

From Brans-Dicke gravity to a geometrical scalar-tensor theoryT. S. Almeida,^{1,*} M. L. Pucheu,^{1,†} C. Romero,^{1,‡} and J. B. Formiga^{2,§}¹*Universidade Federal da Paraíba, Departamento de Física, C. Postal 5008, 58051-970 João Pessoa, Paraíba, Brazil*²*Centro de Ciências da Natureza, Universidade Estadual do Piauí, C. Postal 381, 64002-150 Teresina, Piauí, Brazil*

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We consider an approach to the Brans-Dicke theory of gravity in which the scalar field has a geometrical nature. By postulating the Palatini variation, we find out that the role played by the scalar field consists in turning the space-time geometry into a Weyl integrable manifold. This procedure leads to a scalar-tensor theory that differs from the original Brans-Dicke theory in many aspects and presents some new features.

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I. INTRODUCTION

As is widely known, an important guide to Einstein in the development of his general theory of relativity was what he called the principle of equivalence, which, in mathematical terms, corresponds to the so-called geodesic postulate, i.e., the assumption that free particles under the sole influence of gravity will follow geodesics in a curved space-time. This, clearly, was the first step towards a geometrization of the gravitational interaction. The second step came from setting the field equations, which then establish how matter curves space-time. Or, in the words of American physicist John Wheeler, “space-time tells matter how to move, and matter tells space-time how to curve.” The interesting fact here is that this elegant theoretical scheme, which has set the stage for general relativity, works perfectly with many other metric theories of gravity, including those whose geometrical framework is not *a priori* assumed to be Riemannian or that make use of physical variables other than the metric and matter fields. This is the case, for instance, of one of the most popular alternative theories of gravity, namely, the Brans-Dicke scalar-tensor theory of gravity, a theoretical framework in which the space-time manifold is still assumed to be Riemannian, but the gravitational interaction is described by two fields: the metric tensor $g_{\mu\nu}$ and a scalar field Φ [1]. It turns out, however, that these two fields are of quite a distinct nature. Indeed, while $g_{\mu\nu}$ is essentially geometric, Φ does not appear in the equations of motion of particles and photons. In fact, Φ is neither a matter field nor a geometric field and is traditionally interpreted as the inverse of the gravitational coupling parameter, which in the Brans-Dicke theory is not constant and is considered to be determined by the matter content of the Universe. This nongeometrical character of Φ has led us to speculate on what kind of

gravitational theory would result if Φ were assigned an active geometrical role in the dynamics of the gravitational field as well as in the equations of motion of particles and light. Surely, in this case we would expect that, being part of the geometry, Φ should appear explicitly in the geodesic equations. Moreover, in this new scheme, the gravitational field would be described not only by $g_{\mu\nu}$, but by the pair $(g_{\mu\nu}, \Phi)$. Of course, such features would immediately exclude Riemannian geometry as the mathematical framework to be used to describe space-time. Instead, one would have to look into another geometrical setting which would operate with a geometric scalar field as one of its in-built fundamental constituents. This would then lead us to the question of how to determine, from first principles, the geometry of the space-time. Well, it seems there are at least two ways to answer this question: one is to postulate *a priori* a certain kind of geometry, as in the case of general relativity, the Brans-Dicke theory, and many others. The second way is to choose an action and try to extract the geometry from the action itself by means of a variational principle. As we know, there are essentially two distinct variational principles at our disposal: the one that uses the Hilbert method, in which the field equations are derived by performing variations with respect to the metric, and the so-called Palatini method, which considers independent variations of the affine connection and the metric [2]. It is also well known that, when applied to general relativity, although both methods lead to the same field equations, the latter has the additional advantage of giving a definite specification of the Riemannian character of the space-time. This equivalence, however, is no longer true in the case of more general actions [3]. In view of this special feature, i.e., the ability of the Palatini method to determine the space-time geometry directly from the action, it seems natural to apply this method, and even extend it, to investigate the Brans-Dicke action if we are to assign any geometrical role to the scalar field Φ . In the present work we begin by applying the Palatini variational method to the Brans-Dicke action. However, because the scalar

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field Φ is now regarded as an independent geometric field in its own right, we shall assume that Φ , the metric $g_{\mu\nu}$, and the connection $\Gamma_{\mu\nu}^\alpha$ must be varied independently. As we shall see, the field equations corresponding to the variation of the connection will allow us to identify the space-time geometry as a special case of Weyl geometry, with the scalar field Φ playing the role of the Weyl field [4]. It is worth noting that a close connection between the Brans-Dicke theory and Weyl geometry has already been discovered and may be found in different contexts. In fact, this connection has been shown to exist for any scalar-tensor theory in which the scalar field is nonminimally coupled to the metric [5,6].

It turns out that the change from Hilbert to (an extended) Palatini variational principle when applied to the Brans-Dicke action will lead us to a new scalar-tensor theory of gravity, which presents some distinct features compared with the original Brans-Dicke gravity. For instance, it will be found that the space-time is no longer Riemannian but now has the geometrical structure of what came to be known in the literature as a Weyl integrable space-time (WIST) [7]. Moreover, the usual coupling between matter and gravitation assumed in the Brans-Dicke theory must be modified if we want the equivalence principle to hold in the new theory. These departures from the Brans-Dicke theory lead us to a new scenario, in which the scalar field has a geometrical meaning and plays a fundamental and active role in the motion of particles and light.

The paper is organized as follows. In Sec. 2, we obtain the field equations from the extended Palatini variational method, where the scalar field Φ is now reinterpreted as a purely geometric field, hence being regarded as a fundamental component of the space-time manifold. In Sec. 3, we compare the field equations with those of the Brans-Dicke theory and show that although the two theories are not physically equivalent they bear strong similarities. We proceed to Sec. 4 to show that the field equations viewed in the Riemann frame are formally equivalent to those given by the general relativistic action corresponding to a massless scalar field minimally coupled with the gravitation field. In Sec. 5, this correspondence between the two theories is used to analyze some typical solar-system experiments in the context of the geometrical scalar-tensor theory. In Sec. 6, we briefly discuss the existence of spherically symmetric space-times by simply looking into some corresponding general relativistic solutions, and this seems to suggest that we can view naked singularities and wormholes as geometric phenomena. We conclude with some remarks in Sec. 7.

II. A GEOMETRICAL APPROACH TO SCALAR-TENSOR THEORY

Let us start with the Brans-Dicke action [8]

$$S_G = \int d^4x \sqrt{-g} \left(\Phi R + \frac{\omega}{\Phi} \Phi_{,\alpha} \Phi_{,\alpha} \right), \quad (1)$$

which will be supposed to describe the gravitational field in the absence of matter [1]. Here, we are denoting $R = g^{\mu\nu} R_{\mu\nu}(\Gamma)$, and, in what follows, we shall consider the Ricci tensor $R_{\mu\nu}(\Gamma)$ as being entirely expressed in terms of the affine connection coefficients $\Gamma_{\mu\nu}^\alpha$ through the definition of the curvature tensor [9]. Changing to the new variable ϕ defined by $\Phi = e^{-\phi}$, it is easily seen that (1) becomes

$$S_G = \int d^4x \sqrt{-g} e^{-\phi} (R + \omega \phi^{,\alpha} \phi_{,\alpha}). \quad (2)$$

As we have mentioned above, we want to regard the usual Brans-Dicke scalar field Φ (or, equivalently, $e^{-\phi}$) as possessing an intrinsic geometrical character, which, up to now, is unknown to us. We shall then apply the extended Palatini variational method, which amounts to taking independent variations of the three geometric objects entering in the action (2), namely, Φ , $\Gamma_{\mu\nu}^\alpha$, and $g_{\mu\nu}$. Let us first take the variation of (2) with respect to the affine connection $\Gamma_{\mu\nu}^\alpha$. After simple calculations we obtain

$$\nabla_\alpha (\sqrt{-g} e^{-\phi} g^{\mu\nu}) = 0, \quad (3)$$

which is easily verified to be equivalent to

$$\nabla_\alpha g_{\mu\nu} = g_{\mu\nu} \phi_{,\alpha}. \quad (4)$$

It turns out that the above equation expresses nothing else than the so-called Weyl compatibility condition between the metric and the connection (also called the Weyl non-metricity condition). In this way, we see that the scalar field ϕ acquires a clear geometrical character, while the space-time is naturally endowed with the Weyl integrable space-time [7].

After the determination of the space-time geometry, it seems natural that the next step is to consider a variation of the action (2) with respect to the geometric scalar field ϕ . Strictly speaking, this amounts to proposing an extension of the Palatini variational method, as now we have three independent geometric entities, namely, the affine connection $\Gamma_{\mu\nu}^\alpha$, the metric $g_{\mu\nu}$, and the scalar field ϕ being involved in the process of variation. Let us briefly recall the geometrical role played by these three fields: the metric $g_{\mu\nu}$ is responsible for measuring lengths and angles, and the connection Γ sets the rules for parallel transport and defines the covariant derivatives of vector and tensor fields, whereas the scalar field ϕ defines the nonmetricity, also participating in the parallel transport of vectors, modifying their length at each point of the space-time manifold.

Before going further, some comments about the Weyl geometry are in order [4,10]. Broadly speaking, we can say that the geometry conceived by Weyl is a simple generalization of Riemannian geometry. Indeed, instead of regarding the Levi-Civita compatibility condition, Weyl has extended it to the more general requirement

$$\nabla_{\alpha} g_{\mu\nu} = \sigma_{\alpha} g_{\mu\nu}, \quad (5)$$

where σ_{α} denotes the components of a one-form field σ , globally defined in the manifold. If σ is an exact form, i.e., $\sigma = d\phi$, where ϕ is a scalar field, then we have what has been called a Weyl integrable geometry. In perfect analogy with Riemannian geometry, the condition (4) is sufficient to determine the Weyl connection ∇ in terms of the metric g and the Weyl scalar field. Thus, it is not difficult to verify that the coefficients $\Gamma_{\mu\nu}^{\alpha}$ of the affine connection when expressed in terms of $g_{\mu\nu}$ and ϕ are given by

$$\Gamma_{\mu\nu}^{\alpha} = \left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\} - \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu} \phi_{,\nu} + g_{\beta\nu} \phi_{,\mu} - g_{\mu\nu} \phi_{,\beta}), \quad (6)$$

where $\left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\}$ denotes the Christoffel symbols.

At this point, it is vitally important to note that the Weyl condition (5) remains unchanged when we perform the following simultaneous transformations in g and σ :

$$\bar{g} = e^f g, \quad (7)$$

$$\bar{\sigma} = \sigma + df, \quad (8)$$

where f is a scalar function defined on M . These transformations are known in the literature as Weyl transformations. An important fact that deserves to be mentioned is the invariance of the affine connection coefficients $\Gamma_{\mu\nu}^{\alpha}$ under Weyl transformations, which, in turn, implies the invariance of the affine geodesics.

The set (M, g, ϕ) consisting of a differentiable manifold M endowed with a metric g and a Weyl scalar field ϕ will be referred to as a *Weyl frame*. In the particular case of a Weyl integrable manifold, (8) becomes

$$\bar{\phi} = \phi + f. \quad (9)$$

Note that if we set $f = -\phi$ in the above equation, we get $\bar{\phi} = 0$. In this case, we refer to the set $(M, \bar{g} = e^{-\phi} g, \bar{\phi} = 0)$ as the *Riemann frame*, since in this frame the manifold becomes Riemannian. Incidentally, it can be easily checked that (6) follows directly from $\nabla_{\alpha} \bar{g}_{\mu\nu} = 0$. This simple fact has interesting and useful consequences. One consequence is that, since $\bar{g} = e^{-\phi} g$ is invariant under the Weyl transformations (7) and (9), any geometric quantity constructed exclusively with \bar{g} is invariant. Other geometric objects such as the components of the curvature tensor $R_{\beta\mu\nu}^{\alpha}$, the components of the Ricci tensor $R_{\mu\nu}$, and the scalar $e^{\phi} R$ are evidently invariant under the Weyl transformations (7) and (9). Other invariants that may be constructed with the help of $\bar{g} = e^{-\phi} g$ are, for instance, $e^{2\phi} R_{\mu\nu} R^{\mu\nu}$ and $e^{2\phi} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$. For reasons of consistency we shall adopt these as the natural extensions of the usual Riemannian invariants $\hat{R}_{\mu\nu} \hat{R}^{\mu\nu}$ and $\hat{R}_{\mu\nu\alpha\beta} \hat{R}^{\mu\nu\alpha\beta}$ (the Kretschmann scalar) to a Weyl integrable geometry in a general frame (M, g, ϕ) .

It is important to note here that, because the Weyl transformations (7) and (9) define an equivalence relation between frames (M, g, ϕ) , it seems more natural to focus our attention on the equivalence class of such frames rather than on a particular one. In this regard, a Weyl manifold may be regarded as a frame (M, g, ϕ) that is defined only ‘‘up to a Weyl transformation.’’ Thus Weyl manifolds may be treated by selecting a frame in the equivalence class and applying only invariant constructions to the chosen frame. From this standpoint, it would be more natural to redefine some Riemannian concepts to meet the requirements of Weyl invariance. This viewpoint is analogous to the way one treats *conformal geometry*, a branch of geometry, in which the geometric objects of interest are those that are invariant under conformal transformation, such as, for instance, the angle between two directions [11]. In the same spirit, one should naturally modify the definition of all invariant integrals when dealing with the integration of exterior forms. For instance, the Riemannian p -dimensional volume form defined as $\Omega = \sqrt{-g} dx^1 \wedge \cdots \wedge dx^p$, which is not invariant under Weyl transformations, should be replaced by $\Omega = \sqrt{-g} e^{-\frac{p}{2}\phi} dx^1 \wedge \cdots \wedge dx^p$, and so on. Accordingly, in a Weyl integrable manifold it would be more natural to define the concept of ‘‘length of a curve’’ in an invariant way. As a consequence, our notion of proper time as the arc length of worldlines in four-dimensional Lorentzian space-time should be modified. In view of this, we shall redefine the proper time $\Delta\tau$ measured by a clock moving along a parametrized timelike curve $x^{\mu} = x^{\mu}(\lambda)$ between $x^{\mu}(a)$ and $x^{\mu}(b)$ in such a way that $\Delta\tau$ is the same in all frames. This leads us to the following definition:

$$\Delta\tau = \int_a^b \left(\bar{g}_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \right)^{\frac{1}{2}} d\lambda = \int_a^b e^{-\frac{\phi}{2}} \left(g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \right)^{\frac{1}{2}} d\lambda. \quad (10)$$

It should be noted that the above expression may be also obtained from the special relativistic definition of proper time by using the prescription $\eta_{\mu\nu} \rightarrow e^{-\phi} g_{\mu\nu}$. Clearly, the right-hand side of this equation is invariant under Weyl transformations and reduces to the known expression of the proper time in general relativity in the Riemann frame. We take $\Delta\tau$, as given above, as the extension to an arbitrary Weyl frame of the general relativistic clock hypothesis, i.e., the assumption that $\Delta\tau$ measures the proper time measured by a clock attached to the particle.

It is not difficult to verify that the extremization of the functional (10) leads to the equations

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \left(\left\{ \begin{array}{c} \mu \\ \alpha\beta \end{array} \right\} - \frac{1}{2} g^{\mu\nu} (g_{\alpha\nu} \phi_{,\beta} + g_{\beta\nu} \phi_{,\alpha} - g_{\alpha\beta} \phi_{,\nu}) \right) \times \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda} = 0, \quad (11)$$

where $\{\overset{\mu}{\alpha\beta}\}$ denotes the Christoffel symbols calculated with $g_{\mu\nu}$. Let us recall that in the derivation of the above equations the parameter λ has been chosen such that

$$e^{-\phi} g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = K = \text{const} \quad (12)$$

along the curve, which, up to an affine transformation, permits the identification of λ with the proper time τ . It turns out that these equations are exactly those that yield the affine geodesics in a Weyl integrable space-time, since they can be rewritten as

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0, \quad (13)$$

where $\Gamma_{\alpha\beta}^\mu = \{\overset{\mu}{\alpha\beta}\} - \frac{1}{2} g^{\mu\nu} (g_{\alpha\nu} \phi_{,\beta} + g_{\beta\nu} \phi_{,\alpha} - g_{\alpha\beta} \phi_{,\nu})$, according to (6), may be identified with the components of the Weyl connection. Therefore, the extension of the geodesic postulate by requiring that the functional (10) be an extremum is equivalent to postulating that the particle motion must follow affine geodesics defined by the Weyl connection $\Gamma_{\alpha\beta}^\mu$. It will be noted that, as a consequence of the Weyl compatibility condition (3) between the connection and the metric, (12) holds automatically along any affine geodesic determined by (13). Because both the connection components $\Gamma_{\alpha\beta}^\mu$ and the proper time τ are invariant when we switch from one Weyl frame to the other, Eqs. (13) are invariant under Weyl transformations.

As we know, the geodesic postulate not only makes a statement about the motion of particles but also determines the propagation of light rays in space-time. Because the paths of light rays are null curves, one cannot use the proper time as a parameter to describe them. In fact, light rays are supposed to follow null affine geodesics, which cannot be defined in terms of the functional (10), but, instead, they must be characterized by their behavior with respect to parallel transport. We shall extend this postulate by simply assuming that light rays follow Weyl null affine geodesics.

We have hitherto considered the Brans-Dicke action in vacuum. However, before we proceed with the variation with respect to $g_{\mu\nu}$ and ϕ , it turns out to be more convenient, as part of our reasoning, to complete (1) by adding an action S_m to account for the matter fields. Because we have already discovered that the space-time must be described by two geometric fields, namely, $g_{\mu\nu}$ and ϕ , it is reasonable to expect both to couple with matter, preferably in a frame-independent way. Perhaps a clue to the construction of S_m is given by the fact, mentioned earlier, that any geometric quantity constructed $\bar{g} = e^{-\phi} g$ is invariant under the Weyl transformations (7) and (9). It seems then that the sought-for action will be given by

$$S_m = \kappa^* \int d^4 x \sqrt{-\bar{g}} L_m(\bar{g}_{\mu\nu}, \Psi, \nabla_R^{(\bar{g})} \Psi)$$

or, equivalently,

$$S_m = \kappa^* \int d^4 x \sqrt{-g} e^{-2\phi} L_m(e^{-\phi} g_{\mu\nu}, \Psi, \nabla \Psi), \quad (14)$$

where, as in the Brans-Dicke theory, $\kappa^* = \frac{8\pi}{c^4}$, L_m designates the matter Lagrangian, Ψ stands generically for the matter fields, $\nabla_R^{(\bar{g})}$ denotes the Riemannian covariant derivative with respect to the metric $\bar{g} = e^{-\phi} g$, and ∇ indicates the covariant derivative with respect to the Weyl affine connection [12]. Note that $L_m(g, \phi, \Psi, \nabla \Psi)$ is given by the prescription $\eta_{\mu\nu} \rightarrow e^{-\phi} g_{\mu\nu}$ and $\partial_\mu \rightarrow \nabla_\mu$, where ∇_μ denotes the covariant derivative with respect to the Weyl affine connection. Let us recall here that $L_m(g, \phi, \Psi, \nabla \Psi) \equiv L_m^{\text{sr}}(e^{-\phi} g, \Psi, \nabla \Psi)$, where L_m^{sr} denotes the Lagrangian of the field Ψ in flat Minkowski space-time of special relativity.

With the purpose of obtaining the complete field equations through the variation of the total action $S = S_G + S_m$, we now proceed to the definition of the energy-momentum tensor in this new geometrical setting. From the same arguments that led to the building up of the action (14), it seems natural to define the energy-momentum tensor $T_{\mu\nu}(\phi, g, \Psi, \nabla \Psi)$ of the matter field Ψ , in an arbitrary Weyl frame (M, g, ϕ) , by the formula

$$\begin{aligned} \delta \int d^4 x \sqrt{-g} e^{-2\phi} L_m(g_{\mu\nu}, \phi, \Psi, \nabla \Psi) \\ = \int d^4 x \sqrt{-g} e^{-2\phi} T_{\mu\nu}(\phi, g_{\mu\nu}, \Psi, \nabla \Psi) \delta(e^\phi g^{\mu\nu}), \end{aligned} \quad (15)$$

where the variation on the left-hand side must be carried out simultaneously with respect to both $g_{\mu\nu}$ and ϕ . In order to see that the above definition makes sense, it must be clear that the left-hand side of Eq. (15) can always be put in the same form as the right-hand side of the same equation. This can easily be seen from the fact that $\delta L_m = \frac{\partial L_m}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial L_m}{\partial \phi} \delta \phi = \frac{\partial L_m}{\partial (e^\phi g^{\mu\nu})} \delta(e^\phi g^{\mu\nu})$ and that $\delta(\sqrt{-g} e^{-2\phi}) = -\frac{1}{2} \sqrt{-g} e^{-3\phi} g_{\mu\nu} \delta(e^\phi g^{\mu\nu})$.

We are now ready to perform the variation of the complete action

$$\begin{aligned} S = \int d^4 x \sqrt{-g} e^{-\phi} (R + \omega \phi^\alpha \phi_{,\alpha}) \\ + \kappa^* \int d^4 x \sqrt{-g} e^{-2\phi} L_m(e^{-\phi} g_{\mu\nu}, \Psi, \nabla \Psi) \end{aligned} \quad (16)$$

with respect to the metric $g_{\mu\nu}$. A simple calculation yields

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa^* T_{\mu\nu} - \omega \phi_{,\mu} \phi_{,\nu} + \frac{\omega}{2} g_{\mu\nu} \phi^\alpha \phi_{,\alpha}, \quad (17)$$

where it should be kept in mind that we are denoting by $R_{\mu\nu}$ and R the Ricci tensor and the scalar curvature, respectively, as defined with respect to the Weyl connection (6). Finally, if we now carry out the variation of the action (16) with respect to the scalar field ϕ , we obtain

$$R + 3\omega\phi^\alpha\phi_{,\alpha} + 2\omega\Box\phi = \kappa^*T, \quad (18)$$

where $T = g^{\mu\nu}T_{\mu\nu}$ and \Box denotes the d'Alembert operator defined with respect to the Weyl connection [13]. If we now take the trace of (17), we will get

$$R + \omega\phi^\alpha\phi_{,\alpha} = \kappa^*T, \quad (19)$$

which combined with (18) leads to

$$\Box\phi + \phi^\alpha\phi_{,\alpha} = 0. \quad (20)$$

Of course, we can rewrite all the field equations derived above in a Riemannian form. All we have to do is to express the Weylian geometric quantities $R_{\mu\nu}$ and R in terms of their Riemannian counterparts, which will be denoted by $\hat{R}_{\mu\nu}$ and \hat{R} , both calculated directly from the metric $g_{\mu\nu}$ and the Christoffel symbols $\{\alpha\beta^\mu\}$. In this way, after some straightforward calculations and taking into account (6), we can rewrite (17), (18), and (20), respectively, as

$$\hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R} = -\kappa^*T_{\mu\nu} - \frac{w}{\Phi^2}\left(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{,\alpha}\Phi^{,\alpha}\right) - \frac{\Phi_{,\mu;\nu}}{\Phi}, \quad (21)$$

$$\hat{R} + \frac{w}{\Phi^2}\Phi_{,\alpha}\Phi^{,\alpha} = \kappa^*T, \quad (22)$$

$$\hat{\Box}\Phi = 0, \quad (23)$$

where $w = \omega - \frac{3}{2}$, $\hat{\Box}$ denotes the d'Alembert operator defined with respect to the Riemannian connection, and, in order to make comparisons with the Brans-Dicke field equations, we are working with the field variable $\Phi = e^{-\phi}$.

III. SIMILARITIES WITH BRANS-DICKE THEORY

Equations (17), (18), and (20), which we have derived in the previous section, bear strong similarities to the field equations of the Brans-Dicke theory. In fact, connections between gravity theories based on Weyl integrable geometry and Jordan-Brans-Dicke theories are known to exist and have already been pointed out in the literature (see, for instance, [10]). Let us recall that Brans-Dicke field equations may be written in the form [1]

$$\hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R} = -\frac{\kappa^*}{\Phi}T_{\mu\nu} - \frac{\omega}{\Phi^2}\left(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{,\alpha}\Phi^{,\alpha}\right) - \frac{1}{\Phi}(\Phi_{,\mu;\nu} - g_{\mu\nu}\Box\Phi), \quad (24)$$

$$\hat{R} - 2\omega\frac{\Box\Phi}{\Phi} + \frac{\omega}{\Phi^2}\Phi_{,\alpha}\Phi^{,\alpha} = 0, \quad (25)$$

where we are keeping the notation of the previous section, in which $\hat{R}_{\mu\nu}$ and \hat{R} denote the Ricci tensor and the curvature scalar, respectively, calculated with respect to the metric $g_{\mu\nu}$. By combining (24) and (25) we can easily derive the equation

$$\Box\Phi = \frac{\kappa^*T}{2\omega + 3}, \quad (26)$$

which is the most common form of the scalar field equation usually found in the literature [14]. In this way, we see that in the vacuum case, i.e., when $T_{\mu\nu} = 0$, Brans-Dicke field equations are formally identical to (21) and (23) if we set $w = \omega - \frac{3}{2}$. However, the two theories are not physically equivalent, since in the Brans-Dicke theory test particles follow metric geodesics and not affine Weyl geodesics.

IV. SIMILARITIES WITH EINSTEIN'S GRAVITY

In developing a geometric scalar-field gravity theory, we have hitherto confined ourselves to a generic Weyl frame (M, g, ϕ) , that is, a frame in which space-time is regarded as a differentiable manifold M endowed with a metric g and a non-null Weyl scalar field ϕ . We now wonder how the action and, consequently, the field equations will be affected if we go to the Riemann frame $(M, \bar{g} = e^{-\phi}g, \bar{\phi} = 0)$. To carry out the change of frames, let us apply the Weyl transformations (7) and (9), with $f = -\phi$ to (2). It is not difficult to verify that in the new frame the action reads [15]

$$S = \int d^4x \sqrt{-\bar{g}}(\bar{R}(\bar{g}, 0) + w\phi^\alpha\phi_{,\alpha}) + S_m(\bar{g}, \Psi, \nabla^{\bar{g}}\Psi), \quad (27)$$

where $\bar{R}(\bar{g}, 0) = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu}(\bar{g}, 0)$ are purely Riemannian terms (as $\bar{\phi} = 0$) and we are denoting $\phi^\alpha\phi_{,\alpha} = \bar{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$. It is clear that, by construction, the matter action and the energy-momentum tensor $T_{\mu\nu}$ are invariant with respect to these transformations, that is, $S_m(\bar{g}, 0) = S_m(g, \phi)$ and $T_{\mu\nu}(\bar{g}, 0) = T_{\mu\nu}(g, \phi)$. On the other hand, if we rescale the scalar field ϕ by defining the new field variable $\varphi = \sqrt{w}\phi$, we are finally left with the following equations [16]:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = -\kappa^*T_{\mu\nu} - \varphi_{,\mu}\varphi_{,\nu} + \frac{1}{2}\bar{g}_{\mu\nu}\varphi^\alpha\varphi_{,\alpha} \quad (28)$$

and

$$\bar{\square}\varphi = 0, \quad (29)$$

where $\bar{R}_{\mu\nu}$, \bar{R} , and $\bar{\square}\varphi$ are all defined with respect to the metric $\bar{g} = e^{-\phi}g$. Therefore, the field equations of this geometric scalar-field theory, viewed in the Riemann frame, are given by the general relativistic action corresponding to a massless scalar field minimally coupled with the gravitation field, with the only proviso that the Einstein constant κ must be replaced by κ^* .

V. SPHERICALLY SYMMETRIC SOLUTIONS

Once one has set up a theory of gravity, the first question to be addressed is whether the predictions of the new proposal are in agreement with the so-called solar-system experiments. In the case of the present geometrical approach to the scalar-tensor theory, we have seen in the previous section that the mathematical formalism of Weyl transformations allows us to establish a close connection of the theory with Einstein's gravity minimally coupled with a massless scalar field. We shall take advantage of this fact to briefly investigate the existence of spherically symmetric space-times by simply looking into some corresponding general relativistic solutions already known in the literature.

Scalar fields in general relativity have long been studied with great interest, usually as classical approximation to some effective field theory. Also, many attempts at unifying gravity with other interactions, from Kaluza-Klein theories to superstring models, predict the existence of a massless scalar field, not to mention that, according to the standard model, the Higgs boson is described by a scalar field [17]. Historically, the first static spherically symmetric solution of the coupled Einstein–massless-scalar-field equations was found by Fisher [18]. This solution was later rediscovered by some authors, and now it is often referred to as the Janis-Newman-Winicour solution [19]. A generalization of the Fisher solution to n dimensions ($n \geq 4$) was recently obtained in Ref. [20] and further analyzed in detail in Ref. [21].

The connection between the mathematical framework of the geometrical scalar-tensor theory and that of general relativity sourced by a massless scalar field leads naturally to the question of to what extent the physics described in one framework may be transported to the other. [This point remind us of the controversial issue regarding the equivalence between the so-called Jordan and Einstein frames in scalar-tensor theory and in $f(R)$ cosmology [22].] With regard to physical phenomena that depend solely on the motion of particles moving under the influence of gravity alone or on the propagation of light rays, both descriptions are completely equivalent. The reason for this lies in the fact that geodesics are invariant with respect to Weyl transformations, and hence the causal structure of space-time remains unchanged in all Weyl frames. Moreover, as a consequence of the above-mentioned connection between the two frameworks, all results concerning the classical

solar-system tests of gravity predicted by the Fisher solution may be carried over automatically to the geometrical approach.

Thus, let us consider the static, spherically symmetric vacuum asymptotically flat solution of the field equations (28) and (29). As we have mentioned above, this solution, denoted here by $\bar{g}_{\mu\nu}$, was first found by Fisher, and its line element may be written as

$$d\bar{s}^2 = \left(1 - \frac{r_0}{r}\right)^{\frac{M}{\eta}} dt^2 - \left(1 - \frac{r_0}{r}\right)^{-\frac{M}{\eta}} dr^2 - r^2 \left(1 - \frac{r_0}{r}\right)^{1-\frac{M}{\eta}} (d\theta^2 + \sin^2\theta d\psi^2), \quad (30)$$

$$\varphi = \frac{\Sigma}{\eta\sqrt{2}} \ln \left| 1 - \frac{r_0}{r} \right|, \quad (31)$$

where $r_0 = 2\eta$, $\eta = \sqrt{M^2 + \Sigma^2}$, and $M > 0$ is the body's mass in the center of these coordinates [23]. It turns out that by using the parametrized post-Newtonian formalism it has been shown that for a wide range of values of the massless scalar field Σ the Fisher solution predicts the same effects on solar-system experiments as the Schwarzschild solution does [24]. We therefore conclude that, as far as solar-system experiments are concerned, due to invariance of the geodesics under change of frames, the geometrical scalar-tensor theory yields the same results predicted by general relativity.

VI. NAKED SINGULARITIES AND WORMHOLES AS GEOMETRICAL PHENOMENA

The possibility of converting the present geometrical version of the scalar-tensor theory into general relativity plus a massless scalar field brings up some interesting points. As is well known, it has been shown that the presence of a massless scalar field in general relativity causes the event horizons of Schwarzschild, Reissner-Nordström, and Kerr solutions to be reduced to a point and hence leads to the appearance of naked singularities [25]. In fact, naked singularities, which were predicted to appear in the process of a spherically symmetric collapse of a massless scalar field, has later been found in other systems, such as axisymmetric gravitational waves, radiation, and perfect fluids, and so on [26].

In the case of the Fisher solution, given by (30) and (31), the invariant scalar $\bar{R}(\bar{g}, 0) = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu}(\bar{g}, 0)$ gives [21]

$$\bar{R} = \frac{\Sigma^2}{r^4} \left(1 - \frac{r_0}{r}\right)^{\left(\frac{M}{\eta}-2\right)}, \quad (32)$$

which means we have a naked singularity at $r = r_0$, since $\frac{M}{\eta} = \frac{M}{\sqrt{M^2 + \Sigma^2}} < 1$.

It is important to note that the scalar (32), obtained from (30), may be looked upon as the Weyl invariant $e^\phi R$,

calculated in the Riemann frame $(M, \bar{g}, \bar{\phi} = 0)$. As we have already pointed out in Sec. 2, this scalar is invariant under the Weyl transformations (7) and (9). This means that if we go back to the Weyl frame (M, g, ϕ) , where the field equations are (19) and (20), we still have a space-time singularity at $r = r_0$.

It is interesting to have a look at the Fisher solution when viewed in the Weyl frame (M, g, ϕ) . The Weyl transformation that does this task leads to the metric given by

$$g_{\mu\nu} = e^\phi \bar{g}_{\mu\nu} = e^{\frac{\phi}{\sqrt{w}}} \bar{g}_{\mu\nu} = e^{\frac{\Sigma}{\eta\sqrt{2w}} \ln|1 - \frac{r_0}{r}|} \bar{g}_{\mu\nu},$$

whereas $\phi = \frac{\Sigma}{\eta\sqrt{2w}} \ln|1 - \frac{r_0}{r}|$ is the geometric scalar field in this frame. The line element corresponding to this metric will be written as

$$ds^2 = \left(1 - \frac{r_0}{r}\right)^{\frac{M}{\eta} + \frac{\Sigma}{\eta\sqrt{2w}}} dt^2 - \left(1 - \frac{r_0}{r}\right)^{-\frac{M}{\eta} + \frac{\Sigma}{\eta\sqrt{2w}}} dr^2 - r^2 \left(1 - \frac{r_0}{r}\right)^{1 - \frac{M}{\eta} + \frac{\Sigma}{\eta\sqrt{2w}}} (d\theta^2 + \sin^2\theta d\psi^2). \quad (33)$$

In order to see that we still have a naked singularity in the frame (M, g, ϕ) , let us recall that the area of the surface Γ defined by $t = \text{const}$, $r = r_0$ must be calculated with the invariant integral $A = \int_{\Gamma} e^{-\phi} \sqrt{|h|} d\theta \wedge d\psi$, where h denotes the determinant of the metric on Γ induced by (33). Since A is invariant under Weyl transformations and $A = 0$ in the Riemann frame, we conclude that (33) indeed represents a space-time with a naked singularity.

It is well known that the existence of naked singularities in Fisher space-time is a consequence of the presence of a massless scalar field, a field that is related to a massless particle of zero spin. Up to now, no such particles have been discovered, and all known spin-zero particles are massive; hence, models with massless scalar fields do not seem to be realistic. Also, it is still not clear whether such a solution can be considered as a result of gravitational collapse, thereby representing a violation of the cosmic censorship conjecture [27]. In the present approach, however, it should be noted that the scalar field is not a physical field but should be regarded as an essential part of the geometric structure of space-time. Violation of the cosmic censorship conjecture in this case occurs in quite a different context compared with its general relativistic counterpart.

It should be noted that in deriving the field equations (28) we have implicitly considered $w > 0$. If we do not want to impose any restriction on the value of w , it is preferable to work with the field variable ϕ , and in this case the field equations in the Riemann frame read

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = -\kappa^* T_{\mu\nu} - w \phi_{,\mu} \phi_{,\nu} + \frac{w}{2} \bar{g}_{\mu\nu} \phi^{,\alpha} \phi_{,\alpha}, \quad (34)$$

$$\bar{\square} \phi = 0. \quad (35)$$

These equations, in which the coupling constant w appears explicitly, were first considered by Bergmann and Leipnik [28]. The most general spherically symmetric solution to the coupled Einstein–massless-scalar-field equations were obtained by Wyman and, in fact, includes Fisher’s solution as a particular case [19]. The line element and the scalar field corresponding to Wyman’s solution are given by

$$d\bar{s}^2 = e^{\frac{\alpha}{R}} dt^2 - e^{-\frac{\alpha}{R}} \left[\frac{\frac{\gamma}{R}}{\sinh(\frac{\gamma}{R})} \right]^4 dR^2 - R^2 e^{-\frac{\alpha}{R}} \left[\frac{\frac{\gamma}{R}}{\sinh(\frac{\gamma}{R})} \right]^2 (d\theta^2 + \sin^2\theta d\psi^2), \quad (36)$$

$$V = \frac{1}{R}, \quad (37)$$

where α and γ are constants and V is the scalar function defined by Wyman. The coupling constant μ defined by Wyman is related to α and γ through the expression $\gamma = \frac{(\sqrt{\alpha^2 + 2\mu})}{2}$. It is not difficult to verify that the Fisher solution is a particular case of the Wyman solution for $w > 0$. Indeed, if we define the coordinate R by $e^{-\frac{2\mu}{R}} = 1 - 2\eta/r$, it is straightforward to see that (30) and (31) become, respectively,

$$d\bar{s}^2 = e^{-\frac{2M}{R}} dt^2 - e^{\frac{2M}{R}} \left[\frac{\frac{\eta}{R}}{\sinh(\frac{\eta}{R})} \right]^4 dR^2 - R^2 e^{\frac{2M}{R}} \left[\frac{\frac{\eta}{R}}{\sinh(\frac{\eta}{R})} \right]^2 (d\theta^2 + \sin^2\theta d\psi^2), \quad (38)$$

$$\phi = -\Sigma \sqrt{\frac{2}{w}} \frac{1}{R}, \quad (39)$$

recalling that $\phi = \sqrt{w}\phi$. Therefore, if we set $\alpha = -2M$, $\gamma = \eta$, we see that Wyman’s solution reduces to Fisher’s solution provided that $w > 0$ and $\Sigma = -\sqrt{\frac{\mu}{2}}\lambda$, where λ is an arbitrary constant with units of length [29]. Incidentally, it has been shown that Wyman’s solution leads to three types of space-times according to the value assigned to $w\lambda^2$: a naked singularity ($w > 0$), a Schwarzschild black hole ($w = 0$), and a wormhole solution ($-2M^2 < w\lambda^2 < 0$) [30]. It should be clear that all these configurations carry over to the Weyl frame (M, g, ϕ) .

VII. SUMMARY AND DISCUSSION

The fact that in the Brans-Dicke theory of gravity the scalar field has no geometrical origin, while in general relativity the gravitational sector of the action is purely geometric, has motivated some authors to look for what we might call a geometric scalar-tensor theory of gravitation. Let us briefly comment on some of the attempts that are known to us. In one of them, the scalar field is given a

geometrical interpretation in the spirit of the Rainich-Misner-Dicke geometrization of the electromagnetism, although it is restricted to the vacuum case [31]. In another approach, it is shown that a Brans-Dicke scalar field can be derived from pure geometry if the space-time geometry is assumed to be the Lyra manifold [32]. Weyl integrable geometry also appears in a scalar-tensor theory which is directly obtained from general relativity by writing the gravitational sector of the Einstein-Hilbert action in an arbitrary Weyl frame [33]. (It can be shown, however, that the resulting theory, which does not consider matter couplings, is completely equivalent to general relativity and is also conformally related to the Brans-Dicke theory for $\omega = -\frac{3}{2}$, hence implying that the scalar field has no dynamics [34].) Finally, a geometrical scalar-tensor theory was constructed by using a non-Riemannian geometry, in which the scalar field is related to a scalar torsion field [35,36]. In this case, the theory does not consider the matter coupling, and the vacuum field equations are identical to those of the Brans-Dicke theory written in the Einstein frame.

In this work, we have also developed a scalar-tensor theory in which the scalar field plays a definite geometrical role in the description of the gravitational field. Basically, our procedure consists in considering the original formulation of the Brans-Dicke theory as a starting point and modifying it by letting the Palatini formalism decide what kind of geometry should we assign to space-time. This leads in quite a natural way to Weyl geometry, which possesses an interesting property, namely, the invariance of geodesics under a well defined group of transformations. In fact, this suggests that we are concerned here with a whole class of geometries, or space-time manifolds, that are related by a Weyl transformation. According to this view, it seems natural that the geometric objects of interest are those that are invariant under the invariance group of transformations. By following consistently this idea, we are naturally led to redefine our familiar notions of proper time, space-time singularities, etc., in a way that these notions retain their invariance character; i.e., they must be the same in all frames. Surely, this approach will lead to new physical insights as far as a gravitational theory constructed in this framework is concerned. Consider, for instance, the principle of equivalence. It is clear that it will hold in every Weyl frame inasmuch as geodesics do not change by a Weyl transformation. Of course, we have quite a distinct situation in the original formulation of the Brans-Dicke theory of gravity as regards a change of frames [37]. For example, it is widely known that freely falling particles do not move on geodesics in the so-called Einstein (conformal) frame, and also measurements made by rods and clocks are not invariant under a change of frames [38].

Another comment is in order. It is important to call attention to the fact that, in obtaining Eq. (3) by applying

the Palatini formalism, we have completely ignored the matter action and considered only the action corresponding to the gravitational sector. There is, in fact, a methodological reason to justify this procedure: it is assumed, as a principle, that what really determines the space-time geometry is the gravitational sector. Once the geometry is found, then completing the action by later adding the matter action will not affect (3), since any dependence on the affine connection may be entirely reduced to dependence on the geometric fields g and ϕ through (6). This permits us to proceed with our reasoning without having to make the usual assumption that the matter sector is functionally independent of the (nonmetric) connection [3].

At this point, one might remark that neither the action nor the field equations of the proposed theory are invariant under Weyl transformations, and, in view of the connection of the theory with usual general relativity coupled with a massless scalar field, it would perhaps be desirable, at least from the aesthetic viewpoint, that the whole theory should exhibit Weyl invariance. We believe that this is an important point which should be addressed here. First of all, let us point out that as far as we know no gravitational theory defined in a Weyl integrable space-time is constructed with a Weyl-invariant action (see, for instance, the references in [7]). The reason for this is that, in any theory based on an action with the property of being invariant under Weyl transformations, the scalar field ϕ is completely unnecessary and trivial in the sense that there is always a frame (the Riemann frame) in which ϕ can be made to completely disappear. In this case, the scalar field has no extra degree of freedom and no dynamics, as it may be clearly illustrated by formulating general relativity in a Weyl integrable space-time (see [34].) However, despite this noninvariance of the action, it should be emphasized that all physical and geometric objects of the theory are constructed in such a way that they are automatically invariant under Weyl transformations: the geodesic motion, the proper time of massive particles, all the scalar invariants such as defined in Sec. II, which include the generalized Kretschmann scalar.

Finally, it is interesting to note that the reason why the field equations (21) and (23) derived in Sec. 2 coincide with those of the Brans-Dicke theory only in the case of vacuum is that in the latter the scalar field does not participate directly in the way matter couples with the gravitational field. Indeed, in the Brans-Dicke theory the action describing ordinary matter is postulated to be of the form $S_m = \kappa^* \int d^4x \sqrt{-g} L_m(g_{\mu\nu}, \Psi, \nabla\Psi)$, which is a necessary requirement to ensure that freely falling particles follow Riemannian geodesics. However, in the geometrical scalar-tensor theory we are considering, freely falling particles should follow affine geodesics in a Weyl integrable space-time, and the only matter coupling which is consistent with this requirement is the one given by (14). This can easily be seen, for instance, by considering the field equations (17) in the case where $T_{\mu\nu}$ represents the energy-momentum tensor

of a pressureless perfect fluid (“dust”). Then, it is not difficult to verify that, by taking the covariant divergence (with respect to the Weyl connection) of both sides of (17), we are led to (13) [39].

In conclusion, we remark that scalar-tensor theories have been extensively discussed in the literature. One of the most important areas of their application is cosmology, where the scalar field is sometimes considered as a quintessence field [40]. Scalar-tensor theories have also been investigated in the context of braneworld scenarios [41]. Thus, a natural

followup of the ideas we have discussed in the present article would be an application of the geometric scalar-tensor theory to modern cosmology. We leave this for future work.

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