric of the UV boundary takes the form (20), and the one of IR boundary can be read from (17). They have the same form if z was identified with r. Of course, they are different, but they are related as $z = r_0^2/r$ and the point $r = r_0$ has an important holographic meaning in the bulk. In fact, we find that this point corresponds to the domain wall.

As assured from the metric (17), we could observe that the 4D dual theory living at the boundary $r = 0$ is also the SYM theory in the confinement phase. Furthermore, from the scaling behavior of the metric form of IR boundary (17), the limit of $r = 0$ does not correspond to the IR but to the UV limit of the corresponding 4D theory. Then there are two holographic screens in this case. This implies that the two field theories are described by a common gravity dual, $\widetilde{\mathrm{AdS}}_5$ with the AdS_4 boundary. We notice that we find one boundary at $r = \infty$ and a horizon in the infrared side at finite r for another case of $\widetilde{\mathrm{AdS}}_5$, which has the dS_4 boundary.

The problem in the case of two boundaries is how the bulk manifold $\widetilde{\mathrm{AdS}}_5$ would provide information of the two field theories living on the different boundaries. Is it possible to get precise dynamical information of two theories separately from the common bulk geometry? We could show that the answer is yes for this question in terms of the presence of a sharp domain wall in the bulk. The gravity duals $\widetilde{\mathrm{AdS}}_5$ for the two theories are separated by this wall.

The existence of the domain wall is assured by embedding the fundamental string, D7 brane, and D5 brane in $\widetilde{\mathrm{AdS}}_5$. These objects give us the information of the Wilson loop, quarks, meson spectrum, and baryons of the dual SYM theory. We could find that the embedded regions of these objects are restricted to either side of the bulk separated by the domain wall. In other words, these extended objects cannot be embedded across the domain wall. Then the property of the field theory in one boundary is given by the gravity of one-side bulk divided by the domain wall.

Another interesting embedding problem is found in the calculation of the entanglement entropy $S_{\rm EE}$. This is obtained by the minimal surface Area(γ_A), which is defined as the minimum of the embedded surface whose boundary separates the fixed-time boundary space into two regions. In this calculation, we find the embedded surface never extends across the domain wall as other embedded stringy objects discussed above. This fact implies that the quantum fields in the theories of the two boundary do not affect each other.

As for the entanglement entropy $S_{\rm EE}$ defined in either boundary, it diverges in general and is written as (69). The two coefficients γ_1 and γ_2 of this expression reflect the freedom of the quantum fields of the theory and the geometry of the 4D space-time of the boundary, respectively. The result $\frac{\gamma_1}{2}=N^2$ is common to the one of the case of AdS_5 when we take the area of the sphere of the three space boundary by using the proper distance in the $\widetilde{\mathrm{AdS}}_5$ case with AdS_4 boundary. On the other hand, γ_2 depends on the 4D curvatures and extrinsic curvatures on the 3D sphere. These quantities largely change γ_2 of $\widetilde{\mathrm{AdS}}_5$ from the one of AdS_5. Our result (70) for γ_2 would be important to assure the curvature dependence of $S_{\rm EE}$ in curved space-time. We will discuss this point in the future work.

Finally, we give the following two comments. First, the boundary of the AdS_5 is considered here as the point where a double pole (as given by Witten in [1]) with respect to the fifth coordinate is observed. Two such points are found here at $r=0$ and $r=\infty$ for the (A)dS_4 slice. Of course, another kind of boundary can be considered as discussed in [32]. In [32], the authors have examined the bulk fields near a bulk singularity by supposing the existence of a new CFT there.

As for the boundary as a double pole point, the double boundaries are also observed in the black hole type solutions. In [33], the so-called topological black hole solutions are discussed. While we do not consider this type of geometry, we can see that this case is similar to our solution of the dS_4 slice, since a horizon exists between the two boundaries in both cases.

Second, we should notice the following point. In [15], it is shown that our solutions used here can be rewritten to the form of the topological black hole solutions by a coordinate transformation. This is not surprising, because both solutions are obtained from the same bulk Einstein equations which are derived from the action of the Einstein-Hilbert and 5D cosmological constant as mentioned above. However, this transformation is performed in 5D by a kind of Rindler transformation; then the slice of the 4D space-time and the fifth coordinate are changed. As a result, the properties of the CFT in the sliced 4D space-time are also changed. This point is important and really assured by various holographic methods and quantities. So we think that the dual theory of the topological black hole is different from our present case given in this article. As mentioned in the first paragraph of this section, $\widetilde{\mathrm{AdS}}_5$ is rewritten by AdS_5 through an appropriate coordinate transformation. However, we should notice that we can see the properties of the CFT in 4D space-time, which is deformed from 4D Minkowski space-time, through $\widetilde{\mathrm{AdS}}_5$.

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Appendix A: $\langle T_{\mu\nu}\rangle$ OF THE DUAL THEORY AT $r=\infty$

At first, we show the 4D stress tensor of the boundary theory at $r=\infty$. Previously, it has already been given, so we review it briefly. First we rewrite the 5D part of the metric (4) according to the Fefferman-Graham framework [34–36]. Then it is given as

$$ds^2_{(5)}=\frac{1}{\rho}(-\bar{n}^2dt^2+\bar{A}^2a_0^2(t)\gamma^2(x)(dx^i)^2)+\frac{d\rho^2}{4\rho^2} \quad \text{(A1)}$$

$$=\frac{1}{\rho}\hat{g}_{\mu\nu}dx^\mu dx^\nu+\frac{d\rho^2}{4\rho^2}, \quad \text{(A2)}$$

where $\rho=1/r^2$, $R=1$, and

$$\bar{A}=\left(\left(1-\frac{\lambda}{4\mu^2}\left(\frac{\rho}{R^2}\right)\right)^2+\tilde{c}_0\left(\frac{\rho}{R^2}\right)^2\right)^{1/2}, \quad \text{(A3)}$$

$$\bar{n}=\frac{\left(1-\frac{\lambda}{4\mu^2}\left(\frac{\rho}{R^2}\right)\right)\left(1-\frac{\lambda+\dot{\lambda}a_0/\dot{a}_0}{4\mu^2}\left(\frac{\rho}{R^2}\right)\right)-\tilde{c}_0\left(\frac{\rho}{R^2}\right)^2}{\bar{A}}. \quad \text{(A4)}$$

Next, $\hat{g}_{\mu\nu}$ is expanded as [35]

$$\hat{g}_{\mu\nu}=g_{(0)\mu\nu}+g_{(2)\mu\nu}\rho +\rho^2(g_{(4)\mu\nu}+h_{1(4)\mu\nu}\log\rho+h_{2(4)\mu\nu}(\log\rho)^2)+\cdots, \quad \text{(A5)}$$

where

$$g_{(0)\mu\nu}=(g_{(0)00},g_{(0)ij})=(-1,a_0(t)^2\gamma_{i,j}) \quad \text{(A6)}$$

and

$$g_{(2)\mu\nu}=\frac{\lambda}{2}\left(1+\frac{\frac{a_0}{\dot{a}_0}\dot{\lambda}}{\lambda},-g_{(0)ij}\right), \quad \text{(A7)}$$

$$g_{(4)\mu\nu} = \frac{\tilde{c}_0}{R^4}(3, g_{(0)ij}) + \frac{\lambda^2}{16}\left(-\frac{(\lambda + \frac{a_0}{\dot{a}_0}\dot{\lambda})^2}{\lambda^2}, g_{(0)ij}\right). \quad (A8)$$

Then by using the following formula [34]:

$$\langle T_{\mu\nu}\rangle = \frac{4R^3}{16\pi G_N}\left(g_{(4)\mu\nu} - \frac{1}{8}g_{(0)\mu\nu}((\mathrm{Tr}g_{(2)})^2 - \mathrm{Tr}g_{(2)}^2) - \frac{1}{2}(g_{(2)}^2)_{\mu\nu} + \frac{1}{4}g_{(2)\mu\nu}\mathrm{Tr}g_{(2)}\right), \quad (A9)$$

we find

$$\langle T_{\mu\nu}\rangle = \langle \tilde{T}^{(0)}_{\mu\nu}\rangle + \frac{4R^3}{16\pi G_N^{(5)}}\left\{\frac{3\lambda^2}{16}(1, \beta g_{(0)ij})\right\}, \quad (A10)$$

$$\langle \tilde{T}^{(0)}_{\mu\nu}\rangle = \frac{4R^3}{16\pi G_N^{(5)}}\frac{\tilde{c}_0}{R^4}(3, g_{(0)ij}), \qquad \beta = -\left(1 + \frac{2\frac{a_0}{\dot{a}_0}\dot{\lambda}}{3\lambda}\right), \quad (A11)$$

where $\langle \tilde{T}^{(0)}_{\mu\nu}\rangle$ comes from the conformal YM fields given in Ref. [16], so we find no anomaly for this component:

$$\langle \tilde{T}^{(0)\mu}_{\mu}\rangle = 0. \quad (A12)$$

The second term corresponds to the loop corrections of the YM fields in the curved space-time, and we find the conformal anomaly due to this term as

$$\langle T^{\mu}_{\mu}\rangle = -\frac{3\lambda^2(1 + \frac{\dot{\lambda}}{2\lambda}\frac{a_0}{\dot{a}_0})}{8\pi^2}N^2, \quad (A13)$$

where we used $G_N^{(5)} = 8\pi^3\alpha'^4 g_s/R^5$ and $R^4 = 4\pi N\alpha'^2 g_s$.

The above anomaly (A13) is obtained from the loop corrections of the $\mathcal{N} = 4$ SYM theory in a space-time, $g_{(0)\mu\nu}$, which is given by (A6). For this metric, the curvature squared terms responsible to the anomaly are given as

$$R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} = 12\left(2\lambda^2 + \dot{\lambda}\lambda\frac{a_0}{\dot{a}_0} + \left(\dot{\lambda}\frac{a_0}{2\dot{a}_0}\right)^2\right), \quad (A14)$$

$$R^{\mu\nu}R_{\mu\nu} = 12\left(3\lambda^2 + 3\dot{\lambda}\lambda\frac{a_0}{2\dot{a}_0} + \left(\dot{\lambda}\frac{a_0}{2\dot{a}_0}\right)^2\right), \quad (A15)$$

$$\frac{1}{3}R^2 = 12\left(4\lambda^2 + 4\dot{\lambda}\lambda\frac{a_0}{2\dot{a}_0} + \left(\dot{\lambda}\frac{a_0}{2\dot{a}_0}\right)^2\right). \quad (A16)$$

In general, the conformal anomaly for n_s scalars, n_f Dirac fermions, and n_v vector fields is given as [37,38]

$$\langle T^{\mu}_{\mu}\rangle = -\frac{n_s + 11n_f + 62n_v}{90\pi^2}E_{(4)} - \frac{n_s + 6n_f + 12n_v}{30\pi^2}I_{(4)}, \quad (A17)$$

$$E_{(4)} = \frac{1}{64}(R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2), \quad (A18)$$

$$I_{(4)} = -\frac{1}{64}\left(R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2\right), \quad (A19)$$

where $\Box R$ has been abbreviated since it does not contribute here. For the $\mathcal{N} = 4$ SYM theory, the numbers of the fields are given by $N^2 - 1$ times the number of each fields, which are equivalent to $n_s = 6$, $n_f = 2$, and $n_v = 1$. Then we find, for large N,

$$\langle T^{\mu}_{\mu}\rangle = \frac{N^2}{32\pi^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2\right) = -\frac{3\lambda^2(1 + \frac{\dot{\lambda}}{2\lambda}\frac{a_0}{\dot{a}_0})}{8\pi^2}N^2. \quad (A20)$$

This result (A20) is precisely equivalent to the above holographic one (A13). Thus, we could see that the holographic analysis could give correct results for the energy momentum tensor even if the metric is time dependent as shown previously in Ref. [16].

Appendix B: $\langle T^{\mathrm{IR}}_{\mu\nu}\rangle$ OF THE DUAL THEORY AT $r = 0$

In the IR side, we get $\langle T^{\mathrm{IR}}_{\mu\nu}\rangle$ by the parallel method. By using the above formula (A9), we find

$$\langle T^{\mathrm{IR}}_{\mu\nu}\rangle = \frac{4R^3}{16\pi G_N^{(5)}}(\hat{g}_{(0)00}t_{00}, \hat{g}_{(0)ij}t_{11}), \quad (B1)$$

$$t_{00} = -\frac{3r_0^8}{r^{*4} + \tilde{c}_0}, \qquad t_{11} = -r_0^8\frac{2r^{*4} + (r^* r^*{}_1)^2 + \tilde{c}_0}{(r^{*4} + \tilde{c}_0)((r^* r^*{}_1)^2 - \tilde{c}_0)}. \quad (B2)$$

This result should be interpreted as the vacuum expectation value of the energy momentum tensor of the SYM theory living in the space-time $\hat{g}_{(0)\mu\nu}$ given by (23). In the present case, however, both the metric $\hat{g}_{(0)\mu\nu}$ and the $\langle T_{\mu\nu}\rangle$ are different from the one given at the boundary $r \to \infty$. Then we must check how the two theories on each boundary are different. We perform this for the following three cases.

One expects that the central charges on each boundaries would be different from each other, since the renormalization group flow would be different. The answer for this issue is given by observing the trace anomaly, which is found from the above $\langle T_{\mu\nu}\rangle$ as follows:

$$\langle T^{\mu}_{\mu}\rangle = \frac{4R^3}{16\pi G_N^{(5)}}(t_{00}+3t_{11}) \tag{B3}$$

$$= -\frac{N^2}{2\pi^2}\frac{6r_0^8 r^{*2}((r^{*2}+r^{*}{}_1)^2)}{(r^{*4}+\tilde{c}_0)((r^{*}r^{*}{}_1)^2-\tilde{c}_0)}, \tag{B4}$$

where we used $G_N^{(5)} = 8\pi^3\alpha'^4 g_s/R^5$ and $R^4 = 4\pi N\alpha'^2 g_s$. This is rewritten by using the relation

$$W_{\rm IR} = \left(R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2\right) \tag{B5}$$

$$= -16\frac{6r_0^8 r^{*2}(r^{*2}+r^{*2}_1)}{(r^{*4}+\tilde{c}_0)((r^{*}r^{*}{}_1)^2-\tilde{c}_0)}, \tag{B6}$$

and we obtain

$$\langle T^{\mu}_{\mu}\rangle = \frac{N^2}{32\pi^2}W_{IR}. \tag{B7}$$

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