Self-gravitating branes again

Georgios Kofinas[1,*] and Maria Irakleidou[2,†]

[1]*Department of Information and Communication Systems Engineering, Research Group of Geometry, Dynamical Systems and Cosmology, University of the Aegean, Karlovassi 83200, Samos, Greece*

[2]*Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstrasse 8-10/136, A-1040 Vienna, Austria*

(Received 12 October 2013; published 14 March 2014)

We raise on theoretical grounds the question of the physical relevance of Israel matching conditions and their generalizations to higher codimensions, the standard cornerstone of the braneworld and other membrane scenarios. Our reasoning is based on the incapability of the conventional matching conditions to accept the Nambu-Goto probe limit, the inconsistency of codimension-2 and -3 classical defects for $D = 4$ and the probable inconsistency of high enough codimensional defects for any D since there is no high enough Lovelock density to support them. We propose alternative matching conditions which seem to overcome the previous puzzles. Instead of varying the brane-bulk action with respect to the bulk metric at the brane position, we vary with respect to the brane embedding fields so that the gravitational backreaction is included ("gravitating Nambu-Goto matching conditions"). Here, we consider in detail the case of a codimension-2 brane in 6-dim Einstein-Gauss-Bonnet gravity, prove its consistency for an axially symmetric cosmological configuration and show that the theory possesses richer structure compared to the standard theory. The cosmologies found have the Friedmann behavior and extra correction terms. For a radiation brane one solution avoids a cosmological singularity and undergoes accelerated expansion near the minimum scale factor. In the presence of an induced gravity term, there naturally appears in the theory the effective cosmological constant scale $\lambda/(M_6^4 r_c^2)$, which for a brane tension $\lambda \sim M_6^4$ (e.g. TeV^4) and $r_c \sim H_0^{-1}$ gives the observed value of the cosmological constant.

DOI: 10.1103/PhysRevD.89.065015 PACS numbers: 11.27.+d, 04.40.-b

I. INTRODUCTION

Many physical systems are described by a dynamics which models them as relativistic membranes in an appropriate dimensional spacetime. One example studied extensively during the last decade is the Universe itself within the context of various braneworld and other scenarios. Distributional (thin) as well as finite thickness defects have been studied. However, thin sources are expected to describe the important main features of brane dynamics for an arbitrary family of finite width regularizations in the limit of infinitesimal thickness, and therefore, distributional description is independent from the regularization scheme used.

It is admitted that the phenomenological action describing at lowest order the dynamics of a classical infinitely thin test brane (probe) with tension, moving in a given background spacetime, is proportional to the intrinsic volume of the world sheet, the Nambu-Goto action [1]. Variation of this action with respect to the embedding fields of the brane position gives the Nambu-Goto equations of motion which are geometrically described by the vanishing of the trace of the extrinsic curvature, and therefore, the world sheet swept by the brane is extremal (minimal). Of course, the induced metric on a Nambu-Goto brane is finite (with the possible exception of a set of points with measure zero) which means that the bulk metric is regular at the brane position, since only then, the Nambu-Goto equations of motion are defined. When the gravitational field of the defect is taken into account the situation becomes very different and difficult since now both the bulk metric and the brane position become dynamical. *It is the aim of this communication to examine anew the dynamics of a classical self-gravitating brane with a bulk metric regular on the brane*. The situation with a bulk metric singular on the brane is an interesting (not in the context of a braneworld) but very different story since now the equation of motion of the defect cannot contain the induced metric which is singular. In this case, actually, the full nonperturbative equation of motion is unknown; however, there is a consensus about the equation of motion for the special case of a 0-brane (point particle) in four dimensions and only at the linearized level around an arbitrary background [2,3].

Let us recall the standard or conventional treatment for obtaining the equations of motion for the brane-bulk system. One starts from the gravitational equations containing the Einstein term or some modification on the left-hand side and all the matter content of the bulk on the right-hand side with the localized brane energy-momentum tensor included. Therefore, a source proportional to a delta function of suitable codimension fuels the bulk gravitational equations.

*gkofinas@aegean.gr
†maria.irakleidou@tuwien.ac.at

Whenever a delta function appears on the right-hand side of an equation it always tries to find an analogous distribution on the left-hand side to balance. This balance, when possible, extracts the discontinuity of an appropriate quantity, e.g. in electrodynamics the discontinuities of electric or magnetic field. However, in gravity this discontinuity refers to the (finite) extrinsic curvature of the defect, and since the extrinsic curvature is related to the embedding fields of the defect, it extracts as boundary conditions (matching conditions) the equations of motion of the defect. This is what is said that in gravity everything, bulk dynamics as well as brane equations of motion, is contained in gravity theory itself. By contrast, in electrodynamics one postulates the Lorentz force equation of motion for the point charge. Unfortunately, it is known that Einstein gravity being nonlinear does not succeed the previous balance of distributional terms for all kinds of defects. Thin shells of matter (codimension-1 objects) do make mathematical sense within Einstein gravity and the equations of motion for the domain wall are the well-known Israel matching conditions [4] which are always consistent with bulk gravity equations. Note that thin shells have always regular metrics (analogously to the nonsingular electric potential of a charged plane). However, when a generic distributional stress-energy tensor is supported on a codimension-2 defect the analysis shows that it does not make mathematical sense to consider solutions of Einstein's equations [5–7] (a pure brane tension is a special situation which is consistent [8,9]). Early work on codimension-2 distributional braneworlds also showed that they were pure tension objects [10]. As referred above, possible singular solutions consistent with the underlying symmetry (analogous to the logarithmic electric potential of a charged line) will not be of our interest here [11]. Not surprisingly, the problematic behavior still continues for other higher-codimension branes.

A nice idea to resolve the puzzle of inconsistency was to understand that it was not the defect construction which was problematic; rather the gravity theory itself did not have the relevant differential complexity in order to describe complicated distributional solutions. The clear-cut hint of this point was the notice [12] that upon considering the general second derivative gravity theory in six dimensions [Einstein-Gauss-Bonnet (EGB)] one could have, at least in principle, a nontrivial energy-momentum tensor fueling geometric junction conditions for a codimension-2 conical defect. In [13] the consistency of the whole set of junction plus bulk field equations was explicitly shown for an axially symmetric codimension-2 cosmological brane (not necessarily totally geodesic) in six-dimensional EGB gravity. Presumably this consistency will persist if axial symmetry is abandoned. Analogously, e.g. a 5-brane in eight dimensions is again of codimension-2 and an EGB theory would suffice, but for a 4-brane in eight dimensions (codimension-3) the third Lovelock density [14] would be needed for consistency. However, it is not clear what is the situation when codimensionality is even bigger, e.g. a 1-brane in six dimensions (it is probably inconsistent since the spirit of the proposal is to include higher and higher Lovelock densities to accommodate higher codimension defects and there is no higher than the second Lovelock density in six dimensions). In brief, the generalization of the proposal is that in a D-dimensional spacetime the maximal $[(D-1)/2]$ Lovelock density should be included (possibly along with lower Lovelock densities) and the branes with codimensions $\delta = 1, 2, \ldots, [(D-1)/2]$ should be consistent according to the standard treatment; for yet higher codimensions the situation is not clear and probably inconsistent.

It is evident that the system of gravitational equations with a delta source on the right-hand side, whenever consistent, is equivalent to this system without the localized energy-momentum tensor, plus the corresponding matching conditions. An equivalent way to get all this set of equations is to take the variation of the bulk action with respect to the bulk metric as far as the bulk equations of motion are concerned and to take the variation of the total brane-bulk action with respect to the bulk metric at the brane position as far as the matching conditions are concerned. This method of deriving the matching conditions, although formally equivalent, presents in some cases a few technical differences compared to the initial method where the appropriate distributional terms are isolated. For example, for codimension-1 branes in Einstein gravity the extra Gibbons-Hawking term has to be included in the brane action in order for the variation to be well defined, a term that does not appear in the first method. Note also that contrary to the situation in classical mechanics or field theory where the variation of the coordinates or fields usually vanishes on the boundary, here, the variation of the bulk metric remains arbitrary on the brane since much useful information stems from there. Although the two methods for deriving the matching conditions are formally equivalent, however, there is a subtle but important point concerning branes with codimension larger than one, which makes the variational method superior. To explain, we note that for higher codimension branes there appear more than one kind of distributional terms in the full gravitational equations. For example, for a codimension-2 brane there are (r-independent) terms multiplied with $\delta(r)/r$ and others proportional to $\delta(r)$, where r is the radial coordinate from the brane. One could assume that both kinds of distributional terms should be balanced among the two sides of the distributional equation, arising therefore two different matching conditions. For the six-dimensional EGB gravity the consistency mentioned above includes only the matching conditions related to the wilder distributional terms $\delta(r)/r$, while if additionally the extra matching conditions [15] arising from the distributional terms $\delta(r)$ are included in the whole system of equations, the problem in general becomes inconsistent [13]. An explanation why the mild distributional terms should be ignored is to multiply the initial distributional equation by r; therefore $\delta(r)$ becomes $r\delta(r)$ and vanishes. However, this is not a safe and unambiguous manner to handle distributional equations and therefore this is not the correct reasoning. Because a distribution is defined through integration, the action and the variation of the action naturally provides this integration. In our example of a codimension-2 brane the volume element of integration close to the brane is $rdrd\theta$, and therefore the quantity $r\delta(r)$ naturally and unambiguously appears sending the mild distributional terms to zero.

To make a criticism on the standard treatment for obtaining the equation of motion of a defect that was described in the previous paragraphs, we mention the following points.

(a) A test brane moving in a curved background spacetime traces a minimal surface in the lowest order approximation. When the self-gravitational field of the brane starts to be taken into account it is natural to expect that a small correction should result on top of the background minimal surface motion of the test approximation. However, the equations of motion for a gravitating defect derived following the standard approach do not obey this condition of continuous deformation from the Nambu-Goto probe limit. Indeed, since the matching conditions described so far are nonperturbative, in order to realize the probe limit one should split the brane quantities appearing in these matching conditions (total induced metric, extrinsic curvature and possible deficit angles) to their corresponding background plus perturbed parts. It is not quite obvious what one should do in the probe limit with the brane energy-momentum tensor. Either the total brane energy-momentum including the brane tension (and the possible induced gravity term being an extra brane source) should go to zero, or the brane tension being always nonvanishing should be kept fixed and small (as a sort of regulator) and only the additional brane energy-momentum tensor should go to zero. More precisely, since the brane tension has dimensions of energy per spatial volume, the product of the brane tension with the brane volume should be much smaller than some definition of global energy of the background space. Although there are extreme cases where either the brane volume could be infinite or the gravitational energy attributed to a background space according to some definition of energy could be infinite, however, a criterion of a sort of smallness of the brane tension should be necessary in the probe limit. Under this condition, to suppress the gravitational character of the brane and go back to a probe brane moving in the fixed background, one should formally take to zero the bulk gravity couplings which control the gravitational backreaction of the brane, as well as some extra matter sources which also contribute to the backreaction (then, all the perturbed quantities vanish) and the matching condition should reduce to the Nambu-Goto equation. In a similar context, in the singular case which is not the subject of our discussion here, the linearized equation of motion of a point particle in four dimensions [2,3] is indeed a correction of the geodesic equation of motion on a given background (of course, for a 0-brane the geodesic equation coincides with the Nambu-Goto) and for a two-body system the probe limit is realized when one mass is much smaller than the other. When the brane tension is not small or the brane contains additional energy-momentum, the brane backreacts with the gravitational field and the full equations are needed. Obviously, the Israel matching conditions do not satisfy the requirement of the Nambu-Goto limit since they contain the extrinsic curvature instead of its trace (as expected, the same also happens for the matching condition of a codimension-1 brane in EGB gravity theory [16]). For the codimension-2 matching condition in EGB theory discussed above [12,13,17], the situation is similar. To go one step further, one could oppose by claiming that the correct probe limit of a defect is not the Nambu-Goto equation of motion but the geodesic one, which means that all the extrinsic curvatures have to become zero. This is indeed the case for the Israel matching conditions under the assumption that the total brane matter content goes to zero. However, this reasoning is not correct, because if the geodesic was the correct probe limit, it would be so, independently of the gravitational theory considered (for example, a probe point mass moves on the geodesic of a background solution of any gravitational theory [18,19]), or also independently of the codimension of the defect. But this is not the case, since other than Einstein gravitational theories for codimension-1 or other codimension defects, in the limit of vanishing brane energy-momentum, do not give the geodesic equation [12,16,17].

(b) There is an additional reason why the idea of adding higher Lovelock densities in order to get consistency according to the standard approach cannot be the final word: generic codimension-2 or 3 branes are not allowed to reside in four dimensions since Einstein gravity is insufficient and there are no any higher Lovelock densities in four dimensions to add. Even if four dimensions are not the actual spacetime dimensionality, at certain length and energy scales it has been tested that four-dimensional Einstein gravity represents effectively the spacetime to high accuracy.

(c) Furthermore, it seems most probable that branes of high enough codimension (higher than $[(D-1)/2]$) cannot reside in a D-dimensional spacetime according to the standard treatment since there is no corresponding sufficiently high Lovelock density. In the D-dimensional spacetime where we live (maybe $D = 4$) classical defects of any possible codimension should be compatible. Even if particular branes are not physically interesting or do not appear in nature, they could in principle be constructed in the lab, and therefore the correct mathematical framework should allow all kinds of branes (in analogy, all kinds of charged distributions exist). It does not seem reasonable for some hidden symmetry to prohibit the existence of any possible classical defect. If one disregards this point and sticks to particular branes, the criteria of consistency of a given formulation are reduced. Additionally, since both regular and singular solutions are in general allowed mathematically, it would be peculiar if the regular solutions we investigate here are not permitted. Moreover, it seems impossible for the equations of motion of the various codimensional defects to be derived through different methods, but a unified principle should give the equations of motion of all kinds of defects. Possible direct observational evidence of codimension-1 defects in four dimensions could shed light on the issue of matching conditions.

The present paper studies, instead of the standard, alternative matching conditions which aim to satisfy the previous three points. In particular, concerning the first point above, these alternative matching conditions always have the Nambu-Goto probe limit, independently of the gravitational theory considered, the dimensionality of spacetime or the codimensionality of the defect. As far as the second point is concerned, considering the alternative matching conditions, a codimension-2 brane is consistent in D-dimensional Einstein gravity [20], or in particular in four-dimensional Einstein gravity. Finally, for the third point, the simple case of a codimension-1 brane is still, as in the standard approach, always consistent, independently of the gravitational theory. The codimension-2 brane in EGB theory, examined in the present paper, is also proved to be consistent. Accordingly, it is expected, although we do not have a proof, that any higher codimension brane will also be consistent in either Einstein gravity or any Lovelock extension.

Let us describe now the method for deriving these alternative matching conditions. In the standard method the variation of the bulk action with respect to the bulk metric gives the bulk equations of motion and the variation of the total brane-bulk action with respect to the bulk metric at the brane position gives the matching conditions. Although the bulk equations of motion are not debatable, the sort of variation at the brane position is not unquestionable. The brane defines a sort of boundary (not necessarily of codimension 1) and a boundary is an exceptional place whose position is primarily determined by the embedding fields and secondarily or implicitly by its induced metric or the bulk metric at the brane position. The Nambu-Goto equations of motion arise by varying the Nambu-Goto action with respect to the embedding fields. Typically, the Nambu-Goto action being proportional to the world sheet volume is a function of the induced brane metric which depends explicitly and implicitly (through the bulk metric at the brane position) on the embedding fields. If, furthermore, the brane backreacts with the bulk gravity, gravity will be also present at the brane position and the motion of the brane will be influenced by the gravitational bulk action. A variation of the embedding fields implies a variation of the bulk metric at the brane position, and therefore, additional contributions to the brane equation of motion beyond the Nambu-Goto term arise from the variation of the bulk action with respect to the embedding fields. The distributional terms are responsible for this contribution. Again, as in the standard treatment, everything, bulk dynamics as well as brane equations of motion, is contained in gravity theory itself, but in a different manner than before. Here, the brane equation of motion is the result of the variation of the brane-bulk action at the brane position with respect to the position variables, what can be called "gravitating Nambu-Goto matching conditions." Although the brane energy-momentum tensor is still defined by the variation of the brane action with respect to the bulk metric at the brane position, however, this tensor enters the new matching conditions in a different way than before. The present proposal is not based primarily on the modification of the gravitational theory, but on the modification of the matching conditions. Since the consistency here is not based crucially on the inclusion of the maximal Lovelock density, it is plausible that the consistency will occur even for all higher codimension defects. Here, the distributional terms are still present, not inside a distributional differential equation leading directly to inconsistencies at certain cases, but rather smoothed out inside an integration. Of course, in a higher D-dimensional spacetime higher Lovelock densities should in principle contribute to the bulk action, and actually, the consistency of such a theory, along with the modified matching conditions described above, is the main subject of study in the present paper. Moreover, the inclusion of such extra terms in the action leads probably to physically more interesting and realistic solutions.

Our approach is reminiscent of the "Dirac-style" variation performed in [21] in the study of codimension-1 defects. It also resembles the Regge-Teitelboim brane gravitational theory [22] which is an extension of the Nambu-Goto style of variation, with the crucial difference however that in [22] there are no higher-dimensional gravity terms in the action and the bulk space is prefixed (Minkowski) instead of dynamical which is here. In [21], since the probe limit and the consistency of the various codimensions were not considered, the standard approach was not set into doubt, but rather the derivation was basically suggested as a formal treatment to unify different gravitational theories. In [20], by varying with respect to the embedding fields, the alternative matching conditions of a 3-brane in six-dimensional Einstein gravity were derived and their consistency was shown for an axially symmetric configuration (the same however is true for a codimension-2 brane in any dimension). In the present paper we go one step further and derive the matching conditions of a codimension-2 brane in EGB theory. Their consistency is checked for an axially symmetric cosmological configuration. The derived cosmologies have various differences compared to the corresponding cosmologies derived using the standard matching conditions [13]. The main point stressed in the present paper is that in view of the three points mentioned above, the alternative matching conditions derived by varying with respect to the embedding fields instead of the bulk metric at the brane position could well be closer to the correct direction for deriving realistic matching conditions, compared to the Israel matching conditions and generalizations.

The setup of the paper is as follows: In Sec. II the method is introduced as an extension of the Nambu-Goto variation so that the contribution from the gravitational backreaction is included. In Sec. III we consider a 3-brane in six-dimensional Einstein-Gauss-Bonnet gravity and derive the generic alternative matching conditions and the remaining effective equations on the brane. Similar or identical equations hold for other codimension-2 branes in other spacetime dimensions, but we choose the 3-brane as it can represent our world in the braneworld scenario. In Sec. IV we specialize to the cosmological configuration and demonstrate the consistency of the system. In Sec. V we investigate the cosmological equations and derive solutions for the cosmic evolution. In Sec. VI we study a few special characteristic cases, and we discuss the Einstein limit of the theory and compare with the standard treatment where the conventional matching conditions are used. Finally, in Sec. VII we conclude.

II. GENERAL ARGUMENTS AND INTRODUCTION OF THE METHOD

In order to understand how the proposed variation with respect to the embedding fields of the brane position is performed, we start with a general four-dimensional action of the form

$$s_4 = \int_\Sigma d^4\chi\sqrt{|h|}L(h_{ij}), \tag{2.1}$$

where L is any scalar on Σ built up from the induced metric h_{ij}. The brane coordinates are χ^i ($i, j\ldots$ are coordinate indices on the brane) and the bulk coordinates are x^μ ($\mu, \nu, \ldots$ are D-dimensional indices). In this section D is arbitrary, while in the next sections of the paper we will specialize to $D = 6$. In the present paragraph the bulk metric $g_{\mu\nu}$ is fixed and nondynamical, while the treatment of a backreacted metric will be given in the next paragraphs of this section. The embedding fields are the external (bulk) coordinates of the brane, so they are some functions $x^\mu(\chi^i)$. We could use a capital case letter for the embedding fields to distinguish from the bulk coordinates, but it is better not to do so because first, the embedding fields are the bulk coordinates at the brane position, second, the functions $x^\mu(\chi^i)$ do not define the brane position unless the bulk coordinates x^μ in the full space are given, and third, since our concern is the brane equation of motion, the bulk coordinates away from the brane only incidentally will be considered. Since $h_{ij} = g_{\mu\nu}x^\mu{}_{,i}x^\nu{}_{,j}$ and on the brane it is $g_{\mu\nu}(x^\lambda)$, thus h_{ij} has explicit and implicit dependence on the embedding fields x^μ. Let their variation be described by the displacement vector $\bar\delta x^\mu(x^\nu)$ and the corresponding variation of the various quantities denoted by $\bar\delta_x$. The quantities $x^\mu{}_{,i}$ are tangent vectors on the brane and their variation is $\bar\delta_x(x^\mu{}_{,i}) = (\bar\delta x^\mu)_{,i} \equiv \bar\delta x^\mu{}_{,i}$. We must observe that the variation of $g_{\mu\nu}$ is

$$\bar\delta_x g_{\mu\nu} = g_{\mu\nu,\lambda}\bar\delta x^\lambda, \tag{2.2}$$

so $g_{\mu\nu}$ is considered as a simple scalar function of x^λ ignoring the possible tensorial indices. The reason is that the bulk coordinates, which determine the tensorial behavior of the quantities with spacetime indices, do not change; only the brane position changes—i.e. the embedding fields are varied. So, the spirit of the $\bar\delta_x$ variation is that when an explicit embedding field x^μ is met, it is varied, while when a function of this x^μ is met, the partial derivative is taken. Then,

$$\begin{aligned}\bar\delta_x h_{ij} &= g_{\mu\nu,\lambda}x^\mu{}_{,i}x^\nu{}_{,j}\bar\delta x^\lambda + g_{\mu\nu}x^\mu{}_{,i}\bar\delta x^\nu{}_{,j} + g_{\mu\nu}x^\nu{}_{,j}\bar\delta x^\mu{}_{,i}\\ &= x^\mu{}_{,i}x^\nu{}_{,j}(g_{\mu\nu,\lambda}\bar\delta x^\lambda + g_{\mu\lambda}\bar\delta x^\lambda{}_{,\nu} + g_{\nu\lambda}\bar\delta x^\lambda{}_{,\mu}).\end{aligned} \tag{2.3}$$

If we had wrongly considered the tensorial character of $g_{\mu\nu}$, its variation would be $g'_{\mu\nu}(x^\rho + \bar\delta x^\rho) - g_{\mu\nu}(x^\rho) = -g_{\mu\lambda}(x^\rho)\bar\delta x^\lambda{}_{,\nu} - g_{\nu\lambda}(x^\rho)\bar\delta x^\lambda{}_{,\mu}$, so the variation of h_{ij} would vanish, which is a trivial result expressing no dynamics and is simply due to that $g_{\mu\nu}x^\mu{}_{,i}x^\nu{}_{,j}$ is scalar in the spacetime indices μ, ν. The variation of s_4 is

$$\bar\delta_x s_4 = \int_\Sigma d^4\chi\sqrt{|h|}\tau^{ij}\bar\delta_x h_{ij}, \tag{2.4}$$

where $\tau^{ij} = \frac{\delta L}{\delta h_{ij}} + \frac{L}{2}h^{ij}$. Substituting $\bar\delta_x h_{ij}$ from (2.3), integrating by parts and imposing $\bar\delta x^\mu|_{\partial\Sigma} = 0$, we get

$$\begin{aligned}\bar{\delta}_x s_4 &= -2\int_\Sigma d^4\chi\sqrt{|h|}g_{\mu\sigma}[(\tau^{ij}x^\mu{}_{,i})_{|j} + \tau^{ij}\Gamma^\mu{}_{\nu\lambda}x^\nu{}_{,i}x^\lambda{}_{,j}]\bar{\delta}x^\sigma \\ &= -2\int_\Sigma d^4\chi\sqrt{|h|}g_{\mu\sigma} \\ &\quad\times[\tau^{ij}{}_{|j}x^\mu{}_{,i} + \tau^{ij}(x^\mu{}_{;ij} + \Gamma^\mu{}_{\nu\lambda}x^\nu{}_{,i}x^\lambda{}_{,j})]\bar{\delta}x^\sigma \\ &= -2\int_\Sigma d^4\chi\sqrt{|h|}g_{\mu\sigma}(\tau^{ij}{}_{|j}x^\mu{}_{,i} - \tau^{ij}K^\alpha{}_{ij}n_\alpha{}^\mu)\bar{\delta}x^\sigma, \quad (2.5)\end{aligned}$$

since $x^\mu{}_{|ij} = x^\mu{}_{;ij}$, $x^\mu{}_{;ij} + \Gamma^\mu{}_{\nu\lambda}x^\nu{}_{,i}x^\lambda{}_{,j} = -K^\alpha{}_{ij}n^\mu_\alpha$, where $K^\alpha{}_{ij} = n^\alpha{}_{i;j}$ are the extrinsic curvatures on the brane and $n_\alpha{}^\mu$ ($\alpha = 1,\ldots,\ \delta = D-4$) form a basis of normal vectors to the brane. The covariant differentiations $|$ and; correspond to h_{ij} and $g_{\mu\nu}$, respectively, while $\Gamma^\mu{}_{\nu\lambda}$ are the Christoffel symbols of $g_{\mu\nu}$. Because of the arbitrariness of $\bar{\delta}x^\mu$ it arises

$$\tau^{ij}{}_{|j}x^\mu{}_{,i} - \tau^{ij}K^\alpha{}_{ij}n_\alpha{}^\mu = 0, \quad (2.6)$$

and since the vectors $x^\mu{}_{,i}$, $n_\alpha{}^\mu$ are independent, two sets of equations arise:

$$\tau^{ij}{}_{|j} = 0, \qquad \tau^{ij}K^\alpha{}_{ij} = 0 \Leftrightarrow \tau^{ij}(x^\mu{}_{;ij} + \Gamma^\mu{}_{\nu\lambda}x^\nu{}_{,i}x^\lambda{}_{,j}) = 0 \quad (2.7)$$

(since $n_\alpha{}^\mu n^\beta{}_\mu = \delta^\beta_\alpha$). Note that the previous equivalence of the two expressions, one with free index α and the other with free index μ, is due to that the vectors $x^\mu{}_{;ij} + \Gamma^\mu{}_{\nu\lambda}x^\nu{}_{,i}x^\lambda{}_{,j}$ are normal to the brane. The variation described so far is the same with the one leading to the Nambu-Goto equation of motion. Indeed for $L = 1$, it is $\tau^{ij} = \frac{1}{2}h^{ij}$ and the first equation is empty, while the second becomes $h^{ij}K^\alpha{}_{ij} = 0 \Leftrightarrow \Box_h x^\mu + \Gamma^\mu{}_{\nu\lambda}h^{\nu\lambda} = 0$ (since $-n_\alpha{}^\mu h^{ij}K^\alpha{}_{ij} = \Box_h x^\mu + \Gamma^\mu{}_{\nu\lambda}h^{\nu\lambda}$) which is the Nambu-Goto equation of motion. Note again that the previous equivalence of the two expressions for the Nambu-Goto equation, one with free index α and the other with free index μ, is due to that the vector $\Box_h x^\mu + \Gamma^\mu{}_{\nu\lambda}h^{\nu\lambda}$ is normal to the brane. Similarly, the Regge-Teitelboim equation of motion [22] is a generalization where L collects the four-dimensional terms of (3.1), i.e. $L = \frac{r_c^{D-4}}{2\kappa_D^2}R - \lambda + \frac{L_{\rm mat}}{\sqrt{|h|}}$. It is $\tau^{ij} = -\frac{1}{2}(\frac{r_c^{D-4}}{\kappa_D^2}G^{ij} + \lambda h^{ij} - T^{ij})$, so the first equation becomes the standard conservation $T^{ij}{}_{|j} = 0$ and the second $(\frac{r_c^{D-4}}{\kappa_D^2}G^{ij} + \lambda h^{ij} - T^{ij})K^\alpha{}_{ij} = 0$.

In order to express the backreaction of the brane onto the bulk and vice versa, we now go one step further and consider a general higher-dimensional action of the form

$$s_D = \int_M d^Dx\sqrt{|g|}\mathcal{L}(g_{\mu\nu}), \quad (2.8)$$

where $\mathcal{L}$ is any scalar on M built up from the metric $g_{\mu\nu}$, e.g. $\mathcal{L} = \mathcal{R}(g_{\mu\nu})$. Under an arbitrary variation of the bulk metric $\delta g_{\mu\nu}$ the variation of s_D is $\delta s_D = \int_M d^Dx\sqrt{|g|}\mathrm{E}^{\mu\nu}\delta g_{\mu\nu}$, where $\mathrm{E}^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} + \frac{\mathcal{L}}{2}g^{\mu\nu}$, and the stationarity $\delta s_D = 0$ under arbitrary variations $\delta g_{\mu\nu}$ gives the bulk field equations $\mathrm{E}^{\mu\nu} = 0$. The boundary terms arising from this variation in the presence of a defect disappear by a suitable choice for the boundary condition of $\delta g_{\mu\nu}$, usually by choosing a Dirichlet boundary condition for $\delta g_{\mu\nu}$. However, in the presence of the defect, inside $\mathrm{E}^{\mu\nu}$, beyond the regular terms which obey $\mathrm{E}^{\mu\nu} = 0$, in general there are also nonvanishing distributional terms making the variation δs_D not identically zero. The bulk action knows about the defect through these distributional terms. Since $\mathrm{distr}\,\mathrm{E}^{\mu\nu} \propto \delta^{(\delta)}$, where $\delta^{(\delta)}$ is the δ-dimensional delta function with support on the defect, it is $\mathrm{distr}\,\mathrm{E}^{\mu\nu}\delta g_{\mu\nu} \propto \delta^{(\delta)}\delta g_{\mu\nu} = \delta^{(\delta)}\delta g_{\mu\nu}|_{\rm brane}$, so only the variation of the bulk metric at the brane position contributes to δs_D. More precisely, these distributional terms always appear in the parallel to the brane components and if $\mathrm{distr}\,\mathrm{E}^{ij} = k^{ij}\delta^{(\delta)}$ the variation δs_D gets the form

$$\delta s_D = \int_M d^Dx\sqrt{|g|}k^{ij}\delta^{(\delta)}\delta h_{ij} = \int_\Sigma d^4\chi\sqrt{|h|}k^{ij}\delta h_{ij}. \quad (2.9)$$

Therefore, there is an extra variation of the bulk metric at the brane position $\delta g_{\mu\nu}|_{\rm brane}$ (which in the adapted frame coincides with the variation of the induced metric δh_{ij}) which is independent from the bulk metric variation and this extra variation determines the brane equation of motion (actually, it is sensible for the brane equation of motion not to depend on the variation of the fields away from the brane). The corresponding variation of the total action at the brane position is

$$\delta(s_D + s_4)|_{\rm brane} = \int_\Sigma d^4\chi\sqrt{|h|}(k^{ij} + \tau^{ij})\delta h_{ij}. \quad (2.10)$$

In particular, if all the components of the variation δh_{ij} are independent from each other, the stationarity of the total action at the brane position $\delta(s_D + s_4)|_{\rm brane} = 0$ gives the standard matching conditions (or standard brane equations of motion) $k^{ij} + \tau^{ij} = 0 \Leftrightarrow k^{\mu\nu} + \tau^{\mu\nu} = 0$, where $k^{\mu\nu} = k^{ij}x^\mu{}_{,i}x^\nu{}_{,j}$, $\tau^{\mu\nu} = \tau^{ij}x^\mu{}_{,i}x^\nu{}_{,j}$ are parallel to the brane tensors. Our aim is to consider a variation $\bar{\delta}x^\mu$ of the embedding fields and derive a meaningful nontrivial brane equation of motion in the presence of a higher-dimensional action s_D on top of the four-dimensional action s_4. As we have seen in the previous paragraph, since the bulk coordinates do not change, $g_{\mu\nu}$ at the brane position transforms as (2.2) and the induced metric as (2.3). Now, the corresponding variation of the total action at the brane position is due to $\bar{\delta}x^\mu$ and since $\bar{\delta}_x h_{ij}$ is a special variation of the arbitrary δh_{ij}, it will be

$$\bar{\delta}_x(s_D + s_4)|_{\rm brane} = \int_\Sigma d^4\chi\sqrt{|h|}(k^{ij} + \tau^{ij})\bar{\delta}_x h_{ij}. \quad (2.11)$$

Following the same steps as in the previous paragraph with τ^{ij} replaced by $k^{ij}+\tau^{ij}$, the stationarity $\bar{\delta}_x(s_D+s_4)|_{\rm brane}=0$ gives, due to the arbitrariness of $\bar{\delta}x^\mu$, the brane equations of motion

$$(k^{ij}+\tau^{ij})_{|j}=0,$$
$$(k^{ij}+\tau^{ij})K^\alpha{}_{ij}=0\Leftrightarrow(k^{ij}+\tau^{ij})(x^\mu{}_{;ij}+\Gamma^\mu{}_{\nu\lambda}x^\nu{}_{,i}x^\lambda{}_{,j})=0. \tag{2.12}$$

These can be called "gravitating Nambu-Goto matching conditions" since they collect also the contribution from bulk gravity and they form a schematic summary of our proposal. Nothing *ab initio* assures their consistency with the bulk field equations.

There is an equivalent way to describe this variation and get the same results. Instead of considering the active notion above where the brane is deformed to another position defined by the displacement vector $\bar{\delta}x^\mu$, we consider the passive viewpoint of a bulk coordinate change $x^\mu\to x'^\mu=x^\mu+\delta x^\mu$, where now the brane does not change position but it is described by different coordinates and the change of the embedding fields is δx^μ. Of course, only the value of the variation δx^μ on the brane and not the values δx^μ away from the brane is expected to influence the corresponding variation of the brane-bulk action at the brane position. Although a bulk action s_D is invariant under coordinate transformations, the presence of the defect, i.e. of the distributional terms inside s_D, makes $\delta_x s_D|_{\rm brane}\neq 0$. The crucial point is how the tensor fields $\phi^\mu_\nu(x^\rho)$ are varied. One option which is unsuccessful is the tensorial variation $\tilde{\delta}_x\phi^\mu_\nu=\phi'^\mu_\nu(x'^\rho)-\phi^\mu_\nu(x^\rho)=\phi^\lambda_\nu(x^\rho)\delta x^\mu_{,\lambda}-\phi^\mu_\lambda(x^\rho)\delta x^\lambda_{,\nu}$, where $\delta x^\mu{}_{,\nu}\equiv(\delta x^\mu)_{,\nu}$. The successful option is the *functional* variation which is the change in the functional form of ϕ^μ_ν, i.e.

$$\begin{aligned}\delta_x\phi^\mu_\nu&=\phi'^\mu_\nu(x^\rho)-\phi^\mu_\nu(x^\rho)\\&=\phi^\lambda_\nu(x^\rho)\delta x^\mu_{,\lambda}-\phi^\mu_\lambda(x^\rho)\delta x^\lambda_{,\nu}-\phi^\mu_{\nu,\lambda}\delta x^\lambda\\&=-\pounds_{\delta x}\phi^\mu_\nu.\end{aligned} \tag{2.13}$$

So, the fields transform according to the Lie derivative with generator the infinitesimal coordinate change and $\delta_x\phi^\mu_\nu$ is tensor, contrary to $\tilde{\delta}_x\phi^\mu_\nu$ which is not. In particular, the functional variation of $g_{\mu\nu}$ is

$$\begin{aligned}\delta_x g_{\mu\nu}&=g'_{\mu\nu}(x^\rho)-g_{\mu\nu}(x^\rho)\\&=-(g_{\mu\nu,\lambda}\delta x^\lambda+g_{\mu\lambda}\delta x^\lambda{}_{,\nu}+g_{\nu\lambda}\delta x^\lambda{}_{,\mu})\\&=-\pounds_{\delta x}g_{\mu\nu}.\end{aligned} \tag{2.14}$$

The functional change of the embedding fields $x^\mu(\chi^i)$ is $\delta_x x^\mu=x'^\mu|_{x^\nu}-x^\mu=x^\mu-x^\mu=0$, and for the tangent vectors it is $\delta_x(x^\mu{}_{,i})=0$, or to be more formal $\delta_x(x^\mu{}_{,i})=\delta_x t^\mu_{(i)}=t'^\mu_{(i)}(x^\rho)-t^\mu_{(i)}(x^\rho)=x^\mu{}_{,i}-x^\mu{}_{,i}=0$, where $t^\mu_{(i)}(x^\rho)=x^\mu{}_{,i}$, $t'^\mu_{(i)}(x'^\rho)=x'^\mu{}_{,i}$. Therefore, contrary to the $\bar{\delta}x^\mu$ variation, here the fields $x^\mu{}_{,i}$ are not varied, but the functional variation of $h_{ij}=g_{\mu\nu}x^\mu{}_{,i}x^\nu{}_{,j}$ is again the same

$$\begin{aligned}\delta_x h_{ij}&=(\delta_x g_{\mu\nu})x^\mu{}_{,i}x^\nu{}_{,j}\\&=-x^\mu{}_{,i}x^\nu{}_{,j}(g_{\mu\nu,\lambda}\delta x^\lambda+g_{\mu\lambda}\delta x^\lambda{}_{,\nu}+g_{\nu\lambda}\delta x^\lambda{}_{,\mu}).\end{aligned} \tag{2.15}$$

The variation of the total brane-bulk action at the brane position is $\delta_x(s_D+s_4)|_{\rm brane}=\int_\Sigma d^4\chi\sqrt{|h|}(k^{ij}+\tau^{ij})\delta_x h_{ij}$ and the stationarity $\delta_x(s_D+s_4)|_{\rm brane}=0$ gives, due to the arbitrariness of δx^μ, the same brane equations of motion $(k^{ij}+\tau^{ij})_{|j}=0$, $(k^{ij}+\tau^{ij})K^\alpha{}_{ij}=0\Leftrightarrow(k^{ij}+\tau^{ij})(x^\mu{}_{;ij}+\Gamma^\mu{}_{\nu\lambda}x^\nu{}_{,i}x^\lambda{}_{,j})=0$. Alternatively, instead of directly going to the brane coordinates, we can work with the full spacetime indices and have

$$\begin{aligned}\delta_x s_D&=\int_M d^Dx\sqrt{|g|}\,\mathrm{distr}\,\mathrm{E}^{\mu\nu}\delta_x g_{\mu\nu}\\&=\int_M d^Dx\sqrt{|g|}k^{\mu\nu}\delta^{(\delta)}\delta_x g_{\mu\nu}\\&=\int_\Sigma d^4\chi\sqrt{|h|}k^{\mu\nu}\delta_x g_{\mu\nu}\\&=\int_\Sigma d^4\chi\sqrt{|h|}k^{ij}x^\mu{}_{,i}x^\nu{}_{,j}\delta_x g_{\mu\nu}\\&=\int_\Sigma d^4\chi\sqrt{|h|}k^{ij}\delta_x h_{ij},\end{aligned} \tag{2.16}$$

which is the same result as above. In the next section we will follow this process (actually slightly modified because of the use of Lagrange multipliers) for a codimension-2 defect in EGB bulk gravity and basically try to find k^{ij}.

III. SIX-DIMENSIONAL SETUP, MATCHING CONDITIONS AND EFFECTIVE EQUATIONS

Let us consider the general system of six-dimensional Einstein-Gauss-Bonnet gravity coupled to a localized 3-brane source. The total brane-bulk action is

$$\begin{aligned}S=&\frac{1}{2\kappa_6^2}\int_M d^6x\sqrt{|g|}\\&\times[\mathcal{R}-2\Lambda_6+\alpha(\mathcal{R}^2-4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}+\mathcal{R}_{\mu\nu\kappa\lambda}\mathcal{R}^{\mu\nu\kappa\lambda})]\\&+\int_\Sigma d^4\chi\sqrt{|h|}\left(\frac{r_c^2}{2\kappa_6^2}R-\lambda\right)+\int_M d^6x\mathcal{L}_{\rm mat}\\&+\int_\Sigma d^4\chi L_{\rm mat},\end{aligned} \tag{3.1}$$

where $g_{\mu\nu}$ is the (continuous) bulk metric tensor and $h_{\mu\nu}$ is the induced metric on the brane ($\mu,\nu,\ldots$ are now six-dimensional coordinate indices). The calligraphic quantities refer to the bulk metric, while the regular ones to the brane metric. The brane tension is λ and the induced-gravity term [23], if present, has a crossover length scale r_c. $\mathcal{L}_{\rm mat}$ and $L_{\rm mat}$ are the matter Lagrangians of the bulk and of the brane, respectively.

Varying (3.1) with respect to the bulk metric we get the bulk equations of motion

$$\begin{aligned}\mathcal{G}_{\mu\nu} &+ 2\alpha(\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\kappa}\mathcal{R}_{\nu}{}^{\kappa} - 2\mathcal{R}_{\mu\kappa\nu\lambda}\mathcal{R}^{\kappa\lambda} + \mathcal{R}_{\mu\kappa\lambda\sigma}\mathcal{R}_{\nu}{}^{\kappa\lambda\sigma}) \\ &- \frac{\alpha}{2}(\mathcal{R}^2 - 4\mathcal{R}_{\kappa\lambda}\mathcal{R}^{\kappa\lambda} + \mathcal{R}_{\kappa\lambda\rho\sigma}\mathcal{R}^{\kappa\lambda\rho\sigma})g_{\mu\nu} \\ &= \kappa_6^2\mathcal{T}_{\mu\nu} - \Lambda_6 g_{\mu\nu}, \end{aligned} \tag{3.2}$$

where $\mathcal{G}_{\mu\nu}$ is the bulk Einstein tensor and $\mathcal{T}_{\mu\nu}$ is a regular bulk energy-momentum tensor. We are mainly interested in a bulk with a pure cosmological constant Λ_6, but as it will be seen, the existence of a nonvanishing $\mathcal{T}_{\mu\nu}$ is not very crucial. More precisely, we define the variation $\delta g_{\mu\nu}$ with respect to the bulk metric to vanish on the defect. In this variation, beyond the basic terms proportional to $\delta g_{\mu\nu}$ which give (3.2), there appear, as usually, extra terms proportional to the second covariant derivatives $(\delta g_{\mu\nu})_{;\kappa\lambda}$. Since we are interested in the equations of motion of bulk gravity, we consider a hypersurface which is an infinitely thin "tube" around the codimension-2 defect, and then, the extra bulk integral becomes an integral on this hypersurface with terms proportional to $(\delta g_{\mu\nu})_{;\kappa}$. Adding suitable boundary terms on the hypersurface [16] (analogous to the Gibbons-Hawking term) to cancel the normal derivatives of $\delta g_{\mu\nu}$, the surface integral of the total variation finally consists only of terms proportional to $\delta g_{\mu\nu}$. Considering that the variation of the bulk metric vanishes on this hypersurface, there is nothing left beyond the terms in Eq. (3.2). In the limit where the tube shrinks at the codimension-2 brane, the variation of the bulk metric vanishes on this brane. Note that the boundary terms added on the hypersurface have nothing to do with the equation of motion of the codimension-2 brane derived below, but they are just added to make the variation of the bulk metric well defined. Equation (3.2) in the vicinity of the brane contains, in general, regular terms as well as divergent terms (distributional terms are different and refer to the brane position, not to the bulk space) and will be analyzed further later.

According to the standard method, the interaction of the brane with the bulk comes from the variation $\delta g_{\mu\nu}$ at the brane position of the action (3.1), which is equivalent to adding on the right-hand side of Eq. (3.2) the term $\kappa_6^2\tilde{T}_{\mu\nu}\delta^{(2)}$, where $\tilde{T}_{\mu\nu} = T_{\mu\nu} - \lambda h_{\mu\nu} - (r_c^2/\kappa_6^2)G_{\mu\nu}$. $T_{\mu\nu}$ is the brane energy-momentum tensor, $G_{\mu\nu}$ the brane Einstein tensor and $\delta^{(2)}$ the two-dimensional delta function with support on the defect. This approach has been analyzed in [13] and discussed in the introduction.

Here, we discuss the alternative approach where the interaction of the brane with bulk gravity is obtained by varying the total action (3.1) with respect to δx^μ, the embedding fields of the brane position. The embedding fields are some functions $x^\mu(\chi^i)$ and their variations are $\delta x^\mu(x^\nu)$. While in the standard method the variation of the bulk metric at the brane position remains arbitrary, here the corresponding variation is induced by δx^μ as explained in Sec. II; it is given by Eq. (2.14):

$$\begin{aligned}\delta g_{\mu\nu} &= \delta_x g_{\mu\nu} = g'_{\mu\nu}(x^\rho) - g_{\mu\nu}(x^\rho) \\ &= -(g_{\mu\nu,\lambda}\delta x^\lambda + g_{\mu\lambda}\delta x^\lambda{}_{,\nu} + g_{\nu\lambda}\delta x^\lambda{}_{,\mu}) \\ &= -\pounds_{\delta x} g_{\mu\nu}\end{aligned} \tag{3.3}$$

and is obviously independent from the variation leading to (3.2). The induced metric $h_{ij} = g_{\mu\nu}x^\mu{}_{,i}x^\nu{}_{,j}$ enters the localized terms of the action (3.1) and depends explicitly and implicitly (through $g_{\mu\nu}$) on the embedding fields. Also the various bulk terms of (3.1) contribute implicitly to the brane variation under the variation of the embedding fields. The result of δx^μ variation gives, as we will see, as coefficient of δx^μ a combination of vectors parallel and normal to the brane; therefore, two sets of equations will finally arise as matching conditions instead of one. Instead of directly expressing $\delta g_{\mu\nu}$, δh_{ij} in terms of δx^μ, it is more convenient to include the constraints in the action and vary independently. So, the first constraint $h_{ij} = g_{\mu\nu}x^\mu{}_{,i}x^\nu{}_{,j}$ implies the independent variation of h_{ij}. The variation δx^μ affects the variation of the parallel to the brane vectors $x^\mu{}_{,i}$ which in turn influences the variation of the normal vectors $n_\alpha{}^\mu$. If $n_\alpha{}^\mu$ ($\alpha = 1, 2$) are arbitrary unit vectors normal to the brane and to each other, the additional constraints $n_{\alpha\mu}x^\mu{}_{,i} = 0$, $g_{\mu\nu}n_\alpha{}^\mu n_\beta{}^\nu = \delta_{\alpha\beta}$ have to be added, and $\delta n_{\alpha\mu}$ is another independent variation. Finally, the third variation $\delta g_{\mu\nu}$ depends on δx^μ by (3.3). Therefore, $\delta g_{\mu\nu}$ are independent from $\delta n_{\alpha\mu}$, δh_{ij}, but the various $\delta g_{\mu\nu}$ components are not all independent from each other, so in the end they have to be expressed in terms of δx^μ which are independent. If λ^{ij}, $\lambda^{\alpha i}$, $\lambda^{\alpha\beta}$ are the Lagrange multipliers corresponding to the above constraints, the constraint action added to S is

$$\begin{aligned}S_c = \int_\Sigma d^4\chi\sqrt{|h|}[&\lambda^{ij}(h_{ij} - g_{\mu\nu}x^\mu{}_{,i}x^\nu{}_{,j}) + \lambda^{\alpha i}n_{\alpha\mu}x^\mu{}_{,i} \\ &+ \lambda^{\alpha\beta}(g_{\mu\nu}n_\alpha{}^\mu n_\beta{}^\nu - \delta_{\alpha\beta})].\end{aligned} \tag{3.4}$$

Actually, since in the action (3.1) the normals $n_\alpha{}^\mu$ are not explicitly present, the inclusion of the Lagrange multipliers $\lambda^{\alpha i}, \lambda^{\alpha\beta}$ is not very significant and they will end up vanishing. However, we keep them in order to preserve a uniform treatment with the codimension-1 case where the Gibbons-Hawking term contains the normal vector explicitly. Additionally, such normal vectors $n_\alpha{}^\mu$ will be exploited to settle some geometric identities for the extrinsic curvature. The preservation of the constraint concerning the induced metric, which formally arises by varying with respect to the Lagrange multiplier λ^{ij}, is always valid since it defines the induced metric, but this constraint will not need to be used explicitly during the following derivation.

Variation of $S + S_c$ with respect to $n_{\alpha\mu}, h_{ij}, g_{\mu\nu}$ at the brane gives

$$\delta(S+S_c)|_{\rm brane} = \int_\Sigma d^4\chi\sqrt{|h|}(\lambda^{\alpha i}x^\mu{}_{,i} + 2\lambda^{\alpha\beta}n_\beta{}^\mu)\delta n_{\alpha\mu} + \int_\Sigma d^4\chi\sqrt{|h|}\left[\lambda^{ij} + \frac{1}{2}(T^{ij} - \lambda h^{ij}) - \frac{r_c^2}{2\kappa_6^2}G^{ij}\right]\delta h_{ij}$$
$$- \int_\Sigma d^4\chi\sqrt{|h|}(\lambda^{ij}x^\mu{}_{,i}x^\nu{}_{,j} + \lambda^{\alpha\beta}n_\alpha{}^\mu n_\beta{}^\nu)\delta g_{\mu\nu} - \frac{1}{2\kappa_6^2}\int_M d^6x\sqrt{|g|}\{\mathcal{G}^{\mu\nu} + \alpha\mathcal{J}^{\mu\nu} - \kappa_6^2\mathcal{T}^{\mu\nu} + \Lambda_6 g^{\mu\nu}\}\delta g_{\mu\nu}|_{\rm brane}$$
$$+ \frac{1}{\kappa_6^2}\int_M d^6x\sqrt{|g|}\{g^{\mu[\kappa}g^{\lambda]\nu} + 2\alpha(\mathcal{R}^{\mu\nu\kappa\lambda} + 2\mathcal{R}^{\nu[\kappa}g^{\lambda]\mu} - 2\mathcal{R}^{\mu[\kappa}g^{\lambda]\nu} + \mathcal{R}g^{\mu[\kappa}g^{\lambda]\nu})\}(\delta g_{\nu\kappa})_{;\lambda\mu}|_{\rm brane}, \quad (3.5)$$

where

$$\mathcal{J}^{\mu\nu} = 2\mathcal{R}\mathcal{R}^{\mu\nu} - 4\mathcal{R}^{\mu\kappa}\mathcal{R}^\nu{}_\kappa - 4\mathcal{R}^{\mu\kappa\nu\lambda}\mathcal{R}_{\kappa\lambda} + 2\mathcal{R}^{\mu\kappa\lambda\sigma}\mathcal{R}^\nu{}_{\kappa\lambda\sigma} - \frac{1}{2}(\mathcal{R}^2 - 4\mathcal{R}_{\kappa\lambda}\mathcal{R}^{\kappa\lambda} + \mathcal{R}_{\kappa\lambda\rho\sigma}\mathcal{R}^{\kappa\lambda\rho\sigma})g^{\mu\nu}. \quad (3.6)$$

When $r_c \neq 0$, one should add in (3.1) the integral of the extrinsic curvature k of $\partial\Sigma$ (if $\partial\Sigma$ is not empty) to cancel some terms from the variation δR; this, in general, does not affect the dynamics of Σ [24]. Note that in the $\delta g_{\mu\nu}$ variation of $g_{\mu\nu}n_\alpha{}^\mu n_\beta{}^\nu$, it is $n_{\alpha\mu}$ which is kept fixed and not $n_\alpha{}^\mu$.

The six-dimensional terms in (3.5) have to be integrated out around the brane and finally give a four-dimensional integral which contributes to the matching conditions, as will be explained in the next section. Note that the quantity in curly brackets appearing in the second line of (3.5), though formally identical to that of Eq. (3.2), contains additional information, so this curly bracket does not vanish. Equation (3.2) refers to the bulk and also to the limit as these equations approach the brane, while the corresponding curly bracket in (3.5) refers exactly to the brane, and therefore, it contains extra distributional terms which are not present in (3.2).

Let us consider for simplicity that there is axial symmetry in the bulk, so that the bulk metric ansatz can be written in the brane Gaussian-normal coordinates as

$$ds_6^2 = dr^2 + L^2(\chi,r)d\theta^2 + g_{ij}(\chi,r)d\chi^i d\chi^j. \quad (3.7)$$

The braneworld metric $h_{ij}(\chi) = g_{ij}(\chi,0)$ is assumed to be regular everywhere with the possible exception of isolated singular points. The angle θ has the standard periodicity 2π. Since θ is an angle, close to the brane $r\approx 0$, it has to be $L\propto r$ and the measure of the six-dimensional integration is $\sqrt{|g|}\propto r$. Therefore, only terms proportional to $\frac{\delta(r)}{r}$ inside the two curly brackets of (3.5) contribute to the four-dimensional equations of motion. Let $\mathcal{K}_{ij}(\chi,r) = \frac{1}{2}g'_{ij}(\chi,r)$ (a prime denotes $\partial/\partial r$) be the extrinsic curvature tensor defined everywhere in the bulk. Since the various tensors $\mathcal{G}^{\mu\nu}$, $\mathcal{R}^{\mu\nu}$, …, shown in Appendix A, contain L'', $\mathcal{K}'_{ij}$, the quantities L' or/and $\mathcal{K}_{ij}$ have to be discontinuous at $r = 0$.

Therefore, there are two sources of discontinuity:

(a) cone discontinuity, where the transverse space to the defect is assumed to have the standard conical singularity structure with $L(\chi,r) = \beta(\chi)r + \mathcal{O}(r^2)$ for $r\approx 0$. The conical deficit is $2\pi(1-\beta) > 0$, so it is typically defined $L'(\chi,0) = 1$. The discontinuity of L' arises due to the values $L'(\chi,0^+) = \beta(\chi)$ and $L'(\chi,0) = 1$.

(b) extrinsic curvature discontinuity, where there is a jump in the extrinsic tangential sector. If the extrinsic curvature vanishes on the brane, the corresponding jump is $\mathcal{K}_{ij}(\chi,0) = 0 \neq \mathcal{K}_{ij}(\chi,0^+)$.

Combining these two sources of discontinuity we can have four cases which will give four kinds of matching conditions:

(i) pure cone or topological matching conditions, discussed in [17], which have a geometric origin based on the distributional version of the Chern-Gauss-Bonnet theorem [25]. Here, there is a conical singularity, but the extrinsic curvature has no jump $\mathcal{K}_{ij}(\chi,0) = \mathcal{K}_{ij}(\chi,0^+)$.

(ii) pure extrinsic curvature discontinuity, where there is no cone, so the deficit angle is $\beta = 1$. Although a conical singularity is usually attached to a codimension-2 defect, it is still consistent to appear a discontinuity only in the extrinsic curvature, as in codimension-1 defects.

(iii) cone plus extrinsic curvature discontinuity, introduced in [12], which assumes not only a conical deficit based on the normal geometry but also a jump in the extrinsic tangential sector.

(iv) "smooth" matching conditions, where there is neither conical singularity nor extrinsic curvature discontinuity. Although this full smoothness may look peculiar, it will be seen later that they are consistent backreacted matching conditions with nontrivial solutions.

Introducing the index $\eta = 0, 1$ so that $\mathcal{K}_{ij}(\chi, 0) = \eta K_{ij}(\chi)$, $K_{ij}(\chi) \equiv \mathcal{K}_{ij}(\chi, 0^+)$, the previous matching conditions are described as (i) $\beta \neq 1$, $\eta = 1$, (ii) $\beta = 1$, $\eta = 0$, (iii) $\beta \neq 1$, $\eta = 0$, (iv) $\beta = 1$, $\eta = 1$. More generally, we could consider that the two extrinsic curvatures $\mathcal{K}_{ij}(\chi, 0), \mathcal{K}_{ij}(\chi, 0^+)$ are proportional to each other, i.e. $\mathcal{K}_{ij}(\chi, 0) = \eta \mathcal{K}_{ij}(\chi, 0^+)$ with η a continuous parameter, but in the present paper we are going to restrict ourselves to the two characteristic cases $\eta = 0, 1$. Actually, the case $\eta = 1$ of continuous extrinsic curvature is especially interesting because it sets in a natural way the minimal demand for consistency, either of a pure cone singularity or of a smooth transverse section. Any value of the parameter η, other than $\eta = 1$, carries into the problem the arbitrariness of the matrix $\mathcal{K}_{ij}(\chi, 0)$, so the matching conditions (3.28) and (3.29) do not give the equation of motion of the defect until η is specified. Only for $\eta = 1$ are the equations of motion uniquely defined without any other information. Imposing a value of η, e.g. $\eta = 0$, is similar to the codimension-1 case, where the Israel matching conditions for a general non-Z_2 bulk do not give the equation of motion of the defect, but just the discontinuity of the extrinsic curvature, and only after the additional information of Z_2 symmetry is imposed do the matching conditions define the equation of motion. Analogously to the case $\eta = 1$, for a codimension-1 brane the case of continuous extrinsic curvature (which is consistent in the alternative matching conditions) gives without any extra assumption the equation of motion. To make things worse, if we legitimately imagine that $\mathcal{K}_{ij}(\chi, 0)$, $\mathcal{K}_{ij}(\chi, 0^+)$ are not proportional to each other, but they are unrelated matrices, then the matching conditions (3.28) and (3.29) will carry $\mathcal{K}_{ij}(\chi, 0)$ and the situation will be much more messy.

For $r > 0$ the expansions of $L(\chi, r), \mathcal{K}_{ij}(\chi, r)$ in powers of r are

$$L(\chi, r) = \beta(\chi) r + \frac{1}{2}\beta_2(\chi) r^2 + \mathcal{O}(r^3), \qquad (3.8)$$

$$\mathcal{K}_{ij}(\chi, r) = K_{ij}(\chi) + C_{ij}(\chi) r + \mathcal{O}(r^2). \qquad (3.9)$$

Together with the values $L'(\chi, 0), \mathcal{K}_{ij}(\chi, 0)$, the functions $L'(\chi, r), \mathcal{K}_{ij}(\chi, r)$ are also defined for $r = 0$ and are in general discontinuous. Therefore, the functions $L''(\chi, r)$, $\mathcal{K}'_{ij}(\chi, r)$ obtain distributional parts (beyond the regular ones) given by

$$\mathrm{distr} L''(\chi, r) = -(1 - \beta(\chi)) \delta(r), \qquad (3.10)$$

$$\mathrm{distr} \mathcal{K}'_{ij}(\chi, r) = (1 - \eta) K_{ij}(\chi) \delta(r). \qquad (3.11)$$

For convenience we include the expansion of $g_{ij}(\chi, r)$ which is defined for $r \geq 0$:

$$g_{ij}(\chi, r) = h_{ij}(\chi) + 2K_{ij}(\chi) r + C_{ij}(\chi) r^2 + \mathcal{O}(r^3). \quad (3.12)$$

A. Matching conditions

As usually done when dealing with distributional sources, the matching conditions are derived by integrating around the singular space. Here, the six-dimensional terms in (3.5) are integrated over the (r, θ) transverse disk of radius ϵ in the limit $\epsilon \to 0$. Because of the axial symmetry the angular dependence is factorized. In Appendix A the components of the tensors $\mathcal{R}_{\mu\nu\kappa\lambda}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{G}_{\mu\nu}$ in the Gaussian-normal coordinate system (3.7) are given. Some of these components contain L'', $\mathcal{K}'_{ij}$, which have regular as well as distributional pieces (3.10) and (3.11). Plaguing the expressions of $\mathcal{R}_{\mu\nu\kappa\lambda}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{G}_{\mu\nu}$ in the curly brackets of the six-dimensional integrals of (3.5) and expanding according to (3.8), (3.9), and (3.12) we find four types of terms. First, distributional terms of the form $\frac{\delta(r)}{r}$, arising from the factors $\frac{L''}{L}$ and $\frac{L'}{L}\mathcal{K}'_{ij}$. These terms multiplied by r, coming from the measure of integration, are the surviving terms which contribute to the matching conditions. Second, distributional terms of the form $\delta(r)$, which when multiplied by r vanish. Third, regular terms, i.e. terms with finite values for $r = 0$, either continuous or discontinuous, which when multiplied with the measure of integration give a continuous function vanishing on the brane; therefore their integration obviously vanishes in the limit $\epsilon \to 0$. Finally, singular terms, i.e. finite χ-dependent terms multiplied by $1/r$, which when multiplied by r and integrated obviously vanish. It can be seen that the last line of (3.5), which contains terms multiplied by $(\delta g_{\nu\kappa})_{;\lambda\mu}$, does not contribute to the matching conditions. From the previous terms which are multiplied by $\delta g_{\mu\nu}$, only the parallel to the brane components, i.e. terms multiplied by δg_{ij}, contain distributional terms $\frac{\delta(r)}{r}$ and therefore contribute to the matching conditions. The variational fields $\delta g_{\mu\nu}$ are considered, as usually, smooth functions.

Of course, the correct variational fields have to be dimensionless. Otherwise, since θ is the angle, L has dimension of length and $\delta g_{\theta\theta}$ has dimension of length square. Since $\mathcal{G}^\theta_\theta + \alpha \mathcal{J}^\theta_\theta$ has only regular terms, $\mathcal{G}^{\theta\theta} + \alpha \mathcal{J}^{\theta\theta}$ will have singular terms $1/r^2$. But then, multiplying by r and integrating would give divergence, which is not the case. The correct dimensionless variational field is $g^{\theta\theta}\delta g_{\theta\theta}$ and the corresponding multiplicative term is $g_{\theta\theta}(\mathcal{G}^{\theta\theta} + \alpha \mathcal{J}^{\theta\theta}) = \mathcal{G}^\theta_\theta + \alpha \mathcal{J}^\theta_\theta$ which possesses no singular terms at all. Somehow similarly, if the coordinates χ^i have length dimensions, then g_{ij} and δg_{ij} are dimensionless.

Note also that although the action (3.1) contains distributional terms, S is finite.

The distributional term of $\mathcal{G}^{ij}$ is easily found to be

$$\mathrm{distr}\mathcal{G}^{ij} = -\frac{1-\beta}{\beta} h^{ij} \frac{\delta(r)}{r}. \tag{3.13}$$

The distributional terms of $\mathcal{J}^{ij}$ are more difficult, but they can be combined as $-\frac{4}{L}(L'\mathcal{W}^{ij})' - 4\frac{L''}{L}G^{ij}$, where

$$\mathcal{W}^{ij} = \mathcal{K}^{ik}\mathcal{K}^j_k - \mathcal{K}\mathcal{K}^{ij} - \frac{1}{2}(\mathcal{K}_{k\ell}\mathcal{K}^{k\ell} - \mathcal{K}^2)g^{ij} \tag{3.14}$$

and $\mathcal{K} = \mathcal{K}^i_i$. Then, it is found that

$$\mathrm{distr}\mathcal{J}^{ij} = 4\left(\frac{\eta-\beta}{\beta}W^{ij} + \frac{1-\beta}{\beta}G^{ij}\right)\frac{\delta(r)}{r}, \tag{3.15}$$

where

$$\begin{aligned} W^{ij} &= K^{ik}K^j_k - KK^{ij} - \frac{1}{2}(K_{k\ell}K^{k\ell} - K^2)h^{ij} \\ &= \mathcal{W}^{ij}(r = 0^+). \end{aligned} \tag{3.16}$$

From (3.13) and (3.15), it is seen that the matching conditions (i) and (iii) have distributional terms coming from both $\mathcal{G}^{ij}$ and $\mathcal{J}^{ij}$. The distributional source of matching condition (ii) is only $\mathcal{J}^{ij}$. Finally, matching condition (iv) is not related to distributional terms.

The result for the total variation (3.5) at the brane position is

$$\begin{aligned} \delta(S+S_c)|_{\mathrm{brane}} = &\int_\Sigma d^4\chi\sqrt{|h|}(\lambda^{\alpha i}x^\mu{}_{,i} + 2\lambda^{\alpha\beta}n_\beta{}^\mu)\delta n_{\alpha\mu} \\ &+ \int_\Sigma d^4\chi\sqrt{|h|}\left[\lambda^{ij} + \frac{1}{2}T^{ij} + \frac{2\pi(1-\beta) - \kappa_6^2\lambda}{2\kappa_6^2}h^{ij} - \frac{4\pi\alpha(1-\beta)}{\kappa_6^2}\left(1 + \frac{r_c^2}{8\pi\alpha(1-\beta)}\right)G^{ij} - \frac{4\pi\alpha(\eta-\beta)}{\kappa_6^2}W^{ij}\right]\delta h_{ij} \\ &- \int_\Sigma d^4\chi\sqrt{|h|}(\lambda^{\alpha\beta}n_\alpha{}^\mu n_\beta{}^\nu + \lambda^{ij}x^\mu{}_{,i}x^\nu{}_{,j})\delta g_{\mu\nu}. \end{aligned} \tag{3.17}$$

At this point, had we considered $\delta n_{\alpha\mu}, \delta h_{ij}, \delta g_{\mu\nu}$ independent, three equations algebraic in the Lagrange multipliers would arise, and therefore all Lagrange multipliers would have been zero, leading to the standard matching condition [13] [which consists of vanishing the quantity inside the bracket of Eq. (3.28)]. Instead, as explained above, $\delta n_{\alpha\mu}, \delta h_{ij}$ are independent variations, but $\delta g_{\mu\nu}$ depends on δx^μ which are also independent. So, $\delta(S+S_c)|_{\mathrm{brane}} = 0$ gives

$$\lambda^{\alpha i}x^\mu{}_{,i} + 2\lambda^{\alpha\beta}n_\beta{}^\mu = 0, \tag{3.18}$$

$$\lambda^{ij} = \frac{4\pi\alpha(\eta-\beta)}{\kappa_6^2}W^{ij} + \frac{4\pi\alpha(1-\beta)}{\kappa_6^2}\left(1 + \frac{r_c^2}{8\pi\alpha(1-\beta)}\right)G^{ij} + \frac{\kappa_6^2\lambda - 2\pi(1-\beta)}{2\kappa_6^2}h^{ij} - \frac{1}{2}T^{ij}, \tag{3.19}$$

$$\int_\Sigma d^4\chi\sqrt{|h|}(\lambda^{\alpha\beta}n_\alpha{}^\mu n_\beta{}^\nu + \lambda^{ij}x^\mu{}_{,i}x^\nu{}_{,j})\delta g_{\mu\nu} = 0, \tag{3.20}$$

where $\delta g_{\mu\nu}$ obeys (3.3). Since the vectors $x^\mu{}_{,i}$, $n_\alpha{}^\mu$ are independent, Eq. (3.18) implies $\lambda^{\alpha i} = \lambda^{\alpha\beta} = 0$. Then, Eq. (3.20) with λ^{ij} given by (3.19) takes the form

$$\int_\Sigma d^4\chi\sqrt{|h|}\lambda^{ij}(g_{\mu\nu,\lambda}x^\mu{}_{,i}x^\nu{}_{,j}\delta x^\lambda + 2g_{\mu\nu}x^\mu{}_{,i}x^\lambda{}_{,j}\delta x^\nu{}_{,\lambda}) = 0. \tag{3.21}$$

After an integration of (3.21) by parts and imposing $\delta x^\mu|_{\partial\Sigma} = 0$ it becomes

$$\int_\Sigma d^4\chi\sqrt{|h|}g_{\mu\sigma}[\lambda^{ij}{}_{|j}x^\mu{}_{,i} + \lambda^{ij}(x^\mu{}_{;ij} + \Gamma^\mu{}_{\nu\lambda}x^\nu{}_{,i}x^\lambda{}_{,j})]\delta x^\sigma = 0. \tag{3.22}$$

The covariant differentiation | corresponds to h_{ij} and $\Gamma^\mu{}_{\nu\lambda}$ are the Christoffel symbols of $g_{\mu\nu}$. Because of the arbitrariness of δx^μ and since the extrinsic curvatures of the brane $K^\alpha{}_{ij} = n^\alpha{}_{i;j}$ satisfy $-K^\alpha{}_{ij}n_\alpha{}^\mu = x^\mu{}_{;ij} + \Gamma^\mu{}_{\nu\lambda}x^\nu{}_{,i}x^\lambda{}_{,j}$, Eq. (3.22) is equivalent to

$$\lambda^{ij}{}_{|j}x^\mu{}_{,i} - \lambda^{ij}K^\alpha{}_{ij}n_\alpha{}^\mu = 0. \tag{3.23}$$

Therefore, two matching conditions arise:

$$\lambda^{ij}K^\alpha{}_{ij} = 0, \tag{3.24}$$

$$\lambda^{ij}{}_{|j} = 0. \tag{3.25}$$

Using (3.19), these matching conditions finally take the form

$$\begin{aligned} &\left[W^{ij} + \frac{1-\beta}{\eta-\beta}\left(1 + \frac{r_c^2}{8\pi\alpha(1-\beta)}\right)G^{ij}\right. \\ &\left. + \frac{\kappa_6^2\lambda - 2\pi(1-\beta)}{8\pi\alpha(\eta-\beta)}h^{ij} - \frac{\kappa_6^2}{8\pi\alpha(\eta-\beta)}T^{ij}\right]K_{ij} = 0, \end{aligned} \tag{3.26}$$

$$T^{ij}{}_{|j} = \frac{2\pi}{\kappa_6^2}\beta_{,j}(h^{ij} - 4\alpha G^{ij}) + \frac{8\pi\alpha}{\kappa_6^2}[(\eta-\beta)W^{ij}]_{|j}. \tag{3.27}$$

Since the extrinsic curvature $K_{\alpha ij} = g(\nabla_i n_\alpha, \partial_j) = n_{\alpha i;j}$ (∇ refers, as also ;, to g) in the coordinates (3.7) has $K_{rij} = g'_{ij}/2$, $K_{\theta ij} = 0$, *the matching conditions of codimension-2 Einstein-Gauss-Bonnet gravity* can be rewritten, recovering the manifest normal frame indices

$$\left\{K^{\beta\ell}{}_{\ell}K_{\beta}{}^{ij} - K^{\beta\ell i}K_{\beta}{}^{j}{}_{\ell} - \frac{1}{2}(K^{\beta k}{}_{k}K_{\beta}{}^{\ell}{}_{\ell} - K^{\beta k\ell}K_{\beta k\ell})h^{ij} - \frac{1-\beta}{\eta-\beta}\left(1 + \frac{r_c^2}{8\pi\alpha(1-\beta)}\right)G^{ij} - \frac{\kappa_6^2\lambda - 2\pi(1-\beta)}{8\pi\alpha(\eta-\beta)}h^{ij} + \frac{\kappa_6^2}{8\pi\alpha(\eta-\beta)}T^{ij}\right\} \times K^{\alpha}{}_{ij} = 0, \tag{3.28}$$

$$T^{ij}{}_{|j} = \frac{2\pi}{\kappa_6^2}\beta_{,j}(h^{ij} - 4\alpha G^{ij}) - \frac{8\pi\alpha}{\kappa_6^2}\left\{(\eta-\beta)\left[K^{\alpha\ell}{}_{\ell}K_{\alpha}{}^{ij} - K^{\alpha\ell i}K_{\alpha}{}^{j}{}_{\ell} - \frac{1}{2}(K^{\alpha k}{}_{k}K_{\alpha}{}^{\ell}{}_{\ell} - K^{\alpha k\ell}K_{\alpha k\ell})h^{ij}\right]\right\}_{|j}. \tag{3.29}$$

Indices $\alpha, \beta, \ldots$ are lowered or raised with the matrix $g_{\alpha\beta} = g(n_\alpha, n_\beta)$ and its inverse $g^{\alpha\beta}$. With respect to local rotations $n_\alpha \to O_\alpha{}^\beta(x^\mu)n_\beta$, $K_{\alpha ij}$ transforms as a vector $K_{\alpha ij} \to O_\alpha{}^\beta K_{\beta ij}$; thus Eq. (3.29) is invariant under changes of the normal frame, while (3.28) transforms as a vector.

Equation (3.28) has been brought in a more compact form by dividing by $1-\beta$, $\eta-\beta$. Equations (3.28) and (3.29) are obviously defined for matching conditions (i) and (iii); they are also defined for matching condition (ii) and they are still pretty complicated; finally, for matching condition (iv) they take the simpler form $[\kappa_6^2(T^{ij} - \lambda h^{ij}) - r_c^2 G^{ij}]K^{\alpha}{}_{ij} = 0$, $T^{ij}{}_{|j} = 0$. For a 3-brane, making a general counting, the number of the matching conditions (3.28) and (3.29) is $\delta + 4 = D$ which is 6 here, while the number of the standard matching conditions is 10. In any case the role of the matching conditions is only fulfilled by the inclusion of the bulk field equations.

Equation (3.28) is the algebraic in the extrinsic curvature matching condition. It is a cubic equation in the extrinsic curvature, contrary to the matching condition derived according to the standard method [12,13] which is quadratic in the extrinsic curvature. In the absence of the Gauss-Bonnet term, (3.28) reduces to the matching condition of codimension-2 Einstein gravity [20], which is linear in extrinsic curvature. Equation (3.28) is the generalization of the Nambu-Goto equation of motion when the self-gravitating brane interacts with bulk gravity. In the limiting case of no backreaction, a probe brane with tension λ moving in a fixed background arises. Indeed, in the probe limit, all the geometric quantities h_{ij}, $K_{\alpha ij}$, G_{ij}, β get their background values (most probably the background value of $\beta = 1$) when the bulk gravity couplings go to zero (i.e. $1/\kappa_6^2 \to 0$, $\alpha/\kappa_6^2 \to 0$) and the extra brane sources vanish (i.e. $T_{ij} \to 0$, $r_c^2/\kappa_6^2 \to 0$). Then, Eq. (3.28) becomes $h^{ij}K^{\alpha}{}_{ij} = 0$, which is the Nambu-Goto equation of motion. Inversely, whenever any extra term beyond $\lambda h^{ij}K^{\alpha}{}_{ij}$ (or all terms) appears in (3.28) and (3.29) and these equations are consistent with all the other bulk equations, then these matching conditions are meaningful backreacted matching conditions. In this spirit, matching conditions (iv) with neither conical singularity nor extrinsic curvature discontinuity form an unusual but interesting example. In this case, only the localized matter and four-dimensional gravity terms participate in the brane equations of motion, and although the higher-dimensional bulk terms do not have a direct imprint in these equations, there is still backreaction since the bulk equations have also to be satisfied at the brane position. These "smooth" matching conditions correspond to the Regge-Teitelboim equations of motion [21,22,26] with the crucial difference, however, that there, there are no higher-dimensional gravity terms in the action and the bulk is prefixed (Minkowski). Therefore, possible difficulties discussed in [27] are irrelevant here, since they emanate from the embeddability restrictions in the given nondynamical bulk space, while the matching conditions here dynamically propagate in a nontrivial bulk space. Smooth matching conditions are also meaningful in codimension-1 standard treatment [28], without of course the $K^{\alpha}{}_{ij}$ contraction (where there is no balance of distributional terms between the two sides of the distributional equation, but the right-hand side vanishes on its own), although there, they lose their significance since there is no Nambu-Goto probe limit so that these matching conditions signal a minimal departure from that limit.

Equation (3.29) is the second matching condition and expresses a nonconservation equation of the brane energy-momentum tensor, where the energy exchange between the brane and the bulk is due to the variability along the brane of both the deficit angle and the extrinsic geometry. In the next section, Eq. (3.29) will be written in a more convenient form, from where it will be seen that the possible nonconservation of energy is only due to the variability of β. According to the conventional treatment, a different nonconservation equation is also derived [13], not as a second matching condition, but as a combination of the algebraic matching condition with some bulk equations evaluated on the brane. In the absence of the Gauss-Bonnet term, (3.29) reduces to the nonconservation equation of codimension-2 Einstein gravity, where the energy exchange is due to a variable deficit angle. In the probe limit, Eq. (3.29) is identically satisfied.

B. Effective equations

Having finished with the brane equations of motion arising from the distributional parts of the various quantities, we pass to the bulk equations of motion. These bulk equations are also defined limitingly on the brane, and therefore, additional equations have to be satisfied at the brane position beyond the matching conditions. More precisely, when the nondistributional quantities of Appendix A are substituted in the bulk equations (3.2) (more precisely in the equations with one index up and one down) and expand according to (3.8), (3.9), and (3.12), there appear two kinds of terms: (a) regular terms $\mathcal{O}(1)$, i.e. terms with finite values for $r=0$, and (b) singular terms $\mathcal{O}(1/r)$, i.e. finite χ-dependent terms multiplied by $1/r$. Since the singular terms cannot be canceled by any regular bulk energy-momentum tensor $\mathcal{T}_{\mu\nu}$, their χ-dependent coefficients have to vanish providing some new equations on the brane. Obviously, the regular parts which remain will vanish independently, defining additional equations on the brane. Note that $\mathcal{T}_{\mu\nu}$ cannot blow up close to the distributional singularity; otherwise the singularity would not be distributional.

Regarding the ri components of the bulk equations, their $\mathcal{O}(1/r)$ leading terms come from terms multiplied by L'/L, $L'_{|i}/L$ and yield the equation

$$\frac{\beta_{,j}}{\beta}\left(W^{ij}+G^{ij}-\frac{1}{4\alpha}h^{ij}\right)+W^{ij}{}_{|j}=0. \tag{3.30}$$

Similarly, the $\mathcal{O}(1/r)$ terms of the rr bulk equation are obtained from the terms L'/L and we get the equation

$$\left(W^{ij}+G^{ij}-\frac{1}{4\alpha}h^{ij}\right)K_{ij}=0. \tag{3.31}$$

Finally, in the $\mathcal{O}(1/r)$ terms of the ij components of the bulk equations, not only L'/L, $L'_{|i}/L$ terms, but also terms of the form L''/L, contribute. The corresponding brane equations contain $g_{ij,2}(\chi)=g''_{ij}(\chi,0)$, $\beta_2(\chi)=L''(\chi,0)$ where, of course, the second derivatives refer to the regular pieces of the quantities. These equations are pretty complicated and we will give their explicit form in the case of cosmology.

Using (3.31), the matching condition (3.26) gets the following simpler form which is linear and homogeneous in the extrinsic curvature and it does not contain the deficit angle:

$$(\sigma_1 G^{ij}+\sigma_2 h^{ij}-T^{ij})K_{ij}=0, \tag{3.32}$$

where

$$\sigma_1=\frac{r_c^2}{\kappa_6^2}+\frac{8\pi\alpha(1-\eta)}{\kappa_6^2},\qquad \sigma_2=\lambda-\frac{2\pi(1-\eta)}{\kappa_6^2}. \tag{3.33}$$

Note that for matching conditions (iv), Eq. (3.31) cannot be combined with Eq. (3.26) to eliminate W^{ij} since there is no W^{ij} in (3.26) in this case; however, Eq. (3.32) is still valid since it coincides with (3.26) for $\eta=\beta=1$. Therefore, the proof of the consistency and the investigation of the effective equations will also cover case (iv). It will also be useful to define

$$\sigma=\sigma_2+\frac{\sigma_1}{4\alpha}=\lambda+\frac{r_c^2}{4\alpha\kappa_6^2}, \tag{3.34}$$

which is positive for positive brane tension λ. Note that $\sigma_1=0\Leftrightarrow\eta=1$, $r_c=0\Leftrightarrow\sigma_2=\sigma=\lambda$.

Using (3.30), the conservation equation (3.27) or (3.29) also gets a simpler form:

$$T^{ij}{}_{|j}=-\eta\frac{8\pi\alpha}{\kappa_6^2}\frac{\beta_{,j}}{\beta}\left(W^{ij}+G^{ij}-\frac{1}{4\alpha}h^{ij}\right). \tag{3.35}$$

For $\eta=0$ it is seen that the energy on the brane is strictly conserved, in analogy to the case $\eta=0$ of the standard approach [29]. However, for $\eta=1$ and a varying β, i.e. case (i), the brane radiates in the bulk and there is inevitable energy exchange between the brane and the bulk. Similar nonconservation of energy also occurs in the standard treatment for $\eta=1$; however, the exchange is different [13].

What remains are the regular equations on the brane. The regular parts of the ij equations contain $g_{ij,3}(\chi)=g'''_{ij}(\chi,0)$, $\beta_3(\chi)=L'''(\chi,0)$ and are insignificant for all the other equations. In Appendix B, the regular parts of the rr, ri bulk equations on the brane are derived. Obviously, there are manifestly regular terms inside the tensor components $\mathcal{E}^\mu_\nu=\mathcal{G}^\mu_\nu+\alpha\mathcal{J}^\mu_\nu$ which are obtained by setting formally $\frac{L'}{L}=\frac{L'_{|i}}{L}=\frac{L''}{L}=0$, $L=\beta$ in the unperturbed expressions of $\mathcal{E}$'s. However, there are additional hidden regular terms coming from the expansion of the terms containing $\frac{L'}{L}$, $\frac{L'_{|i}}{L}$, $\frac{L''}{L}$. Therefore, the $\mathcal{O}(1)$ part of the rr bulk equation, after use of (3.31), becomes

$$\begin{aligned}\mathcal{E}^r_r|_{\frac{L'}{L}=0,L=\beta}&+K'-4\alpha(3W^i_j+G^i_j)K^{j\prime}_i-4\alpha G^{i\prime}_j K^j_i\\&=\kappa_6^2\mathcal{T}^r_r-\Lambda_6,\end{aligned} \tag{3.36}$$

where $K^{j\prime}_i$ denotes $\mathcal{K}^{j\prime}_i(\chi,0)$. The $\mathcal{O}(1)$ part of the ri bulk equation because of its complication will be given only for cosmology. Finally, the $\theta\theta$ bulk equation contains only regular $\mathcal{O}(1)$ terms, the corresponding brane equation contains $g_{ij,2}$ and its form is

$$\mathcal{E}^{\theta}_{\theta} = \kappa_6^2 \mathcal{T}^{\theta}_{\theta} - \Lambda_6. \tag{3.37}$$

Its explicit form will be given in the case of cosmology.

To summarize, our system of equations consists of the simplified matching conditions (3.32) and (3.35), the $\mathcal{O}(1/r)$ equations (3.30) and (3.31), the $\mathcal{O}(1/r)$ ij equations, the $\mathcal{O}(1)$ equations (3.36) and (3.37) and the $\mathcal{O}(1)$ ri equation.

IV. COSMOLOGICAL EQUATIONS AND CONSISTENCY

In this section, we will study the cosmological equations for a codimension-2 brane and check their consistency. For this purpose we consider the following bulk cosmological metric:

$$ds_6^2 = dr^2 + L^2(t,r)d\theta^2 - n^2(t,r)dt^2 + a^2(t,r)\gamma_{\hat{i}\hat{j}}(\chi^{\hat{\ell}})d\chi^{\hat{i}}d\chi^{\hat{j}}, \tag{4.1}$$

where $\gamma_{\hat{i}\hat{j}}$ is a maximally symmetric three-dimensional metric characterized by its spatial curvature $k = -1, 0, 1$. The energy-momentum tensor on the brane (beyond that of the brane tension λ) is assumed to be the one of a perfect fluid with energy density ρ and pressure p.

It is very convenient to define the quantities

$$A = \frac{a'}{a}, \qquad N = \frac{n'}{n}, \tag{4.2}$$

$$X = H^2 + \frac{k}{a^2}, \qquad Y = \frac{\dot{H}}{n} + H^2, \qquad H = \frac{\dot{a}}{na}, \tag{4.3}$$

$$\mathcal{X} = X - A^2 + \frac{1}{12\alpha}, \qquad \mathcal{Y} = Y - AN + \frac{1}{12\alpha}, \tag{4.4}$$

where a dot denotes differentiation with respect to t. The cosmic scale factor, lapse function and Hubble parameter arise as the restrictions on the brane of the functions $a(t,r), n(t,r)$ and $H(t,r)$, respectively. Other quantities also have their corresponding values when restricted on the brane, and since all the following equations will refer to the brane position, we will use the same symbols for the restricted quantities without confusion.

For the metric (4.1), Eq. (3.32) becomes

$$N = fA, \tag{4.5}$$

where

$$f = 3\frac{p - \sigma_2 + \sigma_1(X + 2Y)}{\rho + \sigma_2 - 3\sigma_1 X}, \tag{4.6}$$

while Eq. (3.35) becomes

$$\dot{\rho} + 3nH(\rho + p) = -\eta\frac{24\pi\alpha}{\kappa_6^2}\frac{\dot{\beta}}{\beta}\mathcal{X}. \tag{4.7}$$

We now focus on the $\mathcal{O}(1/r)$ parts of the bulk equations. Equations (3.31) and (3.30), which correspond to the rr and rt bulk equations, get the form

$$\left(1 + \frac{N}{A}\right)\mathcal{X} + 2\mathcal{Y} = 0, \tag{4.8}$$

$$2A^2\left[\frac{\dot{A}}{nA} + H\left(1 - \frac{N}{A}\right)\right] = \frac{\dot{\beta}}{n\beta}\mathcal{X}. \tag{4.9}$$

The $\mathcal{O}(1/r)$ part of the tt bulk equation, which was not computed in Sec. III B, is now found to be

$$2\frac{a_2}{a} - \frac{\beta_2}{\beta A}\mathcal{X} = \mathcal{X} + \frac{1}{6\alpha}, \tag{4.10}$$

where $a_2(t) = a''(t,0)$, $n_2(t) = n''(t,0)$, and $\beta_2(t) = L''(t,0)$. Finally, the $\mathcal{O}(1/r)$ part of the $\hat{i}\hat{j}$ bulk equations is

$$\frac{n_2}{n} + \left(1 + \frac{N}{A}\right)\frac{a_2}{a} - (\mathcal{X} + 2\mathcal{Y})\frac{\beta_2}{2\beta A} = \mathcal{Y} + \frac{N}{2A}\mathcal{X} + \frac{1}{12\alpha}\left(2 + \frac{N}{A}\right) - \frac{2\dot{\beta}}{n\beta}\left[\frac{\dot{A}}{nA} + H\left(1 - \frac{N}{A}\right)\right], \tag{4.11}$$

which can be written in a simpler form after using Eqs. (4.5), (4.8), (4.9), and (4.10):

$$\frac{n_2}{n} + f\frac{a_2}{a} - \mathcal{Y}\frac{\beta_2}{\beta A} = \frac{1 + f}{12\alpha} - \mathcal{X} - \left(\frac{\dot{\beta}}{n\beta}\right)^2\frac{\mathcal{X}}{A^2}. \tag{4.12}$$

Up to now, the cosmological equations that have to be satisfied on the brane are (4.5), (4.7), (4.8), (4.9), (4.10), and (4.12). What remains are the regular equations on the brane which contain up to second transverse derivatives. The $\theta\theta$ regular equation (3.37) takes the form

$$(\mathcal{X} + 2\mathcal{Y})\frac{a_2}{a} + \mathcal{X}\frac{n_2}{n} = \mathcal{X}\mathcal{Y} + \frac{\mathcal{X} + \mathcal{Y}}{6\alpha} + 2A^2\left[\frac{\dot{A}}{nA} + H\left(1 - \frac{N}{A}\right)\right]^2 + \frac{1}{12\alpha}\left(\kappa_6^2\mathcal{T}^{\theta}_{\theta} - \Lambda_6 - \frac{5}{12\alpha}\right), \tag{4.13}$$

which, after use of Eqs. (4.5), (4.8), and (4.9), takes the simpler form

$$\frac{n_2}{n} - f\frac{a_2}{a} = \frac{1-f}{12\alpha} + \mathcal{Y} + \left(\frac{\dot\beta}{n\beta}\right)^2 \frac{\mathcal{X}}{2A^2} + \frac{1}{12\alpha\mathcal{X}}\left(\kappa_6^2\mathcal{T}^\theta_\theta - \Lambda_6 - \frac{5}{12\alpha}\right). \tag{4.14}$$

The rr regular equation (3.36) with the help of Appendix B, Eq. (B8), becomes

$$\frac{1}{n\beta}\left(\frac{\dot\beta}{n}\right)^{\cdot}\mathcal{X} + \frac{H\dot\beta}{n\beta}(\mathcal{X}+2\mathcal{Y}) + \mathcal{X}\mathcal{Y} + \frac{1}{6\alpha}(\mathcal{X}+\mathcal{Y}) + 2A^2\left[\left(1+2\frac{N}{A}\right)A' + N'\right] - [(\mathcal{X}+2\mathcal{Y})A' + \mathcal{X}N']$$
$$- [(A+N)X' + 2AY'] = \frac{1}{12\alpha}\left(\Lambda_6 + \frac{5}{12\alpha} - \kappa_6^2\mathcal{T}^r_r\right). \tag{4.15}$$

Finally, the rt regular equation takes the form of (B12)

$$\frac{\dot\beta_2}{n\beta} + \frac{H}{A}\frac{\dot\beta^2}{n^2\beta^2} + \left[\frac{\mathcal{X}'}{\mathcal{X}} + \frac{1}{2A}\left(\mathcal{X} - 2A' + \frac{1}{6\alpha}\right) - N - \frac{\beta_2}{\beta}\right]\frac{\dot\beta}{n\beta} - \frac{2A}{\mathcal{X}}\left[\frac{\dot A'}{n} + (H'+HN)(A-N) + H(A'-N')\right] = \frac{n\kappa_6^2}{12\alpha\mathcal{X}}\mathcal{T}^t_r. \tag{4.16}$$

The $\mathcal{O}(1)$ parts of the tt, $\hat\imath\hat\jmath$ bulk equations contain third derivatives with respect to r and form an algebraic system of two equations for the three quantities $a_3(t) = a'''(t,0)$, $n_3(t) = n'''(t,0)$, and $\beta_3(t) = L'''(t,0)$. Therefore, these two equations are decoupled from all the other equations and do not deserve further study.

In summary, all the cosmological equations on the brane which have to be satisfied simultaneously are (4.5), (4.7)–(4.9), (4.10), (4.12), (4.14), (4.15), and (4.16). In particular, Eqs. (4.10), (4.12), and (4.14) form an algebraic system for the unknown functions a_2, n_2, β_2, which after solved and substituted in (4.15) and (4.16), they make these equations to be satisfied identically when $\mathcal{T}^r_r = \mathcal{T}^\theta_\theta$, $\mathcal{T}^t_r = 0$ on the brane. This calculation is shown in Appendix C and proves the consistency of the whole system for all kinds of matching conditions. Therefore, *the essential equations on the brane are* (4.5), (4.7), (4.8), and (4.9), which is a differential system of four equations for five unknowns a, ρ, β, A, and N.

V. CODIMENSION-2 COSMOLOGICAL EVOLUTION AND SOLUTIONS

We summarize by writing again the coupled system of the essential equations which govern the cosmological evolution. Since the quantity N can be eliminated from Eq. (4.5), the system consists of the three essential equations (4.7)–(4.9) which are written as

$$\dot\rho + 3nH(\rho+p) = -\eta\frac{24\pi\alpha}{\kappa_6^2}\frac{\dot\beta}{\beta}\mathcal{X}, \tag{5.1}$$

$$(1+f)\mathcal{X} + 2\mathcal{Y} = 0, \tag{5.2}$$

$$\frac{\dot A}{nA} + H(1-f) = \frac{\dot\beta}{n\beta}\frac{\mathcal{X}}{2A^2}, \tag{5.3}$$

where

$$\mathcal{X} = X - A^2 + \frac{1}{12\alpha}, \qquad \mathcal{Y} = Y - fA^2 + \frac{1}{12\alpha}, \tag{5.4}$$

$$X = H^2 + \frac{k}{a^2}, \qquad Y = \frac{\dot X}{2nH} + X, \tag{5.5}$$

$$f = 3\frac{p - \sigma_2 + \sigma_1(X+2Y)}{\rho + \sigma_2 - 3\sigma_1 X}. \tag{5.6}$$

Although this is a system of three equations for four unknowns a, ρ, β, and A, it will be seen in a while that it is integrable. Note also that in the system (5.1)–(5.3), β is present only through the derivative factor $\dot\beta/\beta$ and there are no separate β terms. Therefore, whenever one focuses on the case $\beta = \text{const}$, there is no difference between $\beta = \text{const}$ and $\beta = 1$. The effective system of Eqs. (5.1)–(5.3) is blind on the constant value of the cone (of course, the bulk solution should know about this value). Therefore, for $\beta = \text{const}$, matching condition (i) coincides with (iv) and matching condition (iii) coincides with (ii) from the viewpoint of the effective equations. This does not mean that the cone of matching conditions (i) and (iii) disappears. The cone is still there, but its constant throughout the brane deficit angle does not affect the effective equations, so, effectively, it is like "opening" the constant deficit angle to a smooth plane.

From Eqs. (5.2) and (5.3) we take

$$\frac{\dot{\mathcal{X}}}{n\mathcal{X}}+\frac{\dot{\beta}}{n\beta}=-(3+f)H, \qquad 3+f=3\frac{\rho+p+\sigma_1\frac{\dot{X}}{nH}}{\rho+\sigma_2-3\sigma_1 X}, \tag{5.7}$$

and with the help of (5.1), it takes the form

$$\frac{\dot{\mathcal{X}}}{\mathcal{X}}+\frac{\dot{\beta}}{\beta}=\frac{(\rho+\sigma_2-3\sigma_1 X)^{\cdot}}{\rho+\sigma_2-3\sigma_1 X}+\eta\frac{24\pi\alpha}{\kappa_6^2}\frac{\dot{\beta}}{\beta}\frac{\mathcal{X}}{\rho+\sigma_2-3\sigma_1 X}, \tag{5.8}$$

or equivalently

$$\left(\frac{\mathcal{X}}{\rho+\sigma_2-3\sigma_1 X}\right)^{-1}\left(1-\eta\frac{24\pi\alpha}{\kappa_6^2}\frac{\mathcal{X}}{\rho+\sigma_2-3\sigma_1 X}\right)^{-1} \times\left(\frac{\mathcal{X}}{\rho+\sigma_2-3\sigma_1 X}\right)^{\cdot}+\frac{\dot{\beta}}{\beta}=0. \tag{5.9}$$

Equation (5.9) can be integrated giving the general solution

$$\mathcal{X}\left(\eta\frac{24\pi\alpha}{\kappa_6^2}+c\beta\right)=\rho+\sigma_2-3\sigma_1 X, \quad c: \text{integration constant}. \tag{5.10}$$

Equation (5.2) is written in terms of X, A as

$$(1+3f)A^2=\frac{\dot{X}}{nH}+(3+f)\left(X+\frac{1}{12\alpha}\right), \tag{5.11}$$

and $\mathcal{X}$ is found to be

$$(1+3f)\mathcal{X}=2(f-1)\left(X+\frac{1}{12\alpha}\right)-\frac{\dot{X}}{nH}. \tag{5.12}$$

Combining Eqs. (5.10) and (5.12) and replacing the quantity $3+f$ from (5.7), we get the Raychaudhuri equation of the theory containing, however, also β:

$$\blacklozenge\ \frac{\dot{X}}{nH}\left(1+6\sigma_1\frac{X+\frac{1}{12\alpha}}{3\sigma_1 X-\rho-\sigma_2}+\frac{9\sigma_1}{\eta\frac{24\pi\alpha}{\kappa_6^2}+c\beta}\right)+2\left(X+\frac{1}{12\alpha}\right)\left(4+3\frac{\rho+p}{3\sigma_1 X-\rho-\sigma_2}\right)+\frac{9(\rho+p)+8(3\sigma_1 X-\rho-\sigma_2)}{\eta\frac{24\pi\alpha}{\kappa_6^2}+c\beta}=0, \tag{5.13}$$

where $X=H^2+\frac{k}{a^2}$, c is integration constant, $\eta=0$, 1 and $\sigma_1=\frac{r_c^2}{\kappa_6^2}+\frac{8\pi\alpha(1-\eta)}{\kappa_6^2}$, $\sigma_2=\lambda-\frac{2\pi(1-\eta)}{\kappa_6^2}$, $\sigma=\sigma_2+\frac{\sigma_1}{4\alpha}=\lambda+\frac{r_c^2}{4\alpha\kappa_6^2}$. Equation (5.1) with the use of Eq. (5.10) takes the form

$$\blacklozenge\ \dot{\rho}+3nH(\rho+p) =\eta\frac{24\pi\alpha}{\kappa_6^2}\frac{\dot{\beta}}{\beta(\eta\frac{24\pi\alpha}{\kappa_6^2}+c\beta)}(3\sigma_1 X-\rho-\sigma_2). \tag{5.14}$$

Equations (5.13) and (5.14) form the general final two-dimensional system for a, ρ, however, still with the indeterminacy of the function $\beta(t)$. Note that one integration constant c enters the equations, and actually, in the form $c\beta$. Moreover, it is seen from Eqs. (5.13) and (5.14) that under the rescaling of the deficit angle $\beta(t)\to c\beta(t)$, the constant c disappears. However, since this rescaling induces a new angle variable with values in a range different than the standard, we leave c intact in the following. This system of equations is the full information available to us at the brane position and constitutes a nonclosed system. In other words, one needs extra information coming from the bulk geometry in order to fix one of the functions, e.g. β, and then to solve fully the system. The solution in the bulk is no longer unique as in the case of codimension-1 brane cosmology [21,30] and one has a family of bulk solutions parametrized by the angular deficit function β. Not all these bulk solutions will be acceptable since certain of them will inevitably carry singularities away from the brane. Note that matching conditions (ii) and (iv), having $\beta=1$, will form a closed system of equations on the brane without any undetermined function.

After X, ρ have been found, one can use Eqs. (5.10) and (4.5) to find A, N. Only in highly exceptional cases will the extrinsic curvature vanish identically. As it can be seen in Appendix A, the six-dimensional Ricci scalar (as well as other curvature invariants), beyond the distributional terms, contains also singular $1/r$ terms multiplied by the extrinsic curvature. This means that in general the bulk geometry has a genuine curvature singularity at $r=0$ apart from the distributional one. In fact, it is expected from purely geometrical considerations that higher codimension defects will develop curvature singularities in their zero width limit [31]. Moreover, for the standard Schwarzschild solution of a 0-brane (point mass), the metric is singular on the point and the curvature diverges on the defect.

In the special case of $\beta=$ const, there should be no β in Eqs. (5.13) and (5.14) since in this case, there is no β in the initial system (5.1)–(5.3). A seemingly remaining constant β in (5.13) and (5.14) is the result of the integration process. However, the integration process knows to put β in the form $c\beta$; thus a constant β is absorbed in the integration constant c. Therefore, whenever we speak about constant β in the following, we will replace $c\beta$ by c in (5.13) and (5.14).

Equation (5.13), although much complicated, can be brought in a more convenient form defining for $\sigma_1 \neq 0$ the variable ξ as

$$\xi = \frac{\rho+\sigma}{3\sigma_1 X - \rho - \sigma_2} \Leftrightarrow X = \frac{\rho}{3\sigma_1} + \frac{\sigma_2}{3\sigma_1} + \frac{\rho+\sigma}{3\sigma_1}\frac{1}{\xi}. \quad (5.15)$$

It is seen from the last expression that in principle H^2 has the standard Friedmann term linear in ρ, a cosmological constant term and a dark energy contribution attributed to ξ, but more can be said after ξ is found. So, Eq. (5.13), using (5.14), becomes for $p = w\rho$

$$\frac{\dot{\xi}}{nH(\xi+1)} - \eta\frac{24\pi\alpha}{\kappa_6^2}\frac{\dot{\beta}}{nH\beta(\eta\frac{24\pi\alpha}{\kappa_6^2}+c\beta)} = \left[\frac{8}{3} - 3(1+w)\left(1+\frac{\sigma}{\rho}\right)^{-1}\right] \times \frac{3\xi + 3 + 9\sigma_1(\eta\frac{24\pi\alpha}{\kappa_6^2}+c\beta)^{-1}}{2\xi+3+9\sigma_1(\eta\frac{24\pi\alpha}{\kappa_6^2}+c\beta)^{-1}}\frac{\xi}{\xi+1}. \quad (5.16)$$

In order to get some understanding of the behavior of the system, Eq. (5.16) will be integrated in the next section for a characteristic case of β.

In the exceptional case where the denominator $3\sigma_1 X - \rho - \sigma_2$ vanishes, some of the previous equations are not defined. Equation (4.5) gives $\sigma_1(X+2Y) + p - \sigma_2 = 0$, which leads to the conservation equation $\dot{\rho} + 3nH(\rho+p) = 0$ and from Eq. (4.7) we get $\beta = \text{const}$. From Eq. (4.8) we can find $\frac{N}{A}$ as a function of A, ρ and from (4.9) a differential equation for $\frac{dA}{da}$ gives A. The four-dimensional cosmology in this case is the standard FRW cosmology with cosmological constant.

If the deficit angle is time dependent instead of being exactly constant, a time-varying effective four-dimensional gravitational constant will be induced. To make an estimate, from the coefficient of the linear term in (5.15) we have $8\pi G_N = \kappa_4^2 = \frac{\kappa_6^2}{r_c^2 + 8\pi\alpha(1-\beta)}$, which can also be taken from the relative coefficients of the terms G^{ij}, T^{ij} in the matching condition (3.28). The variation of G_N is constrained during the early cosmology by the primordial abundances at the nucleosynthesis epoch approximately by $\frac{|\dot{G}_N|}{G_N H} \lesssim 0.2$ [32]. This constrains the variation of β as $|\frac{\dot{\beta}}{(1-\beta+r_c^2/8\pi\alpha)H}| \lesssim 0.2$ which is not a rather strong constraint. Further constraints come from the fact that the theory with varying β is similar to a scalar-tensor theory, and therefore, there will be strong constraints from solar system observations. Not knowing the full family of solutions in the bulk, we choose to consider in the next two sections V A and V B the case where β is constant. This case will give, at least seemingly, acceptable four-dimensional cosmologies and has the merit that the system of equations is closed and does not depend on undetermined functions. In Sec. V A we derive the general cosmology for $\sigma_1 \neq 0$, while in Sec. V B the general cosmology for the complementary case $\sigma_1 = 0$.

A. Cosmology with $\sigma_1 \neq 0$

As explained above, one characteristic and meaningful way to close the system and capture some features of its behavior is to assume a constant deficit angle $\beta(t) = \text{const}$. Then, Eq. (5.14) gets the standard conservation form

$$\dot{\rho} + 3nH(\rho+p) = 0, \quad (5.17)$$

and Eq. (5.16), using (5.17) to convert the derivative with respect to time to derivative with respect to ρ, becomes integrable:

$$\frac{2\xi+3+9\sigma_1(\eta\frac{24\pi\alpha}{\kappa_6^2}+c)^{-1}}{3\xi+3+9\sigma_1(\eta\frac{24\pi\alpha}{\kappa_6^2}+c)^{-1}}\frac{1}{\xi}d\xi = d\rho\left[\frac{1}{\rho+\sigma} - \frac{8}{9(1+w)}\frac{1}{\rho}\right], \quad (5.18)$$

with general solution

$$\frac{1}{\tilde{c}}\rho^{\frac{8}{3(1+w)}}(\rho+\sigma)^{-3}\xi^3 - 3\xi = 2\gamma, \qquad \tilde{c}:\text{integration constant},$$
$$\gamma = \frac{3}{2} + \frac{9\sigma_1}{2}\left(\eta\frac{24\pi\alpha}{\kappa_6^2}+c\right)^{-1}. \quad (5.19)$$

Equation (5.19) is a cubic for ξ and can be solved analytically giving the function $\xi(\rho)$ and therefore the Hubble evolution $H^2(\rho)$. Since the cubic has various branches, there are also branches for the cosmic evolution.

Branch I: $\tilde{c}(\rho+\sigma) > 0$ *and* $\tilde{c}(\rho+\sigma)^3 \leq \gamma^2\rho^{\frac{8}{3(1+w)}}$.— There is one real solution for ξ:

$$\xi = 2\,\text{sgn}(\gamma)\rho^{-\frac{4}{3(1+w)}}\sqrt{\tilde{c}(\rho+\sigma)^3} \times \cosh\left[\frac{1}{3}\text{arccosh}\left(\frac{|\gamma|\rho^{\frac{4}{3(1+w)}}}{\sqrt{\tilde{c}(\rho+\sigma)^3}}\right)\right]. \quad (5.20)$$

For positive brane tension $\lambda > 0$, it is $\sigma > 0$; therefore the first inequality above becomes $\tilde{c} > 0$ and the Hubble equation is

$$H^2 + \frac{k}{a^2} = \frac{\rho}{3\sigma_1} + \frac{\sigma_2}{3\sigma_1} + \text{sgn}(\gamma)\frac{c_*\rho^{\frac{4}{3(1+w)}}}{6\sigma_1\sqrt{\rho+\sigma}} \times \cosh^{-1}\left[\frac{1}{3}\text{arccosh}\left(\frac{|\gamma|c_*\rho^{\frac{4}{3(1+w)}}}{\sqrt{(\rho+\sigma)^3}}\right)\right], \quad (5.21)$$

where $c_* = \tilde{c}^{-1/2} > 0$ and $\cosh^{-1}x = \frac{1}{\cosh x}$. So, Eq. (5.21) contains two integration constants $c_* > 0, \gamma$. For negative brane tension some slight difference will occur in (5.21) according to (5.20).

As it is seen from the second inequality above, this solution does not accept the regime $\rho \to 0$, but already the linear ρ term of FRW is present. Note that the coefficient of this term is determined by the Gauss-Bonnet coupling α/κ_6^2 and the possible induced gravity coupling r_c^2/κ_6^2, and not by the higher-dimensional gravitational constant $1/\kappa_6^2$ of the Einstein term. The Einstein term contributes only to the effective cosmological constant and to the extra correction term of the Hubble equation. This remark is valid also for the other branches below.

The second inequality above also shows that the energy density ρ cannot become infinite (for $w > -1/9$), which means that an infinite-density singularity $a = 0$ is not encountered for the solution (5.21). This is true independent of the spatial curvature k, or the equation of state. More precisely, for $w = 1/3$, which is the realistic equation of state at the early Universe, the condition for the solution (5.21) to exist is $0 < \tilde{c} < \frac{4\gamma^2}{27\sigma}$ (or for c_* that $c_* > \frac{\sqrt{27\sigma}}{2|\gamma|}$). Note that this effect of avoidance of the infinity is not the same with that of codimension-1 cosmology of the standard approach [33], where the infinite density singularity is removed by the combined effect of the Gauss-Bonnet and the induced gravity term, while neither of the two terms separately can do this. Here, the coupling r_c can be set to zero (along with $\eta = 0$) and still the finite behavior occurs. The maximum energy density ρ_M is found (for $w = 1/3$) by the equation $z^3 + (3 - \ell^2/3)z = \ell - 1 - 2\ell^3/27$, where $z = \frac{\rho_M}{\sigma} + \frac{\ell}{3}$, $\ell = 3 - \frac{\gamma^2}{\tilde{c}\sigma}$ (the previous condition for $\tilde{c}$ means $\ell < -15/4$). In order for ρ_M to be the maximum energy density and not the minimum, the appropriate solution of the cubic equation for z must satisfy $z > 2 + \ell/3$ for any ℓ. To summarize, for a radiation brane the solution (5.21) exists only if $\tilde{c}$ (or c_*) satisfies the previous condition, and then necessarily it has finite energy density ever.

Not only the energy density is finite at early times, but there is a range of the integration constant c (or γ) for which there is accelerated expansion near the minimum scale factor. This, in principle, serves as a geometric form of inflation alternative to the scalar field inflation. More precisely, this happens for $-1/18 < \gamma < 0$ (or in terms of c for $-3\sigma_1 < \eta \frac{24\pi\alpha}{\kappa_6^2} + c < -81\sigma_1/28$). Indeed, using the proper time on the brane, the acceleration parameter $\ddot{a}/a$ is given by the formula $\ddot{a}/a = X + \dot{X}/(2H)$. Using (5.15) to express $\dot{X}$ in terms of $\dot{\xi}$ and (5.16) to substitute $\dot{\xi}$ in terms of $\xi(\rho)$, we find

$$3\sigma_1 \frac{\ddot{a}}{a} = \left[\frac{1+9w}{12}\frac{3\xi + 2\gamma}{\xi(\xi+\gamma)} - \frac{1+3w}{2}\frac{\xi+1}{\xi}\right]\rho + \sigma_2 - \frac{3\xi+\gamma}{\xi(\xi+\gamma)}\frac{\sigma}{3}. \quad (5.22)$$

Equation (5.22) is valid for any w (for example it can be used for $w = 0$) and is true also for the other branches of solutions to be discussed below, not only the current one.

For the solution (5.21) and $w = 1/3$, the value of ξ at the initial scale factor is found from (5.20) to be $\xi_M = 2\gamma$ and the corresponding value of the acceleration is $3\frac{\sigma_1}{\sigma}\frac{\ddot{a}}{a}|_M = (\frac{\sigma_2}{\sigma} - \frac{7}{18\gamma}) + (1 + \frac{1}{18\gamma})(\frac{\ell}{3} - z)$; therefore, for the range of γ given above, $\frac{\ddot{a}}{a}|_M$ is positive and finite. Note that this result also holds independently of the spatial curvature k of the Universe.

Finally, the four-dimensional Ricci scalar on the brane is given by $R/6 = \dot{H} + 2H^2 + k/a^2 = X + \ddot{a}/a$; therefore, from the previous result for $\frac{\ddot{a}}{a}|_M$, we obtain that R is finite and there is no initial singularity.

To conclude, the solution (5.21) with $w = 1/3$, for a range of the integration constants c_*, γ avoids a cosmological singularity (both in density and curvature) and undergoes accelerated expansion near the minimum scale factor. There remains however one crucial caveat: there is no exit from acceleration, as it can be seen that at the minimum energy density there is also acceleration.

Branch II: $\tilde{c}(\rho+\sigma) > 0$ and $\tilde{c}(\rho+\sigma)^3 \geq \gamma^2\rho^{\frac{8}{3(1+w)}}$.— There are three real solutions for ξ:

$$\xi = 2\rho^{-\frac{4}{3(1+w)}}\sqrt{\tilde{c}(\rho+\sigma)^3} \times \cos\left[\frac{1}{3}\arccos\left(\frac{\gamma\rho^{\frac{4}{3(1+w)}}}{\sqrt{\tilde{c}(\rho+\sigma)^3}}\right) + \frac{2\pi m}{3}\right], \qquad m = 0, 1, 2. \quad (5.23)$$

For positive brane tension $\lambda > 0$, it is $\sigma > 0$; therefore the first inequality above becomes $\tilde{c} > 0$ and the Hubble equation is

$$H^2 + \frac{k}{a^2} = \frac{\rho}{3\sigma_1} + \frac{\sigma_2}{3\sigma_1} + \frac{c_*\rho^{\frac{4}{3(1+w)}}}{6\sigma_1\sqrt{\rho+\sigma}} \times \cos^{-1}\left[\frac{1}{3}\arccos\left(\frac{\gamma c_*\rho^{\frac{4}{3(1+w)}}}{\sqrt{(\rho+\sigma)^3}}\right) + \frac{2\pi m}{3}\right], \qquad m = 0, 1, 2, \quad (5.24)$$

where $c_* = \tilde{c}^{-1/2} > 0$ and $\cos^{-1} x = \frac{1}{\cos x}$. So, Eq. (5.24) contains two integration constants $c_* > 0, \gamma$ or alternatively, we could consider the two integration constants $c_* > 0$, $c^* = \gamma c_*$. For negative brane tension some slight difference will occur in (5.24) according to (5.23). The solution (5.24) accepts the regime $\rho \to 0$, where for $w = 0$ it is

$$H^2 + \frac{k}{a^2} \approx \frac{\rho}{3\sigma_1} + \frac{\sigma_2}{3\sigma_1} + \frac{c_*}{6\sigma_1\sqrt{\sigma}}\cos^{-1}\left[\frac{\pi(1+4m)}{6}\right]\rho^{\frac{4}{3}}. \quad (5.25)$$

Depending on the value of $\tilde{c}$ the solution can evolve from an initial big bang to $\rho \to 0$, or from a finite value of the energy density to $\rho \to 0$.

Branch III: $\tilde{c}(\rho+\sigma)<0$.—There is one real solution for ξ:

$$\xi=-2\rho^{-\frac{4}{3(1+w)}}\sqrt{-\tilde{c}(\rho+\sigma)^3}\times\sinh\left[\frac{1}{3}\operatorname{arcsinh}\left(\frac{\gamma\rho^{\frac{4}{3(1+w)}}}{\sqrt{-\tilde{c}(\rho+\sigma)^3}}\right)\right]. \quad (5.26)$$

For positive brane tension $\lambda>0$, it is $\sigma>0$; therefore the inequality above becomes $\tilde{c}<0$ and the Hubble equation is

$$H^2+\frac{k}{a^2}=\frac{\rho}{3\sigma_1}+\frac{\sigma_2}{3\sigma_1}-\frac{c_*\rho^{\frac{4}{3(1+w)}}}{6\sigma_1\sqrt{\rho+\sigma}}\times\sinh^{-1}\left[\frac{1}{3}\operatorname{arcsinh}\left(\frac{\gamma c_*\rho^{\frac{4}{3(1+w)}}}{\sqrt{(\rho+\sigma)^3}}\right)\right], \quad (5.27)$$

where $c_*=|\tilde{c}|^{-1/2}>0$ and $\sinh^{-1}x=\frac{1}{\sinh x}$. So, Eq. (5.27) contains two integration constants $c_*>0,\gamma$ or alternatively, we could consider the two integration constants $c_*>0$, $c^*=\gamma c_*$. For negative brane tension some slight difference will occur in (5.27) according to (5.26). The solution (5.27) accepts the regime $\rho\to 0$, where for $w=0$ it is

$$H^2+\frac{k}{a^2}\approx\frac{\rho}{3\sigma_1}\left(1-\frac{3}{2\gamma}\right)+\frac{1}{3\sigma_1}\left(\sigma_2-\frac{3\sigma}{2\gamma}\right)-\frac{2\gamma c_*^2}{27\sigma_1\sigma^2}\rho^{\frac{8}{3}}. \quad (5.28)$$

Although in principle γ has any sign, the linearized regime (5.28), in comparison to the standard FRW equation, gives the constraints $8\pi G_N=\kappa_4^2=\frac{1}{\sigma_1}(1-\frac{3}{2\gamma})$ and $\Lambda_{\rm eff}=\frac{1}{\sigma_1}(\sigma_2-\frac{3\sigma}{2\gamma})$. Combining these relations we get $\frac{3}{8\alpha\gamma}=\kappa_4^2[\lambda-2\pi(1-\eta)M_6^4]-\Lambda_{\rm eff}$, where $\kappa_6^2=M_6^{-4}$. Since $\kappa_4^2\sim 10^{-31}\ \mathrm{TeV}^{-2}$, $\Lambda_{\rm eff}\sim 10^{-90}\ \mathrm{TeV}^2$, $M_6\gtrsim\mathrm{TeV}$, for any reasonable λ the term $\Lambda_{\rm eff}$ is insignificant. Finally, γ can indeed be either positive or negative depending on the value of λ, and the correction term of (5.27) can also have either sign. Moreover, setting the effective cosmological constant $\Lambda_{\rm eff}$ to zero, in case we want the vacuum to be the Minkowski space, we get $\kappa_4^2=(4\alpha\lambda+r_c^2M_6^4)^{-1}$.

In general, the solutions found above need a more systematic analysis to study their cosmological behavior. Usually one adjusts the parameters of a solution such that in the absence of matter to recover the Minkowski background. However, this is not necessary since any curved spacetime is locally Minkowski and all local physics constraints are satisfied. For example, the Randall-Sundrum fine-tuning in five dimensions assures a Minkowski background by exactly vanishing the effective cosmological constant [30,34]. In this case, the acceleration today would possibly be attributed to some dark energy component. If, on the contrary, the effective cosmological constant of a model does not vanish, the background is de Sitter and it contributes to the acceleration today. For branches I and II above, the effective cosmological constant is $\frac{\sigma_2}{\sigma_1}$ and has no need to vanish (for $\eta=0$) or cannot vanish (for $\eta=1$). Its value, together with the correction term, could possibly define the dark energy today.

Let us finish with an intriguing comment. For a 3-brane in D dimensions, out of the scales $\kappa_D^2=M_D^{2-D}$, λ, r_c is constructed the following scale with dimensions of effective cosmological constant: $\Lambda_{\rm eff}=\frac{\kappa_D^2\lambda}{r_c^{D-4}}=\frac{\lambda}{M_D^{D-2}}\frac{1}{r_c^{D-4}}$. The induced gravity crossover scale r_c distinguishes the four-dimensional from the higher-dimensional regime. For a reasonable value of the brane tension $\lambda\sim M_D^4$ (e.g. $\lambda\sim\mathrm{TeV}^4$ a typical cutoff) which might also have theoretical explanation as it is connected to the fundamental mass scale, it arises $\Lambda_{\rm eff}\sim\frac{1}{(M_Dr_c)^{D-6}}\frac{1}{r_c^2}$. For $D=6$ it is $\Lambda_{\rm eff}\sim\frac{1}{r_c^2}$. If the induced gravity crossover scale is of the cosmic horizon size $r_c\sim H^{-1}$, then $\Lambda_{\rm eff}\sim H^2$; i.e. for today $\Lambda_{\rm eff}$ has the observed magnitude. Since the induced gravity term is generically induced by quantum corrections coming from the bulk gravity and its coupling with matter living on the brane, it would not be absurd to suppose that the accumulative contribution of all the matter inside our horizon gives a connection of r_c to H^{-1}. But if $\Lambda_{\rm eff}\sim H^2$, why only today does $\Lambda_{\rm eff}$ emerge and not in the past? To make another conjecture, only today a reasonable number of galaxies with high energetic interiors exist from where the quantum loops generate the induced gravity term. On the contrary, any other dimensionality D fails to support the observed value of $\Lambda_{\rm eff}$. For $D=5$ it is $\Lambda_{\rm eff}\sim\frac{M_D}{r_c}$ and in order to be $\Lambda_{\rm eff}\sim H_0^2$ it must be $r_c\sim\frac{M_D}{H_0}H_0^{-1}$, which for $M_D\sim\mathrm{TeV}$ gives $r_c\sim 10^{45}H_0^{-1}$ (too high to be realistic). For $D=7$ it is $\Lambda_{\rm eff}\sim\frac{1}{M_Dr_c^3}$ and in order to be $\Lambda_{\rm eff}\sim H_0^2$ it must be $r_c\sim(\frac{H_0}{M_D})^{\frac{1}{3}}H_0^{-1}$, which for $M_D\sim\mathrm{TeV}$ gives $r_c\sim 10^{-15}H_0^{-1}\sim$ A.U. (this may look marginally legitimate but it is probably already excluded since high precision measurements on solar system astronomy occur at distances 30 A.U. which is larger than A.U.; more importantly, a higher value of M_D, which is more reasonable, reduces r_c). For $D=8$, it is $\Lambda_{\rm eff}\sim\frac{1}{M_D^2r_c^4}$ and in order to be $\Lambda_{\rm eff}\sim H_0^2$ it must be $r_c\sim\sqrt{\frac{H_0}{M_D}}H_0^{-1}$, which for $M_D\sim\mathrm{TeV}$ gives $r_c\sim 10^{-27}H_0^{-1}\sim 10^{-12}$A.U. (obviously incompatible). Incompatibility worsens with yet lower r_c for higher D. In all the solutions (5.21), (5.24), (5.27), or even in (5.15), there appears the effective cosmological constant $\Lambda_{\rm eff}=\frac{\sigma_2}{\sigma_1}=\frac{\kappa_6^2\lambda-2\pi(1-\eta)}{r_c^2+8\pi\alpha(1-\eta)}$. Already the scale $\frac{\kappa_6^2\lambda}{r_c^2}$ starts becoming visible, but for $\eta=1$ it is exactly $\Lambda_{\rm eff}=\frac{\sigma_2}{\sigma_1}=\frac{\kappa_6^2\lambda}{r_c^2}$. Therefore, we have a consistent formalism which naturally embodies the effective cosmological constant scale $\frac{\kappa_6^2\lambda}{r_c^2}$

which can suppress the large value of vacuum energy to the small observed cosmological constant [for the standard treatment a relevant scale $\Lambda_{\rm eff}=\frac{\kappa_6^2\lambda-2\pi(1-\beta)}{r_c^2+8\pi\alpha(1-\beta)}$ also arises, as it is seen from (6.37) by setting $\rho=p=0$, $Y=X$]. However, even if all the above arguments have some physical relevance, there is a point that creates a serious phenomenological difficulty. This is the constraint from the four-dimensional Newton's constant G_N. From the solutions (5.21) and (5.25) it has to be $G_N\sim\sigma_1^{-1}=\frac{\kappa_6^2}{r_c^2}$, and then, r_c is constrained by M_6 and all the previous numerology collapses. However, we give an idea how this difficulty might be evaded. In the spirit of Eqs. (5.28) and (5.35), a possible dependence of G_N on some extra integration constant $G_N=G_N(\sigma_1,\mathtt{c})$ [even a whole time dependence of G_N due to $\beta(t)$ would create for today an analogous result] could liberate r_c from M_6 and charge the value of G_N to this integration constant $\mathtt{c}$. For example, for the solution (5.28) it is $8\pi G_N=\kappa_4^2=\frac{\kappa_6^2}{r_c^2}(1-\frac{3}{2\gamma})$, and therefore $\frac{1}{\kappa_4^2}\neq\frac{r_c^2}{\kappa_6^2}$, as opposed to what one usually imagines from the action (3.1). To say it in a different way, in the present theory, Newton's constant is not determined solely from the parameters (M_6,r_c) of the action, but in general depends also from the integration constants of the considered solution. Although in the concrete example of Eq. (5.28) the dependence of G_N on γ does not save the argument, further investigation of the theory and its solutions could shed more light on this issue.

B. Cosmology with $\sigma_1=0$

The case $\sigma_1=0$ corresponds to $\eta=1$, without the induced gravity term $r_c=0$. Therefore, it corresponds to matching conditions (i) and (iv) with $r_c=0$. Then, the conservation equation (5.14) for general $\beta(t)$ can be rewritten as

$$\left(\frac{\rho+\lambda}{\frac{24\pi\alpha}{\kappa_6^2\beta}+c}\right)^{\cdot}+3nH\frac{\rho+p}{\frac{24\pi\alpha}{\kappa_6^2\beta}+c}=0 \qquad (5.29)$$

and displays an interplay between the energy density and the deficit angle during the evolution. This equation looks like the standard energy conservation equation for redefined energy densities of matter and brane tension, i.e. $\rho\to\rho f(\beta)$ and $\rho_\lambda\to\rho_\lambda f(\beta)$, where $\rho_\lambda=\lambda$ and $f(\beta)=(\frac{24\pi\alpha}{\kappa_6^2\beta}+c)^{-1}$. Then, for $p=w\rho$ it is $p\to pf(\beta)$ and since $p_\lambda=-\lambda$ it is $p_\lambda\to p_\lambda f(\beta)$. It is the same to say that the total energy density $\tilde\rho=\rho+\lambda$ is redefined $\tilde\rho\to\tilde\rho f(\beta)$, and therefore, for the total pressure $\tilde p=p-\lambda$ it is $\tilde p\to\tilde p f(\beta)$. Furthermore, the Raychaudhuri equation (5.13) becomes

$$\frac{\dot X}{nH}+2\left(X+\frac{1}{12\alpha}\right)\frac{\rho-3p+4\lambda}{\rho+\lambda}+\frac{\rho+9p-8\lambda}{\frac{24\pi\alpha}{\kappa_6^2}+c\beta}=0. \qquad (5.30)$$

Equation (5.29) is written as

$$\frac{dy}{d\Omega}+3(1+w)y-3(1+w)\lambda\left(\frac{24\pi\alpha}{\kappa_6^2\beta}+c\right)^{-1}=0, \qquad (5.31)$$

where

$$y=\frac{\rho+\lambda}{\frac{24\pi\alpha}{\kappa_6^2\beta}+c},\qquad \Omega=\ln\frac{a}{a_0}, \qquad (5.32)$$

and a_0 is, for example, the today scale factor. Therefore, for $\beta=\beta(a)$, (5.31) is a linear differential equation for y and can be integrated giving $\rho(a)$. Then, (5.30) becomes a linear differential equation for $\tilde X$:

$$\frac{d\tilde X}{d\Omega}+2\frac{(1-3w)\rho+4\lambda}{\rho+\lambda}\tilde X+[(1+9w)\rho-8\lambda]\left(\frac{24\pi\alpha}{\kappa_6^2}+c\beta\right)^{-1}=0,$$
$$\tilde X=X+\frac{1}{12\alpha}, \qquad (5.33)$$

giving $H(a)$ or $H(\rho)$. For example, for a slowly varying β around today, one could assume that $\beta(\Omega)\approx\beta_0+\nu\Omega^2$.

Here, we will restrict ourselves to the case of constant deficit angle during the Universe evolution, $\beta(t)=\mathrm{const}$. The absence, however, of the induced gravity term makes the derived cosmology rather simple. As we have explained, since β is constant, the quantity $c\beta$ in Eqs. (5.29) and (5.30) should be replaced by c. Equation (5.29) becomes the standard conservation law. Equation (5.30) with the use of the conservation equation reduces to the linear differential equation

$$\frac{dX}{d\rho}-\frac{2}{3(1+w)}\frac{(1-3w)\rho+4\lambda}{\rho(\rho+\lambda)}\left(X+\frac{1}{12\alpha}\right)=\frac{1}{\frac{24\pi\alpha}{\kappa_6^2}+c}\frac{(1+9w)\rho-8\lambda}{3(1+w)\rho}, \qquad (5.34)$$

with general solution

$$H^2+\frac{k}{a^2}=\frac{\kappa_6^2\rho}{24\pi\alpha(1-\bar c)}+\left(\frac{\kappa_6^2\lambda}{24\pi\alpha(1-\bar c)}-\frac{1}{12\alpha}\right)+\frac{\tilde c}{(\rho+\lambda)^2}\rho^{\frac{8}{3(1+w)}}, \qquad (5.35)$$

where $\bar c=-\frac{\kappa_6^2}{24\pi\alpha}c$ is a redefinition of the integration constant c, and $\tilde c$ is another integration constant. So, Eq. (5.35) contains two integration constants $\bar{c}, \tilde{c}$. Note that the four-dimensional Newton constant is determined by an integration constant and not solely from the parameters of the theory.

Setting for this cosmology $\kappa_4^2 = \frac{\kappa_6^2}{8\pi\alpha(1-\bar{c})}$ the effective four-dimensional gravitational constant, we have

$$H^2 + \frac{k}{a^2} = \frac{\kappa_4^2}{3}\rho + \left(\frac{\kappa_4^2\lambda}{3} - \frac{1}{12\alpha}\right) + \frac{\tilde{c}}{(\rho+\lambda)^2}\rho^{\frac{8}{3(1+w)}}. \quad (5.36)$$

By fine-tuning the brane tension to the value $\lambda = \frac{1}{4\alpha\kappa_4^2}$, the effective cosmological constant vanishes and (5.36) becomes

$$H^2 + \frac{k}{a^2} = \frac{\kappa_4^2}{3}\rho + \frac{\tilde{c}}{(\rho + \frac{1}{4\alpha\kappa_4^2})^2}\rho^{\frac{8}{3(1+w)}}, \quad (5.37)$$

which includes a nontrivial correction beyond the standard linear to energy density term.

VI. SPECIAL CASES, EINSTEIN LIMIT AND COMPARISON WITH STANDARD APPROACH

A. Special cases

It is instructive to see the limit of a few special cases. These cases need special treatment since several expressions are divided by A, H, etc.

1. Minkowski brane

An exactly Minkowski four-dimensional spacetime has no need to be the solution of a viable gravitational theory since any curved spacetime solution is locally Minkowski, and therefore, satisfies all known local physics. However, the present theory possesses an exact brane Minkowski vacuum, where $H = 0$, $k = 0$, $\rho = 0$. In this case, the two boundary conditions $\eta = 0, 1$ reduce to a single case with vanishing extrinsic curvature $A = N = 0$, the deficit angle β comes out to be an integration constant, while for the second order coefficients it holds $\beta_2 = 0$, $3A' + N' = \kappa_6^2\mathcal{T}^r_r - \Lambda_6$ with $\mathcal{T}^r_r = \mathcal{T}^\theta_\theta$, $\mathcal{T}^t_r = 0$. Here, the regular parts of the ij bulk equations on the brane do not contain a''', n''', as it happens when $A, N \neq 0$, but they contain A', N', L'''; therefore, there is one additional equation for A', N', namely $A' - N' = \frac{\kappa_6^2}{6}(3\mathcal{T}^t_t - \mathcal{T}^{\hat{i}}_{\hat{i}})$, and finally A', N' are uniquely determined $A' = \frac{\kappa_6^2}{24}(6\mathcal{T}^r_r + 3\mathcal{T}^t_t - \mathcal{T}^{\hat{i}}_{\hat{i}}) - \frac{\Lambda_6}{4}$, $N' = \frac{\kappa_6^2}{8}(2\mathcal{T}^r_r - 3\mathcal{T}^t_t + \mathcal{T}^{\hat{i}}_{\hat{i}}) - \frac{\Lambda_6}{4}$. Note that in this special case the brane tension λ does not enter, so irrespectively of its value, a Minkowski brane is achieved without fine-tuning.

If the bulk has only $\Lambda_6 < 0$, one can find the exact bulk solution of EGB theory which is the extension of the above Minkowski brane [13,35]. For $\alpha < \frac{10}{24|\Lambda_6|}$, we start with the static black hole solution of EGB gravity [36] with horizon of toroidal topology

$$ds_6^2 = -\Delta^2(R)d\tau^2 + \frac{dR^2}{\Delta^2(R)} + R^2\delta_{ij}d\zeta^i d\zeta^j,$$
$$\Delta^2(R) = \frac{R^2}{12\alpha}\left[1 - \sqrt{1 + 24\alpha\left(\frac{\Lambda_6}{10} + \frac{\mu}{R^5}\right)}\right] \quad (6.1)$$

(μ is the integration constant) which possesses one singularity at $R = 0$ shielded by an horizon at $R_h = (\frac{10\mu}{-\Lambda_6})^{\frac{1}{5}}$. Making the double Wick rotation $\theta = -i\tau$, $t = -i\zeta^0$ and defining $\chi^{\hat{i}} = \zeta^{\hat{i}}$ and r instead of R by $\frac{dR}{dr} = \Delta(R)$, the solution (6.1) gets the Gaussian-normal form (3.7)

$$ds_6^2 = dr^2 + L^2(r)d\theta^2 + R^2(r)(-dt^2 + d\vec{\chi}_3^2). \quad (6.2)$$

Integrating equation $\frac{dR}{dr} = \Delta(R)$ around $R = R_h$, we find $r \simeq 2\sqrt{\frac{-2}{5\Lambda_6}}\sqrt{1 - \frac{R_h^5}{R^5}}$; therefore the brane should be located at the horizon. Moreover, since the induced metric on the horizon should be the Minkowski metric, it has to be $R_h = 1$, i.e. $\mu = -\Lambda_6/10$, and it is found that $L(r) = \Delta(R(r)) = -\frac{\Lambda_6}{4}r + \mathcal{O}(r^3)$, $R(r) = 1 - \frac{\Lambda_6}{8}r^2 + \mathcal{O}(r^4)$. Comparing the metric (6.2) with (4.1), it is found that $A = N = 0$, $\beta_2 = 0$, $A' = N' = -\frac{\Lambda_6}{4}$, in accordance with the values found above for a Minkowski brane. The constant deficit angle β which from the brane viewpoint of the effective equations remained undetermined; now the embedding of the brane in the exact bulk solution specifies its value to be $\beta = \frac{|\Lambda_6|}{4}$ if the angle θ has the standard normalization $0 \leq \theta < 2\pi$.

For vanishing bulk matter content and $\Lambda_6 = 0$ the bulk solution of Einstein or EGB gravity which is consistent with a Minkowski brane is the locally flat geometry with a constant deficit angle $ds_6^2 = dr^2 + \beta^2 r^2 d\theta^2 - dt^2 + d\vec{\chi}_3^2$. This bulk solution has $A = N = \beta_2 = A' = N' = 0$, as required by the previous brane equations. The only difference of this configuration with the standard treatment is that here the brane tension λ remains an arbitrary parameter, while in the standard approach the equation $\mathcal{G}_{ij} + \alpha\mathcal{J}_{ij} = -\kappa_6^2\lambda h_{ij}\frac{\delta(r)}{2\pi\beta r}$ is more restrictive and implies the well-known fine-tuning $\lambda = \frac{2\pi}{\kappa_6^2}(1-\beta)$ [8,37] (this fine-tuning is the same as in Einstein gravity, since for a Minkowski brane the Gauss-Bonnet term does not contribute). Such a sort of relation we would like to arise as a direct calculation of an appropriately defined energy of the gravitational field, and indeed, one such definition that has successfully passed various other tests is given by the teleparallel representation of Einstein gravity [38]. It gives for the gravitational field of the previous locally flat bulk solution the energy per unit spatial volume of the defect, inside a cylinder of arbitrary radius around the defect, equal to $\varepsilon_g = \frac{2\pi}{\kappa_6^2}(1-\beta)$, which is same with the energy per unit length of a cosmic string. Since the radius of the cylinder is insignificant, it may be concluded that the whole energy is concentrated along the axis $r = 0$. Therefore, in the standard approach the energy is localized on the brane and is due to the brane tension which adjusts β, while in the current approach the energy is again practically localized on the brane in the form of gravitational energy which now adjusts β and has the same value as before. So, now the gravitational energy stored at the defect, instead of the brane tension itself, adjusts the deficit angle and the two descriptions are physically similar.

2. Vanishing extrinsic curvature

Since one of the two matching conditions, Eq. (3.28), is contracted with the extrinsic curvature, in the limit of vanishing extrinsic curvature this equation is satisfied identically. Therefore, the situation is expected rather exceptional. One such example was the Minkowski brane above. For cosmology with $A = N = 0$ the equations on the brane give the exact conservation equation, $\beta = \text{const}$, $\beta_2 = 0$, $\mathcal{T}^r_r = \mathcal{T}^\theta_\theta$, $\mathcal{T}^t_r = 0$. Additionally, the regular parts of the ij bulk equations on the brane do not contain a''', n''', as it happens when $A, N \neq 0$, but they contain A', N', L'''. These two equations, together with the $\theta\theta$ equation (4.13), can be solved for A', N', β_3. Therefore, H remains arbitrary and any four-dimensional cosmology can be a solution, under the condition that it is embedded geodesically in the bulk. Since there is no curvature singularity at $r = 0$ in this case, maybe the stability issue could shed more light on the physical relevance of such geodesic embeddings.

3. Only tension on the brane

This means that $\rho = p = 0$, and therefore, this case cannot be obtained from the solutions of Sec. V. Physically, this situation could approximate periods of the history of the Universe where inflation or dark energy dominate over matter. From Eq. (5.1) it is concluded that $\beta = \text{const}$. Then, Eqs. (5.2) and (5.3) do not contain β; thus, (5.13) and (5.16) should not contain β, and therefore, $c\beta$ is replaced by c in (5.13) and (5.16).

For $\sigma_1 \neq 0$ the equation which governs the evolution is (5.16) and gets the form

$$\frac{\dot{\xi}}{nH} = \frac{4\xi}{3}\frac{3\xi + 2\gamma}{\xi + \gamma}, \tag{6.3}$$

where $\gamma = \frac{3}{2} + \frac{9\sigma_1}{2}(\eta\frac{24\pi\alpha}{\kappa_6^2} + c)^{-1}$, with general solution

$$\frac{1}{\tilde{c}a^8}\xi^3 - 3\xi = 2\gamma \tag{6.4}$$

($\tilde{c}$ integration constant). It is also defined $c_* = |\tilde{c}|^{-1/2} > 0$. Equation (6.4) is a cubic for ξ and can be solved analytically giving the function $\xi(a)$, and then the Hubble evolution $H^2(a)$. Since the cubic has various branches, there are also branches for the cosmic evolution, all containing the two integration constants $c_* > 0, \gamma$. For $\tilde{c} > 0$ and $a^8 \leq \gamma^2/\tilde{c}$, there is one real solution:

$$H^2 + \frac{k}{a^2} = \frac{\sigma_2}{3\sigma_1} + \text{sgn}(\gamma)\frac{\sigma c_*}{6\sigma_1 a^4}\cosh^{-1}\left[\frac{1}{3}\text{arccosh}\left(\frac{|\gamma| c_*}{a^4}\right)\right]. \tag{6.5}$$

For $\tilde{c} > 0$ and $a^8 \geq \gamma^2/\tilde{c}$, there are three real solutions:

$$H^2 + \frac{k}{a^2} = \frac{\sigma_2}{3\sigma_1} + \frac{\sigma c_*}{6\sigma_1 a^4}\cos^{-1}\left[\frac{1}{3}\arccos\left(\frac{\gamma c_*}{a^4}\right) + \frac{2\pi m}{3}\right], \quad m = 0, 1, 2, \tag{6.6}$$

and accept the regime $a \to \infty$

$$H^2 + \frac{k}{a^2} \approx \frac{\sigma_2}{3\sigma_1} + \frac{\sigma c_*}{6\sigma_1}\cos^{-1}\left[\frac{\pi(1+4m)}{6}\right]\frac{1}{a^4}. \tag{6.7}$$

For $\tilde{c} < 0$ there is one real solution:

$$H^2 + \frac{k}{a^2} = \frac{\sigma_2}{3\sigma_1} - \frac{\sigma c_*}{6\sigma_1 a^4}\sinh^{-1}\left[\frac{1}{3}\text{arcsinh}\left(\frac{\gamma c_*}{a^4}\right)\right], \tag{6.8}$$

and accepts the regime $a \to \infty$

$$H^2 + \frac{k}{a^2} \approx \frac{1}{3\sigma_1}\left(\sigma_2 - \frac{3\sigma}{2\gamma}\right) - \frac{2\sigma\gamma c_*^2}{27\sigma_1}\frac{1}{a^8}. \tag{6.9}$$

Note that the previous solutions for $c_* = k = 0$ can give a de Sitter brane.

For $\sigma_1 = 0$, the relevant equation is (5.13) which becomes

$$\frac{\dot{X}}{nH} + 8\left(X + \frac{1}{12\alpha}\right) - \frac{8\lambda}{\frac{24\pi\alpha}{\kappa_6^2} + c} = 0, \tag{6.10}$$

with general solution

$$H^2 + \frac{k}{a^2} = \frac{\tilde{c}}{a^8} + \frac{\kappa_6^2\lambda}{24\pi\alpha(1-\bar{c})} - \frac{1}{12\alpha}, \tag{6.11}$$

where $\bar{c} = -\frac{\kappa_6^2}{24\pi\alpha}c$ is a redefinition of the integration constant c, and $\tilde{c}$ is another integration constant. So, Eq. (6.11) contains two integration constants $\bar{c}, \tilde{c}$. For $\tilde{c} = k = 0$ the solution (6.11) can also give a de Sitter brane, which is alternatively obtained by setting directly $H = \text{const}$ in (6.10).

B. The Einstein limit

In the limit of Einstein bulk gravity the Gauss-Bonnet term is absent: $\alpha = 0$. We give the effective equations for a general axially symmetric configuration. In [20], some terms in the expansion of the regular parts of the bulk equations at the brane position were missed, so we give here an exhaustive account of all the relevant equations at the brane location.

The matching condition (3.28) becomes

$$\{\kappa_6^2 T^{ij} - [\kappa_6^2\lambda - 2\pi(1-\beta)]h^{ij} - r_c^2 G^{ij}\}K_{ij} = 0, \quad (6.12)$$

while the matching condition (3.29) gets the form

$$T^{ij}{}_{|j} = \frac{2\pi}{\kappa_6^2} h^{ij}\beta_{,j}. \quad (6.13)$$

Among the two terms $\frac{L''}{L}$ and $\frac{L'}{L}\mathcal{K}'_{ij}$ which contribute to the distributional terms $\frac{\delta(r)}{r}$, only the first is present in the Einstein tensor. Note that the index η disappears in (6.12) and (6.13). In general, the term $\frac{L'}{L}\mathcal{K}'_{ij}$ attributes to the matching conditions extra terms due to the possible discontinuity of $\mathcal{K}_{ij}$. Since this term is absent here, there is no point to consider $\mathcal{K}_{ij}$ discontinuous; therefore the Einstein limit corresponds to $\eta = 1$.

The $\mathcal{O}(1/r)$ part of the ri bulk equation, namely Eq. (3.30), becomes

$$\beta_{,i} = 0, \quad (6.14)$$

while the $\mathcal{O}(1/r)$ part of the rr bulk equation, i.e. Eq. (3.31), is[1]

$$K = 0. \quad (6.15)$$

The $\mathcal{O}(1/r)$ part of the ij equations, using the expression of $\mathcal{G}_{ij}$ of Appendix A, takes the form

$$K_{ij} - Kh_{ij} - \frac{\beta_2}{\beta}h_{ij} = 0. \quad (6.16)$$

The first matching condition (6.12), using (6.15), takes the form

$$\left(\frac{r_c^2}{\kappa_6^2}G^{ij} - T^{ij}\right)K_{ij} = 0 \quad (6.17)$$

[1] In [39], a self-gravitating string in Einstein gravity was locally described by a thin tube of matter represented by a smoothed conical metric, and under some constraint on the model of the string in the limit where the thickness becomes negligible the central line of the string was shown to follow the Nambu-Goto dynamics.

and coincides with (3.32) if indeed it is set $\eta = 1$ in the constants (3.33). The second matching condition (6.13), using (6.14), becomes

$$T^{ij}{}_{|j} = 0. \quad (6.18)$$

From Eqs. (6.15) and (6.16) we get

$$K_{ij} = \frac{\beta_2}{\beta}h_{ij}. \quad (6.19)$$

Contracting (6.19) with h^{ij} and using again (6.15) we take

$$\beta_2 = 0, \qquad K_{ij} = 0. \quad (6.20)$$

Therefore, the algebraic matching condition (6.17) or (6.12) is trivially satisfied and what remains is $\beta = \text{const}$, the conservation equation (6.18) and Eqs. (6.20). The vanishing of the total extrinsic curvature means that in the Einstein limit the brane is a special case of Nambu-Goto, it is geodesic, i.e. $x^\mu{}_{;ij} + \Gamma^\mu{}_{\nu\lambda}x^\nu{}_{,i}x^\lambda{}_{,j} = 0$. Of course, this happens "on shell"; it is the result of all the equations, not only the matching conditions. On the contrary, in the EGB theory we examined, in general the brane even on shell is not geodesic. It would be interesting to see if in Einstein theory this geodesic result is relaxed when the ansatz of axial symmetry is abandoned. If this is the case, the codimension-2 Einstein gravity will be not only consistent, but also nontrivial. Note that the probe limit of a theory is a different thing and is checked "off shell," from the matching conditions only, since in the probe limit there is no bulk dynamics and the bulk equations are empty.

Continuing with the remaining equations, the regular part of the rr bulk equation is

$$2K' + K^2 - K_{ij}K^{ij} - R + 2\frac{\Box\beta}{\beta} + K\frac{\beta_2}{\beta} = 2\kappa_6^2\mathcal{T}^r_r - 2\Lambda_6, \quad (6.21)$$

where we denote $\mathcal{K}'_{ij} = \mathcal{K}'_{ij}(r=0)$ and Eq. (6.21), due to (6.14) and (6.20), becomes

$$R - 2K' = 2\Lambda_6 - 2\kappa_6^2\mathcal{T}^r_r. \quad (6.22)$$

Similarly, the regular part of the ri bulk equation is

$$K^j_{i|j} - K_{|i} + \frac{\beta_{,j}}{\beta}\left(\frac{\beta_2}{2\beta}\delta^j_i + K^j_i\right) - \frac{\beta_{2,i}}{\beta} = \kappa_6^2\mathcal{T}^r_i, \quad (6.23)$$

which due to (6.14) and (6.20) becomes identically satisfied whenever $\mathcal{T}^r_i = 0$ on the brane. Concerning the $\theta\theta$ bulk equation one obtains

$$2K' + K_{ij}K^{ij} + K^2 - R = 2\kappa_6^2\mathcal{T}^\theta_\theta - 2\Lambda_6, \quad (6.24)$$

which due to (6.20) becomes

$$R - 2K' = 2\Lambda_6 - 2\kappa_6^2 \mathcal{T}^\theta_\theta. \tag{6.25}$$

Equation (6.22) is compatible with Eq. (6.25) whenever $\mathcal{T}^r_r = \mathcal{T}^\theta_\theta$ on the brane. Therefore, all the equations of Einstein gravity on the brane up to now are $\beta = \text{const}$, $\beta_2 = 0$, $T^{ij}{}_{|j} = 0$, $K_{ij} = 0$, $R - 2K' = 2\Lambda_6 - 2\kappa_6^2 \mathcal{T}^r_r$ and what remains is the regular part of the ij bulk equations.

The regular part of the ij bulk equations is

$$\begin{aligned} &G_{ij} + 2K_{ik}K^k_j + KK_{ij} - 2K'_{ij} + \frac{3\beta_2}{2\beta}K_{ij} - \frac{\beta_{|i|j}}{\beta} \\ &\quad + \left(2K' + \frac{1}{2}K_{k\ell}K^{k\ell} + \frac{1}{2}K^2 + \frac{\Box\beta}{\beta} + K\frac{\beta_2}{2\beta} - \frac{\beta_2^2}{2\beta^2} + \frac{\beta_3}{\beta}\right)h_{ij} \\ &= \kappa_6^2 \mathcal{T}_{ij} - \Lambda_6 h_{ij}, \end{aligned} \tag{6.26}$$

where $\beta_3(\chi) = L'''(\chi, 0)$. Equation (6.26), due to (6.14) and (6.20), becomes

$$G_{ij} - 2K'_{ij} + \left(2K' + \frac{\beta_3}{\beta}\right)h_{ij} = \kappa_6^2 \mathcal{T}_{ij} - \Lambda_6 h_{ij}. \tag{6.27}$$

Contracting (6.27) with h^{ij} we find β_3:

$$\frac{4\beta_3}{\beta} = R - 6K' - 4\Lambda_6 + \kappa_6^2 \mathcal{T}_{ij}h^{ij}, \tag{6.28}$$

and substituting back in (6.27) we get

$$G_{ij} - 2K'_{ij} + \frac{1}{4}(R + 2K')h_{ij} = \kappa_6^2 \mathcal{T}_{k\ell}\left(\delta^k_i \delta^\ell_j - \frac{1}{4}h^{k\ell}h_{ij}\right). \tag{6.29}$$

Of course, not all ij equations of (6.29) are independent, but independent are all but one. The nontrivial final equations to be satisfied are (6.22) and (6.29). For cosmology, Eq. (6.22) gets the form

$$3A' + N' - 3(X + Y) = \kappa_6^2 \mathcal{T}^r_r - \Lambda_6 \tag{6.30}$$

and coincides with Eq. (4.13) for $\alpha = 0$. Equation (6.29) for cosmology contains one independent equation which is

$$A' - N' + Y - X = \frac{\kappa_6^2}{6}(3\mathcal{T}^t_t - \mathcal{T}^{\hat{i}}_{\hat{i}}). \tag{6.31}$$

Therefore, the two equations (6.30) and (6.31) can be solved algebraically for A', N' and X, Y remain undetermined; thus, the scale factor remains undetermined. The energy density obeys the standard conservation.

The indeterminacy from the brane viewpoint of one unknown function for cosmology is the result of the codimension-2 geometry. Certainly here, in Einstein gravity, the fact that the scale factor itself remains undetermined is more inconvenient compared to the indeterminacy of the general EGB cosmology which can be rendered to the deficit angle. But still the important thing is the consistency of Einstein gravity in the present formulation, in clear contrast to the inconsistency of Einstein gravity according to the standard treatment [5–7,9,10]. Our main purpose is to raise the interest to the investigation of more realistic, alternative matching conditions, not to give an answer on the selection of the appropriate bulk boundary conditions, or the appropriate codimensionality, or the appropriate gravitational theory in order to pick up the unique final cosmology.

C. Comparison with the cosmology of the standard approach

It was explained thoroughly in the introduction that according to the standard approach the equations of motion of a defect are derived by taking the variation of the brane-bulk action with respect to the bulk metric at the brane position. This is the extension of what is done with the Israel matching conditions and for the codimension-2 Einstein-Gauss-Bonnet theory this was performed in [12,13,17]. The aim here is to compare the cosmological equations of the standard treatment discussed in [13] with the cosmological equations (5.1)–(5.3) of the present analysis.

In [13], cases (i) and (iii) were examined. Together with the matching conditions originating from the distributional terms $\delta(r)/r$, additional matching conditions were considered arising from the distributional terms $\delta(r)$ (also considered in [15]). For case (i), these extra matching conditions are identically satisfied. However, for case (iii) these conditions provide nontrivial constraints. The result of the analysis in [13] for cosmology was that while case (i) is consistent, case (iii) is not consistent with a codimension-2 brane, but an additional codimension-1 brane is needed. However, this result is not correct because the distributional terms $\delta(r)$ should not be considered for deriving extra matching conditions. This becomes obvious from the variational point of view where the volume element $rdrd\theta$ multiplies $\delta(r)$ and vanishes it. Otherwise, looking at an equation containing two sorts of distributions it cannot be said if the correct thing is to leave the equation as it is getting two sets of matching conditions, or to multiply the equation with r getting one set of matching conditions, or why not to multiply the equation with r^2 getting no matching conditions at all. To say it in a similar way, it is not obvious what is the correct regularization and the answer is provided by the variational principle which naturally supplies the correct regularization. Therefore, the correct result is that both cases (i) and (iii) are consistent with a codimension-2 brane in EGB theory according to the standard approach. We can add here that case (ii) is also consistent since mathematically it does not differ significantly from cases (i) and (iii). In the following, the essential cosmological equations of the standard approach are summarized in order to compare with the current treatment. These equations for matching conditions (i), (ii), and (iii) are

$$\dot{\rho}+3nH(\rho+p)=\eta\frac{\dot{\beta}}{\beta(\eta-\beta)}\left(3\frac{r_c^2}{\kappa_6^2}X-\rho-\lambda\right), \quad (6.32)$$

$$A^2=\frac{1}{\eta-\beta}\left(1-\beta+\frac{r_c^2}{8\pi\alpha}\right)X-\frac{\kappa_6^2(\rho+\lambda)}{24\pi\alpha(\eta-\beta)}+\frac{1-\beta}{12\alpha(\eta-\beta)}, \quad (6.33)$$

$$fA^2=\frac{1}{\eta-\beta}\left(1-\beta+\frac{r_c^2}{8\pi\alpha}\right)Y+\frac{\kappa_6^2(\rho+3p-2\lambda)}{48\pi\alpha(\eta-\beta)}+\frac{1-\beta}{12\alpha(\eta-\beta)}, \quad (6.34)$$

$$2A^2\left[\frac{\dot{A}}{nA}+H(1-f)\right]=\frac{\dot{\beta}}{n\beta}\left(X-A^2+\frac{1}{12\alpha}\right), \quad (6.35)$$

where, also here, N has been replaced by $N=fA$. The definitions for X,Y,f are the same with those in Eqs. (5.5) and (5.6).

For matching condition (iv), Eqs. (6.33) and (6.34) are replaced by the cosmological version of the four-dimensional Einstein equations $r_c^2G_{ij}-\kappa_6^2(T_{ij}-\lambda h_{ij})=0$; therefore the strict conservation equation replaces (6.32). Additionally, Eqs. (4.8) and (4.9) are valid with a zero right-hand side. Solving (4.8) for N/A (now we do not have the equation $N=fA$) and substituting in (4.9), there arises a differential equation for A. Although we do not plan here to study exhaustively the second order equations to check the consistency, however, it seems most probable that matching condition (iv) will still be consistent in the standard treatment.

Equations (6.32)–(6.35) form a system of four equations for four unknowns a,ρ,β,A; however, one equation, for example the matching condition (6.34) is redundant, so there is again one indeterminacy. Indeed, differentiating the matching condition (6.33) with respect to time and using (6.32) and (6.35), Eq. (6.34) is obtained. Although redundant, this equation is left inside the set of equations because it will be useful in the following. The conservation equation (6.32) is the corresponding or (5.1), but the two right-hand sides differ significantly. However, for $\eta=0$ or $\beta=\text{const}$ they both reduce to the standard conservation equation. Equation (6.35) is identical to Eq. (5.3). The crucial difference between the two theories is the matching condition (6.33) compared to Eq. (5.2), which is a sort of equivalent to the new, algebraic in extrinsic curvature, matching condition (3.28). While Eq. (6.33) contains only H, Eq. (5.2) contains also $\dot{H}$. In some sense it can be said that in the standard approach the matching condition is already integrated, while on the contrary, the present theory is more complicated and accepts more general solutions with more integration constants.

For example, to be more explicit, for $\beta=\text{const}$ and $\sigma_1=0$, Eq. (6.35) can be integrated for A. Then, Eq. (6.33) gives directly the Hubble parameter in terms of one integration constant [we do not take into account the additional integration constant for ρ from (6.32)] and this solution was obtained in [13]

$$H^2+\frac{k}{a^2}=\frac{\kappa_6^2\rho}{24\pi\alpha(1-\beta)}+\left(\frac{\kappa_6^2\lambda}{24\pi\alpha(1-\beta)}-\frac{1}{12\alpha}\right)+\frac{\tilde{c}}{(\rho+\lambda)^2}\rho^{\frac{8}{3(1+w)}}. \quad (6.36)$$

On the other hand, this same solution for A when substituted in (5.2), a Raychaudhuri equation for $\dot{H}$ is obtained. Its integration gives H in terms of two integration constants and this is the solution (5.35) obtained above. Maybe in this simplified case the two solutions (6.36) and (5.35) look similar, but the main difference has already been pointed out.

In the general case of nonconstant β and any σ_1, for the standard treatment, the Raychaudhuri equation can be easily derived by combining the two algebraic in A matching conditions (6.33) and (6.34):

$$\left(1-\beta+\frac{r_c^2}{8\pi\alpha}\right)(Y-fX)+\frac{\kappa_6^2}{48\pi\alpha}[(1+2f)\rho+3p]+\frac{1-f}{12\alpha}\left(1-\beta-\frac{\kappa_6^2\lambda}{2\pi}\right)=0. \quad (6.37)$$

The analogue of Eq. (6.37) in the alternative approach is Eq. (5.13). The difference is not only that (5.13) already contains one extra integration constant, but the whole structure of the two equations is quite different. To say it in a different way, in the alternative approach, there is only one equation (5.2) algebraic in A (also containing $H,\dot{H}$), instead of the two equations (6.33) and (6.34) algebraic in A in the standard approach. Differentiating this equation with respect to time and using (5.3) to get rid of $\dot{A}$, we could obtain an equation for $\ddot{H}$. Again, the difference with (6.37) is obvious. The successful treatment performed in Sec. V managed to derive Eq. (5.13) which contains only $\dot{H}$, but with the cost of one integration constant c. Equation (6.37) together with the conservation equation (6.32) constitute for the standard approach the two-dimensional system for a,ρ with the indeterminacy of $\beta(t)$. It would be interesting, for example, to be integrated for $\beta=\text{const}$ and $\sigma_1\neq 0$.

VII. CONCLUSIONS

In view of the absence of direct observational evidence of regular gravitating defects, we raise in this paper on theoretical grounds the question of the physical relevance of Israel matching conditions and their generalizations to higher dimensions and codimensions, the standard cornerstone of the braneworld paradigm and other membrane scenarios. While for a self-gravitating brane an equation of the general form bulk gravity tensor equals some smooth matter content or some matter content of a "thick" brane is certainly correct, we claim that it cannot be correct in the shrink limit of distributional branes. A different treatment of the delta function characterizing the defect is needed for extracting its equation of motion. If this is so, the Israel matching conditions, as well as their generalizations where the Einstein bulk gravity tensor is replaced by Lovelock extensions and the branes have differing codimensions, cannot be adequate.

Our reasoning is based on two points: First, the incapability of the conventional matching conditions to accept the Nambu-Goto probe limit. Even the geodesic limit of the Israel matching conditions is not an acceptable probe limit since being the geodesic equation a kinematical fact it should be preserved independent of the gravitational theory or the codimension of the defect, which however is not the case for these matching conditions. Second, in the D-dimensional spacetime where we live (maybe $D = 4$), classical defects of any possible codimension could in principle be constructed (even in the lab), and therefore, they should be compatible. The standard matching conditions fail to accept codimension-2 and -3 defects for $D = 4$ (which represents effectively the spacetime at certain length and energy scales) and most probably fail to accept high enough codimensional defects for any D since there is no corresponding high enough Lovelock density to support them.

We make a proposal that the problem is not the distributional character of the defects, nor the gravitational theory used, but the equations of motion of the defects. The proposed matching conditions might move towards the correct direction of finding realistic matching conditions since they always have the Nambu-Goto probe limit, independently of the gravity theory and independently of the dimension of spacetime or codimension of the brane. Moreover, with these matching conditions, defects of any codimension seem to be consistent for any (second order) gravity theory. These alternative matching conditions arise by promoting the embedding fields of the defect to the fundamental entities. Instead of varying the brane-bulk action with respect to the bulk metric at the brane position and deriving the standard matching conditions, we vary with respect to the brane embedding fields in a way that takes into account the gravitational backreaction of the brane to the bulk ("gravitating Nambu-Goto matching conditions").

In the present paper we have considered in detail the case of a 3-brane in six-dimensional Einstein-Gauss-Bonnet gravity, derived the generic alternative matching conditions and proved the consistency for an axially symmetric cosmological configuration. Of course, same or similar results are true for other codimension-2 defects in other spacetime dimensions. The consistency of such branes in Einstein gravity was also discussed. Additionally, however, a 3-brane could represent our world in the braneworld scenario; therefore, we have investigated the cosmological equations and found solutions for the cosmic evolution. From the technical point of view, compared to the standard equations, the main difference is that the equations here, and accordingly their solutions, have more richness and more complicated structure and contain more integration constants. In all the cosmologies found assuming a constant deficit angle, there is the standard FRW term linear in energy density, a cosmological constant term and an extra correction or dark energy term. In particular, one of these solutions for a radiation brane and for a range of the integration constants avoids a cosmological singularity (both in density and curvature) and undergoes accelerated expansion near the minimum scale factor.

Depending on the existence or not of a conical singularity or of a discontinuous extrinsic curvature, there can be four possible cases as matching conditions. Actually, one of these cases is the smooth matching condition with smooth transverse section (no cone) and smooth extrinsic tangential section, which is still consistent and possesses interesting solutions. In general, for a codimension-2 cosmological configuration, either here or in the standard treatment, the system of the effective equations from the brane viewpoint is nonclosed and we need extra information coming from the bulk geometry in order to fix one of the functions. On the theoretical side it would be interesting to have particular bulk solutions setting the boundary conditions and fixing the deficit angle, whose evolution would then leave an imprint on the cosmological evolution equations. For a pure conical brane such a dynamical deficit angle will make the brane to radiate in the bulk.

Codimension-2 braneworlds in six-dimensional gravity or supergravity have attracted considerable interest in relation to the cosmological constant problem (for a review see [40]). They are proposed to offer a mechanism for understanding the smallness of the vacuum energy since in this scenario, a codimension-2 object induces a conical singularity, and the cancellation occurring between the brane tension λ and the bulk gravitational degrees of freedom gives rise to a vanishing effective cosmological constant. In the approach of the present paper there is no such relation between the brane tension and the conical deficit in order to obtain a Minkowski brane, but a Minkowski brane is obtained by a physically similar balance between the gravitational energy stored at the brane and the deficit angle. However, furthermore, in the presence of the induced gravity term, the proposed formalism embodies naturally the effective cosmological constant scale $\kappa_6^2\lambda/r_c^2$, which for $\lambda \sim M_6^4$ (e.g. TeV^4) and $r_c \sim H_0^{-1}$ gives the observed value H_0^2 of the cosmological constant. The corresponding scale in any other bulk dimension with $\lambda \sim M_D^4$ fails to provide the observed order of magnitude of the cosmological constant. Even if the constraint provided by the four-dimensional Newton's constant G_N is not easily satisfied, there is still hope, since in the present theory G_N is not determined solely from the parameters of the action, but in general depends also on the integration constants of the considered solution.

ACKNOWLEGMENTS

M. I. is supported by START Project No. Y 435-N16 (FWF).

APPENDIX A: GEOMETRIC COMPONENTS

The nonvanishing components of the necessary geometric quantities of the metric (3.7) are

$$\Gamma^r_{\theta\theta} = -LL', \qquad \Gamma^r_{ij} = -\mathcal{K}_{ij}, \qquad \Gamma^\theta_{r\theta} = \frac{L'}{L}, \qquad \Gamma^\theta_{\theta i} = \frac{L_{|i}}{L}, \qquad \Gamma^i_{\theta\theta} = -Lg^{ij}L_{|j}, \qquad \Gamma^i_{rj} = \mathcal{K}^i_j,$$
$$\Gamma^i_{jk} = \frac{1}{2}g^{i\ell}(g_{\ell j,k} + g_{\ell k,j} - g_{jk,\ell}) \tag{A1}$$

(where $\mathcal{K}^i_j = g^{ik}\mathcal{K}_{kj}$),

$$\mathcal{R}_{rirj} = -\mathcal{K}'_{ij} + \mathcal{K}_{ik}\mathcal{K}^k_j, \qquad \mathcal{R}_{\theta i\theta j} = -L(L_{|i|j} + L'\mathcal{K}_{ij}), \qquad \mathcal{R}_{r\theta r\theta} = -LL'', \qquad \mathcal{R}_{ijk\ell} = R_{ijk\ell} + \mathcal{K}_{i\ell}\mathcal{K}_{jk} - \mathcal{K}_{ik}\mathcal{K}_{j\ell},$$
$$\mathcal{R}_{\theta r\theta i} = L(L_{|j}\mathcal{K}^j_i - L'_{|i}), \qquad \mathcal{R}_{rijk} = \mathcal{K}_{ij|k} - \mathcal{K}_{ik|j} \tag{A2}$$

(where $|$ denotes the covariant derivative with respect to the metric g_{ij}),

$$\mathcal{R}_{rr} = -\frac{L''}{L} - \mathcal{K}' - \mathcal{K}_{ij}\mathcal{K}^{ij}, \qquad \mathcal{R}_{\theta\theta} = -L(L'' + \Box L + L'\mathcal{K}), \qquad \mathcal{R}_{ri} = \frac{L_{|j}}{L}\mathcal{K}^j_i - \frac{L'_{|i}}{L} + \mathcal{K}^j_{i|j} - \mathcal{K}_{|i},$$
$$\mathcal{R}_{ij} = R_{ij} - \mathcal{K}'_{ij} + 2\mathcal{K}_{ik}\mathcal{K}^k_j - \mathcal{K}\mathcal{K}_{ij} - \frac{L_{|i|j}}{L} - \frac{L'}{L}\mathcal{K}_{ij} \tag{A3}$$

(where $\mathcal{K} = \mathcal{K}^i_i$ and $\Box$ is the Laplacian operator of the metric g_{ij}),

$$\mathcal{R} = R - 2\frac{L''}{L} - 2\mathcal{K}' - \mathcal{K}_{ij}\mathcal{K}^{ij} - \mathcal{K}^2 - 2\frac{\Box L}{L} - 2\frac{L'}{L}\mathcal{K}, \tag{A4}$$

$$\mathcal{G}_{ri} = \frac{L_{|j}}{L}\mathcal{K}^j_i - \frac{L'_{|i}}{L} + \mathcal{K}^j_{i|j} - \mathcal{K}_{|i}, \qquad \mathcal{G}_{rr} = \frac{1}{2}\mathcal{K}^2 - \frac{1}{2}\mathcal{K}_{ij}\mathcal{K}^{ij} + \frac{\Box L}{L} + \frac{L'}{L}\mathcal{K} - \frac{1}{2}R,$$
$$\mathcal{G}_{\theta\theta} = L^2\left(\mathcal{K}' + \frac{1}{2}\mathcal{K}_{ij}\mathcal{K}^{ij} + \frac{1}{2}\mathcal{K}^2 - \frac{1}{2}R\right),$$
$$\mathcal{G}_{ij} = G_{ij} - \mathcal{K}'_{ij} + 2\mathcal{K}_{ik}\mathcal{K}^k_j - \mathcal{K}\mathcal{K}_{ij} - \frac{L_{|i|j}}{L} - \frac{L'}{L}\mathcal{K}_{ij} + \left(\frac{L''}{L} + \mathcal{K}' + \frac{1}{2}\mathcal{K}_{k\ell}\mathcal{K}^{k\ell} + \frac{1}{2}\mathcal{K}^2 + \frac{L'}{L}\mathcal{K} + \frac{\Box L}{L}\right)g_{ij}. \tag{A5}$$

APPENDIX B: DERIVATION OF REGULAR EQUATIONS

Using the results of Appendix A, we can find analogous but more complicated expressions for the tensor components of $\mathcal{E}^\mu_\nu = \mathcal{G}^\mu_\nu + \alpha\mathcal{J}^\mu_\nu$. In some of these components there are terms proportional to $\frac{L'}{L}, \frac{L'_{|i}}{L}, \frac{L''}{L}$. All the other terms of $\mathcal{E}^\mu_\nu$ are manifestly regular $\mathcal{O}(1)$ terms and can be obtained by setting formally $\frac{L'}{L} = \frac{L'_{|i}}{L} = \frac{L''}{L} = 0$, $L = \beta$ in the $\mathcal{E}^\mu_\nu$ expressions. Now, the terms containing $\frac{L'}{L}, \frac{L'_{|i}}{L}, \frac{L''}{L}$ are certainly the sources of the singular $\mathcal{O}(1/r)$ terms. However, these same terms, when expanded in powers of r, give additional hidden regular $\mathcal{O}(1)$ terms. In this Appendix we derive the regular rr, ri bulk equations on the brane (4.15) and (4.16).

More precisely, in the rr equation there are terms with $\frac{L'}{L}$, while in the ri equation there are terms with $\frac{L'}{L}, \frac{L'_{|i}}{L}$. A typical expansion of these terms is of the form

$$\frac{L'}{L}f(\chi,r)=f\frac{1}{r}+f'+\frac{f\beta_2}{2\beta}+\mathcal{O}(r), \tag{B1}$$

$$\frac{L'_{|i}}{L}f(\chi,r)=f\frac{\beta_{,i}}{\beta}\frac{1}{r}+f'\frac{\beta_{,i}}{\beta}+f\left(\frac{\beta_{2,i}}{\beta}-\frac{\beta_{,i}}{\beta}\frac{\beta_2}{2\beta}\right)+\mathcal{O}(r), \tag{B2}$$

$$\frac{L''}{L}f(\chi,r)=f\frac{\beta_2}{\beta}\frac{1}{r}+\frac{\beta_2}{\beta}\left(f'-f\frac{\beta_2}{2\beta}\right)+f\frac{\beta_3}{\beta}+\mathcal{O}(r), \tag{B3}$$

where f, f' are the values on the brane and $\beta_2(\chi)=L''(\chi,0)$, $\beta_3(\chi)=L'''(\chi,0)$.

In the quantity $\mathcal{E}^r_r$ the term which multiplies $\frac{L'}{L}$ is $f=\mathcal{K}-4\alpha(\mathcal{W}^i_j+G^i_j)\mathcal{K}^j_i$; i.e. at the brane position it is $\mathcal{E}^r_r=\mathcal{E}^r_r|_{\frac{L'}{L}=0,L=\beta}+\frac{L'}{L}f$. Therefore, using the identity $\mathcal{W}^{i\prime}_j\mathcal{K}^j_i=2\mathcal{W}^i_j\mathcal{K}^{j\prime}_i$, we get the $\mathcal{O}(1)$ part of the rr equation:

$$\mathcal{E}^r_r|_{\frac{L'}{L}=0,L=\beta}+K'-4\alpha(3W^i_j+G^i_j)K^{j\prime}_i-4\alpha G^{i\prime}_jK^j_i+[K-4\alpha(W^i_j+G^i_j)K^j_i]\frac{\beta_2}{2\beta}=\kappa_6^2\mathcal{T}^r_r-\Lambda_6, \tag{B4}$$

where $K^{j\prime}_i$ denotes $\mathcal{K}^{j\prime}_i(\chi,0)$. However, from the $\mathcal{O}(1/r)$ part of the rr equation (3.31) it is $f=0$ on the brane and (B4) takes a simpler form:

$$\mathcal{E}^r_r|_{\frac{L'}{L}=0,L=\beta}+K'-4\alpha(3W^i_j+G^i_j)K^{j\prime}_i-4\alpha G^{i\prime}_jK^j_i=\kappa_6^2\mathcal{T}^r_r-\Lambda_6. \tag{B5}$$

For cosmology it is

$$\mathcal{E}^r_r|_{\frac{L'}{L}=0,L=\beta}=-12\alpha\left[\frac{1}{n\beta}\left(\frac{\dot{\beta}}{n}\right)^{\dot{}}\mathcal{X}+\frac{H\dot{\beta}}{n\beta}(\mathcal{X}+2\mathcal{Y})+\mathcal{X}\mathcal{Y}+\frac{1}{6\alpha}\left(\mathcal{X}+\mathcal{Y}-\frac{5}{24\alpha}\right)\right], \tag{B6}$$

so Eq. (B4) is

$$\begin{aligned}&\frac{1}{n\beta}\left(\frac{\dot{\beta}}{n}\right)^{\dot{}}\mathcal{X}+\frac{H\dot{\beta}}{n\beta}(\mathcal{X}+2\mathcal{Y})+\mathcal{X}\mathcal{Y}+\frac{1}{6\alpha}(\mathcal{X}+\mathcal{Y})+2A^2\left[\left(1+2\frac{N}{A}\right)A'+N'\right]-[(\mathcal{X}+2\mathcal{Y})A'+\mathcal{X}N']\\&-[(A+N)X'+2AY']-A\left[\left(1+\frac{N}{A}\right)\mathcal{X}+2\mathcal{Y}\right]\frac{\beta_2}{2\beta}=\frac{1}{12\alpha}\left(\Lambda_6+\frac{5}{12\alpha}-\kappa_6^2\mathcal{T}^r_r\right),\end{aligned} \tag{B7}$$

while its simpler form (B5) is

$$\begin{aligned}&\frac{1}{n\beta}\left(\frac{\dot{\beta}}{n}\right)^{\dot{}}\mathcal{X}+\frac{H\dot{\beta}}{n\beta}(\mathcal{X}+2\mathcal{Y})+\mathcal{X}\mathcal{Y}+\frac{1}{6\alpha}(\mathcal{X}+\mathcal{Y})+2A^2\left[\left(1+2\frac{N}{A}\right)A'+N'\right]-[(\mathcal{X}+2\mathcal{Y})A'+\mathcal{X}N']\\&-[(A+N)X'+2AY']=\frac{1}{12\alpha}\left(\Lambda_6+\frac{5}{12\alpha}-\kappa_6^2\mathcal{T}^r_r\right).\end{aligned} \tag{B8}$$

Using (4.9) it is easy to find $H'=\frac{\dot{\beta}}{2n\beta A}\mathcal{X}-AH$, and then progressively the quantities X', $\dot{\mathcal{X}}$, $\dot{H}'=(H')^{\dot{}}$, Y'. Replacing X', Y' in (B8) and using (4.5) we take the equivalent to the regular rr equation

$$\begin{aligned}&\frac{\dot{\beta}^2}{n^2\beta^2}\left(2\mathcal{X}+\frac{\mathcal{X}^2}{2A^2}\right)+\mathcal{X}\mathcal{Y}+\frac{1}{6\alpha}(\mathcal{X}+\mathcal{Y})+2(1+f)A^2(X+Y)+2A^2[(1+2f)A'+N']-[(\mathcal{X}+2\mathcal{Y})A'+\mathcal{X}N']\\&=\frac{1}{12\alpha}\left(\Lambda_6+\frac{5}{12\alpha}-\kappa_6^2\mathcal{T}^r_r\right).\end{aligned} \tag{B9}$$

For the rt regular equation, due to the complication, we give directly the cosmological expressions, while the $r\hat{i}$ equation vanishes identically. In the quantity $\mathcal{E}^t_r$ the term which multiplies $\frac{L'}{L}$ is $\tilde{f}=-24\alpha\frac{A}{n}[\frac{\dot{A}}{n}+H(A-N)]$, and the term which multiplies $\frac{\dot{L}'}{L}$ is $\bar{f}=\frac{12\alpha\mathcal{X}}{n^2}$; i.e. at the brane position it is $\mathcal{E}^t_r=\mathcal{E}^t_r|_{\frac{L'}{L}=\frac{\dot{L}'}{L}=0,L=\beta}+\frac{L'}{L}\tilde{f}+\frac{\dot{L}'}{L}\bar{f}$. It is

$$\mathcal{E}^t_r|_{\frac{L'}{L}=\frac{\dot{L}'}{L}=0,L=\beta}=12\alpha\left[\frac{\dot{A}}{n^2}+\frac{H}{n}(A-N)\right]\left(\mathcal{X}+\frac{2H\dot{\beta}}{n\beta}+\frac{1}{6\alpha}\right)-12\alpha\mathcal{X}\frac{N\dot{\beta}}{n^2\beta}, \tag{B10}$$

so the $\mathcal{O}(1)$ part of the rt equation is

$$\left[\frac{\dot{A}}{n}+H(A-N)\right]\left(\mathcal{X}+\frac{2H\dot{\beta}}{n\beta}+\frac{1}{6\alpha}+4AN-2A'-\frac{A\beta_2}{\beta}\right)-2A\left[\frac{\dot{A}'}{n}+(H'+HN)(A-N)+H(A'-N')\right]$$
$$+(\mathcal{X}'-3N\mathcal{X})\frac{\dot{\beta}}{n\beta}+\mathcal{X}\left(\frac{\dot{\beta}_2}{n\beta}-\frac{\dot{\beta}}{n\beta}\frac{\beta_2}{2\beta}\right)=\frac{n\kappa_6^2}{12\alpha}\mathcal{T}^t_r. \quad \text{(B11)}$$

Using the $\mathcal{O}(1/r)$ part of the rt bulk equation, i.e. Eq. (4.9), Eq. (B11) gets the simpler form

$$\frac{\dot{\beta}_2}{n\beta}+\frac{H}{A}\frac{\dot{\beta}^2}{n^2\beta^2}+\left[\frac{\mathcal{X}'}{\mathcal{X}}-\frac{\beta_2}{\beta}-\frac{A'}{A}+\frac{1}{2A}\left(\mathcal{X}+\frac{1}{6\alpha}\right)-N\right]\frac{\dot{\beta}}{n\beta}-\frac{2A}{\mathcal{X}}\left[\frac{\dot{A}'}{n}+(H'+HN)(A-N)+H(A'-N')\right]=\frac{n\kappa_6^2}{12\alpha\mathcal{X}}\mathcal{T}^t_r, \quad \text{(B12)}$$

and using $H'=\frac{\dot{\beta}}{2n\beta A}\mathcal{X}-AH$ in (B12), we get the equivalent to the regular rt equation

$$\frac{\dot{\beta}_2}{n\beta}+\frac{H}{A}\frac{\dot{\beta}^2}{n^2\beta^2}+\left[\frac{\mathcal{X}'}{\mathcal{X}}-\frac{\beta_2}{\beta}-\frac{A'}{A}+\frac{1}{2A}\left(\mathcal{X}+\frac{1}{6\alpha}\right)-A\right]\frac{\dot{\beta}}{n\beta}-\frac{2A}{\mathcal{X}}\left[\frac{\dot{A}'}{n}+H(A'-N')-H(A-N)^2\right]=\frac{n\kappa_6^2}{12\alpha\mathcal{X}}\mathcal{T}^t_r. \quad \text{(B13)}$$

APPENDIX C: CONSISTENCY OF THE SYSTEM

In this Appendix we solve algebraically the system of equations (4.10), (4.12), and (4.14) for the unknown a_2, n_2, β_2. Then, we substitute these expressions in Eqs. (4.15) and (4.16) which are the regular parts of the rr, rt equations and check that these are automatically satisfied. This process shows the consistency of the whole system for all kinds of matching conditions. More precisely, Eq. (4.15) is an algebraic equation on the set of variables a_2, n_2, β_2, while on the other hand, Eq. (4.16) is a differential (Bianchi) equation with respect to the second order variables. Of course, the manipulation of the various terms is quite complicated and it needs to be organized systematically.

The solution of Eqs. (4.10), (4.12), and (4.14) for a_2, n_2, β_2 is

$$(1+3f)\frac{a_2}{a}=\frac{1+3f}{12\alpha}+f\mathcal{X}-\frac{3\mathcal{X}}{2A^2}\left(\frac{\dot{\beta}}{n\beta}\right)^2-\frac{1}{12\alpha\mathcal{X}}\left(\kappa_6^2\mathcal{T}^\theta_\theta-\Lambda_6-\frac{5}{12\alpha}\right), \quad \text{(C1)}$$

$$(1+3f)\frac{n_2}{n}=\frac{1+3f}{12\alpha}-\frac{1+4f+f^2}{2}\mathcal{X}+\frac{\mathcal{X}}{2A^2}\left(\frac{\dot{\beta}}{n\beta}\right)^2+\frac{1+2f}{12\alpha\mathcal{X}}\left(\kappa_6^2\mathcal{T}^\theta_\theta-\Lambda_6-\frac{5}{12\alpha}\right), \quad \text{(C2)}$$

$$(1+3f)\frac{\beta_2}{\beta A}=-(1+f)-\frac{3}{A^2}\left(\frac{\dot{\beta}}{n\beta}\right)^2-\frac{1}{6\alpha\mathcal{X}^2}\left(\kappa_6^2\mathcal{T}^\theta_\theta-\Lambda_6-\frac{5}{12\alpha}\right). \quad \text{(C3)}$$

Both the regular rr, rt equations (4.15) and (4.16), or the equivalent equations (B9) and (B13), contain A', N' instead of a_2, n_2. So, it is better to write (C1) and (C2) as

$$(1+3f)A'=(1+3f)\left(\frac{1}{12\alpha}-A^2\right)+f\mathcal{X}-\frac{3\mathcal{X}}{2A^2}\left(\frac{\dot{\beta}}{n\beta}\right)^2-\frac{1}{12\alpha\mathcal{X}}\left(\kappa_6^2\mathcal{T}^\theta_\theta-\Lambda_6-\frac{5}{12\alpha}\right), \quad \text{(C4)}$$

$$(1+3f)N'=(1+3f)\left(\frac{1}{12\alpha}-N^2\right)-\frac{1+4f+f^2}{2}\mathcal{X}+\frac{\mathcal{X}}{2A^2}\left(\frac{\dot{\beta}}{n\beta}\right)^2+\frac{1+2f}{12\alpha\mathcal{X}}\left(\kappa_6^2\mathcal{T}^\theta_\theta-\Lambda_6-\frac{5}{12\alpha}\right). \quad \text{(C5)}$$

From (C4) and (C5) we get

$$fA'-N'=\frac{f-1}{12\alpha}+f(f-1)A^2+\frac{f+1}{2}\mathcal{X}-\frac{\mathcal{X}}{2A^2}\left(\frac{\dot{\beta}}{n\beta}\right)^2-\frac{1}{12\alpha\mathcal{X}}\left(\kappa_6^2\mathcal{T}^\theta_\theta-\Lambda_6-\frac{5}{12\alpha}\right). \quad \text{(C6)}$$

As far as Eq. (B9) is concerned, we use (4.5) and (4.8) to express $\mathcal{Y}$ in terms of $f, \mathcal{X}$ and $X+Y$ in terms of $f, \mathcal{X}, A^2$. We also use (C4) and (C6) to find A', N' in (B9). Finally, we have everything in terms of $f, \mathcal{X}, A^2$ and (B9) becomes identically satisfied if $\mathcal{T}^r_r = \mathcal{T}^\theta_\theta$ on the brane. This shows the nontrivial consistency of the regular rr equation (4.15).

The manipulation of Eq. (B13) is more tricky and the starting point is to take the combination C3 $-\frac{2}{\mathcal{X}}$(C4). Then, we get

$$\frac{\beta_2}{\beta A} - \frac{2}{\mathcal{X}}A' = \frac{2}{\mathcal{X}}\left(A^2 - \frac{1}{12\alpha}\right) - 1. \tag{C7}$$

Differentiating (C7) with respect to time and using the expression for $\dot{\mathcal{X}}$ with respect to $\mathcal{X}, \mathcal{Y}, \beta$, we get some of the difficult terms appearing in (B13):

$$\frac{\dot\beta_2}{n\beta} + \left(\frac{\mathcal{X}'}{\mathcal{X}} - \frac{\beta_2}{\beta}\right)\frac{\dot\beta}{n\beta} - \frac{2A}{\mathcal{X}}\frac{\dot A'}{n} = \frac{\beta_2}{\beta A}\frac{\dot A}{n} + \frac{X'}{\mathcal{X}}\frac{\dot\beta}{n\beta} - \frac{4AA'}{\mathcal{X}^2}H(\mathcal{Y} - \mathcal{X}) - \frac{2A}{\mathcal{X}^2}\frac{\dot{\mathcal{X}}}{n}\left(A^2 - \frac{1}{12\alpha}\right) + \frac{4A^2}{\mathcal{X}}\frac{\dot A}{n}. \tag{C8}$$

Using Eqs. (C6), (C7), (5.2), and (5.3) in (C8) in order to find the coefficient of A', we obtain extra terms of (B13):

$$\begin{aligned}&\frac{\dot\beta_2}{n\beta} + \left(\frac{\mathcal{X}'}{\mathcal{X}} - \frac{\beta_2}{\beta} - \frac{A'}{A}\right)\frac{\dot\beta}{n\beta} - \frac{2A}{\mathcal{X}}\left[\frac{\dot A'}{n} + H(A' - N')\right]\\ &\quad= \left[\frac{1}{\mathcal{X}}\left(6A^2 - \frac{1}{6\alpha}\right) - 1\right]\frac{\dot A}{n} + \frac{2AH}{\mathcal{X}}\left[(1+3f)A' + \frac{\mathcal{X}}{A^2}\frac{\dot\beta^2}{n^2\beta^2}\right.\\ &\qquad\left. - \frac{X}{H}\frac{\dot\beta}{n\beta} - \frac{\dot{\mathcal{X}}}{nH\mathcal{X}}\left(A^2 - \frac{1}{12\alpha}\right) + (1-f)\left(fA^2 + \frac{1}{12\alpha}\right) - \frac{f+1}{2}\mathcal{X} + \frac{1}{12\alpha\mathcal{X}}\left(\kappa_6^2\mathcal{T}^\theta_\theta - \Lambda_6 - \frac{5}{12\alpha}\right)\right].\end{aligned} \tag{C9}$$

Finally, substituting again $\dot A$ from (5.3), $(1+3f)A'$ from (C4) and $\dot{\mathcal{X}}$ in (B13), this becomes an identity, given that $\mathcal{T}^t_r = 0$ on the brane. This shows the nontrivial consistency of the regular rt equation (4.16).

---

[1] B. Carter, J. Geom. Phys. **8**, 53 (1992); arXiv:hep-th/9705172; Int. J. Theor. Phys. **40**, 2099 (2001); R. Capovilla and J. Guven, Phys. Rev. D **51**, 6736 (1995); **52**, 1072 (1995); A. Kehagias and E. Kiritsis, J. High Energy Phys. 11 (1999)022.

[2] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).

[3] T. Quinn and R. Wald, Phys. Rev. D **56**, 3381 (1997); S. Gralla and R. Wald, Classical Quantum Gravity **25**, 205009 (2008).

[4] W. Israel, Nuovo Cimento B **44**, 1 (1966); **48**,463(E) (1967).

[5] W. Israel, Phys. Rev. D **15**, 935 (1977).

[6] R. Geroch and J. Traschen, Phys. Rev. D **36**, 1017 (1987).

[7] D. Garfinkle, Classical Quantum Gravity **16**, 4101 (1999).

[8] A. Vilenkin, Phys. Rev. D **23**, 852 (1981); Phys. Rep. **121**, 263 (1985).

[9] J. A. G. Vickers, Classical Quantum Gravity **4**, 1 (1987); V. P. Frolov, W. Israel, and W. G. Unruh, Phys. Rev. D **39**, 1084 (1989); W. G. Unruh, G. Hayward, W. Israel, and D. Mcmanus, Phys. Rev. Lett. **62**, 2897 (1989); C. J. S. Clarke, J. A. Vickers, and G. F. R. Ellis, Classical Quantum Gravity **7**, 1 (1990); K. Nakamura, Prog. Theor. Phys. **110**, 201 (2003).

[10] J. M. Cline, J. Descheneau, M. Giovannini, and J. Vinet, J. High Energy Phys. 06 (2003)048.

[11] I. Moss and S. Poletti, Phys. Lett. B **199**, 34 (1987).

[12] P. Bostock, R. Gregory, I. Navarro, and J. Santiago, Phys. Rev. Lett. **92**, 221601 (2004).

[13] C. Charmousis, G. Kofinas, and A. Papazoglou, J. Cosmol. Astropart. Phys. 01 (2010)022.

[14] D. Lovelock, J. Math. Phys. (N.Y.) **12**, 498 (1971); B. Zumino, Phys. Rep. **137**, 109 (1986).

[15] S. Kanno and J. Soda, J. Cosmol. Astropart. Phys. 07 (2004) 002.

[16] C. Germani and C. Sopuerta, Phys. Rev. Lett. **88**, 231101 (2002); S. Davis, Phys. Rev. D **67**, 024030 (2003); E. Gravanis and S. Willison, Phys. Lett. B **562**, 118 (2003); R. C. Myers, Phys. Rev. D **36**, 392 (1987).

[17] C. Charmousis and R. Zegers, J. High Energy Phys. 08 (2005) 075.

[18] R. Geroch and P. Jang, J. Math. Phys. (N.Y.) **16**, 65 (1975).

[19] J. Ehlers and R. Geroch, Ann. Phys. (Amsterdam) **309**, 232 (2004).

[20] G. Kofinas and T. Tomaras, Classical Quantum Gravity **24**, 5861 (2007).

[21] A. Davidson and I. Gurwich, Phys. Rev. D **74**, 044023 (2006).
[22] T. Regge and C. Teitelboim, in *Proceedings of the Marcel Grossman Meeting, Trieste, 1975*, edited by R. Ruffini (North-Holland Publishing Company, 1977), p. 77.
[23] G. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000); G. Dvali and G. Gabadadze, Phys. Rev. D **63**, 065007 (2001); C. Deffayet, Phys. Lett. B **502**, 199 (2001).
[24] R. Capovilla and J. Guven, Phys. Rev. D **57**, 5158 (1998).
[25] R. Zegers, J. High Energy Phys. 03 (2008) 066.
[26] B. Carter, Int. J. Theor. Phys. **40**, 2099 (2001); A. Battye and B. Carter, Phys. Lett. B **509**, 331 (2001).
[27] S. Deser, F. A. E. Pirani, and D. C. Robinson, Phys. Rev. D **14**, 3301 (1976).
[28] G. Kofinas, J. High Energy Phys. 08 (2001) 034.
[29] G. Kofinas, Phys. Lett. B **633**, 141 (2006).
[30] P. Binetruy, C. Deffayet, U. Ellwanger, and D. Langlois, Phys. Lett. B **477**, 285 (2000).
[31] R. Gregory, Nucl. Phys. **B467**, 159 (1996); C. Charmousis, R. Emparan, and R. Gregory, J. High Energy Phys. 05 (2001) 026.
[32] C. Bambi, M. Giannotti, and F. L. Villante, Phys. Rev. D **71**, 123524 (2005).
[33] G. Kofinas, R. Maartens, and E. Papantonopoulos, J. High Energy Phys. 10 (2003) 066.
[34] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**,4690 (1999).
[35] C. Charmousis and A. Papazoglou, J. High Energy Phys. 07 (2008) 062.
[36] D. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985); J. T. Wheeler, Nucl. Phys. **B268**, 737 (1986); R. Cai, Phys. Rev. D **65**, 084014 (2002).
[37] J. Gott, Astrophys. J. **288**, 422 (1985); W. Hiscock, Phys. Rev. D **31**, 3288 (1985); B. Linet, Gen. Relativ. Gravit. **17**, 1109 (1985); D. Garfinkle, Phys. Rev. D **32**, 1323 (1985).
[38] K. P. Tod, Classical Quantum Gravity **11**, 1331 (1994); J. Maluf and A. Kneip, J. Math. Phys. (N.Y.) **38**, 458 (1997).
[39] B. Boisseau, C. Charmousis, and B. Linet, Phys. Rev. D **55**, 616 (1997).
[40] C. Burgess, Ann. Phys. (Amsterdam) **313**, 383 (2004); arXiv:hep-th/0510123.