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Gravitational potential of a point mass in a brane world

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In brane-world models, combining the extradimensional field modes with the standard four-dimensional ones yields interesting physical consequences that have been proved from high-energy physics to cosmology. Even some low-energy phenomena have been considered along these lines to set bounds on the brane model parameters. In this work, we extend to the gravitational realm a previous result which gave finite electromagnetic and scalar potentials and self-energies for a source looking pointlike to an observer sitting in a 4D Minkowski subspace of the single brane of a Randall-Sundrum spacetime including compact dimensions. We calculate here the gravitational field for the same type of source by solving the linearized Einstein equations. Remarkably, it also turns out to be nonsingular. Moreover, we use gravitational experimental results of the Cavendish type and the parameterized post-Newtonian coefficients to look for admissible values of the brane model parameters. The anti–de Sitter radius hereby obtained is concordant with previous results based on the Lamb shift in hydrogen. However, the resulting parameterized post-Newtonian parameters lie outside the acceptable value domain.

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I. INTRODUCTION

We are close to celebrating 100 years since the birth of general relativity (GR), one of the most beautiful and spectacular theories in physics ever conceived. GR is beautiful because at the time, it introduced deep and unexpected physical concepts that allowed us to understand the gravitational field and its relation with the geometry of the spacetime. It is spectacular because, despite its age, the GR equations of motion have remained immutable in form, and they describe with great accuracy most of the observable gravitational physics. Over the years, GR has passed most of the experimental tests concerning the theory; however, it is well known that there exist some phenomena escaping an accurate description within the framework of GR and the Standard Model of particle physics, such as dark matter and dark energy. In order to develop a consistent theory that could describe these kinds of phenomena, physicists have tried to modify either GR or quantum mechanics and consider possible extensions of the Standard Model. Indeed, the effort to place limits on possible deviations from the standard formulations of such theories continues today.

Since its inception, there have been many attempts to modify GR with different purposes. Soon after its conception, there were notable proposals with the idea to extend it and incorporate it in a larger unified theory. A relevant example for the purpose of this work is the higher-dimensional theory introduced by Kaluza [1] and refined by Klein [2]. More recently and with the aim to solve the hierarchy problem, the ideas of large extra dimensions [3–5] and brane worlds were introduced [6–10]. Of course, in the literature, there are many other attempts to modify GR (see e.g. Ref. [11] and references therein), and currently people continue exploring the physical consequences predicted from them all and, most importantly, confronting them with experimental data. This work follows the same strategy; we will explore a particular characteristic of the gravitational field, specifically the behavior of the gravitational potential generated by a *pointlike* source in the so-called RSIIp model, which modifies GR by including extra dimensions, and we will confront it with experimental data available today.

The RSIIp model is an extension of the 5D Randall-Sundrum (RS) model with one brane (RSII) extended by p compact extra dimensions (RSIIp) [12–14]. Its construction was motivated by the need to improve the localization properties of matter fields within the standard RS model. Specifically, in the RSII model there exists a problem with localizing spin-1 fields on the brane, and a way out of this problem can be achieved by extending the model with p compact dimensions [15,16]. Thus, the RSIIp setup contains all the nice features of the RS model and additionally has the advantage of localizing every kind of field on the brane. These higher-dimensional models have the property to modify gravity in the low-scale length regime, and a huge amount of physical phenomena have been studied over the years, ranging from particle physics (see e.g.

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Refs. [17,18] and references therein) to cosmology (see e.g. Refs. [19,20] and references therein). Moreover, recently it was shown that an electric source lying in the single brane of a RSIIp spacetime which looks pointlike to an observer sitting in usual 3D space produces a static potential which is nonsingular at the 3D point position [21,22], and furthermore, it matches Coulomb potential outside a small neighborhood. Amusingly, coping with classical singularities goes back to the nonlinear proposal made by Born and Infeld [23]. In regard to the divergences in field theory, over the years there have been many attempts to formulate a theory that avoids the problem, or at least that could improve, for instance, the high-energy behavior of GR. Among them we have, for instance, string theory (see e.g. Ref. [24] and references therein), noncommutative theories (see e.g. Ref. [25] and references therein), and the recent attempt made by Horava [26] at a modified UV theory of gravity.

In this work, we extend the analysis of Ref. [21] to the case of the gravitational field. As we will show, the classical potential due to an effective 4D punctual source becomes regular at the position of the source in an analogous way to the scalar and gauge cases. To complement our study, we compare the consequences of this feature with some experimental observations: in particular, we have chosen to compare the predictions of the model with the experimental data of a Cavendish-type experiment which imposes a bound to the anti–de Sitter (AdS) radius of the bulk AdS metric. We also obtain the parameterized post-Newtonian (PPN) coefficients of the resulting effective theory.

The paper is organized as follows: In Sec. II, we describe briefly the RSIIp scenarios. In Sec. III, we discuss the linearized Einstein equations in the low-energy regime for a massive particle with the topology of a T^p torus, but which is seen as punctual by an observer in our 4D world. In Sec. IV, we obtain the metric perturbations, and in Sec. V, we discuss a Cavendish-type experiment and give the PPN coefficients. We give a short discussion of our results in Sec. VI.

II. RANDALL-SUNDRUM IIp SCENARIOS

The way in which the Randall-Sundrum IIp (RSIIp) scenarios arise has been discussed several times in the literature (see e.g. Refs. [12–16]), so here we just give a short summary including the most important features of the model. The RSIIp setups consist of a $(3+p)$-brane with p compact dimensions and positive tension σ, embedded in a $(5+p)$ spacetime whose metrics are two patches of anti–de Sitter (AdS_{5+p}) having curvature radius ϵ. (For convenience, in some equations we will use, instead of the radius, its inverse: $\kappa \equiv \epsilon^{-1}$.) The model arises from considering the $(5+p)$D Einstein action with bulk cosmological constant Λ and the action of a $(3+p)$-brane

$$S = \frac{1}{16\pi G_{5+p}} \int d^4x dy \prod_{i=1}^{p} R_i d\theta_i \sqrt{|g^{(5+p)}|}(R^{(5+p)} - 2\Lambda) + S_{\mathrm{brane}}, \qquad (1)$$

which leads to the Einstein equations of motion

$$R_{MN} - \frac{1}{2} R g_{MN}^{(5+p)} + \Lambda g_{MN}^{(5+p)} = 8\pi G_{5+p} T_{MN}. \qquad (2)$$

In these equations, we use the following notation for the $5+p$ coordinates: $X^M \equiv (x^\mu, \theta_i, y)$, where $\mu = 0, 1, 2, 3$, and $i = 1, \ldots p$. The four coordinates x^μ denote the coordinates that mimic our Universe, the θ_i's $\in [0, 2\pi]$ denote the p compact coordinates, and the R_i's signal the sizes of the corresponding compact dimensions. Finally, y denotes the noncompact extra dimension. The superscript in the determinant $g^{(5+p)}$ emphasizes the fact that the metric is $(5+p)$D. G_{5+p} is the Newton constant in $(5+p)$D, and the energy-momentum tensor $T_{MN} \equiv \frac{2}{\sqrt{|g^{(5+p)}|}} \frac{\delta S}{\delta g^{MN}}$ corresponds to the one produced by the brane.

With this setup and appropriate fine-tuning between the brane tension σ and the bulk cosmological constant Λ, which are related to κ as follows:

$$\sigma = \frac{2(3+p)}{8\pi G_{5+p}} \kappa,$$
$$\Lambda = -\frac{(3+p)(4+p)}{16\pi G_{5+p}} \kappa^2 = -\frac{(4+p)\sigma}{4} \kappa, \qquad (3)$$

there exists a solution to the $(5+p)$D Einstein equations with the metric

$$ds_{5+p}^2 = e^{-2\kappa|y|} \left[\eta_{\mu\nu} dx^\mu dx^\nu - \sum_{i=1}^{p} R_i^2 d\theta_i^2 \right] - dy^2. \qquad (4)$$

Here, $\eta_{\mu\nu}$ is the 4D Minkowski metric, and without loss of generality it was assumed that the brane is at the position $y = 0$. At $y = \text{constant}$, we have 4D flat hypersurfaces extended by p compact extra dimensions.

The interest in these setups comes from their property of localizing on the brane: scalar, gauge, and gravity fields due to the gravity produced by the brane itself. We emphasize that this property is valid whenever there are p extra compact dimensions [12,13]. In the limiting case $p = 0$, the model localizes scalar and gravity fields but not gauge fields. A short discussion about the consistency of both the KK and the RS compactifications, as well as a discussion of the moduli-fixing mechanisms and stability of the setup can be found, for instance, in Ref. [21]. In the literature, there are already different analyses of low-energy physics effects in these setups, such as the electric charge conservation [12], the Casimir effect between two conductor hyperplates [27–30], the Liennard-Wiechert potentials, the hydrogen

Lamb shift [22], and perturbations to the ground state of the helium atom [31], among others.

III. LOW-ENERGY LINEARIZED EINSTEIN EQUATIONS

In this section, we determine the linearized Einstein equations for the perturbations produced by a static source. In analogy with the scalar and gauge field cases discussed in Ref. [21], we consider a source with the topology of a p-dimensional torus sitting on the $(3+p)$-brane, which is seen as a punctual mass from the perspective of an observer living in the usual 3D low-energy observable part of the brane. In order to solve the equations, we follow closely the technique used in Ref. [32], where authors studied highly energetic particles that leave the 4D brane and propagate into the bulk of the 5D RSII model. The main difference in the physical situation discussed here with respect to the ones previously reported in the literature [32–34] is the inclusion of the p extra compact dimensions.

A. Linearized Einstein equations

Our starting points are the $(5+p)$D Einstein equations [Eq. (2)]. Taking the trace of these equations and replacing the value of R, we obtain the convenient equivalent form

$$R_{MN} = 8\pi G_{5+p}\left(T_{MN} - \frac{1}{3+p}Tg_{MN}\right) + \frac{2}{3+p}\Lambda g_{MN}. \tag{5}$$

In general, the linearized Einstein equations that result from considering metric perturbations h_{MN} to a known metric solution g_{MN}

$$ds^2 = g_{MN}dx^M dx^N + h_{MN}dx^M dx^N \tag{6}$$

and energy-momentum tensor perturbations δT_{MN} to the equations of motion [Eq. (5)] are given by

$$\delta R_{MN} = 8\pi G_{5+p}\left[\delta T_{MN} - \frac{1}{3+p}(h_{MN}T + g_{MN}\delta T)\right] + \frac{2}{3+p}\Lambda h_{MN}, \tag{7}$$

where (see for instance Ref. [35])

$$\delta R_{MN} = -\frac{1}{2}[\nabla_M\nabla_N\hat{h} + \nabla^A\nabla_A h_{MN} - \nabla^A\nabla_M h_{NA} - \nabla^A\nabla_N h_{MA}] \tag{8}$$

and $\hat{h} \equiv g^{MN}h_{MN}$. Following Ref. [32], we will work in Gaussian normal (GN) coordinates. In such a frame, one has

$$h_{yy} = h_{y\bar{M}} = 0, \tag{9}$$

where the coordinates $X^{\bar{M}}$ label the coordinates of the 4D flat brane and the compact dimensions: $X^{\bar{M}} \equiv \{x^\mu, R_i\theta_i\}$. Accordingly, the linearized theory is described by the metric

$$ds^2 = a^2(y)\eta_{\bar{M}\bar{N}}dx^{\bar{M}}dx^{\bar{N}} + h_{\bar{M}\bar{N}}dx^{\bar{M}}dx^{\bar{N}} - dy^2, \tag{10}$$

where $\eta_{\bar{M}\bar{N}} = \mathrm{diag}(1,-1,\ldots,-1)$ is a $(4+p)$D flat metric. We have also introduced the shorthand notation $a(y) \equiv e^{-k|y|}$. It is clear that for the metric of the RSIIp setup, $\hat{h}$ is given simply by $\hat{h} = a^{-2}\eta^{\bar{M}\bar{N}}h_{\bar{M}\bar{N}} \equiv a^{-2}h$.

As for the perturbation of the energy-momentum tensor, we shall consider a static source at the position y_0 with the topology of a pD torus; i.e., we consider that the massive object is located a distance y_0 away from the brane. From these considerations it is clear that during the computation, the perturbed energy-momentum tensor resides entirely on the bulk and is given by

$$\delta T^{MN} = \frac{m^{(5+p)}}{\sqrt{|g^{(5+p)}|}}\frac{dx^M}{ds}\frac{dx^N}{ds}\delta^3(\vec{x}-\vec{x})\delta(y-y_0), \tag{11}$$

where $\frac{dx^M}{ds} = (1,\vec{0})$. A technicality of our calculation is that if $y_0 > 0$ in Eq. (11), it means we are considering an energy-momentum tensor residing to the right of the brane; however, the RSIIp model owns the symmetry $z \to -z$. Then, although we will work entirely only to the right of the brane, it should be understood that matter is symmetric with respect to the brane, and therefore there exists another source located at position $-y_0$. The two symmetrically located sources together with the fact that we are considering only symmetric perturbations to the metric justify the way in which the computation is done [32]. Because we are interested in the gravitational potential produced by a source placed on the brane, after computing the solution to the linearized equations we will consider the limit $y_0 \to 0$, and the perturbations will appear as generated by a source of mass $M = 2m^{(5+p)}$ on the brane.

It is clear that for an energy-momentum tensor on the bulk, the second term on the right-hand side of Eq. (7) vanishes, and the third term becomes $\delta T = a^{-2}\eta^{MN}\delta T_{MN} \equiv a^{-2}\delta T_0^0$. Under these considerations, the nonvanishing linearized Einstein equations on the bulk are

$$\delta R_{yy} = 8\pi G_{5+p}\frac{1}{3+p}\delta T_0^0, \tag{12}$$

$$\delta R_{\bar{M}\bar{N}} - \frac{2\Lambda}{3+p}h_{\bar{M}\bar{N}} = 8\pi G_{p+5}\left[\delta T_{\bar{M}\bar{N}} - \frac{1}{3+p}\eta_{\bar{M}\bar{N}}\delta T_0^0\right], \tag{13}$$

where the variation of the Ricci tensor [Eq. (8)] can be explicitly written as [36]

$$\delta R_{yy} = -\partial_y \left[\frac{\partial_y h}{2a^2}\right], \tag{14}$$

$$\delta R_{\bar{M}\bar{N}} = \frac{1}{2}\partial_y^2 h_{\bar{M}\bar{N}} - \frac{p}{2}\kappa\partial_y h_{\bar{M}\bar{N}} + 2\kappa^2 h_{\bar{M}\bar{N}} - \left(\kappa^2 h + \frac{\kappa}{2}\partial_y h\right)\eta_{\bar{M}\bar{N}} + \frac{1}{2a^2}(\partial^{\bar{L}}\partial_{\bar{M}} h_{\bar{N}\bar{L}} + \partial^{\bar{L}}\partial_{\bar{N}} h_{\bar{M}\bar{L}} - \partial^{\bar{L}}\partial_{\bar{L}} h_{\bar{M}\bar{N}} - \partial_{\bar{N}}\partial_{\bar{M}} h). \tag{15}$$

Notice that the role of the p compact extra dimensions at the level of the variation of the Ricci tensor is given by the second term on the right-hand side of Eq. (15). In the case $p = 0$, we recover the expression of the variation of the Ricci tensor for the standard RS model [32,36].

B. The perturbation in modes

In order to solve the linearized Einstein equations, we start solving Eq. (13) by inserting Eq. (15) into it:

$$\frac{1}{2}h''_{\bar{M}\bar{N}} - \frac{p}{2}\kappa h'_{\bar{M}\bar{N}} + \frac{1}{2a^2}(\partial^{\bar{L}}\partial_{\bar{M}} h_{\bar{N}\bar{L}} + \partial^{\bar{L}}\partial_{\bar{N}} h_{\bar{M}\bar{L}} - \partial^{\bar{L}}\partial_{\bar{L}} h_{\bar{M}\bar{N}} - h_{,\bar{M}\bar{N}}) + 2\kappa^2 h_{\bar{M}\bar{N}} - (4+p)\kappa^2 h_{\bar{M}\bar{N}} = 8\pi G_{p+5}\left[\delta T_{\bar{M}\bar{N}} - \frac{1}{3+p}\eta_{\bar{M}\bar{N}}\delta T^0_0\right] + \left(\kappa^2 h + \frac{\kappa}{2}h'\right)\eta_{\bar{M}\bar{N}}, \tag{16}$$

where the prime denotes the derivative with respect to the y coordinate. At this point it is convenient to introduce a consideration about the mode spectrum of the metric perturbations into the equation, dictated by the geometry of the setup. Formally, we write down the metric perturbation in a Fourier series expansion due to the compact coordinates:

$$h_{\bar{M}\bar{N}}(x,\theta_i,y) = \prod_{k=1}^{p}\frac{1}{\sqrt{2\pi R_k}}\sum_{\vec{n}}(h_{\bar{M}\bar{N}}(x,y))_{(\vec{n})}e^{i\vec{n}\cdot\vec{\theta}}, \tag{17}$$

where $\vec{n}$ denotes the collection of p different indexes $\vec{n} = (n_1, n_2, \ldots, n_p)$ taking values in $\mathbb{Z}$, $\vec{\theta}$ is a p-dimensional vector whose components are the p compact coordinates θ_k: $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)$, and $\sum_{\vec{n}}$ is the collection of p sums $\sum_{\vec{n}} = \sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_p=-\infty}^{\infty}$. The functions $e^{i\vec{n}\cdot\vec{\theta}}$ correspond to the basis of the Fourier decomposition along the compact directions. It is well known that toroidal dimensional reductions *à la* Kaluza-Klein lead to consistent lower-dimensional theories (see e.g. Ref. [37] and references therein) which although they do not come with a mechanism to fix the radii of the T^p torus, by invoking agreement with phenomenology at low enough energy—in particular, agreement with the value of the electron charge—it is possible to set a bound to the radius of the order of the Planck length [2]. In the following, we shall consider a low-energy approximation so that we truncate the massive KK modes of the compact dimensions but keep those that correspond to the noncompact dimension (so far encoded in the y dependence of h_{MN}), meaning that we assume the scale energy of the former is much smaller than that of the latter. Under these considerations, we are performing the dimensional reduction on the T^p torus, or equivalently, we are keeping only the zero mode of the Fourier expansion, i.e.,

$$h_{\bar{M}\bar{N}}(x,\theta_i,y) \approx (h_{\bar{M}\bar{N}}(x,y))_{\vec{0}}. \tag{18}$$

From here onwards, we replace the whole metric perturbation in Eqs. (12) and (13) with its zero mode.

Under this consideration, the Laplacian operator simplifies to $\partial^{\bar{L}}\partial_{\bar{L}} = \Box + \partial^{\theta_i}\partial_{\theta_i} = \Box$, and Eq. (16) can be rewritten as

$$\frac{1}{2}h''_{\bar{M}\bar{N}} - \frac{p}{2}\kappa h'_{\bar{M}\bar{N}} + \frac{1}{2a^2}(\partial^{\bar{L}}\partial_{\bar{M}} h_{\bar{N}\bar{L}} + \partial^{\bar{L}}\partial_{\bar{N}} h_{\bar{M}\bar{L}} - \Box h_{\bar{M}\bar{N}} - h_{,\bar{M}\bar{N}}) + 2\kappa^2 h_{\bar{M}\bar{N}} - (4+p)\kappa^2 h_{\bar{M}\bar{N}} = 8\pi G_{p+5}\left[\delta T_{\bar{M}\bar{N}} - \frac{1}{3+p}\eta_{\bar{M}\bar{N}}\delta T^0_0\right] + \left(\kappa^2 h + \frac{\kappa}{2}h'\right)\eta_{\bar{M}\bar{N}}. \tag{19}$$

Introducing the shorthand definition

$$\xi_{\bar{M}} = h^{\bar{L}}_{\bar{M},\bar{L}} - \frac{1}{2} h_{,\bar{M}}, \tag{20}$$

Eq. (19) takes the form

$$\frac{1}{2} h''_{\bar{M}\bar{N}} - \frac{p}{2}\kappa h'_{\bar{M}\bar{N}} - \frac{1}{2a^2}\Box h_{\bar{M}\bar{N}} - (2+p)\kappa^2 h_{\bar{M}\bar{N}} \\ = 8\pi G_{p+5}\left[\delta T_{\bar{M}\bar{N}} - \frac{1}{3+p}\eta_{\bar{M}\bar{N}} T^0_0\right] \\ + \left(\kappa^2 h + \frac{\kappa}{2} h'\right)\eta_{\bar{M}\bar{N}} - \frac{1}{2a^2}(\xi_{\bar{M},\bar{N}} + \xi_{\bar{N},\bar{M}}). \tag{21}$$

We can consider the following gauge transformation:

$$h_{\bar{M}\bar{N}} = \bar{h}_{\bar{M}\bar{N}} + u_{\bar{M},\bar{N}} + u_{\bar{N},\bar{M}}, \tag{22}$$

where u_μ satisfies

$$u''_{\bar{M}} - p\kappa u'_{\bar{M}} - 2(2+p)\kappa^2 u_{\bar{M}} - \frac{1}{a^2}\Box u_{\bar{M}} = -\frac{1}{a^2}\xi_{\bar{M}}. \tag{23}$$

It follows, then, that $\bar{h}_{\bar{M}\bar{N}}$ should satisfy

$$\frac{1}{2}\bar{h}''_{\bar{M}\bar{N}} - \frac{p}{2}\kappa\bar{h}'_{\bar{M}\bar{N}} - \frac{1}{2a^2}\Box\bar{h}_{\bar{M}\bar{N}} - (2+p)\kappa^2\bar{h}_{\bar{M}\bar{N}} \\ = 8\pi G_{p+5}\left[\delta T_{\bar{M}\bar{N}} - \frac{1}{3+p}\eta_{\bar{M}\bar{N}}\delta T^0_0\right] + \left(\kappa^2 h + \frac{\kappa}{2}h'\right)\eta_{\bar{M}\bar{N}}. \tag{24}$$

The strategy to solve this equation is as follows: We can think the right-hand side of Eq. (24) as an effective energy-momentum tensor $T^{\rm eff}_{\bar{M}\bar{N}}$, in such a way that

$$8\pi G_{p+5}\left[\delta T_{\bar{M}\bar{N}} - \frac{1}{3+p}\eta_{\bar{M}\bar{N}}\delta T^0_0\right] + \left(\kappa^2 h + \frac{\kappa}{2}h'\right)\eta_{\bar{M}\bar{N}} \\ \equiv 8\pi G_{p+5} T^{\rm eff}_{\bar{M}\bar{N}}. \tag{25}$$

Therefore, Eq. (24) takes the form

$$\frac{1}{2}\bar{h}''_{\bar{M}\bar{N}} - \frac{p}{2}\kappa\bar{h}'_{\bar{M}\bar{N}} - \frac{1}{2a^2}\Box\bar{h}_{\bar{M}\bar{N}} - (2+p)\kappa^2\bar{h}_{\bar{M}\bar{N}} \\ = 8\pi G_{p+5} T^{\rm eff}_{\bar{M}\bar{N}}. \tag{26}$$

Solving this equation requires us to know the solutions to the homogeneous equations; once we have these solutions, we can compute the Green function, and with it solve the inhomogeneous Eq. (26). It is also convenient at this point to expand $\bar{h}_{MN}(x,y)$ in terms of the functions $\psi_m(y)$, which correspond to the mode structure of the metric perturbations due to the noncompact dimension y:

$$(h_{\bar{M}\bar{N}}(x,y))_{0^-} = \left(\int (h_{\bar{M}\bar{N}}(x))_m \psi_m(y) dm\right)_{(0^-)}. \tag{27}$$

Plugging this ansatz into the left-hand side of Eq. (26) allows us to perform a separation of variables in the differential operator. Introducing the separation constant m leads us to have an equation for $\psi_m(y)$ of the following form:

$$\left(\partial_y^2 - p\kappa\partial_y - 2(2+p)\kappa^2 + \frac{m^2}{a^2}\right)\psi_m(y) = 0. \tag{28}$$

This equation can be rewritten as a Bessel equation. In order to do this, we perform the variable change $\xi(y) = \epsilon a^{-1}(y)$, and we introduce the rescaled function $\psi(\xi) = \xi^{p/2}\tilde{\psi}(\xi)$, obtaining

$$\left[\partial_\xi^2 + \frac{1}{\xi}\partial_\xi + m^2 - \frac{\alpha^2}{\xi^2}\right]\tilde{\psi} = 0, \tag{29}$$

where the constant $\alpha \equiv 2 + \frac{p}{2}$ contains the information about the number of extra compact dimensions.

For the massless mode ($m = 0$), the solution is

$$\tilde{\psi}_0(\xi) = A_1\xi^\alpha + A_2\xi^{-\alpha} \Rightarrow \psi_0(\xi) = a_1\xi^{p+2} + a_2\xi^{-2}, \tag{30}$$

where a_i are integration constants. We take $a_1 = 0$ in order to have a normalizable solution, which is explicitly given by

$$\psi_0(y) = \sqrt{\left(1 + \frac{p}{2}\right)\kappa e^{-2\kappa y}}. \tag{31}$$

For the massive modes ($m > 0$), we obtain

$$\psi_m(y) = e^{\frac{p\kappa y}{2}}\sqrt{\frac{m}{2\kappa}}\left[a_m J_\alpha\left(\frac{m}{\kappa}e^{\kappa y}\right) + b_m N_\alpha\left(\frac{m}{\kappa}e^{\kappa y}\right)\right], \tag{32}$$

where the constants a_m and b_m are given by

$$a_m = -\frac{A_m}{\sqrt{1 + A_m^2}}, \qquad b_m = \frac{1}{\sqrt{1 + A_m^2}}, \tag{33}$$

with

$$A_m = \frac{N_{\alpha-1}(\frac{m}{\kappa}) - \frac{2\kappa}{m}N_\alpha(\frac{m}{\kappa})}{J_{\alpha-1}(\frac{m}{\kappa}) - \frac{2\kappa}{m}J_\alpha(\frac{m}{\kappa})}. \tag{34}$$

In order to simplify this expression further, it is convenient to take the approximation of light modes $m \ll \kappa^{-1}$; this is plausible because these are the modes contributing the most to the potential. In this approximation,

$$A_m = \frac{\Gamma(\alpha - 1)\Gamma(\alpha)}{\pi}\left(\frac{m}{2\kappa}\right)^{2-2\alpha}, \tag{35}$$

and therefore the coefficients a_m and b_m are given by

$$a_m = -1, \qquad b_m = \frac{\pi}{\Gamma(\alpha-1)\Gamma(\alpha)}\left(\frac{m}{2\kappa}\right)^{2\alpha-2}. \tag{36}$$

Plugging these coefficients into Eq. (32) and considering the same light-modes approximation in the Bessel and Neumman functions, we get

$$\psi_m(0) = \sqrt{\frac{m}{2\kappa}}\frac{1}{\Gamma(\alpha-1)}\left(\frac{m}{2\kappa}\right)^{\alpha-2}, \tag{37}$$

$$\psi_m(y') = -e^{\frac{p}{2}\kappa y'}\sqrt{\frac{m}{2\kappa}}J_\alpha\left(\frac{m}{\kappa}e^{\kappa y'}\right). \tag{38}$$

Notice that we are computing the massive modes at two different points of the y coordinate, because with these functions we are constructing the two-point Green function.

C. The Green function

With the eigenfunctions $\psi_m(y)$, it is straightforward to construct the Green function

$$\begin{aligned} G_R(x,x',y=0,y') &= -\frac{\psi_0(0)\psi_0(y')}{4\pi r} - \int_0^\infty dm\psi_m(0)\psi_m(y')\frac{e^{-mr}}{4\pi r} \\ &= -\frac{1}{4\pi r}\left(1+\frac{p}{2}\right)\frac{1}{\kappa\xi^2} + \frac{\xi^{\frac{p}{2}}}{\Gamma(\alpha-1)2^{\alpha-1}\kappa^{1-\alpha+\frac{p}{2}}}\int_0^\infty dm m^\alpha J_\alpha(m\xi)\frac{e^{-mr}}{m}. \end{aligned} \tag{39}$$

Explicit evaluation of this function depends on the parity of the number p of compact dimensions.

1. p odd

For this case, we have that α takes semi-integer values and the Green function is

$$G_R(x,x',y=0,y') = \frac{1}{4\pi r}\sqrt{\frac{2}{\pi}}\frac{(-1)^{\alpha-\frac{1}{2}}\xi^{\alpha+\frac{p}{2}}}{\Gamma(\alpha-1)2^{\alpha-1}\kappa^{1-\alpha+\frac{p}{2}}}\left(\frac{d}{\xi d\xi}\right)^{\alpha-\frac{1}{2}}\left[\frac{\pi}{2\xi} - \frac{\arctan(\frac{r}{\xi})}{\xi}\right] - \frac{1}{4\pi r}\left(1+\frac{p}{2}\right)\frac{1}{\kappa\xi^2}. \tag{40}$$

Using the relation

$$\left(\frac{d}{\xi d\xi}\right)^{\alpha-\frac{1}{2}}\left[\frac{1}{\xi}\right] = \frac{(-1)^{\alpha-\frac{1}{2}}[2(\alpha-1)]!(\alpha-1)!}{2^{\alpha-\frac{3}{2}}2^{-2\alpha+2}\sqrt{\pi}\xi^{2\alpha}[2(\alpha-1)]!} = \frac{(-1)^{\alpha-\frac{1}{2}}(\alpha-1)!}{2^{-\alpha+\frac{1}{2}}\sqrt{\pi}\xi^{2\alpha}}, \tag{41}$$

the Green function can be written as

$$G_R(x,x',y=0,y') = -\frac{1}{4\pi r}\sqrt{\frac{2}{\pi}}\frac{(-1)^{\alpha-\frac{1}{2}}\xi^{\alpha+\frac{p}{2}}}{\Gamma(\alpha-1)2^{\alpha-1}\kappa}\left(\frac{d}{\xi d\xi}\right)^{\alpha-\frac{1}{2}}\left[\frac{\arctan(\frac{r}{\xi})}{\xi}\right]. \tag{42}$$

The derivative can be evaluated, recalling the relation

$$\frac{d}{\xi d\xi}f(\xi) = 2\frac{d}{d\beta}f\left(\sqrt{\xi^2+\beta}\right)\bigg|_{\beta=0}, \tag{43}$$

which leads to the final form of the Green function:

$$\begin{aligned} G_R(x,x',y=0,y') = &-\frac{1}{4\pi r}\sqrt{\frac{2}{\pi}}\frac{(-1)^{\alpha-\frac{1}{2}}\xi^{\alpha+\frac{p}{2}}}{\Gamma(\alpha-1)2^{\alpha-1}\kappa}2^{\alpha-\frac{1}{2}} \\ &\times\left[-\Gamma\left(\alpha+\frac{1}{2}\right)\frac{r(-1)^{\alpha-\frac{1}{2}}}{2\alpha(r^2+\xi^2)^{\alpha+\frac{1}{2}}}F\left(1,\alpha+\frac{1}{2};\alpha+1;\frac{\xi^2}{r^2+\xi^2}\right)\right. \\ &\left.+\frac{(-1)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}{\sqrt{\pi}}\frac{1}{\xi^{2\alpha}}\arcsin\left(\frac{\xi}{\sqrt{r^2+\xi^2}}\right) + \frac{(-1)^{\alpha-\frac{1}{2}}}{\sqrt{\pi}\xi^{2\alpha}}\Gamma(\alpha)\arctan\left(\frac{r}{\xi}\right)\right]. \end{aligned} \tag{44}$$

2. p even

For even p, α takes integer values and the Green function is

$$G_R(x,x',y=0,y') = -\frac{1}{4\pi r}\left(1+\frac{p}{2}\right)\frac{1}{\kappa\xi^2} + -\frac{1}{4\pi r}\frac{(-1)^\alpha \xi^{\alpha+\frac{p}{2}}}{\Gamma(\alpha-1)2^{\alpha-1}\kappa^{1-\alpha+\frac{p}{2}}}\left(\frac{d}{\xi d\xi}\right)^{\alpha-1}\left[\frac{1}{\xi^2}-\frac{r}{\xi^2\sqrt{r^2+\xi^2}}\right]. \quad (45)$$

In a similar way to the former case, we obtain finally the Green function for even compact dimensions:

$$G_R(x,x',y=0,y') = \frac{1}{4\pi r}\frac{(-1)^\alpha \xi^{\alpha+\frac{p}{2}}}{\Gamma(\alpha-1)2^{\alpha-1}\kappa}\left(\frac{d}{\xi d\xi}\right)^{\alpha-1}\left[\frac{r}{\xi^2\sqrt{r^2+\xi^2}}\right]. \quad (46)$$

IV. SOLUTIONS

We are now in position to compute the solutions to the linearized Einstein equations in the low-energy regime. The order in which the solutions are obtained is the following: We start solving Eq. (12), where the Ricci tensor is given by Eq. (14). This happens because we have to know the expression for the combination $\kappa^2 h(x',y') + \frac{\kappa}{2}\partial_y h(x',y')$ in order to solve for the perturbations $\bar{h}_{MN}$ of Eq. (13).

A. Solution of the *yy* equation

We start by integrating Eq. (12) twice:

$$-\partial_y\left[\frac{\partial_y h}{2a^2}\right] = 8\pi G_{5+p}\frac{1}{3+p}\delta T^0_0. \quad (47)$$

After the first integral, we directly get

$$h' = -2a^2\frac{8\pi G_{5+p}}{3+p}\int_y^\infty dy\delta T^0_0 + 2a^2 C(x), \quad (48)$$

and after the second integral we obtain

$$h = -\int_y^\infty dy\left[2a^2\frac{8\pi G_{5+p}}{3+p}\int_y^\infty dz\delta T^0_0(z) - 2a^2 C(x)\right] + D(x); \quad (49)$$

here $C(x)$ and $D(x)$ are functions to be determined. From the explicit form of $a(y)$, we can evaluate in a straightforward way the second term of the integral in the equation above:

$$\int_y^\infty dy 2a^2(y) = \frac{a^2(y)}{\kappa}, \quad (50)$$

whereas for the first term we use an integration by parts:

$$\int_y^\infty dy2a^2\int_y^\infty dz\delta T^0_0(z) = \frac{a^2}{\kappa}\int_y^\infty dy\delta T^0_0 - \int_y^\infty dy\frac{a^2}{\kappa}\delta T^0_0, \quad (51)$$

obtaining that $h(y)$ is of the form

$$h = -\frac{8\pi G_{5+p}}{(3+p)\kappa}\left[a^2\int_y^\infty dy\delta T^0_0 - \int_y^\infty dy a^2\delta T^0_0\right] + \frac{a^2}{\kappa}C(x) + D(x). \quad (52)$$

So far, we have only considered the perturbation in the bulk. The role played by the brane in the solution appears through the junction conditions

$$K_{\bar{M}\bar{N}} = -\frac{8\pi G_{5+p}}{2}\left(S_{\bar{M}\bar{N}} - \frac{1}{3+p}\eta_{\bar{M}\bar{N}}a^2 S\right), \quad (53)$$

which constitute a connection between the metric perturbations living in the bulk and the matter perturbations confined to the brane ($S_{\bar{M}\bar{N}}$) [38]. In a GN coordinate system, the extrinsic curvature is given by the simple expression

$$K_{\bar{M}\bar{N}} = \frac{1}{2}\partial_y(a^2\eta_{\bar{M}\bar{N}} + h_{\bar{M}\bar{N}}), \quad (54)$$

whereas the energy-momentum tensor on the brane is given by

$$S_{\bar{M}\bar{N}} = -\sigma(a^2\eta_{\bar{M}\bar{N}} + h_{\bar{M}\bar{N}}) + \delta T_{\bar{M}\bar{N}}. \quad (55)$$

In Eq. (53), we are using the definition $S \equiv a^{-2}\eta^{\bar{M}\bar{N}}S_{\bar{M}\bar{N}}$. Plugging the expressions (54) and (55) into Eq. (53) and considering the energy-momentum tensor [Eq. (11)] and the relation between the brane tension and the AdS radius [Eq. (3)], we obtain after taking the trace of the junction condition that

$$\partial_y h + 2\kappa h = \frac{8\pi G_{5+p}}{3+p}\delta T\Big|_{y=0} = \frac{4\pi G_{5+p}\kappa}{3+p}\frac{m^{(5+p)}}{a^{2+p}(y')}\delta(y-y_0)\delta^3(\vec{x}-\vec{x}_0)\Big|_{y=0}. \quad (56)$$

This means that the points $y=0$ and y_0 never coincide, and therefore $\delta(y'-y_0)$ is null. Substitution of expression (52) into Eq. (56) allows us to find the expression for the function $D(x)$, which is given by the equation

$$2\kappa D(x) + 16\pi G_{5+p}\int_0^\infty a^2(y')\delta T(y')dy' = 0. \tag{57}$$

Once we know the value of $D(x)$, we can evaluate the combination of h and h' that appears in the definition [Eq. (25)] of $T^{\rm eff}_{\bar{M}\bar{N}}$:

$$\begin{aligned}\kappa^2 h(x',y') + \frac{\kappa}{2}\partial_{y'}h(x',y') &= -8\pi G_{5+p}\kappa\int_0^{y'} a^2(z)\delta T(z)dz \\ &= -\frac{8\pi G_{5+p}\kappa m^{(5+p)}}{a^{2+p}(y_0)}\theta(y'-y_0)\delta^3(\vec{x}'-\vec{x}_0).\end{aligned} \tag{58}$$

B. The $\bar{h}_{00}$ component

Once we have the solution of Eq. (12) and, as a consequence, the expression for the combination in Eq. (58), we can compute the expressions for the metric perturbations. Using the Green function of Sec. III C, we have that

$$\begin{aligned}\bar{h}_{00}(r,y=0) = 8\pi G_{5+p}\int d^3x'\int dy' G(\vec{x},y=0;\vec{x}',y')\Bigg[\left(\delta T_{00}(x',y') - \frac{1}{3+p}\eta_{00}\delta T^0_0(x',y')\right) \\ + \frac{1}{8\pi G_{5+p}}\left(\kappa^2 h(x',y') + \frac{\kappa}{2}\partial_y h(x',y')\right)\eta_{00}\Bigg],\end{aligned} \tag{59}$$

where according to the energy-momentum tensor [Eq. (11)],

$$\delta T_{00}(x',y') - \frac{1}{3+p}\eta_{00}\delta T^0_0(x',y') = \frac{2+p}{3+p}\frac{m^{(5+p)}}{a^{2+p}(y')}\delta(y'-y_0)\delta^3(\vec{x}'-\vec{x}_0). \tag{60}$$

Plugging expressions (58) and (60) into Eq. (59), we obtain

$$\begin{aligned}\bar{h}_{00}(r,y=0) &= 8\pi G_{5+p}\frac{2+p}{3+p}\frac{m^{(5+p)}}{a^{2+p}(y_0)}G_R(x,x'=x_0,y=0,y'=y_0) \\ &\quad - \frac{8\pi G_{5+p}\kappa m^{(5+p)}}{a^{2+p}(y_0)}\int dy' G_R(x,x'=x_0,y=0,y')\theta(y'-y_0).\end{aligned} \tag{61}$$

As we have discussed, the explicit form of the Green function depends of the parity of the number of compact extra dimensions p, and therefore this also happens for the metric component $\bar{h}_{00}$.

1. p odd

In the case in which p is odd, we use the Green function [Eq. (44)], obtaining

$$\begin{aligned}\bar{h}_{00}(r,y=0) = -8\pi G_{5+p}\frac{2+p}{3+p}\frac{m^{(5+p)}}{a^{2+p}(y_0)}\frac{1}{4\pi r}\sqrt{\frac{2}{\pi}}\frac{(-1)^{\alpha-\frac{1}{2}}\epsilon\xi^{\alpha+\frac{p}{2}}}{\Gamma(\alpha-1)2^{\alpha-1}}\left(\frac{d}{\xi d\xi}\right)^{\alpha-\frac{1}{2}}\left[\frac{\arctan(\frac{r}{\xi})}{\xi}\right]\Bigg|_{\xi=\xi_0} \\ + \frac{8\pi G_{5+p}\kappa m^{(5+p)}}{a^{2+p}(y_0)}\int\frac{d\xi}{k\xi}\frac{1}{4\pi r}\sqrt{\frac{2}{\pi}}\frac{(-1)^{\alpha-\frac{1}{2}}\epsilon\xi^{\alpha+\frac{p}{2}}}{\Gamma(\alpha-1)2^{\alpha-1}}\left(\frac{d}{\xi d\xi}\right)^{\alpha-\frac{1}{2}}\left[\frac{\arctan(\frac{r}{\xi})}{\xi}\right]\theta(\xi-\xi_0).\end{aligned} \tag{62}$$

Example: $p=1$—In particular, if we take the value $p=1$, we have

$$G_R(x,x',y=0,y')^{(1)} = -\frac{1}{4\pi r}\frac{\epsilon\xi^3}{\pi}\left(\frac{d}{\xi d\xi}\right)^2\left[\frac{\arctan(\frac{r}{\xi})}{\xi}\right] = -\frac{1}{4\pi r}\frac{\epsilon}{\pi}\left[\frac{5r}{\xi^3(1+\frac{r^2}{\xi^2})} - \frac{2r^3}{\xi^5(1+\frac{r^2}{\xi^2})^2} + \frac{3}{\xi^2}\arctan\left(\frac{r}{\xi}\right)\right], \tag{63}$$

and $\bar{h}_{00}$ is given by

$$\bar{h}_{00} = \frac{6\pi G_6 m^{(6)}}{a^2(y_0)} \frac{\epsilon}{4\pi^2 r}\left[\frac{5r}{\xi_0^3(1+\frac{r^2}{\xi_0^2})} - \frac{2r^3}{\xi_0^5(1+\frac{r^2}{\xi_0^2})^2} + \frac{3}{\xi_0^2}\arctan\left(\frac{r}{\xi_0}\right)\right]$$
$$- \frac{8\pi G_6 m^{(6)}}{a^2(y_0)} \frac{\epsilon}{4\pi^2 r}\left[-\frac{3}{2}\frac{\arctan(\frac{r}{\xi})}{\xi^2} - \frac{3}{2}\frac{1}{r\xi} - \frac{1}{2}\frac{\arctan(\frac{\xi}{r})}{r^2} + \frac{1}{r\xi(1+\frac{r^2}{\xi^2})}\right]\Bigg|_{\xi=\xi_0}^{\infty}. \tag{64}$$

Taking the limit when $y_0 \to 0$, $\xi_0 = \epsilon$, we obtain for this component

$$\bar{h}_{00} = -\frac{3G_6 m^{(6)}}{2\pi^2}\left[\frac{5}{\epsilon^2(1+\frac{r^2}{\epsilon^2})} - \frac{2r^2}{\epsilon^4(1+\frac{r^2}{\epsilon^2})^2} + \frac{3}{r\epsilon}\arctan\left(\frac{r}{\epsilon}\right)\right]$$
$$+ \frac{2G_6 m^{(6)}}{\pi^2}\left[\frac{3}{2}\frac{\arctan(\frac{r}{\epsilon})}{r\epsilon} + \frac{3}{2}\frac{1}{r^2} + \frac{\epsilon}{2}\frac{\arctan(\frac{\epsilon}{r})}{r^3} - \frac{1}{r^2(1+\frac{r^2}{\epsilon^2})} - \frac{1}{4}\frac{\pi\epsilon}{r^3}\right]. \tag{65}$$

It is illustrative to calculate the short- and the long-distance limits

$$\bar{h}_{00} = -\frac{G_6 m^{(6)}}{\pi^2}\left[\frac{20}{3\epsilon^2} - \frac{44}{5\epsilon^4}r^2 + \cdots\right], \qquad r \to 0, \tag{66}$$

$$\bar{h}_{00} = -\frac{G_6 m^{(6)}}{\pi^2}\left[\frac{3\pi}{4\epsilon}\frac{1}{r} + \frac{\epsilon\pi}{2r^3} + \cdots\right] \sim -\frac{2G_N m}{r}\left(1 + \frac{2\epsilon^2}{3}\frac{1}{r^2}\right), \qquad r \to \infty, \tag{67}$$

where we have defined the effective 4D Newton constant in terms of the 6D one as

$$G_N = \frac{3G_{(6)}}{8\pi\epsilon}. \tag{68}$$

2. *p* even

Example: $p = 2$—For the even case, we give as an example the value $p = 2$. In this case, the Green function is

$$G_R(x, x', y = 0, y')^{(2)} = -\frac{1}{4\pi r}\frac{\epsilon\xi^4}{4}\left(\frac{d}{\xi d\xi}\right)^2\left[\frac{r}{\xi^2\sqrt{r^2+\xi^2}}\right] = -\frac{1}{4\pi}\frac{\epsilon}{4}\left[\frac{8}{\xi^2\sqrt{r^2+\xi^2}} + \frac{4}{(r^2+\xi^2)^{\frac{3}{2}}} + \frac{3\xi^2}{(r^2+\xi^2)^{\frac{5}{2}}}\right], \tag{69}$$

and the potential is given by

$$\bar{h}_{00}^{(2)} = -\frac{4}{5}\frac{8\pi G_7 m^{(7)}}{a^3(y_0)}\frac{1}{4\pi}\frac{\epsilon}{4}\left[\frac{8}{\xi^2\sqrt{r^2+\xi^2}} + \frac{4}{(r^2+\xi^2)^{\frac{3}{2}}} + \frac{3\xi^2}{(r^2+\xi^2)^{\frac{5}{2}}}\right]\Bigg|_{\xi=\xi_0}$$
$$+ \frac{8\pi G_7 m^{(7)}}{\epsilon a^3(y_0)}\int_{\xi_0}^{\infty}\frac{\epsilon d\xi}{\xi}\frac{1}{4\pi}\frac{\epsilon}{4}\left[\frac{8}{\xi^2\sqrt{r^2+\xi^2}} + \frac{4}{(r^2+\xi^2)^{\frac{3}{2}}} + \frac{3\xi^2}{(r^2+\xi^2)^{\frac{5}{2}}}\right]\theta(\xi - \xi_0). \tag{70}$$

Evaluating the integral, we finally have

$$\bar{h}_{00}^{(2)} = -\frac{2G_7 m^{(7)}}{5}\left[\frac{8}{\epsilon\sqrt{r^2+\epsilon^2}} + \frac{4\epsilon}{(r^2+\epsilon^2)^{\frac{3}{2}}} + \frac{3\epsilon^3}{(r^2+\epsilon^2)^{\frac{5}{2}}}\right] + \frac{G_7 m^{(7)}}{2}\frac{4r^2+5\epsilon^2}{\epsilon(\epsilon^2+r^2)^{3/2}}. \tag{71}$$

The short- and long-distance limits for this case are

$$\bar{h}_{00}^{(2)} = -G_7 m^{(7)}\left[\frac{7}{2\epsilon^2} - \frac{21}{4\epsilon^4}r^2 + \cdots\right], \qquad r \to 0, \tag{72}$$

$$\bar{h}_{00}^{(2)} = -G_7 m^{(7)} \left[\frac{6}{5\epsilon}\frac{1}{r} + \frac{\epsilon}{2r^3} + \cdots\right] \sim -\frac{2G_N m}{r}\left(1 + \frac{5\epsilon^2}{12}\frac{1}{r^2}\right), \qquad r \to \infty, \tag{73}$$

where the 4D Newton constant is

$$G_N = \frac{3G_{(7)}}{5\epsilon}. \tag{74}$$

C. The $\bar{h}_{ij}$ components

For the spatial components of the induced metric on the brane, we proceed as before. In this case, the Green function of Sec. III C reads

$$\bar{h}_{ij}(r, y=0) = 8\pi G_{5+p}\int d^3x' \int dy' G(\vec{x}, y=0; \vec{x}', y')\left[\left(\delta T_{ij}(x', y') - \frac{1}{3+p}\eta_{ij}\delta T_0^0(x', y')\right) + \frac{1}{8\pi G_{5+p}}\left(\kappa^2 h(x', y') + \frac{\kappa}{2}\partial_y h(x', y')\right)\eta_{ij}\right], \tag{75}$$

where this time, according to Eq. (11),

$$\delta T_{ij}(x', y') - \frac{1}{3+p}\eta_{ij}\delta T(x', y') = -\frac{\eta_{ij}}{3+p}\frac{m^{(5+p)}}{a^{1+p}(y')}\delta(y' - y_0)\delta^3(\vec{x}' - \vec{x}_0). \tag{76}$$

Thus, in this case, we have in general that

$$\bar{h}_{ij}(r, y=0) = -8\pi G_{5+p}\frac{\eta_{ij}}{3+p}\frac{m^{(5+p)}}{a^{2+p}(y_0)}G_R(x, x'=x_0, y=0, y'=y_0) - \frac{8\pi G_{5+p}\kappa m^{(5+p)}}{a^{2+p}(y_0)}\eta_{ij}\int dy' G_R(x, x'=x_0, y=0, y')\theta(y' - y_0). \tag{77}$$

Again the computations have to be worked out in two separate cases depending on the parity of the number of extra compact dimensions.

1. *p odd*

This case corresponds to having integer values of the parameter α, so the expression of the components $\bar{h}_{ij}$ is given by

$$\bar{h}_{ij}(r, y=0) = 8\pi G_{5+p}\frac{\eta_{ij}}{3+p}\frac{m^{(5+p)}}{a^{2+p}(y_0)}\frac{1}{4\pi r}\sqrt{\frac{2}{\pi}}\frac{(-1)^{\alpha-\frac{1}{2}}\epsilon\xi^{\alpha+\frac{p}{2}}}{\Gamma(\alpha-1)2^{\alpha-1}}\left(\frac{d}{\xi d\xi}\right)^{\alpha-\frac{1}{2}}\left[\frac{\arctan(\frac{r}{\xi})}{\xi}\right]\Bigg|_{\xi=\xi_0} + \frac{8\pi G_{5+p}\kappa m^{(5+p)}\eta_{ij}}{a^{2+p}(y_0)}\int dy' \frac{1}{4\pi r}\sqrt{\frac{2}{\pi}}\frac{(-1)^{\alpha-\frac{1}{2}}\epsilon\xi^{\alpha+\frac{p}{2}}}{\Gamma(\alpha-1)2^{\alpha-1}}\left(\frac{d}{\xi d\xi}\right)^{\alpha-\frac{1}{2}}\left[\frac{\arctan(\frac{r}{\xi})}{\xi}\right]\theta(y' - y_0).$$

Example: $p = 1$—Evaluating the Green function for this case leads us to the expression

$$G_R(x, x', y=0, y')^{(1)} = -\frac{1}{4\pi r}\frac{\epsilon\xi^3}{\pi}\left(\frac{d}{\xi d\xi}\right)^2\left[\frac{\arctan(\frac{r}{\xi})}{\xi}\right] = -\frac{1}{4\pi r}\frac{\epsilon}{\pi}\left[\frac{5r}{\xi^3(1+\frac{r^2}{\xi^2})} - \frac{2r^3}{\xi^5(1+\frac{r^2}{\xi^2})^2} + \frac{3}{\xi^2}\arctan\left(\frac{r}{\xi}\right)\right]. \tag{78}$$

Hence, $\bar{h}_{ij}$ is, after taking the limit $y_0 \to 0$,

$$\bar{h}_{ij} = \frac{G_6 m^{(6)}\eta_{ij}}{2\pi^2}\left[\frac{5}{\epsilon^2(1+\frac{r^2}{\epsilon^2})} - \frac{2r^2}{\epsilon^4(1+\frac{r^2}{\epsilon^2})^2} + \frac{3}{r\epsilon}\arctan\left(\frac{r}{\epsilon}\right)\right] + \frac{2G_6 m^{(6)}\eta_{ij}}{\pi^2}\left[\frac{3}{2}\frac{\arctan(\frac{r}{\epsilon})}{r\epsilon} + \frac{3}{2}\frac{1}{r^2} + \frac{\epsilon}{2}\frac{\arctan(\frac{\epsilon}{r})}{r^3} - \frac{1}{r^2(1+\frac{r^2}{\epsilon^2})} - \frac{1}{4}\frac{\pi\epsilon}{r^3}\right]. \tag{79}$$

For astrophysical applications, it is convenient to calculate the long-distance limit

$$\bar{h}_{ij} = -\frac{G_6 m^{(6)}\eta_{ij}}{\pi^2}\left[-\frac{9\pi}{4\epsilon}\frac{1}{r} + \frac{\epsilon\pi}{2r^3} + \cdots\right] \sim -\frac{2G_N m}{r}\left(-3 + \frac{2\epsilon^2}{3}\frac{1}{r^2}\right)\eta_{ij}, \qquad r\to\infty, \tag{80}$$

where the Newton constant is the same as in Eq. (68).

2. p even

Example: $p=2$—In this case, α is a semi-integer number and the Green function is given by

$$G_R(x,x',y=0,y')^{(2)} = -\frac{1}{4\pi r}\frac{\epsilon\xi^4}{4}\left(\frac{d}{\xi d\xi}\right)^2\left[\frac{r}{\xi^2\sqrt{r^2+\xi^2}}\right] = -\frac{1}{4\pi}\frac{\epsilon}{4}\left[\frac{8}{\xi^2\sqrt{r^2+\xi^2}} + \frac{4}{(r^2+\xi^2)^{\frac{3}{2}}} + \frac{3\xi^2}{(r^2+\xi^2)^{\frac{5}{2}}}\right]. \tag{81}$$

Thus, the potential is written as

$$\bar{h}^{(2)}_{ij} = \frac{G_7 m^{(7)}\epsilon\eta_{ij}}{10a^3(y_0)}\left[\frac{8}{\xi^2\sqrt{r^2+\xi^2}} + \frac{4}{(r^2+\xi^2)^{\frac{3}{2}}} + \frac{3\xi^2}{(r^2+\xi^2)^{\frac{5}{2}}}\right]\Bigg|_{\xi=\xi_0} + \frac{G_7 m^{(7)}\epsilon\eta_{ij}}{2a^3(y_0)}\left[-\frac{4r^2+5\xi^2}{\xi^2(\xi^2+r^2)^{3/2}}\right]\Bigg|^{\infty}_{\xi=\xi_0}. \tag{82}$$

Evaluating the limits explicitly, we have

$$\bar{h}^{(2)}_{ij} = \frac{G_7 m^{(7)}\eta_{ij}}{10}\left[\frac{8}{\epsilon\sqrt{r^2+\epsilon^2}} + \frac{4\epsilon}{(r^2+\epsilon^2)^{\frac{3}{2}}} + \frac{3\epsilon^3}{(r^2+\epsilon^2)^{\frac{5}{2}}}\right] + \frac{G_7 m^{(7)}\eta_{ij}}{2}\frac{4r^2+5\epsilon^2}{\epsilon(\epsilon^2+r^2)^{3/2}}. \tag{83}$$

Taking the long-distance limit ($r\to\infty$), we obtain

$$\bar{h}^{(2)}_{ij} = -G_7 m^{(7)}\eta_{ij}\left[-\frac{14}{5\epsilon}\frac{1}{r} + \frac{\epsilon}{2r^3} + \cdots\right] \sim -\frac{2G_N m}{r}\left(-\frac{7}{3} + \frac{5\epsilon^2}{12}\frac{1}{r^2}\right)\eta_{ij}, \tag{84}$$

where G_N is given by Eq. (74).

V. EXPERIMENTAL TESTS

In this section, we consider two gravitational experiments in order to set bounds to the parameters of the model. First, we look at a Cavendish-type experiment. As a second test, we compare the perturbed induced metric of the model with the generic PPN metric generated by a static nonrotating compact object.

A. Cavendish-type test

In the context of the 5D Randall-Sundrum model, in Ref. [39], authors obtained the relative force corrections to Newton's gravitational force between two massive spheres. The analysis was performed by computing both the exact (considering the whole Kaluza-Klein massive tower) and the approximated gravitational potential (long-distance limit) and comparing them in order to find out where the application of the approximate solution is appropriate. For their analysis, they used the long-distance limit of the potential generated by a massive particle (of mass m) in the RS model, which is of the form

$$\varphi(r) \approx -\frac{mG_N}{r}\left(1+\frac{\alpha}{r^2}\right), \qquad \alpha = l^2/2, \tag{85}$$

where l is proportional to the anti–de Sitter radius. This potential leads to the following gravitational force between two massive spheres:

$$F(r) = \frac{G_N m_1 m_2}{r^2}(1+\delta_F), \tag{86}$$

with

$$\delta_F = -\frac{9\alpha}{8R^3R'^3} \times \left\{\ln\left(\frac{r^2-(R'+R)^2}{r^2-(R'-R)^2}\right)\left[-\frac{1}{4}r^4+\frac{1}{2}r^2(R'^2+R^2) -\frac{1}{4}(R'^2-R^2)^2\right] - r^2R'R + R'^3R + R'R^3\right\}. \tag{87}$$

Here, R and R' are the radii of the spheres. Experimental data to verify this expression of the force is obtained from the Moscow Cavendish-type experiment [40], where one of the spheres was made of platinum with a radius $R \approx 0.087$ cm and mass $m_1 = 59.25 \times 10^{-3}$ g, whereas the second sphere was made of tungsten with a radius $R' \approx 0.206$ cm and mass $m_2 = 706 \times 10^{-3}$ g. The centers of the spheres were separated by a distance of $r = 0.3773$ cm.

To obtain a bound on l, it is necessary to use an accurate value of Newton's gravitational constant. The values given by CODATA in 2010 [41] are

$$\frac{G_N}{10^{-11}}\frac{m^3}{kgs^2} = 6.674215 \pm 0.000092 \quad \text{and} \quad 6.674252 \pm 0.000124; \tag{88}$$

here the relative error $\Delta G_N/G_N$ shows the agreement of the measurements of the gravitational constant with the r^{-2} experiments [39]—i.e., the relation $|\Delta G_N/G_N| = \delta_F$ gives the upper limit for δ_F in order to not detect experimental deviations from Newton's law. In the 5D RS model, this implies that $l \leq 9.067$ μm and $l \leq 10.527$ μm. A second approach using the complete solution gives $l \leq 9.070$ μm and $l \leq 10.531$ μm. For practical use, we can take $l \leq 10$ μm, which, combined with the expressions (67) and (73) that we have obtained for the potentials in the brane, produces a bound to the AdS radius:

$$l^2 = \frac{4\epsilon^2}{3} \Rightarrow \epsilon = \sqrt{\frac{3}{4}}l \simeq 0.86l = 8.6\,\mu\text{m}, \quad \text{for } p=1, \tag{89}$$

$$l^2 = \frac{5\epsilon^2}{6} \Rightarrow \epsilon = \sqrt{\frac{6}{5}}l \simeq 1.09l = 10.9\,\mu\text{m}, \quad \text{for } p=2. \tag{90}$$

These bounds are not in conflict with others previously reported in the literature; nevertheless, the ones obtained here are weaker than, for instance, the ones obtained by the Lamb shift, which gives bounds of the order $\epsilon \sim 10^{-14}$ m for $p = 1$ and $\epsilon \sim 10^{-13}$ m for $p = 2$ [22].

B. The four-dimensional effective metric on the brane

Here we want to obtain the effective metric on the brane and look at the Newtonian and the parametrized post-Newtonian (PPN) limits in order to set some bounds on the parameters of the theory. The PPN limit of metric theories of gravity contains ten real-valued parameters, and to every metric theory of gravitation corresponds a set of values of the PPN parameters. The observational values of the parameters have been measured in the Solar System and also in binary neutron stars [42,43].

The corresponding PPN metric in "standard" spherical coordinates for a nonrotating object is [43]

$$ds^2_{\text{PPN}} = \left[1 - \frac{2G_N m}{\rho} + \frac{2G_N^2 m^2(\beta-\gamma)}{\rho^2} + \cdots\right]dt^2 - \left[1 + \frac{2G_N m\gamma}{\rho} + \cdots\right]d\rho^2 - \rho^2 d\Omega. \tag{91}$$

For this case, only the β and γ parameters appear. The γ parameter measures how much space curvature g_{ij} is produced by a unit rest mass, while β measures how much nonlinearity is there in the superposition law for gravity g_{00}. These two parameters are involved in the astrophysical effects of the perihelion shift and light deflection as follows [44]:

$$\delta_{\text{prec}} = \frac{1}{3}(2+2\gamma-\beta)\left[\frac{6\pi G_N m}{c^2 a(1-e^2)}\right], \tag{92}$$

where a is the orbit's semimajor axis and e is the eccentricity.

$$\delta_{\text{def}} = \frac{1+\gamma}{2}\frac{4G_N m}{c^2 b}; \tag{93}$$

in this case, b is the impact parameter of the light ray.

The four-dimensional effective metric on the brane is given by

$$ds^2 = (1+h_{00})dt^2 + (-\delta_{ij} + h_{ij})dx^i dx^j. \tag{94}$$

For the cases of one and two extra compact dimensions ($p = 1, 2$), taking into account the results in Eqs. (67), (73), (80), and (84), the metric is, to the lowest order that is needed here,

$$ds^2 = \left[1 - \frac{2G_N m}{r}\left(1 + \frac{k_p}{r^2}\right)\right]dt^2 + \left[-1 + \frac{2G_N m}{r} l_p\right]\delta_{ij} dx^i dx^j, \tag{95}$$

with

$$k_1 = \frac{2\epsilon^2}{3}, \qquad l_1 = 3; \qquad k_2 = \frac{5\epsilon^2}{12}, \qquad l_2 = \frac{7}{3}. \tag{96}$$

In spherical coordinates, we have

$$ds^2 = \left[1 - \frac{2G_N m}{r}\left(1 + \frac{k_p}{r^2}\right)\right]dt^2 + \left[-1 - 2l_p \frac{G_N m}{r}\right][dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)]. \tag{97}$$

This metric is given in the isotropic form, and we want it in the "standard" form, which is the one adopted for the calculation of the PPN form of the metric of a static nonrotating compact object. In order to obtain that form for our metric, we take the coordinate transformation

$$\rho = r\left(1 + \frac{l_p G_N m}{r} + \cdots\right). \tag{98}$$

The metric in the new coordinates is

$$ds^2 = \left[1 - \frac{2G_N m}{\rho} - \frac{2l_p G_N^2 m^2}{\rho^2} + \cdots\right]dt^2 - \left[1 + \frac{2l_p G_N m}{\rho} + \cdots\right]d\rho^2 - \rho^2 d\Omega. \tag{99}$$

The corresponding PPN metric is [43]

$$ds^2_{\rm PPN} = \left[1 - \frac{2G_N m}{\rho} + \frac{2G_N^2 m^2(\beta - \gamma)}{\rho^2} + \cdots\right]dt^2 - \left[1 + \frac{2G_N m\gamma}{\rho} + \cdots\right]d\rho^2 - \rho^2 d\Omega. \tag{100}$$

We notice by comparing the metrics that we do have the Newtonian limit. The values of the PPN coefficients β and γ for this theory are

$$\beta = 0; \qquad \gamma = l_p. \tag{101}$$

At the order of approximation considered here, the quantity k_p does not appear, implying that the astrophysical tests do not impose a constraint on the anti–de Sitter length. The obtained values for the PPN parameters for this theory, in the cases where we have one or two extra compact dimensions, disagree with the observed values, since they are very close to 1 (the values for general relativity). We cannot tell if taking more compact dimensions will ameliorate the problem.

VI. DISCUSSION

The perspective on known phenomena changes in light of models of spacetime that include extra dimensions. In particular, brane-world models have provided new possibilities in high-energy physics and cosmology to try to solve some problems like the hierarchy [6,7] or dark matter/energy problems [19,20]. However, little attention has been devoted to low-energy physical effects, which may shed light in regard to the viability of such higher-dimensional scenarios by making reference to known experimental data including the Casimir effect, Lamb shift, and others [21,22,29,30]. In fact, even there, some unexpected results may emerge, as in the case of nonsingular field configurations like those reported here and in previous works [21,22].

In the present work, we studied the gravitational potential produced by a source which looks pointlike to a 4D observer sitting in the single brane of an extended Randall-Sundrum-II scenario. Such source extends along the p compact extra dimensions of the single brane, thus forming a T^p torus touching our usual 3D space at one point. A linear approximation for the hyperdimensional Einstein equations appropriate for such models was used. We obtained a gravitational potential which is nonsingular at the position of the source in 4D. In line with our motivation, we also calculated the gravitational force between two spheres in order to compare it with experimental data. This sets a bound for the AdS radius of the order 10 μm which is consistent with previous more stringent electromagnetic results based on the Lamb shift in hydrogen [22]. On the other hand, we obtained the PPN parameters for the field configuration corresponding to the *pointlike* source. The Newtonian limit is correctly contained in our results, and this was proved explicitly for $p = 1$, 2 extra compact dimensions. However, the PPN values obtained for the parameters of the RSIIp model are out of range of the experimental data. This is not a problem, as long as we do consider our brane model RSIIp to be a test scenario rather than a realistic proposal to describe our world.

Future work along the lines we have developed here can include the following: The gravitational radiation reaction problem may be reanalyzed in a setting similar to the one presented here. This may help us to further understand the role of its specific features that allow us to solve the divergent character of the standard 4D case. In particular, it would be of interest to pinpoint what are the elements relevant for the resolution of the divergence in connection with the source—namely, whether is it linked to its topology, extension, codimension, or something else. Further divergences in field theory may acquire a different form in brane worlds, and we think they deserve some effort. This may be the case, for instance, for quantum field theory in a brane-world background.

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