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Stationary black diholes

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In this paper, we present and analyze the simplest physically meaningful model for stationary black diholes—a binary configuration of counterrotating Kerr-Newman black holes endowed with opposite electric charges—elaborated in a physical parametrization on the basis of one of the Ernst-Manko-Ruiz equatorially antisymmetric solutions of the Einstein-Maxwell equations. The model saturates the Gabach-Clement inequality for interacting black holes with struts, and in the absence of rotation, it reduces to the Emparan-Teo electric dihole solution. The physical characteristics of each dihole constituent satisfy identically the well-known Smarr's mass formula.

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I. INTRODUCTION

In their paper [1], Emparan and Teo constructed and analyzed for the first time an exact electrostatic solution of the Einstein-Maxwell equations describing a nonextremal dihole—a configuration consisting of two nonextremal Reissner-Nordström black holes [2,3] endowed with equal masses and opposite charges of the same magnitude. The black-hole constituents in the Emparan-Teo solution are prevented from falling onto each other by a massless strut (its rough Newtonian analog is a thin rod whose mass can be neglected), the pressure inside of which permits one to get information about the interaction force between the constituents. Various thermodynamical properties of the interacting static black holes were also studied in Ref. [1], which became possible thanks to a known nice physical property of the spacetimes with struts—the struts do not contribute into the total gravitational energy of the system. At the end of their paper, Emparan and Teo mentioned that an extension of their results to the case of rotation might be an interesting work to be done in the future, and the problem of obtaining such an extension seems to have been attacked in a recent paper of Cabrera-Munguia *et al.* [4] who constructed a specific four-parameter exact solution for two oppositely charged counterrotating Kerr-Newman (KN) black holes [5]. Though technically, Ref. [4] is correct, the solution itself, in our opinion, exhibits some unphysical features because of the presence in it of nonvanishing magnetic charges created by rotation of electric charges, which contradicts the known cases of a single KN solution and of the Bretón-Manko (BM) solution [6,7] for a pair of identical counterrotating KN black holes where the electric charges generate a *dipole* magnetic field without magnetic monopoles. As a consequence, the usual Smarr mass formula [8] (not taking into account the contribution of magnetic charges) does not hold for the black-hole constituents comprising that binary configuration, and, moreover, the expression of the important geometric quantity σ obtained in Ref. [4] depends implicitly on the root of a cubic algebraic equation being, therefore, more complicated than, for instance, the analogous quantity of the physically parametrized BM solution.

The present paper aims at working out a unique four-parameter model for stationary diholes with antiparallel rotation of its constituents, in which would be absent not only the total but also individual magnetic charges. To accomplish this task, we shall make use of a five-parameter asymptotically flat specialization of the Ernst-Manko-Ruiz (EMR) equatorially antisymmetric solution [9,10] possessing arbitrary parameters of electric and magnetic dipole moments, which will enable us to eliminate the individual magnetic charges of the constituents by choosing appropriately the values of the latter dipole parameters. This will secure the validity of the standard Smarr formula for each black-hole constituent and, in turn, will enable us to find a remarkably simple expression for σ in terms of the Komar quantities [11] which is a key point for elaborating the physical parametrization of the whole model. After having reached our main objective, we will prove that, similar to the BM model of equally charged counterrotating black-hole constituents, the configuration obtained for KN black holes with opposite electric charges verifies (and actually saturates) the inequality for interacting black holes with struts recently derived by Gabach Clement [12].

II. THE FIVE-PARAMETER ASYMPTOTICALLY FLAT EMR SOLUTION IN σ REPRESENTATION

A key result of Ref. [4] is its expressions (14) for the axis data $e(z)$ and $f(z)$ allowing one to construct in a straightforward manner the corresponding entire metric with the aid of the general formulas of the extended N-soliton solution [13] (we also refer the interested reader to the Appendix of Ref. [10] for a complete set of algebraic relations involved in the construction procedure of the $N=2$ case). However, an explanation the authors of Ref. [4] give to the origin of those expressions—the solution (11) of a complicated system of algebraic

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equations for certain metric functions—raises, in turn, the question of how these formulas (11) were obtained, and below we will give a simple derivation of their data (14) with one additional arbitrary real parameter representing a magnetic dipole moment.

As a starting point of the derivation procedure, we take the following axis data obtained in Ref. [9] for an equatorially antisymmetric spacetime [14] with both electric and magnetic dipole moments:

$$e(z) = \frac{z^2 - b_1 z + b_2}{z^2 + b_1 z + b_2}, \qquad f(z) = \frac{c_2}{z^2 + b_1 z + b_2}, \quad (1)$$

where b_1, b_2, and c_2 are arbitrary complex constants. These data rewritten in an equivalent representation were used in Ref. [9] for constructing the corresponding Ernst potentials [15] $\mathcal{E}$ and Φ of a six-parameter EMR solution. If, nevertheless, one opts to work directly with Eq. (1), then the asymptotic flatness of the solution implies immediately that b_1 is a real constant related to the total mass $2M$ of the binary configuration as $b_1 = 2M$. Choosing then the constant c_2 in the form $c_2 = 2(q + ib)$, the real parameters q and b being associated, respectively, with the electric and magnetic dipole moments, and also formally setting $b_2 = c - i\delta$, we arrive at the five-parameter axis data

$$e(z) = \frac{e_-}{e_+}, \qquad f(z) = \frac{2(q+ib)}{e_+}, \\ e_\mp = z^2 \mp 2Mz + c - i\delta, \quad (2)$$

in which the real constants c and δ should yet be related to some physical or geometrical characteristics.

Recall now that the extended multisoliton solutions involve the constants α_n which satisfy the algebraic equation [16]

$$e(z) + \bar{e}(z) + 2f(z)\bar{f}(z) = 0 \quad (3)$$

(the bar over a symbol means complex conjugation), and in the equatorially antisymmetric case, these can be chosen in the form

$$\alpha_1 = -\alpha_4 = \frac{1}{2}R + \sigma, \qquad \alpha_2 = -\alpha_3 = \frac{1}{2}R - \sigma, \quad (4)$$

where R is a real constant representing the coordinate separation of the sources, and the parameter σ can take non-negative real or pure imaginary values (see Fig. 1). Then instead of the constants c and δ from the axis data (2), one is able to introduce the new parameters R and σ by equating coefficients at the same powers of z on the two sides of the equation

$$\frac{e_-}{e_+} + \frac{\bar{e}_-}{\bar{e}_+} + \frac{4(q^2+b^2)}{e_+\bar{e}_+} = \frac{2\prod_{n=1}^{4}(z-\alpha_n)}{e_+\bar{e}_+}, \quad (5)$$

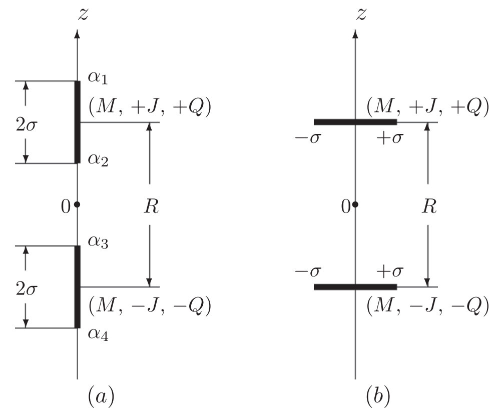

FIG. 1. Location of sources on the symmetry axis for two branches of the dihole solution: (a) a black dihole configuration composed of two KN black holes ($\sigma \geq 0$); (b) a hyperextreme dihole configuration composed of two superextreme KN constituents (pure imaginary σ).

thus, yielding

$$c = 2M^2 - \frac{1}{4}R^2 - \sigma^2, \\ \delta = \sqrt{(R^2 - 4M^2)(M^2 - \sigma^2) - 4(q^2 + b^2)}. \quad (6)$$

Accounting for Eq. (6), the five-parameter axis data (2) finally assume the form

$$e(z) = \frac{e_-}{e_+}, \qquad f(z) = \frac{2(q+ib)}{e_+}, \\ e_\mp = z^2 \mp 2Mz + 2M^2 - \frac{1}{4}R^2 - \sigma^2 - i\delta, \quad (7)$$

and by setting $b = 0$ in Eqs. (6) and (7), one recovers the axis data (14) of Ref. [4] [and, consequently, the quantities $\beta_{1,2}$ and $f_{1,2}$ in Eq. (11) of Ref. [4] via the simple fraction decomposition of $e(z)$ and $f(z)$]. It is worth noting that this procedure of changing parameters in the axis data was already described in application to the case of identical counterrotating uncharged black holes [17] and, moreover, has been recently used for obtaining a physical parametrization of the BM solution [7]. Furthermore, by virtue of the equatorial antisymmetry, the axis condition for the solution defined by the axis data (7) is satisfied automatically, and, therefore, there is no need to solve any additional algebraic equations for the metric functions.

As it is straightforward to elaborate by purely algebraic computing the explicit form of the Ernst potentials defined by the axis data (7), as well as the form of the corresponding metric functions f, γ, and ω entering the stationary axisymmetric line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \quad (8)$$

below we will restrict ourselves to only writing out the final expressions which reproduce and generalize the analogous formulas of Ref. [4]. Then, for $\mathcal{E}$ and Φ, we have

$$\begin{aligned}
&\mathcal{E} = \frac{A-B}{A+B}, \qquad \Phi = \frac{C}{A+B}, \\
&A = R^2[M^2(R^2-4\sigma^2)-4(q^2+b^2)](R_+-R_-)(r_+-r_-)+4\sigma^2[M^2(R^2-4\sigma^2)+4(q^2+b^2)](R_+-r_+)(R_--r_-) \\
&\quad + 2R\sigma(R^2-4\sigma^2)[R\sigma(R_+r_-+R_-r_+)+i\delta(R_+r_--R_-r_+)], \\
&B = 2MR\sigma(R^2-4\sigma^2)[R\sigma(R_++R_-+r_++r_-)-(2M^2-i\delta)(R_+-R_--r_++r_-)], \\
&C = 4(q+ib)R\sigma[(R+2\sigma)(R\sigma-2M^2-i\delta)(r_+-R_-)+(R-2\sigma)(R\sigma+2M^2+i\delta)(r_--R_+)],
\end{aligned} \tag{9}$$

where

$$R_\pm = \sqrt{\rho^2+(z+\frac{1}{2}R\pm\sigma)^2}, \qquad r_\pm = \sqrt{\rho^2+(z-\frac{1}{2}R\pm\sigma)^2}, \tag{10}$$

while the metric functions are given by the expressions

$$\begin{aligned}
&f = \frac{A\bar{A}-B\bar{B}+C\bar{C}}{(A+B)(\bar{A}+\bar{B})}, \qquad e^{2\gamma} = \frac{A\bar{A}-B\bar{B}+C\bar{C}}{16R^4\sigma^4(R^2-4\sigma^2)^2R_+R_-r_+r_-}, \qquad \omega = -\frac{\mathrm{Im}[2G(\bar{A}+\bar{B})+C\bar{I}]}{A\bar{A}-B\bar{B}+C\bar{C}}, \\
&G = -zB+R\sigma\{2R[M^2(R^2-4\sigma^2)-2(q^2+b^2)](R_-r_--R_+r_+)+4\sigma[M^2(R^2-4\sigma^2)+2(q^2+b^2)](r_+r_--R_+R_-) \\
&\quad + M(R+2\sigma)[(R-2\sigma)^2(R\sigma+2M^2-i\delta)-8(q^2+b^2)](R_--r_+) \\
&\quad + M(R-2\sigma)[(R+2\sigma)^2(R\sigma-2M^2+i\delta)+8(q^2+b^2)](R_+-r_-)\}, \\
&I = -zC+4M(q+ib)[R^2(2M^2-2\sigma^2+i\delta)(R_+r_++R_-r_-)+2\sigma^2(R^2-4M^2-2i\delta)(R_+R_-+r_+r_-)] \\
&\quad - 2(q+ib)(R^2-4\sigma^2)\{2M[(R\sigma+2M^2+i\delta)R_+r_--(R\sigma-2M^2-i\delta)R_-r_+] \\
&\quad + R\sigma[(R\sigma+6M^2+i\delta)(R_++r_-)+(R\sigma-6M^2-i\delta)(R_-+r_+)+8MR\sigma]\}.
\end{aligned} \tag{11}$$

The t and φ components of the electromagnetic four-potential are defined by the formulas

$$A_t = -\mathrm{Re}\left(\frac{C}{A+B}\right), \qquad A_\varphi = \mathrm{Im}\left(\frac{I}{A+B}\right), \tag{12}$$

and these complete the general mathematical description of the five-parameter EMR solution in σ parametrization.

At this point, several remarks on the formulas (9)–(12) might be appropriate. First, the above representation of the five-parameter EMR solution is fully equivalent to the known description of that solution worked out in Refs. [9,10] (with the Newman-Unti-Tamburino parameter ν set equal to zero). Second, it is highly important to underline that the arbitrary parameter σ of the solution is not restricted exclusively to real values (contrary to what was assumed in Ref. [4]) but can also take pure imaginary values determining the hyperextreme part of the solution. The significance of this point will be fully understandable later on when we express σ in terms of the Komar quantities. Third, the σ representation of the EMR solution should be only considered as an intermediate parametrization that could be suitable for elaborating the final physical representation in which σ must be replaced by a rotation parameter.

To gain a better insight into the structure of the EMR solution, let us consider its first four Beig-Simon multipole moments [18–20], which can be found with the aid of the Hoenselaers-Perjés procedure [21,22]:

$$\begin{aligned}
&M_0 = 2M, \qquad M_1 = 0, \qquad M_2 = \frac{1}{2}M(R^2-8M^2+4\sigma^2), \qquad M_3 = 0, \\
&J_0 = J_1 = 0, \qquad J_2 = 2M\delta, \qquad J_3 = 0, \\
&Q_0 = 0, \qquad Q_1 = 2q, \qquad Q_2 = 0, \qquad Q_3 = \frac{1}{2}q(R^2-8M^2+4\sigma^2)-2b\delta, \\
&B_0 = 0, \qquad B_1 = 2b, \qquad B_2 = 0, \qquad B_3 = \frac{1}{2}b(R^2-8M^2+4\sigma^2)+2q\delta
\end{aligned} \tag{13}$$

(M_i, J_i, Q_i, and B_i define, respectively, the mass, angular momentum, electric, and magnetic multipole moments), whence it follows the asymptotic flatness of the solution ($J_0 = 0$), the total mass M_0 of the configuration being equal to $2M$, and total angular momentum J_1 being zero due to counterrotation. In the absence of net charges, the parameters q and b define the electric and magnetic dipole moments, respectively, which means that the two sources in the EMR solution are endowed with opposite electric and magnetic charges.

It is clear from the above form of the multipole moments that the special $b = 0$ case of the EMR solution considered in Ref. [4] is characterized by a zero total magnetic dipole moment, and this fact explains the intrinsic presence of magnetic monopoles in that particular solution. Indeed, the magnetic dipole moment $2b$ of the five-parameter EMR solution is a result of the following two nonzero contributions: one coming from the rotating electric charges and the other originated by the opposite magnetic charges. The electric contribution is twice the magnetic dipole moment created by one rotating electric charge, so by demanding $b = 0$, Cabrera-Munguia *et al*. introduced in Ref. [4] a specific nonvanishing magnetic dipole moment due to magnetic charges, antiparallel to that created by electric charges. It would be plausible to suppose that those authors probably confused the case of counterrotating opposite charges with the BM configuration in which the counterrotating charges have the same signs, and, hence, the total magnetic and electric dipole moments are both equal to zero intrinsically. Therefore, a physically meaningful dihole solution arising from the five-parameter EMR configuration must have zero individual magnetic charges and, at the same time, a nonzero magnetic dipole moment generated by counterrotation of opposite electric charges.

The individual magnetic charges in the five-parameter EMR solution can be eliminated by means of the condition [23,24]

$$A_t(\rho = 0, z = \alpha_1) - A_t(\rho = 0, z = \alpha_2) = 0, \qquad (14)$$

which can be easily solved for b. Then, from Eqs. (9), (10), and (12), we get

$$b^2 = \frac{4q^2[(R^2 - 4M^2)(M^2 - \sigma^2) - 4q^2]}{(R^2 - 4M^2)^2 + 16q^2}, \qquad (15)$$

and this condition, together with formulas (9)–(12) with real σ provide one with a σ representation of the physically meaningful four-parameter model for a stationary black dihole whose constituents are counterrotating. It can be verified by a direct calculation that each black-hole constituent of such a model verifies the well-known Smarr mass formula identically.

III. THE FOUR-PARAMETER DIHOLE SOLUTION IN PHYSICAL PARAMETRIZATION

As was already mentioned, the σ representation of the solution is only an intermediate step on the way of obtaining the physical parametrization in terms of the Komar quantities. Once the σ representation is known, our further actions are the following: we must first try to express the parameter q in terms of the individual Komar charge Q of any of the dihole constituents, thus, rewriting the solution in the parameters M, R, Q, and σ and then find the form of σ in terms of M, R, Q, and J, J being the individual Komar angular momentum, from Smarr's mass formula, by considering the latter an algebraic equation for σ. We mention here that although the coordinate distance R is not an invariantly defined quantity, its introduction instead of the proper distance integral $\int \sqrt{f^{-1}\exp(2\gamma)}|_{\rho=0}dz$ is justified by the possibility to obtain simple analytical formulas very suitable for carrying out the physical analysis.

The mass formula for black holes discovered by Smarr [8] relates the mass M, angular momentum J, and charge Q of a black hole to several quantities evaluated on the horizon: the surface gravity κ, horizon's area S and angular velocity Ω^H, and the electric potential Φ^H. The formula reads

$$M = \frac{1}{4\pi}\kappa S + 2J\Omega^H + Q\Phi^H = \sigma + 2J\Omega^H + Q\Phi^H, \quad (16)$$

the Komar quantities M, J, and Q being defined by the integrals [23]

$$M = -\frac{1}{8\pi}\int_H \omega\Omega_{,z} d\varphi dz, \qquad (17)$$

$$J = \frac{1}{8\pi}\int_H \omega\left[-1 - \frac{1}{2}\omega\Omega_{,z} + \tilde{A}_\varphi A'_{\varphi,z} + (A_\varphi A'_\varphi)_{,z}\right] d\varphi dz, \qquad (18)$$

$$Q = \frac{1}{4\pi}\int_H \omega A'_{\varphi,z} d\varphi dz, \qquad (19)$$

with $\Omega = \mathrm{Im}(\mathcal{E})$, $A'_\varphi = \mathrm{Im}(\Phi)$, $\tilde{A}_\varphi = A_\varphi + \omega A_t$ (note that the metric functions ω and γ, as well as the potential $\tilde{A}_\varphi$, take constant values on the horizon), while the form of the constants κ, S, Ω^H, and Φ^H is given by the formulas [23,25]

$$\kappa = \sqrt{-\omega^{-2}e^{-2\gamma}}, \qquad S = 4\pi\sigma\sqrt{-\omega^2 e^{2\gamma}},$$
$$\Omega^H = \omega^{-1}, \qquad \Phi^H = -A_t - \Omega^H A_\varphi. \qquad (20)$$

For our dihole solution, the calculation of the individual charge Q of the upper black-hole constituent, whose horizon is represented by the null hypersurface $\rho = 0, \frac{1}{2}R - \sigma \le z \le \frac{1}{2}R + \sigma$ leads to a cubic equation for q which has to be solved in order to pass from the latter q to the Komar Q in the formulas determining the solution. It is remarkable, however, that the need to solve a cubic equation can be circumvented by an appropriate change of the parameter q. Thus, by introducing a new parameter q via the relation

$$\mathsf{q}^2 = q^2 + b^2, \tag{21}$$

which has some analogy with a duality rotation of the electromagnetic Ernst potential, we find from Eqs. (15) and (21) the form of q and b in terms of q,

$$\begin{aligned} q &= \mathsf{q}(R^2 - 4M^2)/\tau, \qquad b = 2\mathsf{q}\delta'/\tau, \\ \delta' &= \sqrt{(R^2 - 4M^2)(R^2 - \sigma^2) - 4\mathsf{q}^2}, \\ \tau &= \sqrt{(R^2 - 4M^2)(R^2 - 4\sigma^2) - 16\mathsf{q}^2}, \end{aligned} \tag{22}$$

and this redefinition of the parameter q permits us to obtain from Eq. (19) a simple expression for the Komar charge Q in terms of q,

$$Q = -2\mathsf{q}(R + 2M)/\tau, \tag{23}$$

whence we readily get the inverse dependence of q on Q,

$$\mathsf{q} = -\frac{Q\sqrt{(R^2 - 4M^2)(R^2 - 4\sigma^2)}}{2\sqrt{(R + 2M)^2 + 4Q^2}}. \tag{24}$$

The above formula for q permits us, by rewriting the dihole solution in terms of the parameters M, R, Q, and σ, to obtain the quantities Ω^H and Φ^H that we need for finding σ:

$$\begin{aligned} \Omega^H &= \frac{\sqrt{(R - 2M)[(R + 2M)^2 + 4Q^2][(R + 2M)(M^2 - \sigma^2) - Q^2(R - 2M)]}}{(R + 2M)[2M(R + 2M)(M + \sigma) - Q^2(R - 4M - 2\sigma)]}, \\ \Phi^H &= \frac{Q(R - 2M)[(R + 2M)(M + \sigma) + 2Q^2]}{(R + 2M)[2M(R + 2M)(M + \sigma) - Q^2(R - 4M - 2\sigma)]}. \end{aligned} \tag{25}$$

Finally, after the substitution of Eq. (25) into the Smarr formula (16) in which we can put $J = Ma$, a being the angular momentum per unit mass of the upper black hole, we obtain by a simple calculation the desired expression for σ in terms of the physical quantities M, a, Q, and R:

$$\sigma = \sqrt{M^2 - \left(\frac{M^2a^2[(R + 2M)^2 + 4Q^2]}{[M(R + 2M) + Q^2]^2} + Q^2\right)\frac{R - 2M}{R + 2M}}. \tag{26}$$

This formula for σ is the *central* result of our paper. Now the dihole solution can be rewritten in the physical parameters, its Ernst potentials $\mathcal{E}$ and Φ assuming the form

$$\begin{aligned} \mathcal{E} &= \frac{A - B}{A + B}, \qquad \Phi = \frac{C}{A + B}, \\ A &= R^2(M^2 - Q^2\nu)(R_+ - R_-)(r_+ - r_-) + 4\sigma^2(M^2 + Q^2\nu)(R_+ - r_+)(R_- - r_-) \\ &\quad + 2R\sigma[R\sigma(R_+r_- + R_-r_+) + iMa\mu(R_+r_- - R_-r_+)], \\ B &= 2MR\sigma[R\sigma(R_+ + R_- + r_+ + r_-) - (2M^2 - iMa\mu)(R_+ - R_- - r_+ + r_-)], \\ C &= 2C_0R\sigma[(R + 2\sigma)(R\sigma - 2M^2 - iMa\mu)(r_+ - R_-) + (R - 2\sigma)(R\sigma + 2M^2 + iMa\mu)(r_- - R_+)], \end{aligned} \tag{27}$$

where the dimensionless quantities μ, ν, and C_0 are defined as

$$\mu = \frac{R^2 - 4M^2}{M(R + 2M) + Q^2}, \qquad \nu = \frac{R^2 - 4M^2}{(R + 2M)^2 + 4Q^2}, \qquad C_0 = -\frac{Q(R^2 - 4M^2 + 2iMa\mu)}{(R + 2M)(R^2 - 4\sigma^2)}, \tag{28}$$

and the final form of the metric coefficients f, γ, and ω is the following:

$$f = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A+B)(\bar{A}+\bar{B})}, \qquad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{16R^4\sigma^4 R_+ R_- r_+ r_-}, \qquad \omega = -\frac{\mathrm{Im}[2G(\bar{A}+\bar{B}) + C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}},$$
$$G = -zB + R\sigma\{R(2M^2 - Q^2\nu)(R_- r_- - R_+ r_+) + 2\sigma(2M^2 + Q^2\nu)(r_+ r_- - R_+ R_-)$$
$$+ M[(R+2\sigma)(R\sigma - 2M^2 + iMa\mu) + 2(R-2\sigma)Q^2\nu](R_+ - r_-)$$
$$+ M[(R-2\sigma)(R\sigma + 2M^2 - iMa\mu) - 2(R+2\sigma)Q^2\nu](R_- - r_+)\},$$
$$I = -zC + 2C_0M[R^2(2M^2 - 2\sigma^2 + iMa\mu)(R_+ r_+ + R_- r_-) + 2\sigma^2(R^2 - 4M^2 - 2iMa\mu)(R_+ R_- + r_+ r_-)]$$
$$- C_0(R^2 - 4\sigma^2)\{2M[R\sigma(R_+ r_- - R_- r_+) + (2M^2 + iMa\mu)(R_+ r_- + R_- r_+)]$$
$$+ R\sigma[R\sigma(R_+ + R_- + r_+ + r_-) + (6M^2 + iMa\mu)(R_+ - R_- - r_+ + r_-) + 8MR\sigma]\}. \tag{29}$$

It should be noted that σ in Eqs. (27)–(29) is no longer an independent parameter, having conceded that role to the constant a. From Eq. (26), it follows that σ depending on interrelations between the parameters M, a, Q, and R can automatically take on (non-negative) real or pure imaginary values, thus, describing not only the binary configurations of black holes but also of hyperextreme objects. That is why σ's taking pure imaginary values (along with the real ones) in the EMR solution (9) is highly important for the mathematical equivalence of the parameter sets (M, Q, R, σ) and (M, Q, R, a) and, consequently, for the correctness of the entire reparametrization procedure. However, since our primary interest lies in the black-hole sector ($\sigma \geq 0$) of the dihole solution, we may always restrict ourselves to those values of the physical parameters M, Q, R, and a that preserve the reality of σ.

IV. THE LIMITS AND PHYSICAL PROPERTIES OF DIHOLE SOLUTION

The main limits of the dihole solution can be well seen from the formula (26) for σ. Thus, in the absence of rotation ($a = 0$), the solution reduces to the Emparan-Teo electrostatic nonextreme dihole spacetime [1] whose physical form was found in Ref. [26]. In the pure vacuum limit ($Q = 0$,) the solution represents a vacuum specialization of the BM equatorially antisymmetric binary configuration whose physical parametrization was elaborated in Ref. [17] on the basis of Varzugin's expression for the quantity σ [27]. When $R \to \infty$ (no interaction between the dihole constituents), one gets from Eq. (26) $\sigma = (M^2 - a^2 - Q^2)^{1/2}$, which is characteristic of a single KN black hole.

By construction, the upper KN constituent has mass M, angular momentum Ma, and charge Q, whereas the analogous characteristics of the lower constituent are M, $-Ma$, and $-Q$, respectively. The strut separating the two constituents provides us with the information about the interaction force [28], the latter being defined by the expression

$$\mathcal{F} = \frac{1}{4}(e^{-\gamma_0} - 1)$$
$$= \frac{M^2(R+2M)^2 + Q^2R^2}{(R+2M)^2(R^2 - 4M^2)}$$
$$= \frac{1}{R^2 - 4M^2}\left(M^2 + Q^2 - \frac{4MQ^2(R+M)}{(R+2M)^2}\right), \tag{30}$$

where γ_0 is the value of the metric function γ on the strut, and one can see that $\mathcal{F}$ cannot take zero value at any finite separation R of the constituents so that the strut is irremovable generically. It is worth mentioning that formula (30) differs from the analogous expression obtained in Ref. [4], as our $\mathcal{F}$ does not contain a contribution due to magnetic charges. Note also that $\mathcal{F}$ admits a geometrical interpretation in terms of the conical deficit quantity δ via the formula $\mathcal{F} = -\delta/(8\pi)$ and that the absence of the angular momentum J in formula (30) does not mean at all that the spin-spin interaction is not present in our dihole model since, as has already been rightly observed in Ref. [29], the angular momenta contribution (attractive) will enter explicitly into the above formula after the use of the proper distance integral instead of the coordinate distance R.

Turning now to the thermodynamical quantities κ, S, Ω^H, and Φ^H entering the Smarr mass formula (16), it may be observed that these must be calculated only for the upper black-hole constituent because the analogous set for the lower constituent is just κ, S, $-\Omega^H$, and $-\Phi^H$. Then, for the upper constituent, we get

$$\kappa = \frac{R\sigma[(R+2M)^2 + 4Q^2]}{(R+2M)^2[2(M+\sigma)(MR + 2M^2 + Q^2) - Q^2(R-2M)]}, \tag{31}$$

$$S = 4\pi\left(1+\frac{2M}{R}\right)\left(2M(M+\sigma) - \frac{Q^2(R-2M)(R-2\sigma)}{(R+2M)^2+4Q^2}\right)$$
$$= \frac{4\pi}{R(R+2\sigma)}\left((R+2M)^2(M+\sigma)^2 + \frac{M^2a^2(R^2-4M^2)^2}{(MR+2M^2+Q^2)^2}\right), \tag{32}$$

$$\Omega^H = \frac{Ma[2(M-\sigma)(MR+2M^2+Q^2) - Q^2(R-2M)]}{(4M^2a^2+Q^4)(MR+2M^2+Q^2)}, \tag{33}$$

$$\Phi^H = \frac{Q[Q^2(M-\sigma)(MR+2M^2+Q^2) + 2M^2a^2(R-2M)]}{(4M^2a^2+Q^4)(MR+2M^2+Q^2)}, \tag{34}$$

where S is given in two different forms suitable for recovering the known limiting cases. The substitution of Eqs. (31)–(34) into Eq. (16) shows that the above expressions satisfy identically Smarr's formula for black holes.

It has been recently shown [7] that the equally charged black-hole constituents of the BM configuration saturate the Gabach-Clement inequality for black holes with struts [12] which reads

$$\sqrt{1+4\mathcal{F}} \geq \frac{\sqrt{(8\pi J)^2+(4\pi Q^2)^2}}{S}. \tag{35}$$

In this respect, it would be interesting to clarify whether the oppositely charged constituents of our dihole model saturate the inequality (35) too. The saturation means that the extremal dihole constituents must satisfy Eq. (35) with the equality sign. The extremality condition $\sigma = 0$ yields from Eq. (26) the value of a at which the black-hole degeneration occurs:

$$a^2 = \frac{(MR+2M^2+Q^2)^2[M^2(R+2M) - Q^2(R-2M)]}{M^2(R-2M)[(R+2M)^2+4Q^2]}, \tag{36}$$

and by substituting this a into Eqs. (32) and (35), we get ($J = Ma$)

$$S = \frac{4\pi(R+2M)^2[2M^2(R+2M) - Q^2(R-4M)]}{R[(R+2M)^2+4Q^2]},$$
$$(8\pi J)^2 + (4\pi Q^2)^2 = \frac{16\pi^2(R+2M)[2M^2(R+2M) - Q^2(R-4M)]^2}{(R-2M)[(R+2M)^2+4Q^2]}. \tag{37}$$

Taking into account that $\mathcal{F}$ does not depend explicitly on a, it is easy to check that Eqs. (30) and (37) verify the equality in Eq. (35) identically. Therefore, independent of whether the KN black holes in a binary system have equal or opposite charges, the interaction force between them is governed by the Gabach-Clement inequality. We mention here one more common feature shared by the BM and dihole configurations: the extreme limit is achieved in both of them at a larger absolute value of a (for some given M and Q) than in the case of a single KN black hole whose extremality condition is simply $a^2 = M^2 - Q^2$.

We mention that in Ref. [4] the Ernst potentials defining the extreme limit of the four-parameter solution are given with errors. Therefore, we find it useful to give below the expressions for these potentials and corresponding metric functions of the entire five-parameter EMR solution in the extreme limit $\sigma \to 0$:

$$\mathcal{E} = \frac{A-B}{A+B}, \qquad \Phi = \frac{C}{A+B}, \qquad f = \frac{N}{D}, \qquad e^{2\gamma} = \frac{N}{\alpha^8(x^2-y^2)^4}, \qquad \omega = -\frac{4\alpha^2\delta y(x^2-1)(1-y^2)W}{N},$$
$$A = M^2\alpha^2(x^4-1) + \alpha^2(\alpha^2-M^2)(x^2-y^2)^2 + (q^2+b^2)(1-y^4) + 2i\alpha^2\delta(x^2-2x^2y^2+y^2),$$
$$B = 2M\alpha x[\alpha^2(x^2-y^2) - (M^2-i\delta)(1-y^2)], \qquad C = -2(q+ib)y[\alpha^2(x^2-y^2) - (M^2+i\delta)(1-y^2)],$$
$$I = -\alpha xyC - 2(q+ib)(M+\alpha x)(1-y^2)[(M+\alpha x)^2 + (M^2-\alpha^2)y^2 + i\delta(1+y^2)],$$
$$N = [M^2\alpha^2(x^2-1)^2 + \alpha^2(\alpha^2-M^2)(x^2-y^2)^2 - (q^2+b^2)(1-y^2)^2]^2 - 16\alpha^4\delta^2x^2y^2(x^2-1)(1-y^2),$$
$$D = \{M^2\alpha^2(x^4-1) + \alpha^2(\alpha^2-M^2)(x^2-y^2)^2 + (q^2+b^2)(1-y^4) + 2M\alpha x[\alpha^2(x^2-y^2) - M^2(1-y^2)]\}^2$$
$$+ 4\alpha^2\delta^2[\alpha(x^2-2x^2y^2+y^2) + Mx(1-y^2)]^2,$$
$$W = M\alpha^2[(\alpha^2-M^2)(x^2-y^2)(3x^2+y^2) + M^2(3x^4+6x^2-1) + 8M\alpha x^3] - (q^2+b^2)[4\alpha xy^2 - M(1-y^2)^2],$$
$$\delta = \sqrt{M^2(\alpha^2-M^2) - q^2 - b^2}, \qquad \alpha = \frac{1}{2}R, \tag{38}$$

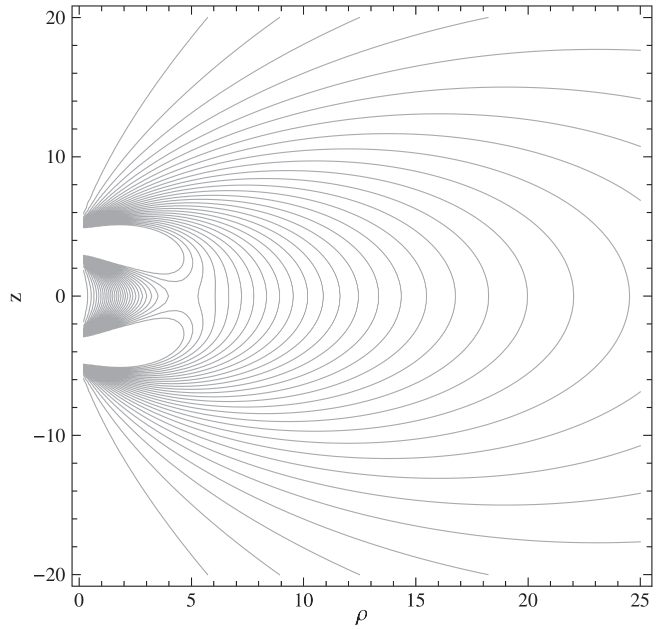

FIG. 2. Magnetic lines of force plotted for the dihole solution in the particular case $R = 8$, $M = 3/2$, $a = 1/8$, $Q = 1/2$.

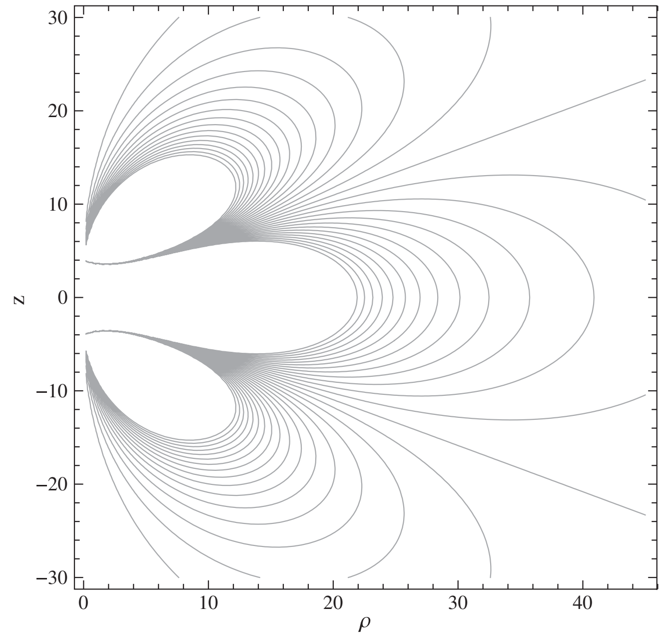

FIG. 3. Magnetic lines of force plotted for the specific EMR solution considered in Ref. [4]; the parameter choice (in the original notation for the charge parameter) is $R = 8$, $M = 3/2$, $\sigma = 1/4$, $Q_0 = 1/2$.

where the prolate spheroidal coordinates (x, y) are related to the cylindrical coordinates (ρ, z) by the formulas

$$x = \frac{1}{2\alpha}(r_+ + r_-), \qquad y = \frac{1}{2\alpha}(r_+ - r_-),$$
$$r_\pm = \sqrt{\rho^2 + (z \pm \alpha)^2}, \tag{39}$$

and where we have also given the explicit extremal form of the function I defining the magnetic potential A_φ via formula (12).

As it follows from Eq. (13), the magnetic field in our dihole solution differs considerably from that of the particular $b = 0$ specialization of the EMR solution considered in Ref. [4]: in the former case, it has a dipole character, while in the latter case, it behaves like a magnetic octupole ($B_3 = 2q\delta$). In Figs. 2 and 3 this difference is illustrated by the plots of magnetic lines of force for two characteristic particular cases.

We end this section by observing that the singularity structure of solution (27), which is determined by zeros of the denominator $A + B$ of the Ernst potentials $\mathcal{E}$ and Φ, is essentially the same as that of the solution considered by Cabrera-Munguia *et al.*, i.e., no ring singularities appear for $\rho > 0$ in the positive mass case $M > 0$ (this has been checked numerically for a wide range of parameters of our solution), while for $M < 0$ there arise two massless ring singularities outside the location of the sources, in agreement with a recent study [30] of the single KN spacetime endowed with negative mass.

V. CONCLUSIONS

In our paper, we succeeded in elaborating a physically consistent four-parameter model for stationary diholes whose exceptionality consists in the fact that it is the *only* four-parameter member of the five-parameter EMR class not containing magnetic charges (whereas, on the other side, there is an infinite number of four-parameter specializations of the latter class with individual magnetic charges of the constituents). This model generalizing the known dihole electrostatic solution earlier obtained by Emparan and Teo is comprised of two identical (up to the sign of charges) counterrotating KN black holes supported from falling onto each other by a massless strut. Curiously, its finding and correct mathematical description has turned out to be a more sophisticated task than in the case of counterrotating equally charged KN black holes represented by the BM solution because the knowledge of a more general five-parameter EMR solution was needed for getting rid of the specific individual magnetic charges initially present in the dihole components. The solution's physical representation was advantageous for a direct check that the binary configuration it describes really saturates the Gabach-Clement inequality for interacting black holes.

Since the aforementioned inequality also takes into account the possibility for the black holes to carry magnetic charges as independent parameters, we would like to mention that our dihole solution can be very easily generalized to the case when the two KN constituents, besides the opposite electric charges, would have arbitrary opposite magnetic charges too, thus, representing a pair of dyons [31]. To introduce an arbitrary magnetic charge $\mathcal{B}$ into our dihole model, one only needs to make the following substitutions in the formulas (26)–(29): change Q to $\mathcal{Q} = Q - i\mathcal{B}$, and Q^2 to $|\mathcal{Q}|^2 = Q^2 + \mathcal{B}^2$ in all the occurrences. For instance, our expression (26) for σ will then assume the form

$$\sigma = \sqrt{M^2 - \left(\frac{M^2a^2[(R+2M)^2 + 4|\mathcal{Q}|^2]}{[M(R+2M) + |\mathcal{Q}|^2]^2} + |\mathcal{Q}|^2\right)\frac{R-2M}{R+2M}}. \tag{40}$$

We underline that the parameter $\mathcal{B}$ thus introduced will be a genuine individual magnetic charge of the upper black hole, and this can be readily verified by means of the formula

$$\mathcal{B} = \frac{1}{4\pi}\int_H \omega A_{t,z} d\varphi dz. \tag{41}$$

It is easy to see that the dyonic dihole model, which is, of course, equivalent to the five-parameter EMR solution, will also saturate the Gabach-Clement inequality because the electric and magnetic charges Q_i and $\mathcal{B}_i$ enter that inequality only in the combination $Q_i^2 + \mathcal{B}_i^2$ [12]. We mention also that the introduction of the magnetic charge does not actually modify seriously the Smarr mass formula (16), provided the substitutions described above are carried out properly in the term $Q\Phi^H$.

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