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Ultraviolet and infrared perturbative finiteness of massless QED_3

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We discuss the ultraviolet and infrared perturbative finiteness of massless QED_3, which is parity and infrared anomaly-free to all orders in perturbation theory.

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The perturbative finiteness is one of the most peculiar properties of topological field theories in three spacetime dimensions [1]. Thanks to a by-product of super-renormalizability and the presence of topological terms, Yang-Mills-Chern-Simons and BF-Yang-Mills theories are also finite at all orders in perturbation theory—in the sense of a vanishing β function [2]. In spite of not being a topological field theory, the massless QED_3 is perturbatively finite, exhibiting quite interesting and subtle properties such as super-renormalizability, parity invariance, and the presence of infrared divergences. The issue of "how super-renormalizable interactions cure their infrared divergences" has been analyzed in [3], and a possible parity breaking at the quantum level, which is called a parity anomaly in the literature, has been discarded [4–6].

The algebraic proof we are presenting in this paper on the ultraviolet and infrared finiteness, and the absence of a parity and infrared anomaly, in the massless QED_3, is based on general theorems of perturbative quantum field theory [7–10], where the Lowenstein-Zimmermann subtraction scheme is adopted. Here we summarize the main results, skipping the intermediate steps of the Lowenstein-Zimmermann subtraction scheme in the framework of the Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein renormalization method [10]. Such a subtraction scheme has to be introduced, thanks to the presence of massless (gauge and fermion) fields, in order to subtract infrared divergences that should arise in the process of the ultraviolet subtractions.

The discussion of the extension of the theory in the tree approximation to all orders in perturbation theory is organized according to two independent parts: In the first step, we study the stability of the classical action. For the quantum theory the stability corresponds to the fact that the radiative corrections can be reabsorbed by a redefinition of the initial parameters of the theory. Next, one computes the possible anomalies through an analysis of the Wess-Zumino consistency condition; then one checks if the possible breakings induced by radiative corrections can be fine-tuned by a suitable choice of noninvariant local counterterms.

The gauge invariant action for the massless QED_3, with the gauge invariant Lowenstein-Zimmermann mass terms added, is given by

$$\Sigma_{\rm inv}^{(s-1)} = \int d^3x \left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}D\!\!\!/\,\psi + \underbrace{\frac{\mu}{2}(s-1)\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - m(s-1)\bar{\psi}\psi}_{\text{Lowenstein-Zimmermann mass terms}} \right\}, \tag{1}$$

where $D\!\!\!/\,\psi \equiv (\partial\!\!\!/ + ieA\!\!\!/)\psi$ and e is a dimensionful coupling constant with mass dimension $\frac{1}{2}$. The Lowenstein-Zimmermann parameter s lies in the interval $0 \le s \le 1$ and plays the role of an additional subtraction variable (such as the external momentum) in the BPHZL renormalization program, such that the massless QED_3 is recovered for $s = 1$.

In the BPHZL scheme a subtracted (finite) integrand $R(p,k,s)$ is written in terms of the unsubtracted (divergent) one, $I(p,k,s)$, as

$$\begin{aligned} R(p,k,s) &= (1 - t^0_{p,s-1})(1 - t^1_{p,s})I(p,k,s) \\ &= (1 - t^0_{p,s-1} - t^1_{p,s} + t^0_{p,s-1}t^1_{p,s})I(p,k,s), \end{aligned}$$

where $t^d_{x,y}$ is the Taylor series about $x = y = 0$ to order d if $d \ge 0$. Thus, for our purposes, by assuming $s = 1$, a subtracted integrand $R(p,k,s)$ reads

$$R(p,k,1) = \underbrace{I(p,k,1)}_{\text{parity-even}} - \underbrace{I(0,k,1)}_{\text{parity-even}} - \underbrace{p^\rho \frac{\partial}{\partial p^\rho} I(0,k,0)}_{\text{parity-odd terms}}.$$

In order to quantize the system (1) one has to add a gauge-fixing action $\Sigma_{\rm gf}$ and an action term $\Sigma_{\rm ext}$, coupling the nonlinear Becchi-Rouet-Stora transformations to external sources:

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$$\Sigma_{\rm gf} = \int d^3x \left\{ b\partial^\mu A_\mu + \frac{\xi}{2}b^2 + \bar{c}\Box c \right\}, \tag{2}$$

$$\Sigma_{\rm ext} = \int d^3x \{ \bar{\Omega} s\psi - s\bar{\psi}\Omega \}. \tag{3}$$

No Lowenstein-Zimmermann mass has to be introduced to the Faddeev-Popov ghosts since they are free fields; therefore, they decouple.

The BRS transformations are given by

$$\begin{aligned} &s\psi = ic\psi, \qquad s\bar{\psi} = -ic\bar{\psi}, \qquad sA_\mu = -\frac{1}{e}\partial_\mu c, \\ &sc = 0, \qquad s\bar{c} = \frac{1}{e}b, \qquad sb = 0, \end{aligned} \tag{4}$$

where c is the ghost, $\bar{c}$ is the antighost, and b is the Lagrange multiplier field.

The complete action $\Sigma^{(s-1)}$ reads

$$\Sigma^{(s-1)} = \Sigma^{(s-1)}_{\rm inv} + \Sigma_{\rm gf} + \Sigma_{\rm ext}. \tag{5}$$

The UV and IR dimensions—those which are involved in the Lowenstein-Zimmermann subtraction scheme [10]—d and r, respectively, as well as the ghost numbers $\Phi\Pi$ and the Grassmann parity GP, of all fields are collected in Table I.

The BRS invariance of the action is expressed in a functional way by the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma^{(s-1)}) = 0, \tag{6}$$

where the Slavnov-Taylor operator $\mathcal{S}$ is defined, acting on an arbitrary functional $\mathcal{F}$, by

$$\begin{aligned} \mathcal{S}(\mathcal{F}) = \int d^3x \bigg\{ & -\frac{1}{e}\partial^\mu c \frac{\delta\mathcal{F}}{\delta A^\mu} + \frac{1}{e} b \frac{\delta\mathcal{F}}{\delta\bar{c}} + \frac{\delta\mathcal{F}}{\delta\bar{\Omega}}\frac{\delta\mathcal{F}}{\delta\psi} \\ & - \frac{\delta\mathcal{F}}{\delta\Omega}\frac{\delta\mathcal{F}}{\delta\bar{\psi}} \bigg\}. \end{aligned} \tag{7}$$

The corresponding linearized Slavnov-Taylor operator reads

$$\begin{aligned} \mathcal{S}_\mathcal{F} = \int d^3x \bigg\{ & -\frac{1}{e}\partial^\mu c \frac{\delta}{\delta A^\mu} + \frac{1}{e} b \frac{\delta}{\delta\bar{c}} + \frac{\delta\mathcal{F}}{\delta\bar{\Omega}}\frac{\delta}{\delta\psi} \\ & + \frac{\delta\mathcal{F}}{\delta\psi}\frac{\delta}{\delta\bar{\Omega}} - \frac{\delta\mathcal{F}}{\delta\Omega}\frac{\delta}{\delta\bar{\psi}} - \frac{\delta\mathcal{F}}{\delta\bar{\psi}}\frac{\delta}{\delta\Omega} \bigg\}. \end{aligned} \tag{8}$$

The following nilpotency identities hold:

TABLE I. UV and IR dimensions (d and r), ghost numbers ($\Phi\Pi$), and Grassmann parity (GP).

	A_μ	ψ	c	$\bar{c}$	b	Ω	$s-1$	s
d	1/2	1	0	1	3/2	2	1	1
r	1/2	1	0	1	3/2	2	1	0
$\Phi\Pi$	0	0	1	−1	0	−1	0	0
GP	0	1	1	1	0	1	0	0

$$\mathcal{S}_\mathcal{F}\mathcal{S}(\mathcal{F}) = 0, \quad \forall\mathcal{F}, \tag{9}$$

$$\mathcal{S}_\mathcal{F}\mathcal{S}_\mathcal{F} = 0 \quad \text{if } \mathcal{S}(\mathcal{F}) = 0. \tag{10}$$

In particular, $(\mathcal{S}_\Sigma)^2 = 0$, since the action $\Sigma^{(s-1)}$ obeys the Slavnov-Taylor identity (6). The operation of $\mathcal{S}_\Sigma$ upon the fields and the external sources is given by

$$\begin{aligned} &\mathcal{S}_\Sigma\phi = s\phi, \quad \phi = \psi, \quad \bar{\psi}, \quad A_\mu, \quad c, \quad \bar{c}, \quad b, \\ &\mathcal{S}_\Sigma\Omega = -\frac{\delta\Sigma^{(s-1)}}{\delta\bar{\psi}}, \quad \mathcal{S}_\Sigma\bar{\Omega} = \frac{\delta\Sigma^{(s-1)}}{\delta\psi}. \end{aligned} \tag{11}$$

The classical action $\Sigma^{(s-1)}$ is moreover characterized by the gauge condition, the ghost equation, and the antighost equation, given by

$$\frac{\delta\Sigma^{(s-1)}}{\delta b} = \partial^\mu A_\mu + \xi b, \tag{12}$$

$$\frac{\delta\Sigma^{(s-1)}}{\delta\bar{c}} = \Box c, \tag{13}$$

$$-i\frac{\delta\Sigma^{(s-1)}}{\delta c} = i\Box\bar{c} + \bar{\Omega}\psi + \bar{\psi}\Omega. \tag{14}$$

The action is invariant also with respect to the rigid symmetry

$$W_{\rm rigid}\Sigma^{(s-1)} = 0, \tag{15}$$

where the Ward operator $W_{\rm rigid}$ is defined by

$$W_{\rm rigid} = \int d^3x \left\{ \psi\frac{\delta}{\delta\psi} - \bar{\psi}\frac{\delta}{\delta\bar{\psi}} + \Omega\frac{\delta}{\delta\Omega} - \bar{\Omega}\frac{\delta}{\delta\bar{\Omega}} \right\}. \tag{16}$$

The classical action for the massless QED_3 ($s = 1$) is also invariant under parity P; its action upon the fields and external sources is fixed as below:

$$\begin{aligned} &x_\mu \xrightarrow{P} x^P_\mu = (x_0, -x_1, x_2), \qquad \psi \xrightarrow{P} \psi^P = -i\gamma^1\psi, \\ &\bar{\psi} \xrightarrow{P} \bar{\psi} \xrightarrow{P} = i\bar{\psi}\gamma^1, \qquad A_\mu \xrightarrow{P} A^P_\mu = (A_0, -A_1, A_2), \\ &\phi \xrightarrow{P} \phi^P = \phi, \qquad \phi = c, \quad \bar{c}, \quad b, \\ &\Omega \xrightarrow{P} \Omega^P = -i\gamma^1\Omega, \qquad \bar{\Omega} \xrightarrow{P} \bar{\Omega}^P = i\bar{\Omega}\gamma^1. \end{aligned} \tag{17}$$

In order to verify if the action in the tree approximation is stable under radiative corrections, we perturb it by an arbitrary integrated local functional (counterterm) $\Sigma^{c(s-1)}$, such that

$$\tilde{\Sigma}^{(s-1)} = \Sigma^{(s-1)} + \varepsilon\Sigma^{c(s-1)}, \tag{18}$$

where ε is an infinitesimal parameter. The functional $\Sigma^c \equiv \Sigma^c|_{s=1}$ has the same quantum numbers as the action in the tree approximation at $s = 1$.

The deformed action $\tilde{\Sigma}^{(s-1)}$ must still obey all the constraints listed above, Eqs. (12)–(15). Then $\Sigma^{c(s-1)}$ is subjected to the following set of constraints:

$$\mathcal{S}_{\Sigma}\Sigma^{c(s-1)} = 0, \tag{19}$$

$$\frac{\delta\Sigma^{c(s-1)}}{\delta b} = \frac{\delta\Sigma^{c(s-1)}}{\delta\bar{c}} = \frac{\delta\Sigma^{c(s-1)}}{\delta c} = 0, \tag{20}$$

$$W_{\mathrm{rigid}}\Sigma^{c(s-1)} = 0. \tag{21}$$

We find that the most general invariant counterterm $\Sigma^{c(s-1)}$, i.e., the most general field polynomial with UV and IR dimensions bounded by $d \leq 3$ and $r \geq \frac{5}{2}$, with ghost number zero and fulfilling the conditions displayed in Eqs. (19)–(21), is given by

$$\Sigma^{c(s-1)} = \int d^3x\{a_1 F^{\mu\nu}F_{\mu\nu} + a_2 i\bar{\psi}\not{D}\psi + a_3\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + a_4\bar{\psi}\psi\}. \tag{22}$$

However, there are other restrictions due to the super-renormalizability of the theory and its parity invariance—the massless QED_3 recovered for $s = 1$. From the super-renormalizability, the coupling constant-dependent power-counting formula [2] is given by

$$\begin{pmatrix}\delta(\gamma)\\ \rho(\gamma)\end{pmatrix} = 3 - \sum_{\Phi}\begin{pmatrix}d_\Phi\\ r_\Phi\end{pmatrix}N_\Phi - \frac{1}{2}N_e, \tag{23}$$

for the UV ($\delta(\gamma)$) and IR ($\rho(\gamma)$) degrees of divergence of a one-particle irreducible Feynman graph γ. Here N_Φ is the number of external lines of γ corresponding to the field Φ; d_Φ and r_Φ are the UV and IR dimensions of Φ, respectively, as given in Table I; and N_e is the power of the coupling constant e in the integral corresponding to the diagram γ. Since the counterterms are generated by loop graphs, they are of order 2 in e at least. Hence, the effective UV and IR dimensions of the counterterm $\Sigma^{c(s-1)}$ are bounded by $d \leq 2$ and $r \geq \frac{3}{2}$; for this reason, $a_1 = a_2 = 0$. Moreover, since the counterterm $\Sigma^c \equiv \Sigma^c|_{s=1}$ is also parity invariant, it yields that $a_3 = a_4 = 0$. It can be concluded that there is no possibility for any local deformation, implying the absence of any counterterm:

$$\Sigma^c = \Sigma^c|_{s=1} = 0. \tag{24}$$

This result means that the usual ambiguities due to the renormalization procedure do not appear in the present model.

Because the classical stability does not imply, in general, the possibility of extending the theory to the quantum level, our purpose now is to show the absence of anomalies. This result, combined with the previous one (24), concerning the absence of counterterms, completes the proof of the perturbative finiteness and absence of a parity anomaly in massless QED_3.

At the quantum level the vertex functional $\Gamma^{(s-1)}$, which coincides with the classical action (5) at order 0 in $\hbar$,

$$\Gamma^{(s-1)} = \Sigma^{(s-1)} + \mathcal{O}(\hbar), \tag{25}$$

has to satisfy the same constraints as the classical action does, namely, Eqs. (12)–(15).

According to the quantum action principle [7,9] the Slavnov-Taylor identity (6) may have a quantum breaking

$$\mathcal{S}(\Gamma^{(s-1)}) = \Delta\cdot\Gamma^{(s-1)}|_{s=1} = \Delta + \mathcal{O}(\hbar\Delta), \tag{26}$$

where $\Delta \equiv \Delta|_{s=1}$ is an integrated local functional, taken at $s = 1$, with ghost number 1 and UV and IR dimensions bounded by $d \leq \frac{7}{2}$ and $r \geq 3$.

The nilpotency identity (9), together with

$$\mathcal{S}_\Gamma = \mathcal{S}_\Sigma + \mathcal{O}(\hbar), \tag{27}$$

implies the following consistency condition for the breaking Δ:

$$\mathcal{S}_\Sigma\Delta = 0. \tag{28}$$

Beyond that, Δ satisfies

$$\frac{\delta\Delta}{\delta b} = \frac{\delta\Delta}{\delta\bar{c}} = \int d^3x\frac{\delta}{\delta c}\Delta = W_{\mathrm{rigid}}\Delta = 0. \tag{29}$$

The Wess-Zumino consistency condition (28) constitutes a cohomology problem in the sector of ghost number 1. Its solution can always be written as a sum of a trivial cocycle $\mathcal{S}_\Sigma\hat{\Delta}^{(0)}$, where $\hat{\Delta}^{(0)}$ has ghost number 0, and of nontrivial elements belonging to the cohomology of $\mathcal{S}_\Sigma$ (8) in the sector of ghost number 1:

$$\Delta^{(1)} = \hat{\Delta}^{(1)} + \mathcal{S}_\Sigma\hat{\Delta}^{(0)}. \tag{30}$$

It should be stressed that it still remains a possible parity violation at the quantum level induced by a parity-odd noninvariant counterterm. Due to the fact that the Lowenstein-Zimmermann subtraction scheme breaks parity during the intermediary steps, the Slavnov-Taylor identity breaking $\Delta^{(1)}$ is not necessarily parity invariant. In any case, $\Delta^{(1)}$ must obey the conditions imposed by Eqs. (28) and (29). The trivial cocycle $\mathcal{S}_\Sigma\hat{\Delta}^{(0)}$ can be absorbed into the vertex functional $\Gamma^{(s-1)}$ as a noninvariant integrated local counterterm $-\hat{\Delta}^{(0)}$. On the other hand, a nonzero $\hat{\Delta}^{(1)}$ would represent an anomaly. If there was any parity-odd $\hat{\Delta}^{(0)}_{\mathrm{odd}}$, a parity anomaly would be present, induced by the noninvariant counterterm $-\hat{\Delta}^{(0)}_{\mathrm{odd}}$.

By analyzing the Slavnov-Taylor operator $\mathcal{S}_\Sigma$ (8) and Eq. (26), one sees that the breaking $\Delta^{(1)}$ has UV and IR dimensions bounded by $d \leq \frac{7}{2}$ and $r \geq 3$. But being an effect of the radiative corrections, the insertion $\Delta^{(1)}$ possesses a factor e^2 at least, and thus its effective dimensions are in fact bounded by $d \leq \frac{5}{2}$ and $r \geq 2$.

From the antighost equation

$$\int d^3x \frac{\delta}{\delta c} \Delta^{(1)} = 0, \tag{31}$$

it can be concluded that $\Delta^{(1)}$ is given by

$$\Delta^{(1)} = \int d^3x (\partial^\mu c) \mathcal{K}_\mu, \tag{32}$$

where $\mathcal{K}_\mu$ has UV and IR dimensions bounded by $d \le \frac{3}{2}$ and $r \ge 1$, and the ghost c is dimensionless. Now, $\Delta^{(1)}$ can be split into two pieces that are even and odd under parity by writing $\mathcal{K}_\mu$ as

$$\mathcal{K}_\mu = r_{\mathrm{v}} \mathcal{V}_\mu + r_{\mathrm{p}} \mathcal{P}_\mu, \tag{33}$$

in such a way that $\mathcal{V}_\mu$ is a vector and $\mathcal{P}_\mu$ a pseudovector. Bearing in mind that $\mathcal{K}_\mu$ has its UV and IR dimensions bounded by $d \le \frac{3}{2}$ and $r \ge 1$, we conclude that there are no $\mathcal{V}_\mu$ satisfying these dimensional constraints; therefore, $\{\mathcal{V}_\mu\} = \varnothing$, which means the absence of parity-even Slavnov-Taylor breaking. However, still remains the odd sector represented by $\mathcal{P}_\mu$, and by a dimensional analysis a candidate for $\mathcal{P}_\mu$ is found. The only candidate which survives all the constraints above is

$$\mathcal{P}_\mu = \tilde{F}_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho}. \tag{34}$$

It turns out that there is only one parity-odd candidate, $\Delta^{(1)}_{\mathrm{odd}}$, which could be a parity anomaly, surviving all the constraints above:

$$\Delta^{(1)} = \Delta^{(1)}_{\mathrm{odd}} = \frac{r_{\mathrm{p}}}{2} \int d^3x (\partial^\mu c) \epsilon_{\mu\nu\rho} F^{\nu\rho}, \tag{35}$$

where integrating by parts shows that

$$\Delta^{(1)} = \Delta^{(1)}_{\mathrm{odd}} \equiv 0. \tag{36}$$

Hence, there are no radiative corrections to the insertion describing the breaking of the Slavnov-Taylor identity, $\{\Delta^{(1)}\} = \varnothing$, which means that there is no possible breaking to the Slavnov-Taylor identity, and neither parity is violated nor infrared anomaly stems by noninvariant counterterms that could be induced due to the Lowenstein-Zimmermann subtraction scheme—which breaks parity.

We finally conclude that the massless QED_3 is infrared and ultraviolet finite (vanishing coupling constant β_e function and anomalous dimensions of the fields), infrared and parity anomaly-free at all orders in perturbation theory. The latter is a by-product of super-renormalizability and the absence of parity-odd noninvariant counterterms.

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