

**Covariant variational approach to Yang-Mills theory**M. Quandt,<sup>\*</sup> H. Reinhardt,<sup>†</sup> and J. Heffner<sup>‡</sup>*Universität Tübingen, Institut für Theoretische Physik, Auf der Morgenstelle 14,  
D-72076 Tübingen, Germany*

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We investigate the low-order Green's functions of  $SU(N)$  Yang-Mills theory in Landau gauge, using a covariant variational principle based on the effective action formalism. Employing an approximation to the Faddeev-Popov determinant established previously in the Hamiltonian approach in Coulomb gauge leads to a closed set of integral equations for the ghost and gluon propagator. We carry out the renormalization and the infrared analysis of this system of equations. Finally, we solve the renormalized system numerically and compare with lattice results and other functional approaches.

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**I. INTRODUCTION**

The low-order Green's functions of Yang-Mills theory have been the focus of many investigations, both in the continuum and on the lattice. Functional methods such as the functional renormalization group (FRG) flow equations [1] and Dyson-Schwinger equations (DSE) [2] initially concentrated their studies on the case of covariant gauges, since this is the natural choice in a covariant setup, and allows us to use Becchi-Rouet-Stora-Tyutin (BRST) symmetry and the ensuing Slavnov-Taylor identities to guide and improve the analysis. Moreover, the Kugo-Ojima criterion [3,4] argues that there should be a direct link between the deep infrared behavior of the gluon and ghost propagator, and the issue of color confinement. Most investigations initially found an infrared vanishing, scaling type of solution for the gluon propagator, which is, however, at odds with high-precision lattice simulations [5–8]. It was only later realized that infrared finite so-called decoupling solutions could also be obtained in the functional approach under certain circumstances [1]. In fact, an infrared finite gluon propagator had been found before in Refs. [9–20].

By contrast, the so-called variational approach to the Hamiltonian formulation of Yang-Mills theory investigated in Refs. [21–25] has been formulated in Coulomb gauge. This noncovariant condition is advantageous in the Hamiltonian approach since it allows for an explicit resolution of Gauß's law. In the Hamiltonian approach of Refs. [21–25] the Schrödinger equation for the Yang-Mills vacuum wave functional is approximately solved by the variational principle, minimizing the energy with suitable trial *Ansätze* for the vacuum wave functional. Compared to functional methods based on the Lagrangian formulation of quantum field theory, like the FRG [1] and DSE [2], the

variational principle of the Hamiltonian approach has the benefit that the relative size of the energy density controls the quality of the approximation made, i.e. of the trial wave functional used. By enlarging the space of trial states the description can be improved. The Hamiltonian approach in Coulomb gauge provides direct access to the so-called Coulomb potential between static charges and gives relatively simple explanations for confinement [26–30] and other low-energy phenomena [31]. Furthermore it yields propagators which are in good agreement with lattice calculations [32–35] and the Gribov-Zwanziger confinement scenario [36,37]. Unfortunately it cannot be generalized directly to covariant gauges and make contact with the alternative studies mentioned above.

In the present paper, we will demonstrate that a conceptually similar variational principle can be established in covariant gauges if one relies on the effective action instead of the energy. We will present a variational approach to quantum field theory which is based on the minimization of the effective action and apply it to Yang-Mills theory in Landau gauge. This approach will result in a system of integral equations for the low-order Green's functions, which we solve using the techniques borrowed from the Coulomb gauge calculations mentioned above. We will also argue that our method gives optimal results for the propagators of the theory, while the extension to realistic vertices presumably requires to go beyond the Gaussian *Ansatz* using Dyson-Schwinger equation techniques [38]; see also Ref. [39] for an alternative approach.

The paper is organized as follows: in the next section, we present the general variational principle for the effective action and explain in Sec. III how it can be applied to Yang-Mills theory in covariant gauges. After presenting our *Ansatz* for the trial path integral measure, we discuss an approximation to the Faddeev-Popov determinant which was introduced in Ref. [24] in the context of the Hamiltonian approach in Coulomb gauge and which facilitates the treatment of the ghost sector. In Sec. IV we apply the variational principle to the effective action of

<sup>\*</sup>markus.quandt@uni-tuebingen.de<sup>†</sup>hugo.reinhardt@uni-tuebingen.de<sup>‡</sup>jan.heffner@uni-tuebingen.de

Yang-Mills theory in Landau gauge and derive a closed set of integral equations for the ghost and gluon propagator. The rather cavalier method to renormalize these equations is put on a solid ground in Sec. V by relating it to the traditional introduction of counterterms. After carrying out the infrared analysis of our renormalized system of equations, we present in Sec. VI their numerical solution and compare with recent high-precision lattice data. Finally, we give a short summary and end with an outlook on further developments.

## II. THE VARIATIONAL PRINCIPLE

Below we recall the variational principle for the effective action in quantum field theory. This material mainly serves to fix some of our notation and introduces various concepts used later on. We start with a theory for a quantum field  $\phi$  defined by an action  $S(\phi)$  in  $d = n + 1$  dimensional Euclidean space time. Expectation values of operators are computed with the normalized measure

$$\begin{aligned} d\mu_0(\phi) &= Z^{-1} \exp[-\hbar^{-1}S(\phi)]d\phi \\ Z &\equiv \int d\phi \exp[-\hbar^{-1}S(\phi)]. \end{aligned} \quad (1)$$

Here  $d\phi$  is the flat (translationally invariant) measure in field space, and an implicit regularization is understood. The obvious analogy with statistical mechanics can now be exploited to define a variational method: The Gibbs-like measure Eq. (1) is the unique probability measure in field space which minimizes the free energy (or rather, free action),

$$F(\mu) \equiv \langle S \rangle_\mu - \hbar \mathcal{W}(\mu) \stackrel{!}{=} \min. \quad (2)$$

Here  $\langle \dots \rangle_\mu \equiv \int d\mu(\phi) \dots$  is the expectation value in the trial measure  $\mu$ , and the normalization is such that  $\langle 1 \rangle_\mu = \int d\mu(\phi) = 1$ . If the measure is written in Radon-Nikodym form  $d\mu(\phi) = d\phi \rho(\phi)$  with a suitable density  $\rho$ , the quantity

$$\mathcal{W}(\mu) \equiv -\langle \ln \rho \rangle_\mu = - \int d\phi \rho(\phi) \ln \rho(\phi) \quad (3)$$

is the entropy which measures the accessible field space for quantum fluctuations. The value of the minimal free action from Eq. (2) attained for the Gibbs measure (1) is  $F(\mu_0) = -\hbar \ln Z$ . Notice that Planck's constant really plays a role similar to the temperature in statistical mechanics, i.e. it controls the balance between the classical action  $\langle S \rangle$  and the fluctuation entropy  $\mathcal{W}(\mu)$ .

Traditionally, one would now take the minimizing measure Eq. (1), or some approximate minimum in a restricted measure space, and compute the Schwinger functions

$$\begin{aligned} G_n(x_1, \dots, x_n) &\equiv \langle \phi(x_1) \cdots \phi(x_n) \rangle \\ &= \int d\mu_0(\phi) \phi(x_1) \cdots \phi(x_n), \end{aligned} \quad (4)$$

or rather the generating functional  $W(j)$  of their connected part. From this, the effective action  $\Gamma$  follows by Legendre transformation.

Alternatively, the effective action can also be characterized directly by a variational principle: To this end, we go back to Eq. (2) and perform the minimization in two steps: first, we define a constrained free action by restricting some operator  $\Omega(\phi)$  to a classical value  $\omega$ , and second, we minimize this action under variation of all trial probability measures, which yields the effective action for the operator  $\Omega$ ,

$$\Gamma(\omega) \equiv \min_\mu F(\mu, \omega) \equiv \min_\mu \{ \langle S \rangle_\mu - \hbar \mathcal{W}(\mu) \mid \langle \Omega \rangle_\mu = \omega \}. \quad (5)$$

It depends, of course, on the choice of the operator  $\Omega$ , and is a functional of the prescribed value  $\omega$  both explicitly (via the constraint) and implicitly (via the  $\omega$ -dependence of the solving measure). In most cases, the constraint is chosen as the vev of the quantum field itself, so that

$$\Gamma(\varphi) \equiv \min_\mu \{ \langle S \rangle_\mu - \hbar \mathcal{W}(\mu) \mid \langle \phi \rangle_\mu = \varphi \}. \quad (6)$$

The variational principle Eq. (2) is now equivalent to  $\Gamma(\varphi) \stackrel{!}{=} \min$ , which is a problem in classical field theory that can be considered as solved. We now have two distinct definitions of the effective action:

(i) *Functional*: Define the Gibbs measure as the (unique) solution of Eq. (2), compute the Green functions, or their generating functional, from Eq. (4), and finally construct the effective action as generating functional of 1PI correlators by means of a Legendre transformation.

(ii) *Linear response*: Compute the effective action directly from the variational principle Eq. (6) and obtain the 1PI proper  $n$ -point functions as derivatives at  $\varphi = 0$ .

It is not hard to see that the two descriptions agree, cf. Appendix A, but this identity only holds for the *exact* solution of the variation problem, when no restrictions are placed on the trial measures  $\mu$ . This is rarely ever the case. In practice, a viable variation scheme will have to restrict the space of trial measures  $\{\mu\}$  to those candidates for which the expectation values in Eq. (2) or Eq. (6) can be computed explicitly. In such a restricted variation, the effective action determined from (i) and (ii) will differ. In Appendix B, we sketch the two approaches for a simple  $\phi^4$  theory with Gaussian trial measures, and also compare with the variational solution of the Schrödinger equation in the Hamilton formalism.

As with all variational methods, it is not *a priori* clear which of the two formulations above will give the better

approximation to the true system, although general arguments [40] indicate that the linear response approach (ii) is of higher order in the difference between true and approximate minimal measure  $\mu_0$ . (However, higher order does not always mean higher accuracy.) In any case, the representation Eq. (6) is conceptually simpler and automatically ensures that  $\Gamma(\varphi)$  is a convex *upper bound* to the true effective action whenever restrictions are placed on the measure  $d\mu(\phi)$ .

### III. APPLICATION OF THE VARIATIONAL PRINCIPLE TO YANG-MILLS THEORY IN COVARIANT GAUGES

Since our variational principles are covariant, it is natural to study  $SU(N)$  Yang-Mills theory in covariant gauges, in particular Landau gauge. Ignoring Gribov copies, the exact measure for this problem is

$$d\mu_0(A) = Z^{-1} \mathcal{J}(A) \exp[-\hbar^{-1} S_{\text{gf}}(A)] dA \quad (7)$$

$$Z = \int dA \mathcal{J}(A) \exp[-\hbar^{-1} S_{\text{gf}}(A)]$$

$$S_{\text{gf}} = \frac{1}{2} \|F_A\|^2 + \frac{1}{2\xi} \|d^\dagger A\|^2, \quad (8)$$

where  $A$  and  $F_A$  are the differential forms for the gauge connection and its field strength, respectively,

$$A = A_\mu dx^\mu = A_\mu^a T^a dx^\mu$$

$$F_A = dA + gA \wedge A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} F_{\mu\nu}^a T^a dx^\mu \wedge dx^\nu, \quad (9)$$

and  $g$  is the bare coupling strength. As usual, Feynman gauge ( $\xi = 1$ ) simplifies the Lorentz structure of the propagator, while Landau gauge ( $\xi = 0$ ) yields transversal gluons that can be compared directly to lattice investigations. The prefactors in the action and the inner product of Lie-Algebra valued forms on 4D euclidean space  $M$ ,

$$(\eta, \omega) \equiv (-2) \text{tr} \int_M \eta \wedge * \omega = \int_M \eta^a \wedge * \omega^a$$

are consistent with anti-Hermitian generators of the Lie-algebra, normalized according to  $\text{tr} T^a T^b = -\frac{1}{2} \delta^{ab}$ . Moreover, the measure factor  $\mathcal{J}(A)$  in Eq. (8) is the (normalized) Faddeev-Popov determinant

$$\mathcal{J}(A) \equiv \text{Det}[-\partial_\mu \hat{D}_\mu^{ab}] / \text{Det}[-\square \delta^{ab}]$$

$$= \text{Det}[-\square \delta^{ab} - g \partial_\mu f^{abc} A_\mu^c] / \text{Det}[-\square \delta^{ab}] \quad (10)$$

with  $\mathcal{J}(0) = 1$ . It is helpful to interpret  $\mathcal{J}(A)$  as the weight of the gauge orbit through  $A$ , i.e. the canonical volume form on this orbit in field space.

### A. Modified variational principle

The appearance of the Faddeev-Popov determinant in the measure Eq. (8) requires some modification of the basic variational principle Eq. (2), because the latter only holds for measures of the Gibbs form Eq. (1). One obvious solution is to transfer the Faddeev-Popov determinant into the action,

$$S_{\text{gf}} \rightarrow \bar{S} = S_{\text{gf}} - \hbar \ln \mathcal{J}.$$

Then the variational principle based on  $\bar{S}$  takes the standard form  $F(\mu) \stackrel{!}{=} \min$ , where

$$F(\mu) = \langle \bar{S} \rangle_\mu - \hbar \mathcal{W}(\mu)$$

$$= \langle S_{\text{gf}} \rangle_\mu - \hbar \langle \ln \mathcal{J}(A) \rangle_\mu + \hbar \langle \ln \rho(A) \rangle_\mu. \quad (11)$$

Here,  $\rho$  is the deviation of the trial measure from the flat measure (excluding the Faddeev-Popov determinant),  $d\mu = dA \rho(A)$ . The rhs of Eq. (11) suggests to rewrite the variational principle by redefining the entropy,

$$F(\mu) = \langle S_{\text{gf}} \rangle_\mu - \hbar \bar{\mathcal{W}}(\mu)$$

$$\bar{\mathcal{W}}(\mu) \equiv \mathcal{W}(\mu) + \langle \ln(\mathcal{J}) \rangle_\mu = -\langle \ln(\rho/\mathcal{J}) \rangle_\mu = -\langle \ln \bar{\rho} \rangle_\mu. \quad (12)$$

Here,  $\bar{\rho}$  is now the deviation from the standard measure *including* the Faddeev-Popov determinant, which is the natural metric on the space of gauge orbits,

$$d\mu = dA \rho(A) = dA \mathcal{J}(A) \bar{\rho}(A). \quad (13)$$

The redefined entropy Eq. (12) coincides with the usual notion of the *relative entropy* of the trial volume form  $dA \rho(A)$  compared to the standard weight  $dA \mathcal{J}(A)$  on the space of gauge orbits. As a consequence, the general variational approach sketched above remains valid for YM theory in covariant gauges, if we only replace the entropy  $\mathcal{W}$  by the *relative entropy*  $\bar{\mathcal{W}}$ .

### B. Gaussian trial measure

In the next step, we have to choose a class of suitable trial measures which is simple enough to allow for the necessary integrals to be performed, but still captures the essential physics of the system. For this purpose, we note

- (i) gluons are only weakly interacting in the ultra-violet due to asymptotic freedom
- (ii) gluon configurations near the Gribov horizon ( $\mathcal{J} = 0$ ) are assumed to play a dominant role in the infrared, and the self-interactions of gluons in such configurations may become sub-dominant.

The overall picture is (i) an (almost) noninteracting constituent gluon with (ii) an enhanced weight near the Gribov horizon. It is precisely this picture which is

supported by the variational calculation in the Hamiltonian approach to Yang-Mills theory in Coulomb gauge developed in Refs. [23–25]. The first condition implies that the trial action should be (close to) Gaussian, while the enhancement at the horizon is controlled by the volume form on the gauge orbit: Since the natural volume form is  $\mathcal{J}(A)$ , replacing it in the trial measure by  $\mathcal{J}^{1-2\alpha}$  with  $\alpha \geq 0$  will enhance the weight of near-horizon configurations by a relative factor  $\mathcal{J}(A)^{-2\alpha} \gg 1$ .

We will thus attempt a variational approach based on trial measures of the form ( $\hbar = 1$ )

$$d\mu(A) = \mathcal{N}_\alpha \cdot dA \mathcal{J}(A)^{1-2\alpha} \times \exp \left[ -\frac{1}{2} \int d^d(x, y) A_\mu^a(x) \omega_{\mu\nu}^{ab}(x, y) A_\nu^b(y) \right], \quad (14)$$

where  $\mathcal{N}_\alpha$  is the overall normalization.<sup>1</sup> Note that for  $\alpha = 0$  the Gaussian represents the relative weight  $\bar{\rho}(A)$  while for  $\alpha = \frac{1}{2}$  it gives the full weight  $\rho(A)$ , cf. Eq. (12). We will treat  $\alpha$  as a variational parameter, although we shall find below that the exact value of  $\alpha$  is immaterial, at least up to two loop order in a formal loop counting scheme introduced in the next subsection.

The measure (14) is unconstrained and thus appropriate for the functional approach discussed in Sec. II. (We present the necessary modifications to comply with the constraint  $\langle A \rangle = \mathcal{A}$  in Eq. (26) below.) In the absence of an external classical field  $\mathcal{A}$  the variational method maintains global color and Lorentz symmetry, and the kernel  $\omega$  can be chosen diagonal and transversal up to the covariant gauge fixing term from Eq. (8),

$$\omega_{\mu\nu}^{ab}(x, y) = \int \frac{d^d k}{(2\pi)^4} e^{ik(x-y)} \omega_{\mu\nu}^{ab}(k) \\ \omega_{\mu\nu}^{ab}(k) = \delta^{ab} \left[ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} (1 - \xi^{-1}) \right] \omega(k) \equiv \delta^{ab} t_{\mu\nu}(k) \omega(k). \quad (15)$$

The normalization in Eq. (14) is such that the bare gluon propagator

$$D_{\mu\nu}^{ab}(x, y) \equiv \langle A_\mu^a(x) A_\nu^b(y) \rangle = \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} D_{\mu\nu}^{ab}(k) \quad (16)$$

reduces for  $g \rightarrow 0$  (and hence  $\mathcal{J} \rightarrow 1$ ) to

<sup>1</sup>The normalization factor  $\mathcal{N}_\alpha$  will, in general, depend on the variational kernel  $\omega$ . In the case  $\alpha = \frac{1}{2}$ , for instance, we have  $\mathcal{N}_{\frac{1}{2}} = \det[\omega/(2\pi)]^{\frac{1}{2}}$ .

$$D_{\mu\nu}^{ab}(k) \xrightarrow{\mathcal{J} \rightarrow 1} [\omega(k)^{-1}]_{\mu\nu}^{ab} = \delta^{ab} t_{\mu\nu}^{-1}(k) \frac{1}{\omega(k)} \stackrel{g \rightarrow 0}{=} \delta^{ab} t_{\mu\nu}^{-1}(k) \frac{1}{k^2}, \quad (17)$$

i.e.  $\omega(k) \rightarrow k^2$ . This is also the UV limit of  $\omega(k)$  due to asymptotic freedom. Here, the Lorentz structure is given by the inverse

$$t_{\mu\nu}^{-1}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} (1 - \xi); \quad (18)$$

for Landau gauge ( $\xi = 0$ ) this becomes the transversal projector.

A possible caveat against the trial measure Eq. (14) is that it does not respect the BRST symmetry of the full theory, nor any of the identities that follow from it. This is, in a sense, unavoidable in a variational *Ansatz*, because the simplest nontopological action with full BRST symmetry is already the full YM theory, so that *any* truncation will necessarily break the Slavnov-Taylor identities to a certain extent. For Landau gauge, however, recent lattice calculations favor a soft BRST breaking massive gluon propagator in the deep infrared, and such *decoupling* solutions were also found under certain assumptions within functional approaches [9–18]. Since a dynamical mass generation is one of the main virtues of variational methods, the *Ansatz* Eq. (14) seems therefore justified for covariant gauges.

### C. Curvature approximation

The Faddeev-Popov determinant Eq. (10) and its expectation value in the trial measure (14) cannot be computed in closed form. In the following, we will adopt an approximation that has been shown to be correct up to two-loop order in the energy functional within the variational Hamiltonian approach [24] in *Coulomb gauge*: Since  $(\ln \mathcal{J})$  and  $\delta \ln \mathcal{J} / \delta A$  both vanish at  $A = 0$ , we can write

$$\ln \mathcal{J}[A] = -\frac{1}{2} \int d^d(x, y) K_{\mu\nu}^{ab}(x, y) A_\mu^a(x) A_\nu^b(y) \quad (19)$$

with a symmetric kernel  $K[A]$  that may depend arbitrarily on the gauge connection  $A$ . As a consequence,

$$K(1, 2) = -\frac{\delta^2 \ln \mathcal{J}}{\delta A(1) \delta A(2)} - \left\{ \frac{\delta K(1, 3)}{\delta A(2)} + \frac{\delta K(2, 3)}{\delta A(1)} \right\} A(3) \\ - \frac{1}{2} \frac{\delta K(3, 4)}{\delta A(1) \delta A(2)} A(3) A(4), \quad (20)$$

where each digit stands for the combination of color, Lorentz and spacetime indices, and repeated indices are summed or integrated over. If we now *formally* introduce a loop counting parameter in the exponent of the trial measure (14), we find from Eqs. (19) and (20)

$$\begin{aligned}
\langle \ln \mathcal{J}[A] \rangle &= \left\langle \frac{1}{2} \frac{\delta^2 \ln \mathcal{J}}{\delta A(1) \delta A(2)} A(1) A(2) \right\rangle \\
&+ \left\langle \frac{1}{2} \left\{ \frac{\delta K(1,3)}{\delta A(2)} + \frac{\delta K(2,3)}{\delta A(1)} \right\} A(1) A(2) A(3) \right\rangle \\
&+ \left\langle \frac{1}{4} \frac{\delta K(3,4)}{\delta A(1) \delta A(2)} A(1) A(2) A(3) A(4) \right\rangle \\
&= \left\langle \frac{1}{2} \frac{\delta^2 \ln \mathcal{J}}{\delta A(1) \delta A(2)} \right\rangle \langle A(1) A(2) \rangle + \dots,
\end{aligned}$$

where the expectation value is taken with the trial measure, Eq. (14), and the dots indicate higher orders in the loop counting scheme. To leading loop order, we can therefore replace the kernel  $K[A]$  by the average curvature on the gauge orbit<sup>2</sup>,

$$K_{\mu\nu}^{ab}(x, y) \rightarrow \chi_{\mu\nu}^{ab}(x, y) \equiv - \left\langle \frac{\delta^2 \ln \mathcal{J}}{\delta A_\mu^a(x) \delta A_\nu^b(y)} \right\rangle. \quad (21)$$

In total, this approximation then yields the formula [24]

$$\ln \mathcal{J}[A] \approx - \frac{1}{2} \int d^d(x, y) \chi_{\mu\nu}^{ab}(x, y) \cdot A_\mu^a(x) A_\nu^b(y) \quad (22)$$

which differs, on average, from the exact expression (19) only by a higher-loop effect.<sup>3</sup> However, the omitted higher-loop terms need not be negative definite and we cannot guarantee that our approximate effective action is always a strict upper bound to the true effective action. We will refer to Eq. (22) in the following as the curvature approximation.

The salient point of this approximation is now that the curvature Eq. (21) is easier to compute than the expectation value of the full Faddeev-Popov determinant. As shown in Sec. IV C, the curvature can be related to the ghost propagator (although we do not explicitly introduce ghosts), which in turn can be evaluated from a Dyson equation that involves the kernel  $\omega$  and the full ghost-gluon vertex. In the rainbow-ladder approximation, this full vertex is replaced by the bare one and  $\chi$  becomes a well-defined function of the ghost propagator and the variation parameters  $\omega$  and  $\alpha$ ; for further details cf. Sec. IV C. This approximation is further supported by the fact that the ghost-gluon vertex in Landau gauge is not renormalized [41] and shows little dressing in lattice simulations [42], i.e. the radiative corrections to the vertex tend to cancel.

We can now use the curvature approximation Eq. (22) directly in the trial measure (14). From global color and Lorentz symmetry of our variational *Ansatz* (in the functional formulation without an external classical field  $\mathcal{A}$ ), the

<sup>2</sup>Besides being defined in  $d = 4$  Euclidean dimensions, the curvature  $\chi$  introduced in Eq. (21) differs from the one defined in Ref. [24] by a factor of 2; the same is true for the kernel  $\omega$ .

<sup>3</sup>In Ref. [24], it was shown by explicit calculation that Eq. (22) is, in fact, exact up to including two loops in the effective action.

expectation value in the definition Eq. (21) entails that the curvature has the same simple Lorentz structure as the variation kernel

$$\chi_{\mu\nu}^{ab}(x, y) = \delta^{ab} t_{\mu\nu}(k) \chi(k), \quad (23)$$

where  $\chi(k)$  is known as the scalar curvature. The trial measure Eq. (14) thus depends on the curvature and the parameter  $\alpha$  only in the combination

$$\bar{\omega}(k) \equiv \omega(k) + (1 - 2\alpha)\chi(k). \quad (24)$$

This has the same effect as putting  $\alpha = \frac{1}{2}$  and replacing  $\omega \rightarrow \bar{\omega}$  in Eq. (14), i.e. the Faddeev-Popov determinant drops out from our variational *Ansatz* within the curvature approximation which then becomes a simple Gaussian with kernel  $\bar{\omega}(k)$ , and hence results in the gluon propagator [cf. Eq. (16)],

$$D_{\mu\nu}^{ab}(k) = \delta^{ab} t_{\mu\nu}^{-1}(k) \frac{1}{\bar{\omega}(k)}. \quad (25)$$

This observation greatly simplifies the computation of expectation values, since we then have *Wick's theorem* at our disposal. It must be stressed, however, that the variation is still with respect to  $\omega(k)$ , *not*  $\bar{\omega}(k)$ , because the curvature  $\chi(k)$  is, in principle, a dependent quantity.

So far, we have mainly discussed the unconstrained measure for the functional approach, when no external classical field  $\mathcal{A}$  is prescribed. As discussed in Appendix B, this is entirely sufficient to investigate the propagators, since these agree in the functional and linear response approach. Although we will not study the nontrivial vertex corrections arising in the linear response formulation in any detail, we still want to show at least how the variation problem can be set up in this case: First, we have to adjust the trial measures introduced above to comply with the constraint  $\langle A \rangle = \mathcal{A}$  imposed by the classical field. This can always be achieved by *shifting* the gauge field  $A_\mu \rightarrow A_\mu - \mathcal{A}_\mu$  in the (full) density  $\rho(A)$  of the trial measure.<sup>4</sup> In the present case, the density  $\rho(A)$  is Gaussian after applying the curvature approximation to Eq. (14), so that the final form

<sup>4</sup>This statement is not restricted to Gaussian measures: since the full density  $\rho(A)$  multiplies, by definition, the flat measure  $dA$ , the proposed shift in  $\rho(A)$  leads to

$$\begin{aligned}
\langle A \rangle &= \int dA \rho(A) \cdot A \rightarrow \int dA \rho(A - \mathcal{A}) \cdot A \\
&\stackrel{(*)}{=} \int dA \rho(A) \cdot A + \mathcal{A} \int dA \rho(A) \\
&= \langle A \rangle + \mathcal{A},
\end{aligned} \quad (26)$$

where we used the translation invariance of the flat measure in (\*). In the absence of a classical field, Lorentz invariance entails  $\langle A \rangle = 0$ , and the constraint  $\langle A \rangle = \mathcal{A}$  follows.

of our trial probability measure for the linear response approach is

$$\rho(A) = \text{Det} \left( \frac{\bar{\omega}}{2\pi} \right)^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \int d^d(x, y) (A_\mu^a(x) - \mathcal{A}_\mu^a(x) - \bar{\omega}_{\mu\nu}^{ab}(x, y) (A_\nu^b(y) - \mathcal{A}_\nu^b(y))) \right]. \quad (27)$$

It should be emphasized again that the optimal kernel  $\bar{\omega}_{\mathcal{A}}$  determined from Eq. (27) will depend implicitly on the classical field  $\mathcal{A}$ , which is externally prescribed and thus arbitrary. As a consequence, we can no longer assume the simple color and Lorentz structure (15) valid in the functional approach; instead we would have to deal with the full matrix gap equation in position space, cf. Appendix C.

$$\begin{aligned} \langle A_\mu^a(x) \rangle_\mu &= \mathcal{A}_\mu^a(x) \\ \langle A_\mu^a(x) A_\nu^b(y) \rangle_\mu &= \mathcal{A}_\mu^a(x) \mathcal{A}_\nu^b(y) + [\bar{\omega}^{-1}]_{\mu\nu}^{ab}(x, y) \\ \langle A_\mu^a(x) A_\nu^b(y) A_\alpha^c(z) \rangle_\mu &= ([\bar{\omega}^{-1}]_{\mu\nu}^{ab}(x, y) \mathcal{A}_\alpha^c(z) + 2 \text{ perm.}) + \mathcal{A}_\mu^a(x) \mathcal{A}_\nu^b(y) \mathcal{A}_\alpha^c(z) \\ \langle A_\mu^a(x) A_\nu^b(y) A_\alpha^c(z) A_\beta^d(u) \rangle_\mu &= ([\bar{\omega}^{-1}]_{\mu\nu}^{ab}(x, y) [\bar{\omega}^{-1}]_{\alpha\beta}^{cd}(z, u) + 2 \text{ perm.}) + ([\bar{\omega}^{-1}]_{\mu\nu}^{ab}(x, y) \mathcal{A}_\alpha^c(z) \mathcal{A}_\beta^d(u) + 5 \text{ perm.}) \\ &\quad + \mathcal{A}_\mu^a(x) \mathcal{A}_\nu^b(y) \mathcal{A}_\alpha^c(z) \mathcal{A}_\beta^d(u). \end{aligned} \quad (28)$$

Next, we have to insert this into  $\langle S_{\text{gf}} \rangle$  and combine it with the relative entropy (cf. Eq. (32) below) to obtain the free action  $F(\omega, \mathcal{A})$ . The resulting expressions are, however, rather complicated because the kernel  $\bar{\omega}$  in the linear response approach does not have the color and Lorentz symmetry Eq. (15). On the other hand, this complication is unnecessary: as argued in Appendix B, it is sufficient to use the functional approach with  $\mathcal{A} = 0$  and the symmetric kernel Eq. (15), as long as we are only interested in the propagators of the theory. For completeness, we present the full expression for the free action  $F(\omega, \mathcal{A})$  as well as the ensuing gap equation in Appendix C, but we do

## IV. THE EFFECTIVE ACTION OF YANG-MILLS THEORY

We are now in a position to determine the effective action of Yang-Mills theory from the variational principle using the trial probability measure (27).

### A. The free action

To evaluate the free action of the trial measure (27), we first expand the classical Yang-Mills Lagrangian including the gauge fixing term,

$$\begin{aligned} \mathcal{L}_{\text{gf}} &= \frac{1}{2} A_\mu^a [-\square \delta_{\mu\nu} + (1 - \xi^{-1}) \partial_\mu \partial_\nu] A_\nu^a \\ &\quad + g f^{abc} (\partial_\mu A_\nu^a) A_\mu^b A_\nu^c + \frac{g^2}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A_\mu^d A_\nu^e. \end{aligned}$$

The relevant correlators in the measure Eq. (27) can easily be worked out using Wick's theorem,

not investigate the resulting vertex corrections in more detail.

For the remainder of this paper, we therefore concentrate on the propagators, i.e. we set  $\mathcal{A} = 0$  and use translational invariance to transform to momentum space as in Eq. (15). The global color symmetry  $\bar{\omega}^{ab} \sim \delta^{ab}$  combined with the antisymmetry of the structure constants  $f^{abc}$  then allows to perform all color traces by means of the  $SU(N)$  relations

$$f^{abc} f^{abd} = N \delta^{cd} \quad \Rightarrow \quad f^{abc} f^{abc} = N(N^2 - 1).$$

The result for the average classical action is

$$\begin{aligned} \langle S_{\text{gf}} \rangle_{\bar{\omega}} &\equiv \langle S_{\text{gf}} \rangle_{\bar{\omega}, \mathcal{A}=0} = S_{\text{gf}}(\mathcal{A}=0) + \frac{1}{2} V_d (N^2 - 1) b_0 \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{\bar{\omega}(k)} \\ &\quad + \frac{g^2}{4} V_d N (N^2 - 1) \left\{ b_1 \Omega^2 - b_2 \int \frac{d^d(p, k)}{(2\pi)^{2d}} \frac{(p \cdot k)^2}{p^2 k^2} \frac{1}{\bar{\omega}(p) \bar{\omega}(k)} \right\}. \end{aligned} \quad (29)$$

Here,  $V_d$  is the spacetime volume and the numerical factors are

$$\begin{aligned} b_0 &= d_\xi \\ b_1 &= d^2 - 3d + 3 + 2(d-2)\xi + \xi^2 \\ b_2 &= (1 - \xi)^2, \end{aligned} \quad (30)$$

where the symbol  $d_\xi$  in the first line means  $d$  for all  $\xi \neq 0$ , and  $(d-1)$  for  $\xi = 0$ . The kernel  $\bar{\omega}$  appears in Eq. (29) both explicitly and also within the momentum-independent expressions

$$\begin{aligned}\Omega_{\mu\nu} &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{k^2} \frac{1}{\bar{\omega}(k)}, \\ \Omega &\equiv \Omega_{\mu\mu} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\bar{\omega}(k)}.\end{aligned}\quad (31)$$

Next we need the entropy of the Gaussian measure, which can be calculated in the same way as for the  $\phi^4$  case (cf. Appendix B). The additional color indices on the kernel  $\bar{\omega}$  in the measure simply yield an overall factor of  $N^2 - 1$ , so that

$$\begin{aligned}\mathcal{W} &= \frac{1}{2} \text{Tr} \left\{ \mathbb{1} - \ln \left( \frac{\omega}{2\pi\hbar} \right) \right\} \\ &= -\frac{1}{2} (N^2 - 1) V_d \int \frac{d^d k}{(2\pi)^d} \\ &\quad \times \text{tr} \ln \left[ \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} (1 - \xi^{-1}) \right) \bar{\omega}(k) \right] + \text{const},\end{aligned}\quad (32)$$

where the trace  $\text{tr}$  is with respect to the Lorentz indices. For  $\xi \neq 0$  we can now use the following identity for  $(d \times d)$  matrices,

$$\begin{aligned}\text{tr} \ln(t_{\mu\nu} \cdot \bar{\omega}) &= \ln \det(\bar{\omega} \cdot t_{\mu\nu}) = \ln(\bar{\omega}^d \cdot \det t_{\mu\nu}) \\ &= d \ln \bar{\omega} + \ln \det t_{\mu\nu}.\end{aligned}\quad (33)$$

The last term is  $\omega$  independent and may thus be dropped, so that<sup>5</sup>

$$\mathcal{W} = -(N^2 - 1) \frac{1}{2} d_\xi V_d \int \frac{d^d k}{(2\pi)^d} \ln \bar{\omega}(k) + \text{const}.\quad (34)$$

This is the full entropy of the measure Eq. (27) with  $\mathcal{A} = 0$  and the symmetric kernel  $\bar{\omega}$ . As pointed out earlier, the free action in the YM case must, however, be based on the *relative* entropy  $\bar{\mathcal{W}}$  Eq. (12), which differs from Eq. (34) by the expectation value of the Faddeev-Popov determinant

$$\bar{\mathcal{W}} = -(N^2 - 1) \frac{1}{2} d_\xi V_d \int \frac{d^d k}{(2\pi)^d} \ln \bar{\omega}(k) + \langle \ln \mathcal{J} \rangle_{\bar{\omega}}.\quad (35)$$

We will later treat the last term in curvature approximation, but for now we keep it general. Eventually, we find the free action as the difference between the average action Eq. (29) and the relative entropy Eq. (35),

$$F(\bar{\omega}) = \langle S_{\text{gf}} \rangle_{\bar{\omega}} - \bar{\mathcal{W}}(\bar{\omega}).\quad (36)$$

<sup>5</sup>The formula also holds for Landau gauge  $\xi = 0$  if the factor of  $d$  is replaced by  $(d-1)$ .

## B. The gap equation

Our task is to minimize the free action Eq. (36) with respect to the kernel  $\omega(k)$ . From Eq. (24) and the fact that the curvature  $\chi(k)$  is, in principle, a  $\omega$ -dependent quantity, we have

$$\frac{\delta}{\delta\omega(k)} = \frac{\delta}{\delta\bar{\omega}(k)} + (1 - 2\alpha) \int dp \frac{\delta\chi(p)}{\delta\omega(k)} \cdot \frac{\delta}{\delta\bar{\omega}(p)}.\quad (37)$$

The second term describes the implicit change of the curvature with the variation kernel. While an integral type of equation for this quantity can, in principle, be written down, it represents a higher order effect and will thus be neglected within the present approximation scheme. It also vanishes for  $\alpha = \frac{1}{2}$ . Since we will later find that the effective action depends only on  $\bar{\omega}$  and is thus independent of  $\alpha$  we can safely put  $\alpha = \frac{1}{2}$ . The remaining derivative acting on Eq. (36) gives the gap equation in the form

$$\begin{aligned}0 = \frac{\delta F}{\delta\bar{\omega}(k)} &= -\frac{V_d}{(2\pi)^d} \frac{1}{2\bar{\omega}(k)^2} \cdot \left\{ (N^2 - 1) d_\xi [k^2 - \bar{\omega}(k)] \right. \\ &\quad \left. + g^2 C_2 \int \frac{d^d p}{(2\pi)^d} \left[ b_1 - b_2 \frac{(k \cdot p)^2}{k^2 p^2} \right] \frac{1}{\bar{\omega}(p)} \right\} \\ &\quad - \frac{\delta}{\delta\bar{\omega}(k)} \langle \ln \mathcal{J} \rangle_{\bar{\omega}}.\end{aligned}\quad (38)$$

For the last term, we resort again to the curvature approximation Eq. (22). With the correlators from Eq. (28), we have

$$\begin{aligned}\langle \ln \mathcal{J} \rangle_{\bar{\omega}} &\approx -\frac{1}{2} \int d(x, y) \chi_{\mu\nu}^{ab}(x, y) \langle A_\mu^a(x) A_\nu^b(y) \rangle_{\bar{\omega}, \mathcal{A}} \\ &= -\frac{1}{2} \int d(x, y) \chi_{\mu\nu}^{ab}(x, y) \left[ \mathcal{A}_\mu^a(x) \mathcal{A}_\nu^b(y) \right. \\ &\quad \left. + [\bar{\omega}^{-1}]_{\mu\nu}^{ab}(x, y) \right] \\ &\stackrel{\mathcal{A}=0}{=} -\frac{1}{2} V_d d_\xi (N^2 - 1) \int \frac{d^d p}{(2\pi)^d} \chi(k) \bar{\omega}(k)^{-1}.\end{aligned}\quad (39)$$

Taking the derivative with respect to  $\bar{\omega}(k)$  and neglecting again the implicit dependence of  $\chi$  on  $\bar{\omega}$ , we obtain

$$-\frac{\delta}{\delta\bar{\omega}(k)} \langle \ln \mathcal{J} \rangle_{\bar{\omega}} \approx -\frac{1}{2} \frac{V_d}{(2\pi)^d} d_\xi (N^2 - 1) \cdot \frac{\chi(k)}{\bar{\omega}(k)^2}.\quad (40)$$

We can now use this result in Eq. (38) and finally obtain the gap equation in the form

$$\begin{aligned}\bar{\omega}(k) &= k^2 + \chi(k) + \frac{Ng^2}{d_\xi} \int \frac{d^d p}{(2\pi)^d} \\ &\quad \times \left[ b_1 - b_2 \frac{(k \cdot p)^2}{k^2 p^2} \right] \frac{1}{\bar{\omega}(p)}.\end{aligned}\quad (41)$$

When the solution of Eq. (41) is used in the traditional way to compute Schwinger functions and eventually the effective action, the result *must* be

$$\Gamma(\mathcal{A}) = \frac{1}{2} \int d(x, y) \mathcal{A}_\mu^a(x) \mathcal{A}_\nu^b(y) \cdot \bar{\omega}_{\mu\nu}^{ab}(x, y), \quad (42)$$

because the gluon propagator at  $\mathcal{A} = 0$  is  $\bar{\omega}^{-1}$  according to Eq. (28), and the functional approach has no higher vertices.

Due to the Lorentz invariance of our approach at  $\mathcal{A} = 0$ , the kernel  $\bar{\omega}(p)$  is a function of  $|p|$  only and does not single out a direction, so that

$$\int \frac{d^d p}{(2\pi)^d} \frac{p_\mu p_\nu}{p^2} \frac{1}{\bar{\omega}(p)} = \frac{\delta_{\mu\nu}}{d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\bar{\omega}(p)}.$$

The gap equation (41) can now be written in a very compact form,

$$\bar{\omega}(k) = k^2 + M^2 + \chi(k), \quad (43)$$

where the gluon mass is dynamically generated through the nonlinear integral equation,

$$M^2 = C \cdot Ng^2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + M^2 + \chi(p)}, \quad (44)$$

with  $C \equiv (b_1 - b_2/d)/d_\xi$ . The gap equation (43) is very transparent: the dynamical mass  $M^2$  is generated from the 4-gluon vertex, while the curvature  $\chi(k)$  describes the coupling to the Faddeev-Popov ghost fields. As we will see shortly  $\chi(k)$  is just the ghost loop, see Eq. (51) below. In fact the gap equation can be interpreted as the dispersion relation of a relativistic particle with mass  $M$  and a self-energy given by the curvature  $\chi(k)$ .

Let us finally emphasize that the functional approach and the gap equation (43) can be interpreted in a slightly different way: If we go back to the general definition Eq. (5) of the effective action, but this time constrain the gluon propagator  $\langle A_\mu^a(-p) A_\nu^b(p) \rangle = \delta^{ab} t_{\mu\nu}(p) \bar{\omega}(p)^{-1}$  instead of the gluon field itself, we can set  $\mathcal{A} = 0$  in our trial measure Eq. (27) and interpret the variational parameter  $\bar{\omega}$  as the classical value for the inverse gluon propagator. The free action  $F(\bar{\omega})$  from Eq. (36) therefore coincides with the effective action  $\Gamma(\bar{\omega})$  for the (inverse) gluon propagator. This entails that the optimal value for  $\bar{\omega}$  is given by  $\delta\Gamma/\delta\bar{\omega} = \delta F(\bar{\omega})/\delta\bar{\omega} = 0$ , which is exactly Eq. (41) and hence the gap Eq. (43). Thus, the gap equation yields the best match with the exact gluon propagator (in the sense of the effective action) which can be achieved within our variational *Ansatz*. No such argument exists for the vertex corrections in the linear response approach, i.e. while this formulation is able to produce radiative corrections for higher-order Green functions (even with a Gaussian

measure), a realistic description of higher vertices presumably requires to go beyond the Gaussian *Ansatz* [38]. We will therefore restrict our investigations to the propagators of the theory.

### C. Ghost DSE and the curvature

The gap equation (43) contains the curvature  $\chi(k)$  (21), which is nothing but the ghost loop and will be calculated below. The ghost propagator is the expectation value (in our trial measure) of the inverse Faddeev-Popov operator

$$G^{-1} = -\hat{D}_\mu \partial_\mu = -(\partial_\mu + g\hat{A}_\mu) \partial_\mu \equiv G_0^{-1} - h, \quad (45)$$

where  $G_0 = -\square$  is the free ghost propagator and  $h = g\hat{A}\partial$  describes the interaction with the gluon.<sup>6</sup> From the usual resolvent identities, we obtain first  $G = G_0 + G_0 h G$  and thus

$$\langle G \rangle = G_0 + G_0 \langle hG \rangle =: G_0 + G_0 \Sigma \langle G \rangle, \quad (46)$$

where we have introduced the ghost self energy  $-\Sigma = \langle G \rangle^{-1} - G_0^{-1}$ . Following Ref. [23] one can derive for  $\Sigma$  the expression

$$\Sigma \langle G \rangle \equiv \langle hG \rangle = \int D\Gamma_0 \langle G \rangle \Gamma, \quad (47)$$

which involves the gluon propagator  $D$  from Eq. (25) as well as the free ( $\Gamma_0$ ) and full ghost-gluon vertex  $\Gamma$  defined by

$$\langle G\Gamma_0 G \rangle = \langle G \rangle \Gamma \langle G \rangle. \quad (48)$$

From Dyson-Schwinger and flow equation approaches to Yang-Mills theory in Landau gauge, it is well known that the dressing of the full ghost-gluon vertex is a subleading effect, so that the full vertex in the last equation can be replaced by the bare one (rainbow-ladder approximation). This has the advantage that the ghost self-energy  $\Sigma$  (and thus the full ghost propagator  $G$ ) can be expressed through the kernel  $\bar{\omega}$  alone. To do so, we introduce the ghost form factor  $\eta$  via

$$\langle G \rangle = G_0 \cdot \eta. \quad (49)$$

Note that both  $G_0$  and  $\langle G \rangle$  are color diagonal, so that  $\eta = \eta(x, y)$  has no color index. Using Eq. (46), it is easy to see that  $\eta^{-1} = 1 - \Sigma G_0$ . If we now use the rainbow-ladder approximation in Eq. (47) for  $\Sigma$  and re-express  $\langle G \rangle$  through the form factor via Eq. (49), we obtain a closed integral equation for the ghost form factor in momentum space,

<sup>6</sup>Unless stated otherwise, all operators in this subsection are adjoint color and spacetime matrices, for instance  $G = G^{ab}(x, y)$ .



$$\begin{aligned}\eta(k)^{-1} &= 1 - Ng^2 I_\eta(k) \\ &\equiv 1 - Ng^2 \int \frac{d^d q}{(2\pi)^d} [1 - (\hat{k} \cdot \hat{q})^2] \frac{\eta(k-q)}{(k-q)^2 \bar{\omega}(q)},\end{aligned}\quad (50)$$

where the explicit form Eq. (25) of the gluon propagator was inserted.

As for the curvature Eq. (21), taking two functional derivatives of  $(\ln \mathcal{J}) = \text{Tr} \ln G^{-1}$  and using Eq. (48) yields

$$\chi_{\mu\nu}^{ab}(x, y) = -\text{Tr}[\langle G \rangle \Gamma_\mu^a(x) \langle G \rangle [\Gamma_\nu^b(y)]]. \quad (51)$$

Using the rainbow-ladder approximation again and contracting Lorentz indices and colors, we can express the scalar curvature Eq. (23) in momentum space through the ghost form factor,

$$\begin{aligned}\chi(k) &= Ng^2 I_\chi(k) \\ &\equiv Ng^2 \cdot \frac{1}{d-1} \int \frac{d^d q}{(2\pi)^d} [1 - (\hat{k} \cdot \hat{q})^2] \frac{\eta(k-q)\eta(q)}{(k-q)^2}\end{aligned}\quad (52)$$

The DSE (50) for the ghost form factor depends explicitly on the kernel  $\bar{\omega}$  of the trial measure (27), determined by the gap equation (43). Eqs. (52), (50), (44) and (43) form a closed system to determine the ghost form factor  $\eta(k)$ , the curvature  $\chi(k)$ , the mass  $M$ , and the variational kernel  $\bar{\omega}(k)$ .

## V. RENORMALIZATION

### A. Counterterms

To complete our analysis, we have to determine the high momentum behavior of our Green functions and renormalize the corresponding integral equations. We begin with the ghost DSE (50). For large momenta,  $\bar{\omega}(k) \sim k^2$  and  $\eta(k) \sim 1$  (up to logarithmic corrections), so that dimensional analysis implies for the logarithmic divergence in  $I_\eta(k)$  (50)

$$\begin{aligned}\eta^{-1}(k) &= 1 - \Sigma G_0 = 1 - Ng^2 I_\eta(k) \\ &= 1 - Ng^2 [a_0 \ln(\Lambda^2/M^2) + \text{finite}],\end{aligned}$$

where  $\Lambda$  is a suitable UV cutoff and  $a_0$  a finite numerical constant that depends on the cutoff procedure. Since  $G_0(k) = 1/k^2$ , the corresponding counterterm  $\delta\Sigma$  for the ghost self-energy is proportional to  $k^2$ , i.e. it is a ghost field renormalization. (No ghost mass term is induced by the theory.) The same conclusion could be reached if we introduced explicit ghost fields  $\{c, \bar{c}\}$ , because  $\langle \bar{c}c \rangle = \langle G \rangle = G_0 \eta$  and the field renormalization  $c \rightarrow \sqrt{Z_c} c$  is equivalent to  $\eta \rightarrow Z_c \eta$ . To the given loop order, this field renormalization leads to

$$\frac{1}{\eta(k)} = 1 - Ng^2 I_\eta(k) - \delta Z_c + \text{two loops}.$$

In terms of explicit ghost fields, the counterterm  $\delta Z_c = (\delta\Sigma)G_0$  (or  $\delta\Sigma = k^2 \delta Z_c$ ) would hence correspond to the local expression

$$\delta Z_c \int d^4 x \partial_\mu \bar{c} \partial^\mu c \quad (53)$$

in the exponent of the trial measure (27). We can now adjust the finite pieces in  $\delta Z_c$  such that

$$1 - \delta Z_c = \eta(\mu)^{-1} + Ng^2 I_\eta(\mu) \quad (54)$$

where  $\eta(\mu)$  is an arbitrary finite constant, because Eq. (54) is independent of  $k$  and the (logarithmic)  $\Lambda$ -divergences on both sides agree. This prescription leads directly to the renormalized ghost equation

$$\eta(k)^{-1} = \eta(\mu)^{-1} - Ng^2 [I_\eta(k) - I_\eta(\mu)], \quad (55)$$

which could also be obtained by simply subtracting the bare equation (50) at  $k = \mu$ . Notice that Eq. (54) may be a rather unusual field normalization, but any other prescription for  $Z_c$  can only differ by a finite constant. Notice also that the renormalized Eq. (55) is independent of  $g$ , as can be seen e.g. by rescaling  $\eta \rightarrow \tilde{\eta} \equiv g\eta$ .

Next, we study the mass and curvature equations (44) and (52), respectively, which are quadratically divergent by power counting,

$$\begin{aligned}M^2 &= Ng^2 [a_1 \Lambda^2 + b_1 M^2 \ln(\Lambda^2/M^2) + \text{finite}] \\ \chi(k) &= Ng^2 [a_2 \Lambda^2 + b_2 M^2 \ln(\Lambda^2/M^2) \\ &\quad + ck^2 \ln(\Lambda^2/M^2) + \text{finite}],\end{aligned}\quad (56)$$

with numerical factors  $a_i, b_i$  and  $c$  that depend on the cutoff procedure.<sup>7</sup> The subtraction is a bit more complicated in this case because of the sub-leading logarithmic divergence. We begin by subtracting the  $k$ -independent contributions with counterterms for  $M^2$  and  $\chi(k)$ , respectively,

$$\begin{aligned}\delta M_1^2 &\equiv -Ng^2 [a_1 \Lambda^2 + b_1 M^2 \ln(\Lambda^2/M^2) + \text{finite}] \\ \delta \chi_1 &\equiv -Ng^2 I_\chi(\mu) + \chi_{\text{fin}},\end{aligned}\quad (57)$$

<sup>7</sup>In particular, the coefficients  $a_i$  could vanish in a gauge-invariant regularization scheme such as proper time or dim. reg. This does not mean, however, that the divergence would be absent, and the local counterterm for the gluon will always be of the form (61) below. Since the counterterms modify the trial measure (27) rather than the Yang-Mills action, gauge invariance of the regularization is not an issue, and we will later use an  $O(4)$ -invariant cutoff for our numerical procedure. Equation (56) is also consistent with Ref. [43], where the divergencies of the Faddeev-Popov determinant were identified within a gradient expansion.

where  $\chi_{\text{fin}}$  is an arbitrary finite constant. The mass equation (44) is now finite with a dynamically induced mass  $M^2$  whose absolute value is undetermined because of the finite pieces in the corresponding counterterm  $\delta M_1^2$ . As a consequence, we do not need to solve Eq. (44) but rather take  $M^2$  as a finite parameter that can be chosen at will. The subtracted curvature equation takes the form

$$\chi(k) = Ng^2[I_\chi(k) - I_\chi(\mu)] + \chi_{\text{fin}}. \quad (58)$$

This is not yet finite because the difference of the two integrals contains the subleading logarithmic divergence [cf. Eq. (56)]

$$[I_\chi(k) - I_\chi(\mu)] = c(k^2 - \mu^2) \ln(\Lambda^2/\mu^2) + \text{finite}. \quad (59)$$

We must therefore add a second,  $k$ -dependent counterterm  $\delta\chi_2(k)$  which equals the negative of the divergence on the rhs of Eq. (59). In order to associate these subtractions with local terms in the exponent of the trial measure Eq. (27), we isolate the  $k$ -dependent pieces in  $\delta\chi_2(k)$  and write the total curvature counterterm as  $\delta\chi(k) \equiv \delta\chi_1 + \delta\chi_2(k) = \delta\chi_0 + k^2\delta Z_A$  with

$$\begin{aligned} \delta\chi_0 &\equiv -Ng^2I_\chi(\mu) + c\mu^2 \ln(\Lambda^2/\mu^2) + \chi_{\text{fin}} \\ \delta Z_A &\equiv -Ng^2c \ln(\Lambda^2/\mu^2) + z(\mu), \end{aligned} \quad (60)$$

where  $z(\mu)$  is again an arbitrary finite piece that can be added to the counterterm. If we now replace  $\chi(k) \rightarrow \chi(k) + \delta\chi(k)$  as well as  $M^2 \rightarrow M^2 + \delta M_1^2$ , Eqs. (44) and (52) will be finite. From the gap equation (43), it is clear that the  $k$ -independent subtractions can be combined to a gluon mass counterterm  $\delta M^2 = \delta\chi_0 + \delta M_1^2$ , while the  $k$ -dependent counterterm  $k^2\delta Z_A$  rescales the kinetic energy, i.e. it represents a gluon field renormalization. These subtractions correspond to local gluon counterterms

$$\frac{1}{2}\delta M^2 \int d^4x (A_\mu^a)^2 + \frac{1}{4}\delta Z_A \int d^4x (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2. \quad (61)$$

It should be emphasized once again that these counterterms modify the trial measure used to compute expectation values, i.e. they add to the exponent of Eq. (27) (with  $\mathcal{A} = 0$ ), *not* to the Yang-Mills action. As a consequence, soft BRST breaking mass terms as the ones observed on the lattice are allowed (cf. Sec. VI B), as well as counterterms for them.

The renormalized curvature equation is now

$$\begin{aligned} \chi(k) &= Ng^2[I_\chi(k) - I_\chi(\mu) - c(k^2 - \mu^2) \ln(\Lambda^2/\mu^2)] \\ &\quad + z(\mu)(k^2 - \mu^2) + \chi(\mu), \end{aligned} \quad (62)$$

where  $\chi(\mu) = \chi_{\text{fin}} + z(\mu)\mu^2$  is finite.

Equation (62) is not very suitable for numerical investigations. To put it in a manageable form, we need to determine the constant  $c$  and devise a way to perform the necessary subtractions under the integral  $I_\chi(k)$ . Unfortunately,  $I_\chi(k)$  and hence the factor  $c$  depend on the ghost profile  $\eta(k)$  and its derivatives, which are only known numerically. Thus, we proceed as follows: At any stage in the iterative solution of the integral equation system, we perform the angular integration in  $I_\chi(k)$  with the current solution of the ghost form factor  $\eta(q)$  to find an expression

$$I_\chi(k) = \int_0^\Lambda dq f(k, q), \quad (63)$$

with a complicated function  $f$  only known numerically. Our construction of the counterterms above translates into the asymptotics

$$f(k, q) - f(\mu, q) \xrightarrow{q \rightarrow \infty} \frac{(k^2 - \mu^2)}{q} \cdot c + \mathcal{O}(q^{-2}). \quad (64)$$

This can be verified numerically: in the left panel of Fig. 1, we have plotted  $q[f(k, q) - f(\mu, q)]$  as a function of

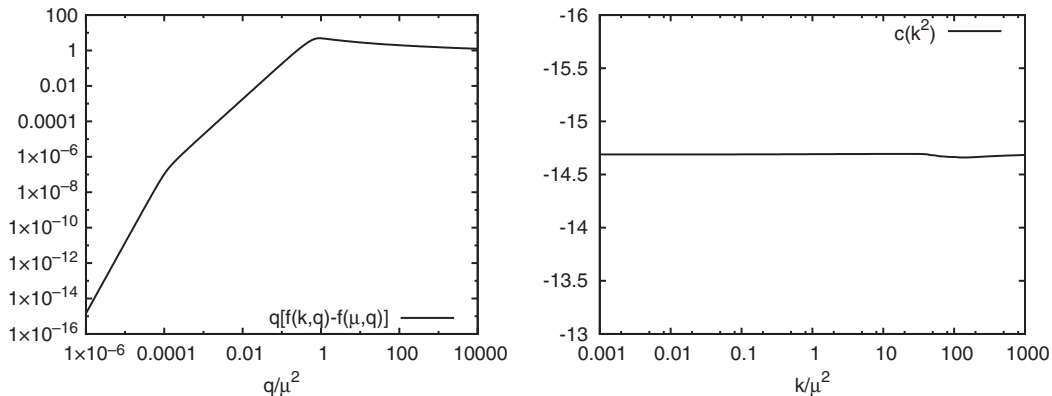


FIG. 1. *Left panel:* The scaled integrand  $q[f(k, q) - f(\mu, q)]$  of the subtracted loop integral  $I_\chi(k)$  as a function of  $q$ , cf. Eq. (64). *Right panel:* The asymptotic value of the subtracted integrand in  $I_\chi(k)$ , as a function of  $k$ , cf. Eq. (65).

$q$  and observe that it approaches a constant value<sup>8</sup> for large  $q$ . The value of that constant depends on  $k$  in the expected way: in the right panel of Fig. 1, we have plotted the value

$$\frac{q}{k^2 - \mu^2} [f(k, q) - f(\mu, q)]|_{q=\Lambda} \quad (65)$$

as a function of  $k$  and observe that this quantity is independent of  $k$ , as expected from Eq. (64). The constant value is exactly the factor  $c$  used in Eq. (62) above. Thus, at any stage of the iterative solution with the current form of  $\eta(k)$ , we first perform the angle integrations in  $I_\chi(k)$  to obtain the integrand  $f(k, q)$  in Eq. (63) and then evaluate the curvature  $\chi(k)$  from Eq. (62) written in the form

$$\chi(k) = Ng^2 \int_0^\infty dq \left[ f(k, q) - f(\mu, q) - \frac{\Lambda}{q} [f(k, \Lambda) - f(\mu, \Lambda)] \right] + z(\mu)(k^2 - \mu^2) + \chi(\mu). \quad (66)$$

This procedure determines the counterterm coefficient  $c$  from Eq. (62) iteratively and renders the system of integral Eqs. (43), (55) and (66) finite. It also leads to a stable solution which is numerically independent of the cutoff  $\Lambda$  if the latter is chosen large enough. The system involves four renormalization constants  $\eta(\mu)$ ,  $\chi(\mu)$ ,  $z(\mu)$  and  $M^2$  which could even be defined at different scales  $\mu$ . One of the constants is, however, redundant because finite changes in  $\chi(\mu)$  can be absorbed in  $M^2$  as discussed above. In fact, only the gluon mass parameter<sup>9</sup>  $M_A^2(\mu) \equiv M^2 + \chi(\mu) - z(\mu)\mu^2$  will appear in the renormalized equations, and we can therefore drop either  $\chi(\mu)$  or  $M^2$  in favor of the other without loss of generality. The remaining renormalization constants  $\eta(\mu)$  and  $z(\mu)$  fix the prefactor of the kinetic energy for ghost and gluon, respectively, i.e. they fix the scale of the ghost and gluon field. (No vertex renormalization is induced by the theory.)

The three independent counterterms are also summarized in Table I. Clearly, the gauge and ghost fields are multiplicatively renormalized, and the same is true for the gluon mass term, if we *formally* introduce such a mass in the bare kernel  $\omega$  (which we are entitled to do since we never have to specify the bare kernel explicitly). In this sense, our approach is multiplicatively renormalizable.

<sup>8</sup>There are still small deviations from a constant due to numerical issues that prevent us from going to very large cutoffs while preserving sufficient accuracy in the iterative solution of the integral equation system. However, extracting  $c$  as in Eq. (65) yields finite iterative solutions that become cutoff-independent long before the numerics become delicate.

<sup>9</sup>To avoid problems with possible infrared singularities, we renormalize the gluon propagator at  $\mu > 0$  so that  $M_A^2(\mu) = M^2 + \chi(\mu) - z(\mu)\mu^2$  is merely a renormalization constant without a direct interpretation as a mass. If the curvature (and thus the gluon propagator) happens to be finite at  $k = 0$ , the intercept  $\bar{\omega}(0) = M^2 + \chi(0) = M_A^2(0)$  can be interpreted as a (constituent) gluon mass.

TABLE I. Renormalization of the final system of integral equations.

Renormalization	Counterterm	Renormalization constant
Gluon field	$\delta Z_A$	$z(\mu)$
Gluon mass	$\delta M^2 = \delta\chi_0 + \delta M_1^2$	$M_A^2(\mu)$
Ghost field	$\delta Z_c$	$\eta(\mu)$

## B. Numerical treatment

Let us briefly comment on the numerical treatment of the integral equation system, and in particular on the role of the various renormalization constants. First, we note that the coupling constant  $g$  can be eliminated from all equations by rescaling  $\tilde{\eta} = g\eta$ . If we insert the curvature equation (66) directly into the gap equation (43) and further introduce the gluon mass parameter  $M_A^2(\mu) \equiv M^2 + \chi(\mu) - z(\mu)\mu^2$  as before, we obtain the renormalized system

$$\begin{aligned} \bar{\omega}(k) &= [1 + z(\mu)]k^2 + M_A^2(\mu) \\ &\quad + N[I_\chi(k) - I_\chi(\mu) - c(k^2 - \mu^2) \ln(\Lambda^2/\mu^2)] \\ \bar{\eta}(k)^{-1} &= \bar{\eta}(0)^{-1} - N[I_{\bar{\eta}}(k) - I_{\bar{\eta}}(0)]. \end{aligned} \quad (67)$$

Recall that the coefficient  $c$  in the first equation is determined during the iterative solution of the system such that it becomes cutoff-independent.

Ideally, we would choose the renormalization point  $\mu$  for the gluon and ghost field at a large Euclidean scale  $\mu \gg 1$  far away from possible singularities, where asymptotic freedom provides natural renormalization conditions. Unfortunately, such a procedure does not lead to a stable numerical solution in the deep infrared, because the system has a family of solutions which are qualitatively different at low momenta, but cannot be discriminated in the ultraviolet. In order to stabilize the integration and bring the distinct solutions to the fore we must, at least for the ghost equation, choose a renormalization point in the deep infrared, or even at  $\mu = 0$ , which we did in all numerical investigations. The gluon renormalization scale  $\mu$ , by contrast, can be chosen finite and arbitrary, and we use it to rationalize all dimensionfull quantities within our numerical treatment. Thus, we find that the shape of the rescaled (dimensionless) integral equation system only depends on the three dimensionless parameters  $\bar{\eta}(0)^{-1}$ ,  $z(\mu)$ , and  $M_A(\mu)/\mu$ .

We have already mentioned above that the first of these three parameters,  $\bar{\eta}(0)^{-1}$ , discriminates between the scaling and decoupling solutions. By imposing a condition on  $\bar{\eta}(0)^{-1}$ , we are thus *enforcing* a specific type of solution *by hand*, as is done in all other functional approaches. However, the variational method allows, in principle, to determine the correct solution from first principles, because the true form of the propagators is the one which leads to minimal effective action. This investigation is fairly

complex and we have left it for future studies; below, we will instead compare to lattice data to determine which of the possible solutions is realized.

The second parameter  $M_A(\mu)/\mu$  has no effect on the infrared behavior of the scaling solution, while it determines the  $k=0$  limit of the gluon propagator for the decoupling solution, i.e. the constituent gluon mass. Finally, the gluon field renormalization factor  $z(\mu)$  changes the overall size of the kernel  $\bar{\omega}(k)$ . More precisely, any change  $z(\mu) \rightarrow z'(\mu)$  for the scaling type of solution simply leads to an overall rescaling

$$\bar{\omega}'(k) = \frac{1+z'}{1+z} \bar{\omega}(k), \quad \tilde{\eta}'(k) = \sqrt{\frac{1+z'}{1+z}} \tilde{\eta}(k). \quad (68)$$

For the decoupling solution, this simple rescaling also holds if we simultaneously change the finite renormalization constant  $\tilde{\eta}^{-1}(0)$  as in Eq. (68).

## VI. RESULTS

### A. Infrared analysis

The infrared analysis of the renormalized system of integral equations (67) can be carried out much as in the Coulomb gauge case [44,45], see also Ref. [2] and [46]. If we assume a power-like behavior in the infrared,

$$\bar{\omega}(k) \sim (k^2)^\alpha, \quad \eta(k) \sim (k^2)^{-\beta} \quad (69)$$

the nonrenormalization of the ghost-gluon vertex [41] implies the *sum rule*

$$-\alpha + 2\beta = \frac{d}{2} - 1. \quad (70)$$

Furthermore, if  $\bar{\omega}$  is infrared divergent it follows from the gap equation (43) that the curvature  $\chi(k)$  (52) also has the same infrared exponent  $\alpha$ . Depending on the choice of our finite renormalization constants we obtain two different types of solutions:

- (i) Implementing the horizon condition [47]  $\eta^{-1}(0) = 0$  we find the so-called critical or scaling solution with the infrared exponents

$$\beta = \frac{1}{98} (93 \mp \sqrt{1201}) \approx \{0.5954, 1.3025\}, \quad (71)$$

which both entail  $\alpha > 0$ , i.e. an infrared vanishing gluon propagator. These are the same infrared exponents found from the one loop DSEs in Landau gauge [2]. Only the first of these scaling solutions could be found in our numerical calculation, cf. Sec. VI B. It should also be mentioned that the value of the gluon mass parameter  $M_A^2(\mu)$  is irrelevant, as long as the curvature  $\chi(k)$  is infrared divergent, i.e. as

long as  $\beta > 1/2$ , which is the case for both exponents given in Eq. (71).

- (ii) Assuming an infrared finite ghost form factor  $\eta^{-1}(0) > 0$ , i.e.  $\beta = 0$ , the sum rule (70) would yield  $\alpha = -1$ . However, in this case a nonzero mass parameter  $M_A^2(\mu)$  in the gap equation dominates the infrared behavior, which invalidates the sum rule. For fixed mass parameter  $M_A(\mu)/\mu$  and fixed gluon wave function renormalization  $z(\mu)$ , we obtain a one-parameter family of solutions labeled by the ghost renormalization constant  $\eta^{-1}(0) \neq 0$ . These are the so-called subcritical or decoupling solutions already found from the DSEs in Landau gauge [2]. The intercept  $\bar{\omega}(0) = M^2 + \chi(0)$  can be interpreted as a constituent gluon mass.

In the special case  $M_A^2(\mu) = 0$  there is also a solution with an infrared finite ghost form factor  $\eta(k)$  and an infrared vanishing gluon kernel  $\bar{\omega}(k)$ , which formally obeys the sum rule with  $\alpha = -1$  and  $\beta = 0$ . This solution is, however, definitely ruled out by all existing lattice data, in which the ghost form factor always diverges at  $k \rightarrow 0$ . This leaves us with just the two type of solutions listed above. In the next subsection we compare these solution with high-precision lattice results.

### B. Numerical results

As discussed above the discriminating criterion for the two type of solutions is whether we choose the ghost form factor  $\eta(0)$  finite (decoupling solutions) or infinite (scaling solution). Let us briefly describe our numerical findings for both type of solutions:

*Scaling solution:* With  $\eta(0)^{-1} = 0$  we have enforced an IR diverging ghost form factor. In our numerical solution we could only find the less divergent of the two possible exponents determined in the IR analysis Eq. (71). The best fit to our numerical data suggests  $\beta = 0.595(3)$  for the ghost form factor and  $\alpha = 0.191$  for the gluon kernel  $\bar{\omega}$ , which are both in excellent agreement with the analytic results Eq. (71). Furthermore the sum rule Eq. (70) is satisfied numerically to better than  $10^{-3}$  accuracy. To compare with lattice data, we have to adjust the finite gluon wave function renormalization  $z(\mu)$  at an arbitrary scale  $\mu > 0$  to match the overall scale of the propagator. The remaining renormalization parameter  $M_A^2(\mu)$  is immaterial since it has no effect on the final solution. (This is because the IR diverging curvature dominates all constant terms in the gap equation at  $k \rightarrow 0$ .) In Fig. 2, the critical solution is compared to high-precision lattice data taken from Ref. [7]. It is obvious that the scaling type of solution describes the UV behavior fairly well, but it severely deviates in the deep infrared.

*Decoupling solutions:* These solutions have no scaling behavior, and instead exhibit the emergence of a soft BRST breaking mass scale in the deep infrared, so that all Green's functions remain finite at  $k \rightarrow 0$ . In this case, the

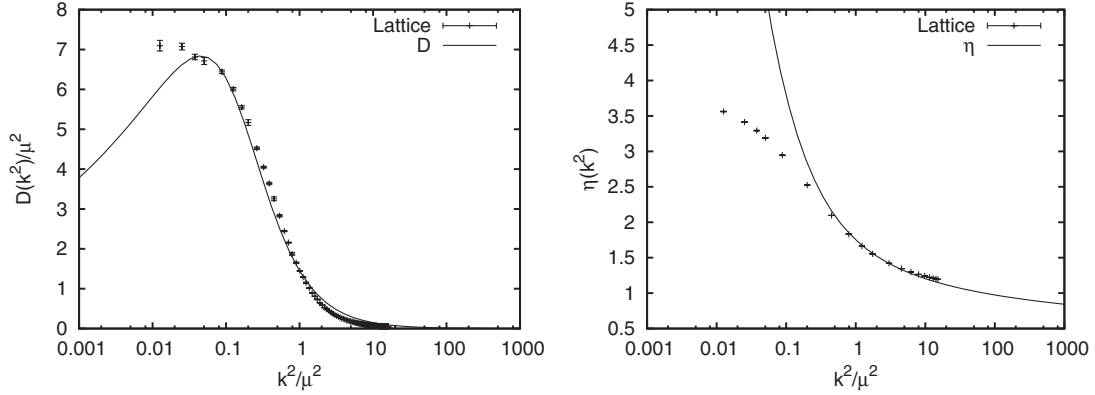


FIG. 2. Critical solution for the gluon propagator (left panel) and the ghost form factor (right panel) for  $SU(2)$  Yang-Mills theory in Landau gauge. The crosses denote the lattice results from Ref. [7].

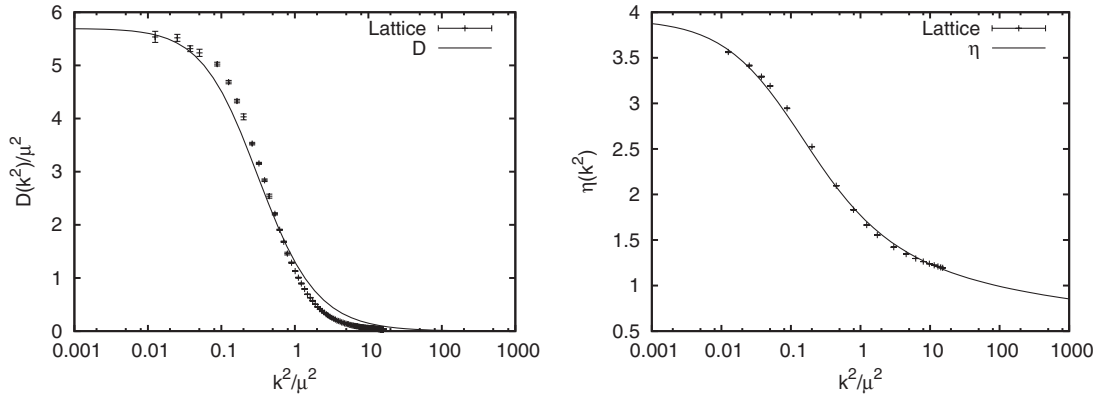


FIG. 3. Subcritical solution for the gluon propagator (left panel) and the ghost form factor (right panel) for  $SU(2)$  Yang-Mills theory in Landau gauge. The crosses denote the lattice results from Ref. [7].

renormalization constant  $M_A^2(\mu)$  obviously matters as it dominates the gluon propagator in the deep infrared. In addition, we can also adjust the intercept  $\eta(0)$  for the ghost form factor, and the overall scale  $z(\mu)$  of the gluon propagator at some nonzero scale  $\mu > 0$ . (The combination of  $M_A^2(\mu)$  and  $z(\mu)$  determines the slope of the gluon propagator in the transition region around  $k = \mu$ .) The decoupling solution is shown in Fig. 3, along with lattice data [7] for comparison. As can be seen from the plot, this type of solution is in good agreement with the lattice data for the entire momentum range. In particular, the agreement with the ghost data is almost perfect.<sup>10</sup>

Unfortunately, the critical and subcritical solution differ only at very low momenta (which is the reason for the numerical instability when imposing renormalization conditions at large scales), and it requires large lattices to definitely rule out one or the other. As explained above, the

<sup>10</sup>We have optimized the renormalization parameters in the ghost sector, which incurs slight deviations for the gluon propagator in the transition region  $k \approx \mu$ , cf. Fig. 3. This could be mitigated by a more balanced approach that tries to optimize the parameters for both propagators on average.

available lattice data now clearly favors the decoupling solution. Analytical approaches, by contrast, always exhibit both kind of solutions, and it becomes a matter of “boundary conditions”  $\eta^{-1}(0)$  to select one or the other. In Dyson-Schwinger or functional renormalization group approaches, there is no compelling reason to prefer one boundary condition over the other. This is different in our variational approach, since it is always the solution with the lowest effective action that must be realized. To determine the correct solution, we would thus have to insert the various (numerical) solutions  $\bar{\omega}(k)$  in the free action  $F$  from Eq. (36). This procedure requires first a full renormalization of the effective (or free) action, which is left for future work.

## VII. SUMMARY AND CONCLUSIONS

In this paper, we have investigated the low-order Green’s functions in  $SU(N)$  Yang-Mills theory, using Landau gauge and a covariant variational principle based on the effective action. The formalism leads to a set of integral equations which, after proper renormalization, could be solved numerically over a wide range of momenta. We obtain the two types of solutions also found in other

functional approaches: (i) a critical or scaling solution in which the ghost form factor diverges in the infra-red by a power law with an exponent  $\beta = 0.5953$  while the gluon propagator vanishes with a weak infrared exponent  $\alpha = 0.191$ , and (ii) a subcritical or decoupling solution where both quantities remain finite at low momenta. Recent high-precision lattice data compares favorably with both solutions in the UV and into the transition region, although detailed studies prefer the subcritical (decoupling) solution in the infrared. Our numerical treatment is on par with the best analytical studies, and our decoupling solution, in particular, agrees very well with the available lattice data over the entire momentum range. In addition, the variational principle used here offers a physical criterion to distinguish between the two type of solutions which is not based on arbitrary boundary conditions, namely the solution with the lowest free action *must* be realized. This question will be studied in a future investigation.

The method presented here works with Euclidean path integrals only, so that it naturally generalizes to all extensions that can be formulated within a path integral. In particular, we can include fermions and study finite temperatures and chemical potentials without conceptual problems. These issues will also be subject to future work.

### ACKNOWLEDGMENTS

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### APPENDIX A: THE QUANTUM EFFECTIVE ACTION

We want to show that the effective action defined by the variational principle (6) coincides with the traditional generating functional for 1PI proper functions. We fix  $\varphi(x)$  and start directly from the definition (6). First, we have the following upper bound for arbitrary currents  $j(x)$ ,

$$\begin{aligned} \Gamma(\varphi) &\stackrel{(6)}{\equiv} \inf_{\mu} [\langle S \rangle_{\mu} - \hbar \mathcal{W}(\mu) | \langle \phi \rangle_{\mu} = \varphi] \\ &= \inf_{\mu} [\langle S \rangle_{\mu} - \hbar \mathcal{W}(\mu) + (j, \langle \varphi - \phi \rangle_{\mu}) | \langle \phi \rangle_{\mu} = \varphi] \\ &\geq \inf_{\mu} [\langle S \rangle_{\mu} - \hbar \mathcal{W}(\mu) + (j, \langle \varphi - \phi \rangle_{\mu})] \\ &= (j, \varphi) + \inf_{\mu} [\langle S - (j, \phi) \rangle_{\mu} - \hbar \mathcal{W}(\mu)] \\ &\equiv (j, \varphi) - W(j). \end{aligned} \quad (\text{A1})$$

(The inequality follows because the constrained minimum is always larger than the unconstrained one.) Next we want to show that  $\Gamma(\varphi)$  is in fact the smallest upper bound, i.e. the supremum,

$$\Gamma(\varphi) = \sup_j [(j, \varphi) - W(j)]. \quad (\text{A2})$$

To see this, it is sufficient to find a current  $j = j_{\varphi}$  for which the bound in Eq. (A1) is saturated,

$$\Gamma(\varphi) = (j_{\varphi}, \varphi) - W(j_{\varphi}), \quad (\text{A3})$$

because then the upper bound in Eq. (A1) is a maximum (and hence a supremum). From the derivation of Eq. (A1), it is clear that  $j_{\varphi}$  should be chosen such as to obey the constraint  $\langle \phi \rangle_{\mu_j} = \varphi$  for the Gibbs measure  $\mu_j$  that solves the minimization problem on the rhs of Eq. (A1). This is because the unconstrained minimum happens to obey the constraint and is therefore also the constrained minimum; i.e., the inequality in the third line of Eq. (A1) becomes an equality. The relevant equation  $\langle \phi \rangle_{\mu_j} = \varphi$  for  $j = j_{\varphi}$  is, however, merely the extremality condition

$$\begin{aligned} 0 &= \frac{\delta}{\delta j(x)} [(j, \varphi) - W(j)] = \varphi(x) - \frac{\delta W}{\delta j(x)} \\ &= \varphi(x) - \hbar \frac{\delta Z / \delta j(x)}{Z(j)} = \varphi(x) - \langle \phi(x) \rangle_{\mu_j}, \end{aligned}$$

which we assume to always have a solution.<sup>11</sup> Thus, the effective action  $\Gamma(\varphi)$  defined by the variational principle (6) is the Legendre transformation of the functional  $W(j)$  defined in the last line of Eq. (A1).

It remains to show that  $W(j)$  agrees with the generating functional of connected Green's functions. To see this, recall that the Gibbs measure Eq. (1) is the unique solution of the unconstrained variational principle Eq. (2), irrespective of the actual form of the action. Since we have an additional linear term in the action for the  $\mu$  minimization in the definition of  $W(j)$  Eq. (A1), the solution of this  $\mu$  minimization must be a modified Gibbs measure with the additional linear term in the action,

$$d\mu_j(\phi) = Z(j)^{-1} d\phi \exp\{-\hbar^{-1}(S(\phi) - (j, \phi))\} \quad (\text{A4})$$

$$Z(j) = \int d\phi \exp\{-\hbar^{-1}(S(\phi) - (j, \phi))\}. \quad (\text{A5})$$

The corresponding value of the minimum in Eq. (A1) is then given by the modified partition function with a linear term in the action,

$$\begin{aligned} -W(j) &\equiv \inf_{\mu} [\langle S - (j, \phi) \rangle_{\mu} - \hbar \mathcal{W}(\mu)] \\ &= \langle S - (j, \phi) \rangle_{\mu_j} - \hbar \mathcal{W}(\mu_j) = -\hbar \ln Z(j). \end{aligned} \quad (\text{A6})$$

<sup>11</sup>This assumption is implicit in the traditional definition of the generating functional, and we do not touch the more subtle question of what happens if no such  $j_{\varphi}$  exists and the supremum (A2) is *not* a maximum.

Clearly, this identifies  $W(j)$  as the usual generating functional of connected Green's functions and hence the Legendre transform Eq. (A2) as the generator of 1PI proper functions. The inverse Legendre transformation

$$W(j) = \sup_{\varphi} [(j, \varphi) - \Gamma(\varphi)] \quad (\text{A7})$$

implies  $\delta\Gamma/\delta\varphi(x) = j_{\varphi}(x)$ . Finally, Eqs. (A2) and (A7) entail that both functionals  $W(j)$  and  $\Gamma(\varphi)$  must be convex.

## APPENDIX B: THE VARIATIONAL PRINCIPLE FOR $\phi^4$ THEORY

In this appendix, we will sketch the application of the variational principle to  $\phi^4$  theory. This is a standard application of the variational method and dates back at least to Ref. [48], but we will repeat it here within our formulation to clarify the distinction between the functional and linear response approach. We start with the Lagrangian (in Euclidean space)

$$\mathcal{L} = \frac{1}{2}(\partial_{\mu}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \quad (\text{B1})$$

Our trial measures will be Gaussians characterized by a variation kernel  $\omega(x, y)$ ,

$$\rho(\varphi) \sim \exp \left[ -\frac{1}{2} \int d^d x d^d y (\varphi(x) - \phi(x)) \omega(x, y) \times (\varphi(y) - \phi(y)) \right]. \quad (\text{B2})$$

We have centered the Gaussian at the classical field  $\varphi(x)$  in order to obey the constraint  $\langle\phi\rangle = \varphi$  for the linear response approach; in the functional approach based on the unconstrained free action, we can set  $\varphi = 0$ , cf. Sec. II. Next, we compute the average action given by

$$\langle S \rangle_{\mu} = \int d^d x \left[ \frac{1}{2} (-\square_x + m^2) \langle \phi(x)^2 \rangle_{\mu} + \frac{\lambda}{4!} \langle \phi(x)^4 \rangle_{\mu} \right].$$

The relevant correlators can easily be worked out using Wick's theorem

$$\begin{aligned} \langle \phi(x) \rangle_{\mu} &= \varphi(x) \\ \langle \phi(x)\phi(y) \rangle_{\mu} &= \varphi(x)\varphi(y) + \hbar\omega^{-1}(x, y) \\ \langle \phi(x)^4 \rangle_{\mu} &= \varphi(x)^4 + 6\hbar\omega^{-1}(x, x)\varphi(x)^2 + 3\hbar^2\omega^{-2}(x, x). \end{aligned} \quad (\text{B3})$$

The average action thus takes the form<sup>12</sup>

<sup>12</sup>The action of the Laplace operator is understood as  $\square_x \omega^{-1}(x, x) \equiv \partial_{\mu}^x \partial_{\mu}^y \omega^{-1}(x, y)|_{y=x}$ .

$$\begin{aligned} \langle S \rangle_{\mu} &= S(\varphi) + \frac{\hbar}{2} \int d^d x \left[ -\square_x + m^2 + \frac{\lambda}{2} \varphi(x)^2 \right] \omega^{-1}(x, x) \\ &\quad + \frac{\hbar^2 \lambda}{8} \int d^d x \omega^{-2}(x, x). \end{aligned} \quad (\text{B4})$$

The calculation of the entropy is slightly more involved. From the explicit form of the Gaussian measure and the definition, Eq. (3), we have formally

$$\begin{aligned} \mathcal{W}(\mu) &= \langle -\ln \rho \rangle_{\mu} \\ &= \int d\mu(\phi) \left[ \ln Z + \frac{1}{2\hbar} \int d^d(x, y) (\phi(x) - \varphi(x)) \times \omega(x, y) (\phi(y) - \varphi(y)) \right] \\ &= \frac{1}{2} \ln \det \left( \frac{\omega}{2\pi\hbar} \right) + \frac{1}{2\hbar} \int d^d(x, y) [\omega(x, y) \times \langle \phi(x)\phi(y) \rangle_{\mu} - 2\varphi(x)\omega(x, y)\langle \phi(y) \rangle_{\mu} \\ &\quad + \varphi(x)\omega(x, y)\varphi(y)]. \end{aligned}$$

(An implicit regularization is understood.) Using Wick's theorem for the correlators, all dependence on the classical field  $\varphi$  drops out,<sup>13</sup>

$$\mathcal{W}(\mu) = \frac{1}{2} \text{Tr} \left\{ \mathbb{1} - \ln \left( \frac{\omega}{2\pi\hbar} \right) \right\}. \quad (\text{B5})$$

The free action (2) for the Gaussian measure in  $\phi^4$  theory is now  $F(\omega, \varphi) = \langle S \rangle_{\mu} - \hbar\mathcal{W}(\mu)$  with Eqs. (B4) and (B5) providing the details. At this point, the functional and linear response approach start to deviate.

### 1. The functional approach

As explained in Sec. II, we simply take the unconstrained free action  $F(\omega, \varphi = 0)$  and vary with respect to the kernel  $\omega$ , or rather, its inverse. After a short calculation, one finds

$$\begin{aligned} \frac{\delta F(\omega, \varphi = 0)}{\delta \omega^{-1}(x, y)} &= \frac{\hbar}{2} \left[ -\square_x + m^2 + \frac{\hbar\lambda}{2} \omega^{-1}(x, y) \right] \delta(x, y) \\ &\quad - \frac{\hbar}{2} \omega(y, x) \stackrel{!}{=} 0. \end{aligned} \quad (\text{B6})$$

Due to the absence of a classical field, the solution of this gap equation in the functional approach has global translation and rotation invariance, i.e. the kernel has the Fourier representation

$$\omega(x, y) = \int \frac{d^d k}{(2\pi)^d} e^{-ik \cdot (x-y)} \omega(k), \quad (\text{B7})$$

<sup>13</sup>This is expected since the entropy encodes the available phase space for quantum fluctuations, which is not affected by the field shift  $\varphi$ .

where  $\omega(k)$  depends only on the modulus  $k = |k_\mu|$ . The gap equation in momentum space takes the simple form

$$\omega(k) = k^2 + m^2 + \frac{\hbar\lambda}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\omega(p)}. \quad (\text{B8})$$

The solution is the covariant dispersion relation for a free boson of a dynamically generated mass,

$$\omega(k) = k^2 + M^2, \quad (\text{B9})$$

where  $M$  is implicitly determined by the nonlinear equation

$$M^2 = m^2 + \frac{\hbar\lambda}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + M^2}. \quad (\text{B10})$$

Finally, we can now compute all connected Green functions of the solving Gaussian measure, construct a generating functional  $W(j)$  and Legendre transform to obtain the effective action  $\Gamma(\varphi)$ . Obviously, all proper vertices vanish and we get

$$\Gamma(\varphi) = \int d^d x \left[ \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{M^2}{2} \varphi^2 \right]. \quad (\text{B11})$$

It should finally be noted that the  $p_0$  integration in the covariant mass equation (B10) can be performed explicitly. For  $d = 4$ , we obtain

$$M^2 = m^2 + \hbar \frac{\lambda}{4} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{\mathbf{p}^2 + M^2}}, \quad (\text{B12})$$

which coincides exactly with the result of the traditional variational approach in the Hamiltonian picture based on a Gaussian wave functional.

## 2. The linear response approach

In this case, we have to retain the classical field in order to comply with the constraint  $\langle \phi \rangle = \varphi$ . From the variation of the constrained free action  $F(\omega, \varphi)$ , we obtain the modified gap equation

$$\begin{aligned} \frac{2}{\hbar} \frac{\delta F(\omega, \varphi)}{\delta \omega^{-1}(x, y)} &= \left[ -\square_x + m^2 + \frac{\lambda}{2} \varphi(x)^2 + \frac{\hbar\lambda}{2} \omega^{-1}(x, y) \right] \\ &\times \delta(x, y) - \omega(y, x) = 0. \end{aligned} \quad (\text{B13})$$

As expected in the linear response approach, the optimal kernel  $\omega_\varphi$  determined from this equation depends implicitly on the classical field  $\varphi$ , while the effective action is simply  $\Gamma(\varphi) = F(\omega_\varphi, \varphi)$ . Since the external fields are arbitrary, the implicit  $\varphi$ -dependence of the optimal kernel  $\omega_\varphi$  spoils the translational or rotational invariance. As a consequence, we *cannot* go to momentum space as we did in Eq. (B7) within the functional approach, which complicates the solution of Eq. (B13) considerably.

One possible approach is to attempt a solution of Eq. (B13) by expanding in powers of the classical field. Upon comparing the two gap equations (B6) and (B13), we conclude that

$$\omega_\varphi = \omega + \mathcal{O}(\varphi^2), \quad (\text{B14})$$

where  $\omega$  on the rhs is the translationally invariant solution of Eq. (B6) found earlier. If we insert the expansion back into the free action, we find the effective action

$$\Gamma(\varphi) = \int d^d x \left[ \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{M^2}{2} \varphi^2 + \mathcal{O}(\varphi^4) \right]. \quad (\text{B15})$$

Notice that the quadratic pieces in  $\Gamma$  are identical for the functional and linear response approach. This is generally true because the Gaussian *Ansatz* for our trial measure is already quadratic in the fields, and the kernels of the two approaches agree at  $\varphi = 0$ . Thus we arrive at the important conclusion:

*For Gaussian trial measures, the functional and linear response approach always lead to the same quadratic pieces in the effective action, i.e. to the same propagators.*

Differences arise in the vertices, i.e. the higher powers of the classical field: while all vertices vanish in the functional approach, the linear response formulation allows for  $\mathcal{O}(\varphi^4)$  pieces in Eq. (B15) even though we only used Gaussian trial measures.

To examine the structure of these vertex corrections in more detail, we can restrict our investigation to constant classical fields whence the effective action reduces to the effective potential. This has the benefit that Eq. (B11) will again be translationally invariant and can thus be solved in momentum space. In fact, we can simply repeat the calculation of the functional approach with the replacement  $m^2 \rightarrow m^2 + \frac{\lambda}{2} \varphi^2$ . To obtain the effective potential, we now have to insert the expression for  $\omega_\varphi$  back into the free action  $F(\omega, \varphi)$ . The resulting effective potential is rather complicated and cannot be expressed in closed form,

$$\begin{aligned} U_{\text{eff}}(\varphi) &= \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \int \frac{d^d k}{(2\pi)^d} \frac{k^2 + m^2 + \frac{\lambda}{2} \varphi^2}{k^2 + M^2(\varphi)} \\ &+ \frac{\hbar^2 \lambda}{8} \left[ \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + M^2(\varphi)} \right]^2 \\ &+ \frac{\hbar}{2} \int \frac{d^d k}{(2\pi)^d} \ln[k^2 + M^2(\varphi)], \end{aligned} \quad (\text{B16})$$

where  $M^2(\varphi)$  is the solution of

$$M^2(\varphi) = m^2 + \frac{\lambda}{2} \varphi^2 + \hbar \frac{\lambda}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + M^2(\varphi)}. \quad (\text{B17})$$

All these formulas are understood to be properly regularized and subject to subsequent renormalization. (We will



not discuss the necessary counterterms in more detail.) The effective potential (B16) has an interesting structure: The first two terms are the classical potential *including* the bare  $\varphi^4$  vertex, while the remaining quantum corrections contain all powers of  $\hbar$  and  $\lambda$ . When expanding in powers of the fields, we obtain

$$U_{\text{eff}}(\varphi) = \frac{M^2}{2}\varphi^2 + \frac{\lambda}{4!}[1 + \mathcal{O}(\hbar)]\varphi^4 + \mathcal{O}(\varphi^6), \quad (\text{B18})$$

where  $M^2$  is the same field-independent dynamical mass Eq. (B10) as in the functional approach. We will not further investigate the vertex corrections and their renormalization.

### APPENDIX C: THE GAP EQUATION IN THE LINEAR RESPONSE APPROACH

As mentioned in Sec. IV A, the optimal kernel  $\bar{\omega}_{\mathcal{A}}$  in the linear response approach is no longer translationally invariant and has no special color or Lorentz symmetry. As a consequence, we cannot reduce the free action and the ensuing gap equation to a system for a single-component function  $\omega(k)$  in momentum space. Instead, we have to work with the complete matrix  $\bar{\omega}_{\mu\nu}^{ab}(x, y)$  in position space. The corresponding formulas are listed below for completeness, although they are only necessary to compute radiative corrections to higher vertices, which we do not consider.

The average action follows by inserting the correlators (28) into the classical action. After some lengthy calculation, we obtain

$$\begin{aligned} \langle S_{\text{gf}} \rangle_{\mu} &= S_{\text{gf}}(\mathcal{A}) + \frac{1}{2} \int d(x, y) \bar{\omega}_{\mu\nu}^{aa}(x, y) [-\square_x \delta_{\mu\nu} + (1 - \xi^{-1}) \partial_{\mu}^x \partial_{\nu}^x] \delta(x, y) \\ &+ g f^{abc} \int d(x, y) \delta(x, y) \partial_{\mu}^x \{ [\bar{\omega}^{-1}]_{\nu\mu}^{ab}(x, y) \mathcal{A}_{\nu}^c(y) + [\bar{\omega}^{-1}]_{\nu\nu}^{ac} \mathcal{A}_{\mu}^b(y) + [\bar{\omega}^{-1}]_{\mu\nu}^{bc} \mathcal{A}_{\nu}^a(x) \} \\ &+ \frac{g^2}{4} f^{cab} f^{cde} \int dx \{ [\bar{\omega}^{-1}]_{\mu\nu}^{ab}(x, x) [\bar{\omega}^{-1}]_{\mu\nu}^{de}(x, x) + [\bar{\omega}^{-1}]_{\mu\mu}^{ad}(x, x) [\bar{\omega}^{-1}]_{\nu\nu}^{be}(x, x) + [\bar{\omega}^{-1}]_{\mu\nu}^{ae}(x, x) [\bar{\omega}^{-1}]_{\nu\mu}^{bd}(x, x) \\ &+ [\bar{\omega}^{-1}]_{\mu\nu}^{ab}(x, x) \mathcal{A}_{\mu}^d(x) \mathcal{A}_{\nu}^e(x) + [\bar{\omega}^{-1}]_{\mu\mu}^{ad}(x, x) \mathcal{A}_{\nu}^b(x) \mathcal{A}_{\nu}^e(x) + [\bar{\omega}^{-1}]_{\mu\mu}^{ae}(x, x) \mathcal{A}_{\nu}^b(x) \mathcal{A}_{\mu}^d(x) \\ &+ [\bar{\omega}^{-1}]_{\nu\mu}^{bd}(x, x) \mathcal{A}_{\nu}^a(x) \mathcal{A}_{\mu}^e(x) + [\bar{\omega}^{-1}]_{\nu\mu}^{be}(x, x) \mathcal{A}_{\mu}^a(x) \mathcal{A}_{\mu}^d(x) + [\bar{\omega}^{-1}]_{\mu\nu}^{de}(x, x) \mathcal{A}_{\mu}^a(x) \mathcal{A}_{\nu}^b(x) \}. \end{aligned} \quad (\text{C1})$$

To find the free action, we have to subtract the relative entropy, cf. Eqs. (32) and (39),

$$\bar{\mathcal{W}} = \frac{1}{2} \text{Tr} \left\{ \mathbb{1} - \ln \left( \frac{\bar{\omega}}{2\pi} \right) \right\} + \langle \ln \mathcal{J} \rangle_{\bar{\omega}, \mathcal{A}} = \frac{1}{2} \text{Tr} \left\{ \mathbb{1} - \ln \left( \frac{\bar{\omega}}{2\pi} \right) \right\} - \frac{1}{2} \int d(x, y) \chi_{\mu\nu}^{ab}(x, y) [\mathcal{A}_{\mu}^a(x) \mathcal{A}_{\nu}^b(y) + [\bar{\omega}^{-1}]_{\mu\nu}^{ab}(x, y)]. \quad (\text{C2})$$

It is now straightforward, though very cumbersome, to derive the gap equation by minimizing  $F = \langle S \rangle_{\mu} - \bar{\mathcal{W}}$  with respect to the kernel  $\bar{\omega}$ , or rather its inverse. The result can be put in the form

$$\begin{aligned} \bar{\omega}_{\mu\nu}^{ab}(x, y) &= \delta^{ab} [-\square_x \delta_{\mu\nu} + (1 - \xi)^{-1} \partial_{\mu}^x \partial_{\nu}^x] \delta(x, y) + \chi_{\mu\nu}^{ab}(x, y) + g^2 \{ f^{cab} f^{cde} [\bar{\omega}^{-1}]_{\mu\nu}^{de}(x, x) \\ &+ f^{cad} f^{cbe} (\delta_{\mu\nu} [\bar{\omega}^{-1}]_{\alpha\alpha}^{de}(x, x) - [\bar{\omega}^{-1}]_{\mu\nu}^{de}(x, x)) \} \delta(x, y) + 2g f^{abc} \{ (\partial_{\mu} \mathcal{A}_{\nu}^c(x)) - \partial_{\nu} \mathcal{A}_{\mu}^c(x) + \partial_{\alpha} \mathcal{A}_{\alpha}^c(x) \} \delta(x, y) \\ &+ g^2 \{ f^{cab} f^{cde} \mathcal{A}_{\mu}^d(x) \mathcal{A}_{\nu}^e(x) + f^{cad} f^{cbe} \delta_{\mu\nu} \mathcal{A}_{\alpha}^d(x) \mathcal{A}_{\alpha}^e(x) \} \delta(x, y). \end{aligned} \quad (\text{C3})$$

Notice that the derivatives in the third line also act on the  $\delta$  function, except for the first term in the curly brackets, and the curvature also enters with its full color and Lorentz structure.

Upon setting  $\mathcal{A} = 0$ , only the first two lines on the rhs of Eq. (C3) survive. This is just the position space equivalent to the gap equation (41), to which it reduces when the symmetric form Eq. (15) is assumed for  $\bar{\omega}$  and  $\chi$ . An exact solution of Eq. (C3) combined with the full curvature equation is not feasible. However, the  $\mathcal{A}$ -dependent terms on the rhs are independent of the kernel  $\bar{\omega}$  and thus act like an inhomogeneity. This suggests an iterative solution by expanding in powers of the classical field. When inserted back into the free action, the  $\mathcal{O}(\mathcal{A}^2)$  correction to  $\bar{\omega}$  gives rise to vertices up to  $\mathcal{O}(\mathcal{A}^4)$  etc. Thus, the computation of the vertex corrections should at least be feasible for the lowest nontrivial orders.

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