

Transforming gravity: From derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian

Miguel Zumalacárregui^{1,2} and Juan García-Bellido¹¹*Instituto de Física Teórica IFT-UAM-CSIC, Universidad Autónoma de Madrid, Nicolás Cabrera 13-15, Cantoblanco, 28049 Madrid, Spain*²*Institut für Theoretische Physik, Ruprecht-Karls-Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany*

(Received 24 October 2013; published 19 March 2014)

We study the structure of scalar-tensor theories of gravity based on derivative couplings between the scalar and the matter degrees of freedom introduced through an effective metric. Such interactions are classified by their tensor structure into conformal (scalar), disformal (vector), and extended disformal (traceless tensor), as well as by the derivative order of the scalar field. Relations limited to first derivatives of the field ensure second-order equations of motion in the Einstein frame and hence the absence of Ostrogradski ghost degrees of freedom. The existence of a mapping to the Jordan frame is not trivial in the general case, and can be addressed using the Jacobian of the frame transformation through its eigenvalues and eigentensors. These objects also appear in the study of different aspects of such theories, including the metric and field redefinition transformation of the path integral in the quantum mechanical description. Although second-order in the Einstein frame, generic disformally coupled theories are described by higher-order equations of motion in the Jordan frame. This apparent contradiction is solved by the use of a hidden constraint: the contraction of the metric equations with a Jacobian eigentensor provides a constraint relation for the higher field derivatives, which allows one to express the dynamical equations in a second-order form. This signals a loophole in Horndeski's theorem and allows one to enlarge the set of scalar-tensor theories which are Ostrogradski stable. The transformed Gauss-Bonnet terms are also discussed for the simplest conformal and disformal relations.

DOI: 10.1103/PhysRevD.89.064046

PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k, 98.80.Cq

I. INTRODUCTION

Current cosmological observations agree on the fact that the Universe is undergoing a late phase of accelerated expansion [1–5], analogous to the early-time, high-energy inflationary mechanism that is believed to have set the conditions necessary for big bang cosmology [6]. The simplest explanation for such an acceleration in an otherwise matter-dominated universe is provided by the inclusion of a cosmological constant, which is however very small compared to other energy scales present in the standard model of particle physics and which are expected to contribute to the Universe's acceleration [7,8]. This puzzle has triggered the revival and proposal of a number of alternative theories, which attempt to explain the surprising behavior of the Universe on Hubble scales [9,10].

Such theories are generally described by an action functional

$$S[g_{\mu\nu}, \psi^{(i)}] = \int d^4x \sqrt{-g} \mathcal{L}[g_{\mu\nu}, \psi^{(i)}], \quad (1)$$

in which the Lagrangian density \mathcal{L} is a Lorentz scalar that depends locally on the metric and matter fields ($g_{\mu\nu}$, $\psi^{(i)}$) and their derivatives. The classical dynamics followed by such fields are given by the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \psi^{(i)}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \psi^{(i)})} + \nabla_\mu \nabla_\nu \frac{\partial \mathcal{L}}{\partial (\nabla_\nu \nabla_\mu \psi^{(i)})} = 0, \quad (2)$$

obtained by varying the action with respect to the fundamental fields. It has been further assumed that the action contains up to their second derivatives.

Any alternative theory has to fulfill a number of requirements in order to be satisfactory. A very strong limitation to the space of possible theories is given by Ostrogradski's theorem [11,12]: for a nondegenerate theory whose Lagrangian contains second or higher derivatives with respect to time, their associated Hamiltonian is unbounded from below, making the system unstable and lacking a well-defined vacuum state. Degenerate theories are those for which Ostrogradski's construction does not apply, as is the case for any theory described by second-order equations of motion. Such is the case of general relativity (GR), whose Lagrangian contains second derivatives of the metric, but with the right degenerate structure to be described by second-order equations of motion. In fact, it is the only four dimensional, Lorentz-covariant local theory of a metric tensor which fulfills this requirement [13].¹

¹Higher-order terms are also acceptable if they represent perturbative corrections (e.g. to a low-energy effective theory) and this perturbative nature is enforced in the solutions [14].

TABLE I. Notation used in the text. Quantities with a bar or a tilde are constructed using the barred or tilde metric. All metrics have the signature $(-, +, +, +)$ signature, and the Riemann tensor is defined by $2\nabla_{[\mu}\nabla_{\nu]}v^\alpha \equiv R^\alpha{}_{\beta\mu\nu}v^\beta$. Parenthesis/brackets between indices will denote symmetrization/antisymmetrization $t_{(\mu\nu)} = \frac{1}{2}(t_{\mu\nu} + t_{\nu\mu})$, $t_{[\mu\nu]} = \frac{1}{2}(t_{\mu\nu} - t_{\nu\mu})$. The symbol $\supset^{(l)}$ will be used to denote the higher time derivatives of an expression. The word “frame” (or physical frame) refers to the set of variables on which the variation is performed (e.g. Einstein/Jordan frame). Units in which $c = 1$ will be used throughout.

Scalar Fields ϕ, π

$X = -\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} \rightarrow$ Canonical kinetic term for the scalar

$\Phi_{\mu\nu} = \phi_{;\mu\nu}$, $\Phi_{\mu\nu}^n = \phi_{;\mu\alpha_1}\phi^{;\alpha_1;\alpha_2}\dots\phi^{;\alpha_{n-1};\nu} \rightarrow$ Contraction of second derivatives of the scalar (n fields)

$[\Phi^n] = g^{\mu\nu}\Phi_{\mu\nu}^n$, e.g. $[\Phi] = \phi^{;\mu}{}_{;\mu} \equiv \square\phi$, $[\Phi^2] = \phi_{;\alpha\beta}\phi^{;\alpha\beta}\dots \rightarrow$ Traces with the metric

$\langle\Phi^n\rangle = \phi^{;\mu}\Phi_{\mu\nu}^n\phi^{;\nu}$, e.g. $\langle\Phi\rangle = \phi_{,\alpha}\phi^{;\alpha\beta}\phi_{,\beta}$, $\langle\Phi^2\rangle = \phi_{,\alpha}\phi^{;\alpha\lambda}\phi_{,\lambda\beta}\phi^{;\beta}$ $\dots \rightarrow$ Traces with the field derivatives

Tensor Fields $g_{\mu\nu}, \bar{g}_{\mu\nu}, \tilde{g}_{\mu\nu}\dots$

$g_{\mu\nu} \rightarrow$ Dynamical metric, i.e. its dynamics determined by $\delta S/\delta g_{\mu\nu} = 0$.

$\bar{g}_{\mu\nu}, \tilde{g}_{\mu\nu} \rightarrow$ Effective metric with scalar field dependence, (cf. Table II)

$\nabla_\mu, \Gamma_{\mu\nu}^\alpha \rightarrow$ Torsion-free covariant derivative and connection compatible with $g_{\mu\nu}$. Also $u^\mu{}_{;\nu} = \nabla_\nu u^\mu$

$\bar{\nabla}_\mu, \bar{\Gamma}_{\mu\nu}^\alpha, \tilde{\nabla}_\mu, \tilde{\Gamma}_{\mu\nu}^\alpha \rightarrow$ *idem* for $\bar{g}_{\mu\nu}, \tilde{g}_{\mu\nu}$ cf. Eq. (35). All barred/tilde quantities constructed out of $\bar{g}_{\mu\nu}, \tilde{g}_{\mu\nu}$

Curvature $R^\alpha{}_{\beta\mu\nu}$ cf. (35)

$[R_{\mu\nu}] = R_{\mu\nu}g^{\mu\nu}$, $\langle R_{\mu\nu} \rangle = \phi^{;\nu}\phi^{;\mu}R_{\mu\nu}$, $[R_{\mu\nu}^2] = R_{\mu\nu}R^{\mu\nu}$, $\langle R_{\mu\nu}^2 \rangle = \phi^{;\alpha}R_{\alpha\mu}R^{\mu\beta}\phi_{,\beta}\dots$

$\langle R_{\mu\nu}R^{\alpha\mu\beta\nu} \rangle = R_{\mu\nu}R^{\alpha\mu\beta\nu}\phi_{,\alpha}\phi_{,\mu}$, $\langle\langle R_{\mu\alpha\nu\beta}\Phi^{\alpha\lambda}\Phi^{\beta\sigma}\rangle\rangle = \phi^{;\mu}\phi^{;\nu}R_{\mu\alpha\nu\beta}\Phi^{\alpha\lambda}\Phi^{\beta\sigma}\phi_{,\lambda}\phi_{,\sigma}$

When one considers scalar-tensor theories of gravitation, Horndeski’s theorem [15] determines the most general four dimensional, Lorentz-covariant, local scalar-tensor theory for which the variation (2) produces second-order equations. It is described by a Lagrangian density of the form $\mathcal{L}_H = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5$, with

$$\mathcal{L}_2 = G_2(X, \phi), \quad (3)$$

$$\mathcal{L}_3 = G_3(X, \phi)[\Phi], \quad (4)$$

$$\mathcal{L}_4 = G_4(X, \phi)R + G_{4,X}([\Phi]^2 - [\Phi^2]), \quad (5)$$

$$\mathcal{L}_5 = G_5(X, \phi)G_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}([\Phi]^3 - 3[\Phi][\Phi^2] + 2[\Phi^3]), \quad (6)$$

plus a matter Lagrangian (See [16,17] for modern rederivations). The notation used in the above equations and throughout the article is presented in Table I. The key to the degeneracy of the above theory is that second derivatives of the scalar field appear in antisymmetric combinations, so that higher derivatives cancel in the Euler-Lagrange variation (2) if the free functions G_3, G_4, G_5 depend on the field derivatives through X . A very important advantage of the Horndeski Lagrangian is that it contains many interesting physical theories (see [18] for a summary) and allows for a systematic study of their properties. Such a general approach has been applied to cosmological dynamics [19,20], compatibility with cosmological observations [21–23], inflationary mechanisms [24] and screening

modifications of gravity [25,26] and the effective cosmological constant [27]. Besides general relativity ($G_2 = G_3 = G_5 = 0, G_4 = 1/16\pi G$), the best known class of theories contained in \mathcal{L}_H are Jordan-Brans-Dicke theories [28], for which $G_3 = G_5 = 0, G_4 = f(\phi)/16\pi G$ and $G_2 = X/\omega(\phi) - V(\phi)$.

An important aspect of Jordan-Brans-Dicke theories is that the coupling between the scalar field and the curvature, given by $G_4(\phi)$, can be eliminated by a conformal transformation in which the metric is rescaled by a function of the field $g_{\mu\nu} \rightarrow G_4^{-1}g_{\mu\nu}$. This allows one to obtain different representations of the same theory, usually known as “frames,” depending on which variables are considered dynamical. The original formulation is known as the “Jordan frame”: the field ϕ and the Ricci scalar couple directly, but the matter Lagrangian only involves the metric, with no direct interaction between ϕ and the matter degrees of freedom. Alternatively, one may perform the aforementioned conformal transformation to the Einstein frame in which the gravitational Lagrangian has the Einstein-Hilbert form ($G_4 = 1/16\pi G$) but the matter sector is directly affected by the scalar field, which mediates an additional force. Both representations are equivalent at the classical level (cf. [29–31] and references therein), and each of them offers useful insight into their characteristics and behavior.

A natural question is whether generalizations of the conformal relation can offer further insights into the general class of scalar-tensor theories given by Eqs. (3)–(6). This can be done in some special cases, the simplest of them being the Dirac-Born-Infeld Galileons, which describe induced gravity on four-dimensional branes embedded in five-dimensional space [32]. For a quartic DBI Galileon

with $G_4 = \sqrt{1 - 2X/M^4}$, $G_5 = 0$, it is possible to eliminate the nonminimal coupling to the Ricci scalar by means of a special disformal transformation,

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = C(\phi)g_{\mu\nu} + D(\phi)\phi_{,\mu}\phi_{,\nu}, \quad (7)$$

with $C = 1$, $D = -1/M^4$ [18]. More generally, Horndeski's theory has been shown to be formally invariant under special disformal transformations, which amount to a redefinition of the free functions $G_2 - G_5$ [33]. Therefore, it is natural to consider (7) as an integral part of Horndeski's theory.

When the matter sector is considered, the Einstein frame representation of the DBI Galileons introduces a derivative coupling between the scalar field and the matter degrees of freedom. This has important phenomenological consequences, as it modifies the relative causal structure between $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. Additionally, the derivative coupling allows for the disformal screening mechanism [34], which can hide the scalar-mediated additional force in high-density environments. This effect might be related to the Vainshtein screening mechanism [35], which hides the scalar force within a certain radius of point sources due to the nonlinear derivative self-interactions of the field caused by the degenerate terms (4–6), as both theories are classically equivalent [18]. The inclusion of derivative interaction also allows for shift symmetry, i.e. invariance of the action under the transformation $\phi \rightarrow \phi + c$. Exact (or softly broken) shift symmetry can be used to prevent large contributions to the field mass term or interactions with matter arising from quantum corrections. Shift symmetry plays an important role in certain scalar-tensor theories, such as Higgs-dilaton cosmology [36–38], and is a particular case of Galilean symmetry $\phi \rightarrow \phi + c + b_\mu x^\mu$, which provides further improvement on the quantum properties of the theory [39] (for cosmological applications of Galileons see [40,41]).

Disformal relations were originally introduced by Bekenstein in a more general form in which C and D are also allowed to depend on X [42],

$$\tilde{g}_{\mu\nu} = C(X, \phi)g_{\mu\nu} + D(X, \phi)\phi_{,\mu}\phi_{,\nu}. \quad (8)$$

Such relations have turned out to be very fruitful in the construction of alternatives to general relativity, to a large extent because they can distort the causal structure between the two space-times since the line elements are now related by $d\tilde{s}^2 = Cds^2 + D(\phi_{,\mu}dx^\mu)^2$. Assuming $C > 0$, a four-vector that is null with respect to $g_{\mu\nu}$ will be spacelike or timelike with respect to $\tilde{g}_{\mu\nu}$ depending on whether D is positive or negative locally. Applications of the disformal relation (8) include inflation [43], varying speed of light theories [44–46], gravitational alternatives to dark matter (DM) [47–49], screening modifications of gravity [18,34,50], violation of Lorentz invariance [51], massive gravity [52,53], dark energy [54,55], DM candidates from

extra dimensions [56,57] or string theory [58,59] and exotic DM interactions [60,61]. Disformal relations also appear in generalized Palatini gravities [62], provide new symmetries of Maxwell's [63] and Horndeski's [33] theories, allow for the Lorentzian signature of the metric to emerge dynamically [64,65] and might be used in the construction of renormalizable theories of gravity [66]. Unlike conformal relations, disformal couplings have non-trivial effects on radiation and can affect photons, a possibility that has been studied in the context of laboratory tests [67] and cosmological implications [68–70].

Could a disformal transformation be used to remove the derivative couplings between the scalar field and the curvature from the higher Horndeski's terms \mathcal{L}_4 , \mathcal{L}_5 ? The fact that a disformal coupling to matter of the form (8) does not introduce second-order terms in the dynamical equations suggests that the equivalent Jordan frame representation belongs to Horndeski's theory. However, Bettoni and Liberati have shown that this is not the case [33]: the action of a general disformal transformation (8) on the gravitational sector generates terms that can not be expressed in the form (3–6) unless the transformation is of the special type (7). One of the purposes of this work is to examine the apparent contradiction between the second-order nature of the disformally coupled theory, versus its apparently higher-order nature in the Jordan frame.

In Sec. II we study generalizations of the disformal relation. These can be classified by their tensor structure into conformal (scalar), disformal (vector) and extended disformal (traceless tensor), as well as by the number of derivatives of the fields that take part in the transformation. For relations involving a scalar and a metric tensor field, Bekenstein's relation (8) turns out to be a fairly natural choice, for which only two free functions are allowed and the equations of motion contain at most second derivatives of the field. Terms constructed out of second field derivatives might be also considered. However, they allow the construction of infinitely many terms which would generically lead to higher-order terms in the equations.

The existence of a Jordan frame is examined in Sec. III by studying under which conditions it is possible to find an inverse mapping for the disformal relation. The existence of such an inverse transformation is not trivial in the general case, and can be determined by studying the determinant of the Jacobian associated with (8), seen as a function of $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}$: an inverse map exists around any point for which the Jacobian determinant is nonzero. The invertibility is studied in detail for a general conformal and disformal case using the eigenvalues and eigenvectors of the Jacobian, which are in turn related to other aspects of frame transformation, including the transformation properties of the path integral. Several examples of inverse mappings with and without singular points are discussed.

Having obtained the conditions for an inverse transformation to exist, we proceed to analyze the Jordan frame

TABLE II. Possible relations between metrics, cf. Eq. (9). The columns classify the possible tensor structures that can be considered in the transformation (middle columns) as well as the possible dependences of the free functions (last column), i.e. all the Lorentz-scalars that can be constructed out of the objects introduced. The last column indicates the possible dependences of the functions C, D, E . Here $\Phi_{\mu\nu}^1 = \phi_{;\mu\nu}$ and $\Phi_{\mu\nu}^n = \phi_{;\mu\alpha_1}\phi^{;\alpha_1;\alpha_2}\cdots\phi^{;\alpha_{n-1};\nu}$ for $n > 1$ (n indicates the number of twice differentiated fields). The table considers both the general case and the case of scalar-tensor theories in which zero, one or two field derivatives are allowed. Allowing second field derivatives in the disformal relation allows for an (in principle) arbitrary number of terms to be added, due to the possibility of constructing contractions of $\phi_{;\mu\nu}$ with free indices. The last row displays the possible terms arising from a vector field and its first derivatives.

	Conformal	Disformal	Tensor	Dependence of C, D, E
		Vector		
General	$Cg_{\mu\nu}$	$Dd_\mu d_\nu$	$E_{(\mu\nu)}$	$C, d^2, [E^n], d \cdot E^n \cdot d$
ϕ	$Cg_{\mu\nu}$	–	–	ϕ
$\phi_{;\mu}$	$Cg_{\mu\nu}$	$D\phi_{;\mu}\phi_{;\nu}$	–	ϕ, X
$\phi_{;\mu\nu}$	$Cg_{\mu\nu}$	$D\phi_{;\mu}\phi_{;\nu} + \sum_{n,m} D_{m,n} \phi^\alpha \Phi_{;\alpha(\mu}^m \Phi_{;\nu)\beta}^n \phi^{\beta}$	$E\phi_{;\mu\nu} + \sum_l E_l \Phi_{;\mu\nu}^l$	$\phi, X, [\Phi^n], \langle \Phi^n \rangle$
$v_{\mu;\nu}$	$Cg_{\mu\nu}$	$Dv_\mu v_\nu + \sum_{n,m} D_{m,n} v^\alpha v_{\alpha;(\mu}^n v_{;\nu)\beta}^m v^{\beta}$	$E v_{(\mu;\nu)} + \sum_l E_l v_{(\mu;\nu)}^l$	$v^2, [v_{\mu\nu}^n], d \cdot v_{\mu\nu}^n \cdot d$

theory in Sec. IV, first in the case of a nontrivial conformal transformation and then for the general case, focusing on a gravitational sector which is of the Einstein-Hilbert form in the original frame. The terms generated by the transformation do not belong to Horndeski's theory and their variation leads to equations which contain up to fourth time derivatives of the field and third time derivatives of the metric. However, by contracting the metric equations with a Jacobian eigentensor, a relation is derived which can be used to remove the higher derivatives from the equations. The hidden, second-order nature of disformally coupled theories in the Jordan frame signals a loophole in Horndeski's theorem, as its derivation does not take into account the possibility of using combinations of the (initially higher-order) dynamical equations to construct a second-order theory.

Section V contains a discussion of the main results, open questions and possible applications of the methods developed. Appendix A contains equations that arise from the general disformal relation (8) and were to long to be included in the text. Appendix B presents certain equations for special disformal transformations, including the transformation rules for the Einstein-Hilbert and the Horndeski Lagrangians, as well as the Gauss-Bonnet terms in the pure special conformal and pure special disformal case.

II. DERIVATIVE COUPLINGS TO MATTER

Let us start by examining some of the properties of the theories of gravity formulated in a frame in which the matter sector contains disformal couplings between the matter degrees of freedom, the gravitational metric and the scalar field. An immediate question is what extensions of the original disformal relation (8) can be proposed and whether they are physically viable. We can classify the transformations of the metric according to two different criteria:

- (i) By the tensor structure. In order to obtain a symmetric tensor, it is possible to consider a conformal term proportional to the original metric, a disformal term constructed out of a vector d_μ and an extended disformal term consisting of a rank-two symmetric tensor $E_{(\mu\nu)}$,

$$\tilde{g}_{\mu\nu} = Cg_{\mu\nu} + Dd_\mu d_\nu + E_{(\mu\nu)}. \quad (9)$$

The first term preserves the causal structure associated to both metrics (i.e. null vectors are null with respect to the two metrics), while the second and third terms do not. The main difference between the nonconformal terms is that $d_\mu d_\nu$ introduces a privileged direction along d_μ .² To eliminate the degeneracy associated with the extended disformal term, we must project away the former terms so that $d^\mu d^\nu E_{(\mu\nu)} = g^{\mu\nu} E_{(\mu\nu)} = 0$.

- (ii) By the derivative order, i.e. how many derivatives of the variables are allowed in the transformation. Here we will restrict to relations in which no derivatives of the metric are introduced, except through covariant derivatives.³ The highest derivatives allowed are important for the character of the dynamical equations describing the theory, which might become higher than second order if second derivatives are included. Furthermore, higher derivatives also provide further tensor structures, as they allow arbitrary contractions using the same objects.

Table II summarizes the result of this classification for metrics constructed out of a scalar field.

²One may as well include several disformal terms $d_\mu^{(i)} d_\nu^{(i)}$, or even terms made out of spinors, $d_\mu = \bar{\psi} \gamma_\mu \psi, \bar{\psi} \gamma_\mu \gamma_5 \psi, \bar{\psi} \nabla_\mu \psi, \bar{\psi} \gamma_5 \nabla_\mu \psi, E_{(\mu\nu)} = \bar{\psi} \gamma_\mu \gamma_\nu \psi$ (here $\bar{\psi} = \psi^\dagger \gamma^0$), as long as they are consistent with the parity and tensor structure of the metric.

³An example of theories featuring two metrics whose relation involves derivatives are C and D theories, which allow a conformal dependence on R and a disformal dependence with the tensor structure of $R_{\mu\nu}$ [71].

In the original work, Bekenstein disregarded theories including higher than second derivatives of the scalar because he expected that such theories would lead to higher than second-order equations of motion and unbounded Hamiltonians. In addition, Table II makes clear that the introduction of objects with two indices allows for a potentially infinite set of different contractions.⁴ Finally, if (covariant) derivatives of nonscalar objects are considered, the relation generically introduces derivatives of the metric tensor through the Christoffel connection. For the sake of simplicity according to the above considerations, we will restrict our attention to metrics constructed only out of first field derivatives, as in the original disformal relation (8).

Besides mapping $g_{\mu\nu}$ into another rank-two symmetric tensor $\tilde{g}_{\mu\nu}$, other physical requirements are necessary for the disformal relation to be physically reasonable. In order to have a well defined inverse metric $\tilde{g}^{\mu\nu}$ and provide an invariant integration volume, $\tilde{g}_{\mu\nu}$ needs to have a non-vanishing determinant,

$$\tilde{g} = C^3(C - 2DX)g \neq 0, \quad (10)$$

(see Appendix C of Ref. [48] for a derivation of the above expression). The authors of Ref. [72] suggest that the functions C, D have to be chosen such that the previous condition is satisfied for all possible values of X , i.e. $C > 0$ and $C > 2DX$. However, it has been observed that the second condition is maintained dynamically in cosmological models, as the field slows down whenever X approaches $C/(2D)$. This happens both in the case of disformal couplings to matter [18,34] and scalar field self coupling [55] for $D = D(\phi) > 0$, suggesting that it is not necessary to tailor the functions C, D . Moreover, theories formulated in terms of $\tilde{g}_{\mu\nu}$ should be required to have a well posed initial value problem and give rise to second-order evolution equations.

A. Matter-scalar interaction

Theories in which the matter Lagrangian is formulated in terms of a tilde metric (8, 9) introduce interactions between the matter and scalar degrees of freedom. If $S_m = \int d^4x \sqrt{-\tilde{g}} \tilde{\mathcal{L}}_m(\tilde{g}, \psi^{(m)})$, the invariance of S_m under coordinate transformations $x^\mu \rightarrow x^\mu + \xi^\mu$ implies that $\delta S_m = 0$ and hence

$$\begin{aligned} & \int d^4x \sqrt{-\tilde{g}} \left(\frac{1}{\sqrt{-\tilde{g}}} \frac{\delta(\sqrt{-\tilde{g}} \tilde{\mathcal{L}}_m)}{\delta \tilde{g}_{\mu\nu}} \delta_\xi \tilde{g}_{\mu\nu} + \frac{\delta \tilde{\mathcal{L}}_m}{\delta \psi^{(m)}} \delta_\xi \psi^{(m)} \right) \\ &= \int d^4x \sqrt{-\tilde{g}} \left(\frac{1}{2} \tilde{\nabla}_\mu \tilde{T}^{\mu\nu} \right) \xi_\nu = 0, \end{aligned} \quad (11)$$

⁴Some of these terms or their combinations might give rise to total derivatives, which do not contribute to the equations of motion. Note also that a field redefinition $\varphi = \varphi(Y, \pi)$ with $Y = -\frac{1}{2}(\partial\pi)^2$ adds a vector-like, second derivative term to the standard disformal structure $\phi_{,\mu}\phi_{,\nu} = (\phi_{,Y})^2 \pi^\alpha \pi^\beta \pi_{,\alpha\mu} \pi_{,\beta\nu} - 2(\phi_{,Y}\phi_{,\pi}) \pi^\alpha \pi_{,\alpha(\mu} \pi_{,\nu)} + (\phi_{,\pi})^2 \pi_{,\mu} \pi_{,\nu}$.

where the coefficient of $\delta \tilde{g}_{\mu\nu}$ is proportional to the energy momentum tensor defined with respect to the tilde metric, the matter equations of motion $\delta \tilde{\mathcal{L}}_m / \delta \psi_i = 0$ and the metric transformation $\delta_\xi \tilde{g}_{\mu\nu} = \tilde{\nabla}_{(\mu} \xi_{\nu)}$ [73] have been used, and $\tilde{\nabla}_\mu$ is a torsion-free covariant derivative compatible with $\tilde{g}_{\mu\nu}$ (cf. Eq. (35) and Appendix A1). Therefore, as a direct application of Noether's theorem, energy-momentum is covariantly conserved as long as $\tilde{g}_{\mu\nu}$ is used consistently in the equations

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0. \quad (12)$$

This derivation is valid as long as the theory is invariant under coordinate transformations and the motion for ψ_i are fully determined from $\delta \tilde{\mathcal{L}}_m$ alone. Note that no assumptions about the form of the gravitational sector and/or the tilde metric have been made in the derivation.

For disformal couplings to matter containing the field and its first derivatives (8), the contribution of the matter Lagrangian to the scalar field equation reads

$$\begin{aligned} \frac{1}{\sqrt{-\tilde{g}}} \frac{\delta \sqrt{-\tilde{g}} \tilde{\mathcal{L}}_m}{\delta \phi} &= -\tilde{T}^{\mu\nu} \tilde{\nabla}_\mu (D\phi_{,\nu}) + \frac{1}{2} \tilde{\nabla}_\alpha (\tilde{T}^{\mu\nu} \phi^\alpha (C_{,X} g_{\mu\nu} \\ &+ D_{,X} \phi_{,\mu} \phi_{,\nu})) + (C_{,\phi} g_{\mu\nu} + D_{,\phi} \phi_{,\mu} \phi_{,\nu}) \tilde{T}^{\mu\nu}, \end{aligned} \quad (13)$$

where the energy momentum tensor in the first term has “escaped” the derivative by virtue of tilde energy conservation (12). As a consequence of the chain rule $\tilde{T}^{\mu\nu}$ appears contracted with the partial derivatives of the tilde metric, $\tilde{g}_{\mu\nu,X}$ and $\tilde{g}_{\mu\nu,\phi}$. The above equation implies that special disformal relations (7) do not introduce derivatives of $\tilde{T}^{\mu\nu}$ in the field equations, as $C_{,X}, D_{,X} = 0$. In this case, equations (12) and (13) are equivalent to the equations derived in Refs. [18,34] in terms of untilde quantities (using the appropriate connection (A3) and contracting tilde matter conservation (12) with $\tilde{g}_{\lambda\nu}$). This simplification may also be related to the fact that the Horndeski Lagrangian (3–6) is formally invariant under special disformal transformations (7) [33].

Note that all the terms contributed by (13) to the field equations of motion are at most second order in field derivatives, and therefore do not introduce Ostrogradski instabilities. This does not generally hold if second field derivatives are allowed, as in the relation (9) with the terms described in Table II. The variation of the matter Lagrangian w.r.t. the scalar field then reads

$$\begin{aligned} & \frac{2}{\sqrt{-\tilde{g}}} \frac{\delta \sqrt{-\tilde{g}} \tilde{\mathcal{L}}_m}{\delta \phi} \\ &= \tilde{T}^{\mu\nu} \frac{\partial \tilde{g}_{\mu\nu}}{\partial \phi} - \tilde{\nabla}_\alpha \left(\tilde{T}^{\mu\nu} \frac{\partial \tilde{g}_{\mu\nu}}{\partial \phi_{,\alpha}} \right) + \tilde{\nabla}_\beta \tilde{\nabla}_\alpha \left(\tilde{T}^{\mu\nu} \frac{\partial \tilde{g}_{\mu\nu}}{\partial \phi_{,\alpha\beta}} \right) \\ & \quad - \tilde{\nabla}_\lambda \left(\tilde{T}^{\mu\nu} \frac{\partial \tilde{g}_{\mu\nu}}{\partial \phi_{,\alpha\beta}} \mathcal{K}^{\lambda}_{\alpha\beta} \right), \end{aligned} \quad (14)$$

where $\mathcal{K}^\lambda_{\alpha\beta} \equiv \tilde{\Gamma}^\lambda_{\alpha\beta} - \Gamma^\lambda_{\alpha\beta}$ is the difference between the connections for the field dependent and dynamical metrics, as given by Eq. (35) below. Even if it is possible to choose the coefficients of the disformal relation shown in Table II to achieve second-order equations of motion (including the difference between the connections $\mathcal{K}^\lambda_{\alpha\beta}$), second derivatives of the field also introduce second derivatives of the energy momentum tensor. Although such theories are not a priori ruled out, they can lead to phenomenological problems (see Ref. [74] for a discussion of such terms in the gravitational equations).

III. FRAME TRANSFORMATIONS: THE JACOBIAN

Let us now study under which conditions disformally coupled theories can be re-expressed in the *Jordan frame*, i.e. using the metric to which matter couples minimally as a fundamental variable. Ultimately, finding the Jordan frame requires inverting the relation between the two metrics and transforming the gravitational and matter sector accordingly. In the most thoroughly explored case of a special disformal relations (7), such an inverse can be obtained trivially

$$g_{\mu\nu} = \frac{1}{C} \tilde{g}_{\mu\nu} - \frac{D}{C} \phi_{,\mu} \phi_{,\nu}, \quad (15)$$

and is well defined as long as $C \neq 0$.

In more general cases, it is possible to address the existence of inverse map between the metrics by using the *inverse function theorem* [75]. Given a continuous differentiable function (that may depend on several variables), the theorem ensures the existence of an inverse, continuous differentiable function in a neighborhood of a point whenever the Jacobian determinant of the transformation is different from zero at that point. When applied to a function $\tilde{g}_{\mu\nu}(g_{\alpha\beta})$, it implies that an inverse $g_{\mu\nu}(\tilde{g}_{\alpha\beta})$ exists around any point for which

$$\left| \frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}} \right| \neq 0, \quad (16)$$

where we think of $\partial \tilde{g}_{\mu\nu} / \partial g_{\alpha\beta}$ as a linear mapping of (0,2) symmetric tensors into (0,2) symmetric tensors (antisymmetric tensors are mapped to zero due to index symmetry). Note that the condition (16) only ensures the existence of an inverse map for the covariant metric. The existence of a contravariant metric $\tilde{g}^{\mu\nu}$ satisfying $\tilde{g}^{\alpha\lambda} \tilde{g}_{\lambda\beta} = \delta^\alpha_\beta$ requires that $\tilde{g}_{\mu\nu}$ has nonvanishing determinant, cf. (10). This issue may be addressed by studying the *contravariant Jacobian* $\partial \tilde{g}^{\mu\nu} / \partial g^{\alpha\beta}$. Demanding the absence of zeros might provide additional conditions for the frame transformation to exist, but these will not be considered here. The theorem also determines that the Jacobian determinants of both mappings are the inverse of each other, $|\partial \tilde{g}_{\mu\nu} / \partial g_{\alpha\beta}| = |\partial g_{\mu\nu} / \partial \tilde{g}_{\alpha\beta}|^{-1}$.

The Jacobian determinant can be evaluated in terms of the eigenvalues of the Jacobian $|\partial \tilde{g}_{\mu\nu} / \partial g_{\alpha\beta}| = \prod_n \lambda_n$, where each eigenvalue satisfies

$$\left(\frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}} - \lambda_i \delta^\alpha_\mu \delta^\beta_\nu \right) \xi_{\alpha\beta}^{(i)} = 0, \quad (17)$$

for its associated eigentensor $\xi_{\alpha\beta}^{(i)}$. It is easy to check that for the special disformal transformation the only eigenvalue is $\lambda_C = C$, as expected from the explicit inverse metric (15). The Jacobian and its eigenvalues will be studied in the nontrivial conformal case and for a general disformal metric (8) in the following subsections. However, the formalism presented above is general as long as the transformations of the metric tensor depends on the metric algebraically, and could be applied to more general relations such as (9).⁵

Besides determining the existence of an inverse transformation, the Jacobian occurs naturally at different points in the analysis of theories which can be formulated in different frames:

- (i) *Energy-Momentum Tensor in different frames.* The Jacobian $\frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}}$ determines the relationship of the energy momentum tensor in different frames, which are related by the associated transformation

$$T^{\mu\nu} = \sqrt{\frac{\tilde{g}}{g}} \frac{\partial \tilde{g}_{\alpha\beta}}{\partial g_{\mu\nu}} \tilde{T}^{\alpha\beta}, \quad (19)$$

where $T^{\mu\nu}$ and $\tilde{T}^{\alpha\beta}$ are defined as the variation of the matter action with respect to the untilde and tilde metric respectively, as in Eq. (13). The energy momentum tensor obtained from the matter metric has the usual interpretation in terms of the energy fluxes seen by observers, while the one obtained with respect to the dynamical metric sources gravitational equations. The Jacobian can be used to analyze the relation between them.

- (ii) *Dynamical equations in the Jordan frame.* The Jacobian also appears in the equations of motion in the Jordan frame through the chain rule

$$\delta S_G = \frac{\delta S_G}{\delta \tilde{g}_{\alpha\beta}} \frac{\partial \tilde{g}_{\alpha\beta}}{\partial g_{\mu\nu}} \delta g_{\mu\nu} + \dots, \quad (20)$$

where the remaining terms in the variation have been omitted. This will be studied in detail in Sec. IV,

⁵If derivatives of $\tilde{g}_{\mu\nu}$ are introduced in the relation, the Jacobian may be generalized to

$$\frac{\delta \tilde{g}_{\mu\nu}}{\delta g_{\alpha\beta}} = \frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}} - \partial_\lambda \frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta,\lambda}}, \quad (18)$$

but in this case it can not be used to determine the existence of an inverse map, which would be given by a differential equation.

where the Jacobian will be used to re-write the theory in terms of second-order equations of motion.

- (iii) *Quantum mechanical formulation.* The variational principle (2) obeyed by classical systems can be understood as a consequence of quantum mechanics in terms of the path integral

$$Z(J) = \int \mathcal{D} Q_i \exp\left(-\frac{i}{\hbar} \left(S[Q_i] + \int d^4x J^i Q_i \right)\right), \quad (21)$$

where J^i is an external source and the integration is performed over all possible configurations of the dynamical variables, collectively denoted Q_i . In this interpretation, the imaginary exponent weights the probability associated to any given process. For configurations away from the classical solution the integrand oscillates rapidly and the amplitudes interfere destructively. As the classical solution minimizes the exponent, it will provide the only nonzero probability in the limit $\hbar \rightarrow 0$.

If one considers an initial theory with a given set of fundamental variables and wishes to change the physical frame, e.g. from $\tilde{g}_{\mu\nu}, \phi$ to $g_{\mu\nu}, \phi$, then the integration element in field space transforms as

$$\mathcal{D} Q_i = \mathcal{D}\phi \mathcal{D}\tilde{g}_{\mu\nu} \rightarrow \mathcal{D}\phi \mathcal{D}g_{\alpha\beta} \left| \frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}} \right|. \quad (22)$$

By using the relation $\det M = \exp(\text{tr}(\log(M)))$ it is possible to argue that the classical action picks up extra terms from the Jacobian. This is analogous to the occurrence of quantum anomalies, which typically lead to total derivatives and do not contribute to the classical equations of motion (e.g. [76]). This is in agreement with the classical, but not necessarily quantum, equivalence between frames (see also Refs. [29,77]).

The transformation properties of the path integral are hence essentially linked to the problem of equivalence between physical frames. However, the quantum mechanical formulation of derivatively coupled scalar-tensor theories lies beyond the scope of this work, and in what follows we will consider all fields as classical and all frames as physically equivalent. The consequences of nontrivial metric transformations for quantum mechanics will be addressed elsewhere.

A. Derivative conformal relation

For a conformal relation depending on the field derivatives, $\tilde{g}_{\mu\nu} = C(X, \phi)g_{\mu\nu}$, the Jacobian reads

$$\frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}} = C \delta_\mu^\alpha \delta_\nu^\beta + \frac{1}{2} C_{,X} g_{\mu\nu} \phi^\alpha \phi^\beta. \quad (23)$$

As the dependence on ϕ does not alter the form of the equations, it will be omitted from them in the following. The equation for the eigenvalues (17) then follows

$$(C - \lambda) \xi_{\mu\nu} + \frac{1}{2} C_{,X} g_{\mu\nu} \langle \xi_{\alpha\beta} \rangle = 0, \quad (24)$$

with $\langle \xi_{\alpha\beta} \rangle \equiv \phi^\alpha \xi_{\alpha\beta} \phi^\beta$. The set of eigenvalues and eigentensors can be readily found from the above relation:

$$\lambda_C = C, \quad \xi_{\mu\nu}^C = v_{(\mu}^{(1)} v_{\nu)}^{(2)}, \quad \text{with } v_\mu^{(n)} \phi^\mu = 0, \quad (25)$$

$$\lambda_K = C - C_{,X} X, \quad \xi_{\mu\nu}^K = C_{,X} g_{\mu\nu}. \quad (26)$$

The conformal eigenvalue (25) was already found in the discussion of the special disformal relations (7) and their inverse (15). It is degenerate with multiplicity 9, as there are 3^2 directions orthogonal to $\phi_{,\nu}$. The new, characteristic feature of conformal derivative couplings comes from the kinetic eigenvalue (26), which is associated with the nontrivial dependence on X . It is nondegenerate with multiplicity 1, and becomes equal to λ_C whenever C_X or X are zero (the normalization has been chosen for consistency with the general case studied in the next section). Note that there is no eigentensor proportional to $\phi_{,\mu} \phi_{,\nu}$, as this is no privileged direction for $\tilde{g}_{\mu\nu}$.

The kinetic eigenvalue (26) has an associated eigentensor proportional to $g_{\mu\nu}$. This has important implications for the relation between the energy-momentum tensors defined with respect to the tilde and untilde metric, as given by Eq. (19), and in particular for their traces, which fulfill

$$g_{\mu\nu} T^{\mu\nu} \propto (C - C_{,X} X) \tilde{g}_{\mu\nu} \tilde{T}^{\mu\nu}. \quad (27)$$

This type of relation might be used to analyze energy conditions in derivatively coupled scalar-tensor theories, analogously to similar studies in other alternative theories of gravity (e.g. [78]).

The inverse transformation $\tilde{g}_{\mu\nu}(g_{\mu\nu})$ is not well defined around points for which either C or $C - C_{,X} X$ are equal to zero. Wherever the inverse exist, it will be of the conformal form in order for both metrics to represent the same space-time, with unaltered null geodesics. We can hence use $g_{\mu\nu} = A(\tilde{X}) \tilde{g}_{\mu\nu}$, $g^{\mu\nu} = A(\tilde{X})^{-1} \tilde{g}^{\mu\nu}$ as an ansatz for the inverse metric. Direct substitution in $\tilde{g}_{\mu\nu} = C(X) g_{\mu\nu}$, $X = -\frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}$ yields the following condition on the form of $A(\tilde{X})$,

$$C(\tilde{X}/A(\tilde{X})) A(\tilde{X}) = 1, \quad (28)$$

with $X = \tilde{X}/A(\tilde{X})$, or alternatively $A(\tilde{X}) = 1/C(X)$ (by contracting $\phi_{,\mu} \phi_{,\nu}$ with both metrics one obtains the equalities $\tilde{X} = X/C(X)$ and $X = \tilde{X}/A(\tilde{X})$). Note that $A(\tilde{X})$ can be a multivalued function, as shown in Fig. 1.

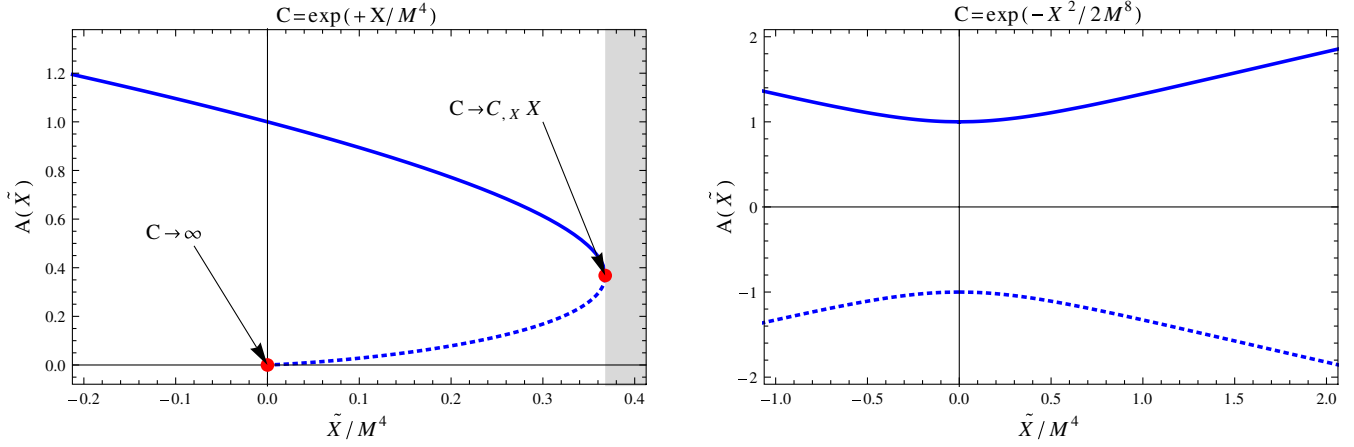


FIG. 1 (color online). Inverse transformation $g_{\mu\nu} = A(\tilde{X})\tilde{g}_{\mu\nu}$ for $\tilde{g}_{\mu\nu} = e^{X/M^4}g_{\mu\nu}$ (left) and $\tilde{g}_{\mu\nu} = e^{-X^2/(2M^8)}g_{\mu\nu}$ (right). The inverse conformal factor $A(\tilde{X})$ is obtained implicitly through Eq. (28), and can be multivalued for certain ranges of X , giving rise to two branches characterized by the kinetic eigenvalue λ_K , given by Eq. (26). The branches with positive (solid blue) and the negative (dotted blue) values of λ_K meet or end at singular points, in which either one of the Jacobian eigenvalues (25, 26) vanishes. This is seen explicitly in the case of the exponential function (left), which becomes bi-valued for $\tilde{X}/M^4 > 0$. Both branches meet at $\tilde{X}/M^4 = e^{-1}$ (corresponding to $X/M^4 = 1$) for which the kinetic eigenvalue becomes zero. The singular point $A(\tilde{X}) = 0$ corresponds to $C(X) \rightarrow \infty$. The gray shaded region, $\tilde{X}/M^4 \geq e^{-1}$, is forbidden. In the Gaussian case (right) both eigenvalues are always positive. Therefore there are no singular points and the two branches are not connected.

In order to see this formalism at work, let us consider an exponential derivative conformal factor $C = \exp(X/M^4)$. This choice ensures that $C \neq 0$ for any finite X , but the additional eigenvalue $\lambda_K = C(1 - X/M^4)$ spoils the existence of an inverse around points where $X = M^4$. The relation for the inverse conformal factor, Eq. (28), reduces to $\tilde{X}/M^4 = -A(\tilde{X}) \log(A(\tilde{X}))$. It is possible to obtain $A(\tilde{X})$ implicitly, as shown in the left panel of Fig. 1, where it is clear that $A(\tilde{X})$ becomes bi-valuate for $\tilde{X}/M^4 \in [0, e^{-1}]$. The point $\tilde{X}/M^4 = e^{-1}$ is an upper bound on the field gradient, which corresponds to the singular point $X = M^4$ at which $\lambda_K = 0$.

As an example of a better behaved relation, one may consider a Gaussian function $C = \exp(-\frac{1}{2}X^2/M^8)$. Unlike in the previous case, this choice of the conformal factor ensures that neither of the eigenvalues (25, 26) vanish, as $C > 0$ and $C - C_{,X}X = C(1 + X^2/M^8) > 0$. The inverse relation (28) satisfies $\tilde{X}^2/M^4 = A^2(\tilde{X}) \log(A^2(\tilde{X}))$, and therefore for any solution $A(\tilde{X})$, another solution $-A(\tilde{X})$ exists. However, the two branches do not meet, as shown in the right panel of Fig. 1. These examples show how the Jacobian analysis can provide a valuable tool to analyze scalar-tensor theories with general derivative couplings to matter.

B. General disformal relation

The arguments can be straightforwardly generalized to the disformal relation (8). The Jacobian reads

$$\frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}} = C \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} + \frac{1}{2} C_{,X} g_{\mu\nu} \phi^{\alpha} \phi^{\beta} + \frac{1}{2} D_{,X} \phi_{,\mu} \phi_{,\nu} \phi^{\alpha} \phi^{\beta}, \quad (29)$$

where we have again omitted the possible dependence on ϕ , which does not modify the equations. The equation for the eigenvalues (17) reads

$$(C - \lambda) \xi_{\mu\nu} + \frac{1}{2} (C_{,X} g_{\mu\nu} + D_{,X} \phi_{,\mu} \phi_{,\nu}) \langle \xi_{\alpha\beta} \rangle = 0, \quad (30)$$

and yields the following set of eigentensors:

$$\lambda_C = C, \quad \xi_{\mu\nu}^C = v_{(\mu}^{(1)} v_{\nu)}^{(2)}, \quad \text{with } v_{\mu}^{(n)} \phi^{\mu} = 0, \quad (31)$$

$$\lambda_K = C - C_{,X}X + 2D_{,X}X^2, \quad \xi_{\mu\nu}^K = C_{,X}g_{\mu\nu} + D_{,X}\phi_{,\mu}\phi_{,\nu}. \quad (32)$$

The conformal eigenvalue and its associated eigentensor (31) have the same expression as it was found in the previous section for a pure conformal coupling (25), while the kinetic eigenvalue and eigentensor (32) are modified if $D_{,X} \neq 0$. Just as in the conformal case, λ_C is degenerate with multiplicity 9 and λ_K is nondegenerate unless X or $C_{,X}, D_{,X}$ are zero. Note that $\xi_{\mu\nu}^K$ coincides with the partial derivative of $\tilde{g}_{\mu\nu}$ with respect to X .

Any values of X for which λ_C, λ_K become zero indicate the lack of existence of an inverse transformation. The term introduced by the disformal part of the transformation is proportional to X^2 rather than linear in X . Part of the difficulty in finding a suitable (purely) conformal function in the previous section was that X can have either sign depending on whether ϕ_{μ} is timelike (+) or spacelike (-). Adding a disformal factor with $D_{,X} > C_{,X}/X - C/X^2$ may prevent the singular points from occurring. In particular, a

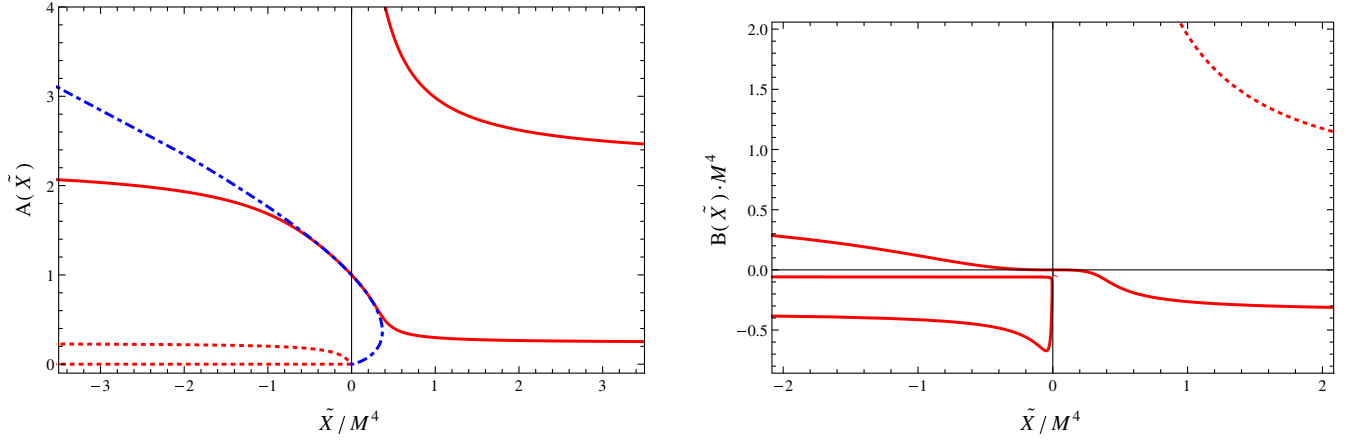


FIG. 2 (color online). Inverse map for $\tilde{g}_{\mu\nu} = \exp(X/M^4)g_{\mu\nu} + \frac{1}{2}X^3/M^{16}\phi_{,\mu}\phi_{,\nu}$ given by Eq. (33) with the inverse conformal function $A(\tilde{X})$ on the left panel and the inverse disformal function $B(\tilde{X})$ on the right. Both functions admit four branches with positive (solid) and negative (dotted) values of $C(X) - 2D(X)X$, due to the fact that $\tilde{X}(X)$ has multiple poles. The inverse conformal factor of $\tilde{g}_{\mu\nu} = \exp(X/M^4)g_{\mu\nu}$ (left panel of Fig. 1 is shown for comparison (blue dash-dotted). A branch with $C(X) - 2D(X)X > 0$ exists for large values of X , but is difficult to plot because the inverse conformal factor tends to zero rapidly.

purely disformal transformation ($C = 1$) has viable eigenvalues provided that $D_{,X}$ is positive. However, disformal relations with $\lambda_K \neq 0$ might still be problematic if the determinant of the metric (10) vanishes, in which case the contravariant metric $\tilde{g}^{\mu\nu}$ (and therefore $g^{\mu\nu}$) is not well defined.

An inverse map for the lowercase metric, $g_{\mu\nu}(\tilde{g}_{\alpha\beta})$ might be found around any nondegenerate point. It should be of the disformal type to ensure that the causal distortion induced by the transformation is proportional to $\phi_{,\mu}$. Therefore the ansatz for the inverse is $g_{\mu\nu} = A(\tilde{X})\tilde{g}_{\mu\nu} + B(\tilde{X})\phi_{,\mu}\phi_{,\nu}$, $g^{\mu\nu} = \frac{1}{A(\tilde{X})}(\tilde{g}^{\mu\nu} - \frac{B(\tilde{X})}{A(\tilde{X}) - 2B(\tilde{X})\tilde{X}}\phi^{,\mu}\phi^{,\nu})$ and $\tilde{X} = -\frac{1}{2}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$. Substituting the original metric in this expression yields the conditions

$$A(\tilde{X}) = \frac{1}{C(X)}, \quad B(\tilde{X}) = -\frac{D(X)}{C(X)}, \quad \tilde{X} = \frac{X}{C - 2DX}. \quad (33)$$

This relation for the field's kinetic terms obtained with respect to the two metrics becomes singular when the relation between the determinants (10) does. Figure 2 shows the inverse of a transformation exhibiting this type of singularity, which is related to its multivalued character. It is still possible that these singularities are dynamically avoided, and the inverse map remains within a certain branch. This has been found for special disformal relations, as it was discussed after Eq. (10).

IV. THE JORDAN FRAME: SECOND-ORDER THEORIES BEYOND THE HORNDESKI LAGRANGIAN

In previous sections it was shown that (i) a scalar field coupled to matter through a disformal metric (8) is described

by second-order equations (13) and (ii) an inverse map to the Jordan frame exists, except around points for which (31) or (32) vanish. These results seem in contradiction with the finding of non-Horndeski terms introduced by general disformal transformations of the gravitational sector, unless these are of the special disformal type (7) [33].

The purpose of this section is to examine this apparent contradiction by explicitly computing the Jordan frame action and equations. For the sake of concreteness, the analysis will be restricted to theories in which the original frame, for which the equations of section II hold, is actually the Einstein frame. The action is then given by

$$S_J[g_{\mu\nu}, \phi, \psi_m] = \int d^4x \left\{ \sqrt{-\bar{g}} \frac{\bar{R}[\bar{g}_{\mu\nu}]}{16\pi G} + \sqrt{-g} \mathcal{L}_M[g_{\mu\nu}, \psi_m] + \sqrt{-g} \mathcal{L}_\phi[g_{\mu\nu}, \phi] \right\}, \quad (34)$$

where $g_{\mu\nu}$, ϕ and the matter fields ψ_m are the dynamical variables which determine the field equations, and the gravitational metric $\bar{g}_{\mu\nu}[g_{\mu\nu}, \phi]$ is given by the inverse mapping of the metric to which matter couples [e.g. Eq. (8)], as discussed in the previous section. It will be assumed that the scalar field Lagrangian is described by terms of the form (3), (4), on which a disformal transformation only changes the form of the free functions (cf. (B11, B12) for a simpler case).

The transformed gravitational action can be written in terms of the difference between the covariant derivatives associated with $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$,

$$\mathcal{K}^\alpha{}_{\mu\nu} \equiv \bar{\Gamma}^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\mu\nu} = \bar{g}^{\alpha\lambda} \left(\nabla_{(\mu} \bar{g}_{\nu)\lambda} - \frac{1}{2} \nabla_\lambda \bar{g}_{\mu\nu} \right), \quad (35)$$

which transforms as a tensor [73]. The barred Riemann tensor can then be defined from the commutator of covariant

derivatives acting on a vector $2\bar{\nabla}_{[\mu}\bar{\nabla}_{\nu]}v^\alpha \equiv \bar{R}^\alpha{}_{\beta\mu\nu}v^\beta$, which allows one to write

$$\bar{R}^\alpha{}_{\beta\mu\nu} = R^\alpha{}_{\beta\mu\nu} + 2\nabla_{[\mu}\mathcal{K}^\alpha{}_{\nu]\beta} + 2\mathcal{K}^\alpha{}_{\gamma[\mu}\mathcal{K}^\gamma{}_{\nu]\beta}, \quad (36)$$

as well as its contractions, such as the Ricci scalar,

$$\bar{R} \equiv \bar{g}^{\mu\nu}\bar{R}^\alpha{}_{\mu\alpha\nu}, \quad (37)$$

where the barred metric has to be used self-consistently, e.g. $\bar{R}_{\alpha\beta\mu\nu} \equiv \bar{g}_{\alpha\lambda}\bar{R}^\lambda{}_{\beta\mu\nu}$, $\bar{R}^{\alpha\beta\mu\nu} \equiv \bar{g}^{\beta\lambda}\bar{g}^{\mu\sigma}\bar{g}^{\nu\kappa}\bar{R}^\alpha{}_{\lambda\sigma\kappa}$. It is possible to express the unbarred covariant derivative in the second term of Eq. (36) as a barred covariant derivative using the relation $(\bar{\nabla}_\alpha - \nabla_\alpha)\mathcal{K}^\lambda{}_{\mu\nu} = \mathcal{K}^\lambda{}_{\alpha\sigma}\mathcal{K}^\sigma{}_{\mu\nu} - 2\mathcal{K}^\sigma{}_{\alpha[\mu}\mathcal{K}^\lambda{}_{\nu]\sigma}$. As $\bar{\nabla}_\alpha\bar{g}^{\mu\nu} = 0$, an equivalent expression for the Jordan frame action (34) can be obtained,

$$S_J[g_{\mu\nu}, \phi, \psi_m] = \int d^4x \left\{ \frac{\sqrt{-\bar{g}}}{16\pi G} (\bar{g}^{\mu\nu}(R^\alpha{}_{\mu\alpha\nu} - 2\mathcal{K}^\alpha{}_{\gamma[\alpha}\mathcal{K}^\gamma{}_{\mu]\nu}) + \bar{\nabla}_\alpha\xi^\alpha) + \sqrt{-g}\mathcal{L}_M + \sqrt{-g}\mathcal{L}_\phi \right\}, \quad (38)$$

where the dependences on the right-hand side have been dropped and $\xi^\alpha \equiv \mathcal{K}^\alpha{}_{\mu\nu}\bar{g}^{\mu\nu} - \bar{g}^{\alpha\mu}\mathcal{K}^\nu{}_{\mu\nu}$ enters through a total derivative and thus does not contribute to the equations of motion. This expression has the advantage of not introducing derivatives of the connection, and therefore keeps only second derivatives of the metric and the scalar field in the Lagrangian. Let us consider the case of a derivative conformal relation before tackling the general case.

A. Derivative conformal relation

The simplest case that can be considered is the gravitational Lagrangian of a pure conformal relation

$$\bar{g}_{\mu\nu} = \Omega^2(X, \phi)g_{\mu\nu}, \quad (39)$$

(the conformal factor is squared in order to simplify the equations and facilitate the comparison with the literature). The action in the Jordan Frame is given by $S_C[g_{\mu\nu}, \phi, \psi_m] = \int d^4x\mathcal{L}_C$, with

$$\mathcal{L}_C = \frac{\sqrt{-g}}{16\pi G}(\Omega^2 R + 6\Omega_{,\alpha}\Omega^{,\alpha}) + \sqrt{-g}(\mathcal{L}_\phi + \mathcal{L}_m), \quad (40)$$

after integration by parts as in Eq. (38). As it was noted in Ref. [33], the second term contains $6(\Omega_{,X})^2\langle\Phi\rangle$, which can not be written in the appropriate Horndeski form (5). As a consequence, its variation with respect to the field contains up to fourth field derivatives: $\frac{\delta\mathcal{L}_C}{\delta\phi} \supset^{(!)} \nabla_\mu\nabla_\nu\frac{\partial\mathcal{L}_C}{\partial\phi_{,\mu\nu}} \supset^{(!)} (\Omega_{,X})^2\phi_{,\sigma}\phi^{,\sigma\mu}{}_{;\mu\nu}\phi^{,\nu}$, where $\supset^{(!)}$ denotes that only the higher derivatives are shown in the right-hand side.

Let us examine the full equations of motion in detail. Variation of the Jordan frame Lagrangian (40) yields

$$\Omega^2 G_{\mu\nu} + 2\Omega(g_{\mu\nu}\square\Omega - \Omega_{;\mu\nu}) + (6\square\Omega - \Omega R)\Omega_{,X}\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\Omega_{,\alpha}\Omega^{,\alpha} + 4\Omega_{,\mu}\Omega_{,\nu} = 8\pi G(T_{\mu\nu}^\phi + T_{\mu\nu}^m), \quad (41)$$

$$\nabla_\mu(\Omega_{,X}\phi^{;\mu}(\Omega R - 6\square\Omega)) + \Omega_{,\phi}(\Omega R - 6\square\Omega) + \frac{1}{2}\frac{\delta\mathcal{L}_\phi}{\delta\phi} = 0, \quad (42)$$

where it has been used that $\delta\Omega = \Omega_{,\phi}\delta\phi - \Omega_{,X}(\phi^\alpha\nabla_\alpha\delta\phi + \frac{1}{2}\phi_{,\alpha}\phi_{,\beta}\delta g^{\alpha\beta})$, $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and $\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\square\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}$. Although the field equation (42) does indeed contain the expected fourth order term, a relation between $\square\Omega$ and R can be obtained by taking the trace of the gravitational equation (41):

$$(6\square\Omega - \Omega R)(\Omega - 2\Omega_{,X}X) = 8\pi GT, \quad (43)$$

with $T = g^{\mu\nu}(T_{\mu\nu}^\phi + T_{\mu\nu}^m)$. This relation motivates the definition of the *kinetic mixing factor* for a conformal transformation,

$$T_K \equiv \frac{8\pi G\Omega_{,X}T}{\Omega - 2\Omega_{,X}X}. \quad (44)$$

Using this definition, Eq. (43) can be substituted back into the original equations, yielding a rather simple result,

$$\Omega^2 G_{\mu\nu} + 2\Omega(g_{\mu\nu}\square\Omega - \Omega_{;\mu\nu}) + T_K\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\Omega_{,\alpha}\Omega^{,\alpha} + 4\Omega_{,\mu}\Omega_{,\nu} = 8\pi GT_{\mu\nu}^{\text{tot}}, \quad (45)$$

$$\nabla_\mu(\phi^{;\mu}T_K) + \frac{\Omega_{,\phi}}{\Omega_{,X}}T_K - \frac{1}{2}\frac{\delta\mathcal{L}_\phi}{\delta\phi} = 0. \quad (46)$$

The equations do not contain higher than second derivatives of the dynamical variables when written in this form. The field equation (46) is manifestly second order, since the term in parentheses contains at most first derivatives of ϕ (as long as $\nabla_\mu(T^\phi + T^m)$ and $\frac{\delta\mathcal{L}_\phi}{\delta\phi}$ are themselves second order). Third-order time derivatives in the gravitational equations might arise from the second term in (45). To show that this is not the case, we can examine its temporal, mixed and spatial components,

$$g_{\mu\nu}\square\Omega - \Omega_{;\mu\nu} = \left(\begin{array}{c|c} g^{k\alpha}\Omega_{;k\alpha} & -\Omega_{;0i} \\ \hline -\Omega_{;0i} & g_{ij}\square\Omega - \Omega_{;ij} \end{array} \right), \quad (47)$$

where sums over $\alpha = 0-3$, $k = 1-3$ are implicit. As $\Omega_{;\mu\nu} \supset^{(!)} \Omega_{,X}\phi^\alpha\phi_{,\alpha\mu\nu}$, the only third time derivatives of the field would occur in the spatial part of the tensor through $\square\Omega$, which is second order by virtue of Eq. (43). The cancellation of the high time derivatives in the $(0, \mu)$ components is to be expected, as they represent constraint equations for $\bar{G}_{\mu\nu}$ [73].

B. General disformal relation

Let us now study the disformal case in which the Jordan frame action is given by Eq. (38) with

$$\bar{g}_{\mu\nu} = A(X, \phi)g_{\mu\nu} + B(X, \phi)\phi_{,\mu}\phi_{,\nu}, \quad (48)$$

the inverse map of (8), as discussed in Sec. III. For transformations beyond the derivative conformal and special disformal cases, the action expressed in terms of the dynamical variables $g_{\mu\nu}, \phi$ acquires many terms and the equations become very difficult to handle in practice. However, it is still possible to address the second-order nature of the evolution in relatively simple terms if both barred and unbarred quantities are allowed in the equations.⁶ The Jordan frame form of the resulting theory will be shown in Appendix A 2.

The variation of the gravitational sector (34) can be expressed in terms of barred quantities, in which it has the usual general relativistic form,

$$\delta(\sqrt{-\bar{g}}\bar{R}) = -\sqrt{-\bar{g}}(\bar{G}^{\mu\nu}\delta\bar{g}_{\mu\nu} + (\bar{g}^{\mu\nu}\bar{\square} - \bar{\nabla}_\mu\bar{\nabla}_\nu)\delta\bar{g}_{\mu\nu}). \quad (49)$$

The second term is a total derivative and hence does not contribute to the equations of motion.⁷ We can now write the equations in terms of the dynamical variables. The variation of the barred metric reads

$$\begin{aligned} \delta\bar{g}_{\mu\nu} = & \frac{\partial\bar{g}_{\alpha\beta}}{\partial g_{\mu\nu}}\delta g_{\alpha\beta} - (A_{,X}g_{\mu\nu} + B_{,X}\phi_{,\mu}\phi_{,\nu})\phi^\alpha(\delta\phi)_{,\alpha} \\ & + 2B\phi_{,(\mu}(\delta\phi)_{,\nu)} + (A_{,\phi}g_{\mu\nu} + B_{,\phi}\phi_{,\nu}\phi_{,\mu})\delta\phi. \end{aligned} \quad (50)$$

Note that the metric part of the variation is just given by the Jacobian (29) and the first term in parentheses is proportional to the kinetic eigentensor $\xi_{\mu\nu}^K$, given by Eq. (32). The last terms are the ones arising from a special disformal relation. The connection between the variation and the Jacobian has strong consequences for the structure of the equations.

The equations for gravity and the scalar field can then be written using (49), (50) in terms of the Jacobian (29) and the kinetic eigentensor $\xi_{\mu\nu}^K$ (32),

$$\bar{G}^{\alpha\beta}\frac{\partial\bar{g}_{\alpha\beta}}{\partial g_{\mu\nu}} = 8\pi G\sqrt{\frac{\bar{g}}{g}}(T_m^{\mu\nu} + T_\phi^{\mu\nu}), \quad (51)$$

$$\begin{aligned} \bar{\nabla}_\alpha(\bar{G}^{\mu\nu}\xi_{\mu\nu}^K\phi^\alpha) - \bar{G}^{\mu\nu}\bar{\nabla}_\mu(B\phi_{,\nu}) + \bar{G}^{\mu\nu}(A_{,\phi}g_{\mu\nu} + B_{,\phi}\phi_{,\nu}\phi_{,\mu}) \\ - \sqrt{\frac{\bar{g}}{g}}\frac{\delta\mathcal{L}_\phi}{\delta\phi} = 0, \end{aligned} \quad (52)$$

⁶This is similar to the analysis of Ref. [79] for inflationary scenarios.

⁷The structure of the variation (49) strongly suggests that the appropriate boundary term necessary to ensure that $\nabla_\alpha\delta g_{\mu\nu} = 0$ on the boundary [80,81], would be reproduced from the original one when transforming to the Jordan frame. See Ref. [82] for the study of this term in $f(R)$ and Gauss-Bonnet gravity.

where in the field equation the barred Bianchi identity $\bar{\nabla}_\mu\bar{G}^{\mu\nu} = 0$ has been used on the second term. The analogue of Eq. (43) which allows us to solve for the higher derivatives can be obtained by contracting (51) with the kinetic eigentensor,

$$\bar{G}^{\mu\nu}\xi_{\mu\nu}^K = 8\pi G\sqrt{\frac{\bar{g}}{g}}\frac{T_{\text{tot}}^{\mu\nu}\xi_{\mu\nu}^K}{\lambda_K} \equiv T_K, \quad (53)$$

where the kinetic eigenvalue λ_K comes from the action of the Jacobian on its eigentensor $\xi_{\mu\nu}^K$. This equation provides the generalization of the kinetic mixing factor obtained for the purely conformal case (44), which played a central role in the reduction of the equations to a second-order expression in the previous section. Divided by G , it provides a dimensionless measure of the kinetic mixing between the scalar field and matter due to the X dependence of the barred metric: it vanishes identically both in vacuum or for theories with $A_{,X}, B_{,X} = 0$. In addition, it diverges at points in which either the kinetic eigenvalue or the barred metric become singular. Note that it is still possible to have kinetic mixing and $T_K = 0$ if the metric is of the special disformal type (7).

Equation (53) allows one to write the field equation (52) in a manifestly second-order form,

$$\begin{aligned} \bar{\nabla}_\alpha(T_K\phi^\alpha) - \bar{G}^{\mu\nu}\bar{\nabla}_\mu(B\phi_{,\nu}) + \bar{G}^{\mu\nu}(A_{,\phi}g_{\mu\nu} + B_{,\phi}\phi_{,\nu}\phi_{,\mu}) \\ - \sqrt{\frac{\bar{g}}{g}}\frac{\delta\mathcal{L}_\phi}{\delta\phi} = 0, \end{aligned} \quad (54)$$

since T_K only contains first derivatives of the field. It remains to show that the third time derivatives present in $\bar{G}^{\mu\nu}$ can be solved away using (53). The higher-order structure of the barred Ricci tensor is given by the $2\nabla_{[\lambda}\mathcal{K}^{\lambda}_{\beta\alpha]}$ terms from (36), which can be expanded,

$$\begin{aligned} \bar{R}_{\alpha\beta} \supset^{(1)} \frac{1}{2}\bar{g}^{\lambda\sigma}(\bar{g}_{\sigma\beta,\alpha\lambda} + \bar{g}_{\sigma\alpha,\beta\lambda} - \bar{g}_{\alpha\beta,\lambda\sigma} - \bar{g}_{\lambda\sigma,\alpha\beta}) \\ \supset^{(1)} \frac{1}{2}\bar{g}^{\lambda\sigma}(\xi_{\sigma\beta}^K X_{,\alpha\lambda} + \xi_{\sigma\alpha}^K X_{,\beta\lambda} - \xi_{\alpha\beta}^K X_{,\lambda\sigma} - \xi_{\lambda\sigma}^K X_{,\alpha\beta}). \end{aligned} \quad (55)$$

The second relation follows from introducing $\bar{g}_{\beta\sigma,\alpha\lambda} \supset^{(1)} 2B\phi_{,(\sigma}\phi_{,\beta)\alpha\lambda} + \xi_{\sigma\beta}^K X_{,\alpha\lambda}$ in the first term in Eq. (55), where the contribution proportional to B cancels due to antisymmetry in β, λ .⁸ The higher derivative structure of the barred Einstein tensor is given by

⁸The reason why special disformal transformations do not introduce non-Horndeski terms in the action [33] (while X -dependent disformal maps do) is that higher field derivatives in the barred Ricci tensor are proportional to $\xi_{\mu\nu}^K$. In the special disformal case the equations of motion only contain up to second field derivatives without the need to use an implicit relation such as Eq. (53), as they depend on $\bar{R}_{\mu\nu}$. Theories for which $\xi_{\mu\nu}^K = 0$ must therefore belong to the set of theories described by Horndeski's theorem.

$$\begin{aligned} \bar{G}^{\alpha\beta} \supset^{(1)} \bar{G}_{(1)}^{\alpha\beta} \equiv & \frac{1}{2} \{ 2\bar{\xi}^{\delta(\alpha} \bar{g}^{\beta)\gamma} - \bar{\xi}^{\gamma\alpha} \bar{g}^{\beta\delta} - \bar{\xi}^{\alpha\beta} \bar{g}^{\gamma\delta} \\ & - \bar{g}^{\alpha\beta} (\bar{\xi}^{\gamma\delta} - \bar{\xi}^{\delta\gamma}) \} X_{,\gamma\delta}, \end{aligned} \quad (56)$$

where $\bar{\xi}^{\alpha\beta} = \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} \xi_{\mu\nu}^K$ and $\bar{\xi} = \bar{g}^{\mu\nu} \xi_{\mu\nu}^K$. Third time derivatives in the above expression occur only through $X_{,00} = \phi^{,\sigma} \phi_{,\sigma 00}$. The value of $X_{,00}$ can be solved in terms of lower derivatives using the constraint equation (53) together with (56) (problems associated to points at which the coefficient of $X_{,00}$ approaches zero might occur, analogous to the Jacobian eigenvalues becoming zero). Although the third time derivatives of the field can be generically removed from the equations in theories with $\xi_{\mu\nu}^K \neq 0$, Eq. (56) still contains third field derivatives involving spatial directions. This feature might be relevant for the initial value problem in this type of non-Horndeski theories, although such derivatives are absent in the Einstein frame equations, cf. Sec. II. The initial value formulation in disformally coupled theories deserves a detailed study, which will be presented elsewhere.

C. A loophole in Horndeski's theorem: Hidden constraints

The computations in the last subsections show how the higher time derivatives of the scalar field can be eliminated from the Jordan frame equations through the use of implicit constraint relations, rendering the system second order. A direct implication is the incompleteness of Horndeski's theorem in the identification of Ostrogradski stable scalar-tensor theories, since it only identifies the maximal set of theories with a second-order Euler-Lagrange equation (2), regardless of implicit constraint relations that allow to solve for the higher derivatives. The possibility of rewriting apparently higher-order theories in a second-order form is not explored in the different proofs of Horndeski's theorem [15–17]. Alternatively, the loophole can be regarded as the possibility of allowing the scalar field to directly modify the matter metric, e.g. allowing a general disformal coupling to matter (8).

The use of implicit (constraint) relations to remove the higher time derivatives of the dynamical variables is related to a recent result found in [83], which allows one to “exorcise Ostrogradski's ghost in higher-order derivatives with constraints.” It states that theories with constraints which reduce the dimensionality of the phase space of the system do not have a linear instability in the Hamiltonian, even if the original Lagrangian includes second or higher derivatives with respect to time. This result is obtained in the context of one dimensional, higher-order theories with constraints in the form of Lagrangian multipliers. The extension to four dimensional field theories with implicit constraints seems plausible, as the constraints determine the value of the field's third and fourth derivatives, effectively reducing the dimension of the phase space by two. This strongly suggests that

Ostrogradski's theorem does not apply to the Jordan frame representation of disformally coupled theories, in agreement with the second-order description in the Einstein frame shown in Sec. II A.

Another well-known example of a naively unstable field theory is given by $f(R)$ gravity, to which Ostrogradski's theorem would in principle apply [12]. The Euler-Lagrange equations for such theories yield up to quartic derivatives of the metric through the terms $\nabla_\mu \nabla_\nu f_{,R} - g_{\mu\nu} \square f_{,R}$. As higher derivatives of the metric always occur in terms of first or second derivatives of $f_{,R}$, the loophole to Ostrogradski's theorem comes from the identification of $f_{,R}$ as a new propagating scalar degree of freedom, which is not associated to an unbounded Hamiltonian. Then, although $f(R)$ gravity does not formally belong to Horndeski's theory, it can be shown to be equivalent to a scalar-tensor theory specified by $G_{4,X} = G_3 = G_5 = 0$, $G_2 \neq 0$ by means of a Legendre transformation [84,85].

Another example of an implicit constraint occurs in “veiled” general relativity, in which both the matter and the gravitational sector are conformally transformed by $g_{\mu\nu} \rightarrow A(\phi)g_{\mu\nu}$ (see [30]). In the transformed frame, the equation for the conformal factor $A(\phi)$ reduces to the trace of the metric equations. This shows that the dynamics of the scalar field are redundant, as expected from the fact that the scalar field was introduced artificially (and not present in the original frame). The case of “veiled” GR is analogous (although simpler) to the Jordan frame version of disformally coupled theories: taking the trace of the metric equations is equivalent to the contraction with the kinetic eigentensor discussed in Secs. IV A and IV B. In disformally coupled theories only the higher-order dynamics are artificial, and can consequently be eliminated by a similar procedure.

A hint about the incompleteness of Horndeski's theory has been obtained in the framework of effective field theory for (linear) cosmological perturbations, in the form of an operator combination which leads to second-order equations but can not be obtained from any of the terms in \mathcal{L}_H (3)–(6) [86]. Certain generalized Galileon and massive gravity theories in the decoupling limit also contain a scalar field with a higher derivative Lagrangian, which nonetheless does not introduce additional or ghostly degrees of freedom [87]. This is shown by writing down the Hamiltonian and finding the primary constraints, leading to a theory with higher *spatial* derivatives in the action. This is analogous to what is found in the theories here studied after applying the implicit constraints, cf. (47).

The second-order nature of the equations has been established under the assumption of an Einstein-Hilbert form for the metric in the original frame. A natural question is whether this assumption is relevant to the procedure, i.e. if the reduction to second order can be applied to the transformed version of more general Lagrangians. This

seems plausible in the case of Horndeski's theory, as the gravitational equations do not contain derivatives of the curvature tensor. Therefore, the variation with respect to the dynamical metric is proportional to the Jacobian, as in Eq. (49). The contraction with the kinetic eigentensor will then be proportional to the kinetic eigenvalue, providing an analogue of Eq. (53) to substitute the higher derivatives in the field equation with second-order terms from the metric equations.

V. DISCUSSION

In this paper we have examined scalar-tensor theories with derivative couplings to matter, which enter the Lagrangian through an effective metric that depends on the scalar field. Apart from providing a generalization of Jordan-Brans-Dicke theories, they also offer interesting phenomenological possibilities, such as allowing for complete or softly broken shift symmetry and mixing derivatives of the scalar and matter degrees of freedom in the dynamical equations. At a more fundamental level, redefinitions of the physical variables can establish equivalences between classical theories, which can be used to simplify the analysis and provide additional understanding of underlying structures.

The possible relations between the matter and the gravitational metrics can be classified by their tensor structure into three types: conformal (a scalar times the gravitational metric), disformal (the tensor product of a vector) and extended disformal (a rank-two tensor whose contraction with the former terms is zero). A complementary classification is provided by the order of the scalar field derivatives introduced in the matter metric. Allowing the relation to depend on second (covariant) derivatives of the scalar introduces a number of difficulties: an (*a priori*) infinite number of tensor structures can be included, due to the possibility of contracting the field with itself. Covariance requires the introduction of derivatives of the metric through the connection coefficients, which invalidates the algebraic treatment performed here. Finally, such a metric coupling will generically lead to higher derivatives in the equations of motion. This set of arguments single out the disformal relation originally introduced by Bekenstein (8) as the most reasonable choice.

Further physical insight on these theories can be gained by expressing the action in the Jordan frame, i.e. using the metric to which matter couples minimally as a dynamical variable. The mapping to the Jordan frame amounts to inverting the disformal relation, whose existence can be determined studying the Jacobian of the transformation. The inverse map fails to exist around points at which its determinant vanishes, i.e. when one eigenvalue of the Jacobian is equal to zero. This happens when the conformal factor vanishes, but in the case of general disformal transformations can also occur under different circumstances due to the additional dependence of the free

functions on the metric, characterized by the kinetic eigenvalue (32). The simplicity of the Jacobian analysis makes it a natural starting point in the study of concrete models, as was shown for several examples.

The Jacobian of the mapping between frames also appears in the study of different aspects of the theory. It determines the relation between the energy-momentum tensor that represents the matter energy density and momentum fluxes seen by observers (obtained by variation with respect to the matter metric) and the source of the gravitational equations (obtained by variation with respect to the gravitational metric). The Jacobian and its kinetic eigentensor also appears in the Jordan frame equations for the metric and the scalar field. Finally, we expect the Jacobian to play a role at the quantum mechanical level by producing extra surface terms due to the transformation rules of the path integral, in a manner analogous to the occurrence of quantum anomalies [76]. The analysis of this feature might shed some light on the problem of the classical equivalence and quantum inequivalence of physical frames, and is left for a future publication.

Disformally coupled theories expressed in the Jordan frame produce terms that do not pertain to the Horndeski Lagrangian, and hence their Euler-Lagrange variation introduces higher derivatives in the equations of motion (unlike in the original frame). However, it is possible to obtain a relation for the higher derivatives by contracting the metric equations with the kinetic eigentensor of the Jacobian. This implicit constraint can be then used to rewrite the dynamics in terms of second-order equations, without higher derivatives with respect to time and hence free of Ostrogradski instabilities. The case of a derivative-dependent conformal transformation is particularly simple to analyze, as the higher derivatives can be eliminated by taking the trace of the metric equations. The study of the general case makes clear why special disformal transformations avoid all these difficulties and incarnate a formal invariance of Horndeski's theory [33]: if the free functions only depend on ϕ , the Jordan frame equations (52) remain second order as a consequence of the Bianchi identities for the field dependent metric, while in the Einstein frame the equations (13) simplify due to stress-energy conservation with respect to the field-dependent metric. This is analogous to the much simpler structure of $\mathcal{L}_4, \mathcal{L}_5$ in Horndeski's theory (5, 6), when G_4, G_5 are functions of ϕ only.

The analysis of the equations uncovers a loophole in Horndeski's theorem: certain theories, whose variation contains higher derivatives of the fields, might be rendered second order by the existence of hidden constraints in the dynamical equations. Such theories provide further ways to overcome the difficulties generically caused by higher derivative Lagrangians, including the existence of Ostrogradski's instability. This situation shares essential analogies with $f(R)$ gravity (which can be reduced to a second-order form by identifying $f_{,R}$ as a scalar degree of

freedom) and general relativity expressed in a different conformal frame (which introduces redundant equations). The reduction of the fields phase space due to constraints has been explicitly shown to eliminate Ostrogradki's ghosts in one dimensional systems [83], strongly suggesting that this will also be the case for the scalar-tensor theories under consideration. The sanity of disformally coupled theories is further supported by the second-order nature of the equations in the Einstein frame.

The most immediate question is whether disformally coupled theories represent the most general set of second-order theories beyond the Horndeski Lagrangian.⁹ Classical equivalence between frames implies that any theory that is second order in a given frame will remain second order under field redefinitions. An extension of Horndeski's theorem might be then found in the context of extended disformal transformations that depend on second field derivatives, *if* the free functions displayed in Table II can be suitably tuned to produce second-order equations (although such finely tuned coefficients would be unnatural in a quantum mechanical description if they are not protected by a symmetry). The methods presented here can also be applied to study frame transformations in other alternative theories of gravity, including vector-coupled theories such as TeVeS [48], conformal vector screening [93] and other generalizations, e.g. [94]. It is also possible that further scalar-tensor theories with hidden constraints can be obtained by modifications in the gravitational sector which cannot be absorbed by a redefinition of the metric. A first step in this direction is the study of the transformed Gauss-Bonnet term, which is presented in Appendix B 3, where it is shown that such terms belong to \mathcal{L}_H for special conformal transformations, but not for special disformal transformations. These and other possibilities (e.g. nonpolynomial dependence on second field derivatives) might provide an even larger class of sensible scalar-tensor theories beyond the Horndeski Lagrangian.

Theories with implicit constraints therefore constitute a new class of scalar-tensor theories, essentially different from those for which the Euler-Lagrange variation is directly second order, such as Horndeski's theory. The fact that the gravitational equations involve third derivatives of the field (although not third time derivatives) from the barred Einstein tensor might be relevant for the initial value formulation of such theories, even though such a difficulty seems to be absent in the Einstein frame. Lorentz invariance plays a crucial role, for it forces the field derivatives and the metric to occur together and eventually provide the right implicit constraints. More importantly, the dynamical

character of space-time is essential for the existence of implicit constraints, which are lost if a flat background is imposed. This is in stark contrast with Horndeski theories, for which the Minkowski limit is described by second-order equations. In a broader scope, degenerate field theories might provide new theoretical challenges and phenomenological applications in gravitation and cosmology, as time and time again the search for loopholes in no-go theorems has proved to be a very constructive way to expand the horizons in fundamental physics.

ACKNOWLEDGMENTS

We are very thankful to Luca Amendola, Diego Blas, Margarita García Pérez, Antonio González-Arroyo, Kurt Hinterbichler, Tomi S. Koivisto, and Slava Mukhanov for discussions at different stages of this work, to Jose Beltrán Jimenez for a very detailed reading of the first manuscript and to Dario Bettoni, Tsutomu Kobayashi, Jeremy Sakstein, Ignacy Sawicki, Filippo Vernizzi and Felix del Teso for correspondence and comments on the first version. The authors also acknowledge financial support from the Madrid Regional Government (CAM) under the program HEPHACOS S2009/ESP-1473-02, from the Spanish MICINN under Grant No. AYA2009-13936-C06-06 and Consolider-Ingenio 2010 PAU (CSD2007-00060), from the MINECO, Centro de Excelencia Severo Ochoa Programme, under Grant No. SEV-2012-0249, as well as from the European Union Marie Curie Initial Training Network UNILHC PITN-GA-2009-237920. The computations presented in the Appendixes have been checked with the xACT package for MATHEMATICA [95,96].

Note added.—A few weeks before the first version of this manuscript was released, a preprint appeared that had some overlap with some of our results [33]. In particular, their authors present the transformation rules for the Horndeski Lagrangian for special disformal transformations (with arbitrary functions of the field) and show that disformal transformations that depend on X produce non-Horndeski terms. Our computation of the action of special disformal transformations on Horndeski's theory is presented in Appendix.

APPENDIX A: GENERAL DISFORMAL RELATIONS

In this section we will present some relations for disformal relations of the type proposed by Bekenstein,

$$\bar{g}_{\mu\nu} = Ag_{\mu\nu} + B\phi_{,\mu}\phi_{,\nu}, \quad \bar{g}^{\mu\nu} = \frac{1}{A}(g^{\mu\nu} - \gamma^2 B\phi^{,\mu}\phi^{,\nu}), \quad (\text{A1})$$

with $\gamma^2 = (A - 2BX)^{-1}$, $X = \frac{1}{2}\phi_{,\mu}\phi^{,\mu}$.

1. Connection

The connection for a field-dependent metric can be computed directly from the usual definition,

⁹Other ways to generalize Horndeski's theory without introducing ghosts include multiscalar field theories [88–90], nonlocal gravity theories [91] and nonlinear extensions of Horndeski functions [92]

$$\begin{aligned}\bar{\Gamma}^\alpha_{\mu\nu} = & \Gamma^\alpha_{\mu\nu} + \delta^\alpha_{(\mu} \log A_{\nu)} - \frac{1}{2} \log A^\alpha g_{\mu\nu} - \frac{B\gamma^2}{A} \phi^\alpha \left[A_{,(\mu} \phi_{,\nu)} - \frac{1}{2} \phi^{,\lambda} A_{,\lambda} g_{\mu\nu} \right] + B\gamma^2 \phi^\alpha \phi_{;\mu\nu} + \gamma^2 \phi^\alpha B_{,(\mu} \phi_{,\nu)} \\ & - \frac{1}{2A} \phi_{,\mu} \phi_{,\nu} (B^\alpha - B\gamma^2 \phi^\alpha \phi^{,\lambda} B_{,\lambda}),\end{aligned}\quad (\text{A2})$$

where A, B are general scalar functions. The difference between connections can be also written as (35). For A, B depending on ϕ, X , the above expression can be expanded

$$\begin{aligned}\mathcal{K}^\alpha_{\mu\nu} = & +(\log A)_{,\phi} \left\{ \phi_{,(\mu} \delta^\alpha_{\nu)} - B\gamma^2 \phi^\alpha \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} A\gamma^2 \phi^\alpha g_{\mu\nu} \right\} \\ & + (\log A)_{,X} \left\{ -\phi^\sigma \phi_{;\sigma(\mu} \delta^\alpha_{\nu)} + B\gamma^2 \phi^\alpha \phi^\sigma \phi_{;\sigma(\mu} \phi_{,\nu)} + \frac{1}{2} [\phi_{,\sigma} \phi^{;\sigma\alpha} - B\gamma^2 \phi^\alpha \langle \Phi \rangle] g_{\mu\nu} \right\} \\ & + B\gamma^2 \phi^\alpha \phi_{;\mu\nu} + \frac{1}{2} B_{,\phi} \gamma^2 \phi^\alpha \phi_{,\mu} \phi_{,\nu} - B_{,X} \gamma^2 \phi^\alpha \phi^\sigma \phi_{;\sigma(\mu} \phi_{,\nu)} + \frac{B_{,X}}{2A} \phi_{,\mu} \phi_{,\nu} [\phi_{,\sigma} \phi^{;\sigma\alpha} - B\gamma^2 \phi^\alpha \langle \Phi \rangle].\end{aligned}\quad (\text{A3})$$

2. Non-Horndeski terms

Starting with the Jordan frame in the form (38) and plugging the X -dependent terms from the barred connection (A3), the higher-order part of the action is given by $S_J[g_{\mu\nu}, \phi] = \int d^4x \frac{\sqrt{-g}}{16\pi G} \mathcal{L}_{\text{disf}}$ with

$$\begin{aligned}\mathcal{L}_{\text{disf}} = & \sqrt{A} \sqrt{A - 2BX} R - \frac{\sqrt{AB}}{\sqrt{A - 2BX}} \langle R_{\mu\nu} \rangle + \frac{\sqrt{AB}^2 (\langle \Phi^2 \rangle - \langle \Phi \rangle [\Phi])}{(A - 2BX)^{3/2}} - \frac{A_{,X} B (\langle \Phi^2 \rangle - \langle \Phi \rangle [\Phi]) (A - 5BX)}{\sqrt{A} (A - 2BX)^{3/2}} \\ & + B_{,X} \left(\frac{\sqrt{AB} (\langle \Phi^2 \rangle - \langle \Phi \rangle [\Phi]) X}{(A - 2BX)^{3/2}} - \frac{A_{,X} (A - 3BX) (\langle \Phi \rangle^2 + 2\langle \Phi^2 \rangle X)}{\sqrt{A} (A - 2BX)^{3/2}} \right) \\ & + \frac{A_{,X}^2 (3A^2 \langle \Phi^2 \rangle + 8B^2 X (\langle \Phi \rangle^2 + 2\langle \Phi^2 \rangle X) - 3AB (\langle \Phi \rangle^2 + 4\langle \Phi^2 \rangle X))}{2A^{3/2} (A - 2BX)^{3/2}} + \mathcal{L}_3, \mathcal{L}_2 \text{ terms} \propto A_{,\phi}, B_{,\phi}.\end{aligned}\quad (\text{A4})$$

It is important to remember that the above expression only contains non-Horndeski terms. Terms involving one instance of $A_{,\phi}, B_{,\phi}$ can at most depend on $\langle \Phi \rangle, [\Phi]$, and therefore contribute to \mathcal{L}_3 (4). Terms involving two instances of $A_{,\phi}, B_{,\phi}$ do not contain second derivatives, and therefore belong to \mathcal{L}_2 (3). These terms have been indicated schematically in the last line.

APPENDIX B: FRAME TRANSFORMATION FOR SPECIAL DISFORMAL MAPPINGS

Let us now explore the transformation rules for gravitational theories under special disformal relations (7). Let us first consider the transformations of the Einstein-Hilbert and the Horndeski Lagrangians for the purely disformal case. Then the Gauss-Bonnet term will be presented in both the purely conformal and purely disformal cases.

1. Einstein-Hilbert Lagrangian

Let us consider a normalized, pure disformal relation,

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \pi_{,\mu} \pi_{,\nu}, \quad \bar{g}^{\mu\nu} = g^{\mu\nu} - \gamma_0^2 \pi^{,\mu} \pi^{,\nu}, \quad (\text{B1})$$

with $\gamma_0^2 = \frac{1}{1-2X_\pi}$, $X_\pi = -\frac{1}{2} \pi_{,\mu} \pi^{,\mu}$, where the free function has been absorbed by a field redefinition $\pi = \int B(\phi) d\phi$. This simple form suffices to relate DBI Galileons to disformally coupled theories [18]. The field dependence will be restored in the final result.

The connection (A3) and curvature tensor (36) for the above relation are

$$\mathcal{K}^\alpha_{\mu\nu} = \bar{g}^{\alpha\lambda} (\pi_{,\lambda} \pi_{;\mu\nu}) = \gamma_0^2 \pi^{,\alpha} \pi_{;\mu\nu}. \quad (\text{B2})$$

$$\bar{R}^\alpha_{\beta\mu\nu} = \bar{g}^{\alpha\lambda} (R_{\lambda\beta\mu\nu} + \gamma_0^2 \pi_{;\lambda[\mu} \pi_{;\nu]\beta}). \quad (\text{B3})$$

Note that the form (B1) has been assumed in the last expression to factor out the inverse barred metric (the first index can be then straightforwardly lowered, $\bar{R}_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} + \gamma_0^2 \pi_{;\alpha[\mu} \pi_{;\nu]\beta}$). The Ricci tensor and scalar are given by

$$\bar{R}_{\mu\nu} \equiv \bar{R}^\lambda{}_{\mu\lambda\nu} = R_{\mu\nu} - \gamma_0^2 R_{\alpha\mu\beta\nu} \pi^{;\alpha} \pi^{;\beta} + \gamma_0^2 \{ [\Pi] \pi_{;\mu\nu} - \pi_{;\nu\alpha} \pi^{;\alpha}{}_{;\mu} \} - \gamma_0^4 \{ \langle \Pi \rangle \pi_{;\mu\nu} - \pi^{;\alpha} \pi_{;\alpha\mu} \pi^{;\beta} \pi_{;\beta\nu} \}, \quad (\text{B4})$$

$$\bar{R} \equiv \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} = R - 2\gamma_0^2 \langle R_{\mu\nu} \rangle + \gamma_0^2 ([\Pi]^2 - [\Pi^2]) - 2\gamma_0^4 ([\Pi] \langle \Pi \rangle - \langle \Pi^2 \rangle). \quad (\text{B5})$$

The transformed Einstein-Hilbert Lagrangian density can be obtained from Eq. (38),

$$\begin{aligned} \sqrt{-\bar{g}} \bar{R} &= \sqrt{-g} \left(\frac{1}{\gamma_0} R - \gamma_0 \langle R^{\mu\nu} \rangle - \gamma_0^3 ([\Pi] \langle \Pi \rangle - \langle \Pi^2 \rangle) \right) \quad (\text{B6}) \\ &= \sqrt{-g} \left(\frac{1}{\gamma_0} R - \gamma_0 ([\Pi]^2 - [\Pi^2]) + \nabla_\alpha \xi^\alpha \right), \quad (\text{B7}) \end{aligned}$$

in terms of a total derivative that does not contribute to the bulk equations of motion.¹⁰ The above Lagrangian has the right Horndeski form (5) with $G_4 = \gamma_0^{-1} \equiv \sqrt{1 - 2X_\pi}$, $G_{4,X_\pi} = -\gamma_0$ and $X_\pi = -\frac{1}{2} \pi_{;\mu} \pi^{;\mu}$, therefore producing second-order equations of motion.

It is possible to restore the field dependence in the disformal relation through a field redefinition $\pi = \int \sqrt{B(\phi)} d\phi$ in the Lagrangian density (B7). Then $\pi_{;\mu} = \sqrt{B} \phi_{;\mu}$, $\pi_{;\mu\nu} = \sqrt{B} \phi_{;\mu\nu} + \frac{1}{2} \frac{B_{;\phi}}{\sqrt{B}} \phi_{;\mu} \phi_{;\nu}$ and the transformed Einstein-Hilbert term becomes

$$\sqrt{-\bar{g}} \bar{R} = \frac{1}{\gamma_b} R - B \gamma_b ([\Phi]^2 - [\Phi^2]) + B_{;\phi} \gamma_b (2X[\Phi] + \langle \Phi \rangle), \quad (\text{B9})$$

with $\gamma_b \equiv (1 - 2BX)^{-1/2}$, $X = -\frac{1}{2} \phi_{;\mu} \phi^{;\mu}$. As these expressions contain no square roots of B , they are valid for negative values and recover the special case $B = -1$. Note that allowing B to depend on ϕ adds lower order Horndeski terms, which are proportional to $B_{;\phi}$. These can be simplified by the addition of a total derivative¹¹

¹⁰This can be shown by partial integration of the last term in (B6),

$$-\gamma_0^3 ([\Pi] \langle \Pi \rangle - \langle \Pi^2 \rangle) = (\nabla_\alpha \gamma_0) (\pi^\alpha \square \pi - \pi^{;\alpha\beta} \pi_{;\beta})$$

$$= -\gamma_0 ([\Pi]^2 - [\Pi^2]) + \gamma_0 \langle R_{\mu\nu} \rangle + \nabla_\alpha \xi^\alpha, \quad (\text{B8})$$

with $\xi^\alpha = \gamma_0 (\pi^\alpha \square \pi - \pi^{;\alpha\beta} \pi_{;\beta})$. The first equality uses the fact that $\nabla_\mu \gamma_0 = -\gamma_0^3 \pi^{;\alpha} \pi_{;\alpha\mu}$, and the second follows after integration by parts and noting that $\pi^{;\beta} \nabla_{[\alpha} \nabla_{\beta]} \pi^{;\alpha} = 2R_{\alpha\beta} \pi^{;\alpha} \pi^{;\beta}$.

¹¹It is possible to remove the $f(\Phi)$ term in Eq. (B9) by adding $\nabla_\alpha (g \phi^{;\alpha}) = g[\Phi]_{;\alpha} g_{;\alpha} \langle \Phi \rangle - 2X g_{;\phi}$ with $g = \int f dX + s(\phi)$. Choosing $g = -\frac{B_{;\phi}}{\gamma_b}$ allows us to obtain Eq. (B10).

$$\begin{aligned} \sqrt{-\bar{g}} \bar{R} &= \frac{1}{\gamma_b} R - B \gamma_b ([\Phi]^2 - [\Phi^2]) + \frac{B_{;\phi}}{\gamma_b B} (\gamma_d^2 - 2)[\Phi] \\ &\quad + 2X \left(\frac{B_{;\phi}}{\gamma_b B} \right)_{;\phi}, \quad (\text{B10}) \end{aligned}$$

which corresponds to $G_4 = \sqrt{1 - 2BX}$, $G_3 = \frac{B_{;\phi}}{\gamma_b B} (\gamma_d^2 - 2)$ and $G_2 = 2X \left(\frac{B_{;\phi}}{\gamma_b B} \right)_{;\phi}$ in the original Horndeski form (3)–(5).

2. Horndeski Lagrangian

The transformation rules for the Horndeski Lagrangian (3)–(6) under special disformal maps are presented in Ref. [33]. In this section we derive the transformation rules for a normalized, pure disformal relation (B1) in detail for $\mathcal{L}_2, \mathcal{L}_3$ and \mathcal{L}_4 . The lowest-order term is trivial to compute,

$$\mathcal{L}_2 = G_2(X_\pi, \pi) \rightarrow \frac{1}{\gamma} G_2(\gamma^2 X_\pi, \pi) \equiv \frac{1}{\gamma} \bar{G}_2, \quad (\text{B11})$$

where the γ^{-1} factor arises from the barred volume element and a bar over a function means that the factor γ^2 has been reabsorbed into the definition of the function $\bar{G}_i \equiv G_i(\gamma^2 X_\pi, \pi)$. Implicit dependence on X_π, π of the Horndeski functions will be assumed in the following. The next term is also simple to transform, noting that $\pi_{;\mu\nu} \rightarrow \bar{\nabla}_\mu \bar{\nabla}_\nu \pi = \gamma^2 \pi_{;\mu\nu}$,

$$\mathcal{L}_3 = G_3[\Pi] \rightarrow \gamma \bar{G}_3([\Pi] - B\gamma^2 \langle \Pi \rangle). \quad (\text{B12})$$

See the footnote before Eq. (B10) on how to write cubic terms in Horndeski's form.

The quartic term is more complicated, but its Jordan frame counterpart can be easily restored to a canonical Horndeski by noting that $[\Pi]^2 - [\Pi^2] \rightarrow \gamma^4 \{ [\Pi]^2 - [\Pi^2] - 2\gamma^2 ([\Pi] \langle \Pi \rangle - \langle \Pi^2 \rangle) \}$, $G_{4,X_\pi} \rightarrow \bar{G}_{4,\bar{X}_\pi} = \bar{G}_{4,X_\pi} (\partial \bar{X}_\pi / \partial X_\pi)^{-1} = \gamma^{-4} \bar{G}_{4,X_\pi}$ and following the same considerations used to transform (B6) into (B7),

$$\begin{aligned} \mathcal{L}_4 &= G_4 R + G_{4,X_\pi} ([\Pi]^2 - [\Pi^2]) \rightarrow \frac{\bar{G}_4}{\gamma} R + \left(\frac{\bar{G}_4}{\gamma} \right)_{;\bar{X}_\pi} ([\Pi]^2 \\ &\quad - [\Pi^2]) + 2\gamma \bar{G}_{4,\pi} (\langle \Pi \rangle + 2X_\pi [\Pi]). \quad (\text{B13}) \end{aligned}$$

It can be seen that on top of a redefinition $G_4 \rightarrow \frac{\bar{G}_4}{\gamma}$, if $G_{4,\pi} \neq 0$ a part of the Lagrangian is projected onto the lower order contribution \mathcal{L}_3 (last term).

3. Gauss-Bonnet term

Besides the Ricci scalar present in the Einstein-Hilbert action, Lovelock's theorem allows for higher curvature terms whose variation gives second-order equations of motion [13]. The following is the Gauss-Bonnet (GB) term, which does not contribute to the equations of motion

in four dimensions. In this section we will compute the transformed GB term,

$$\bar{\mathcal{G}} = \bar{R}^2 - 4\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \bar{R}_{\mu\nu\alpha\beta}\bar{R}^{\mu\nu\alpha\beta}. \quad (\text{B14})$$

for a special conformal and a normalized special disformal mapping. Note that these results are essentially different from the projection of the bulk GB term into a codimension one submanifold, which is the usual approach in brane-world gravity [97].

a. Pure conformal relation

Under a conformal transformation of the metric,

$$\bar{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}, \quad (\text{B15})$$

one finds the following transformation of the quadratic contractions,

$$\bar{R}^2 = \Omega^{-4}[R^2 - 12R\Omega^{-1}(\square\Omega) + 36\Omega^{-2}(\square\Omega)^2], \quad (\text{B16})$$

$$\begin{aligned} \bar{R}_{\mu\nu}\bar{R}^{\mu\nu} = & \Omega^{-4}[R_{\mu\nu}R^{\mu\nu} - 2\Omega^{-1}(2R_{\mu\nu}\Omega^{;\mu\nu} + R\square\Omega) + \Omega^{-2}(8R_{\mu\nu}\Omega^{;\mu}\Omega^{;\nu} - 2R(\partial\Omega)^2 + 4\Omega_{;\mu\nu}\Omega^{;\mu\nu} + 8(\square\Omega)^2) \\ & - \Omega^{-3}(4\Omega_{;\mu\nu}\Omega^{;\mu}\Omega^{;\nu} - (\square\Omega)(\partial\Omega)^2) + 12\Omega^{-4}(\partial\Omega)^2], \end{aligned} \quad (\text{B17})$$

$$\begin{aligned} \bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} = & \Omega^{-4}[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 8\Omega^{-1}R_{\mu\nu}\Omega^{;\mu\nu} + 4\Omega^{-2}((\square\Omega)^2 + 2\Omega_{;\mu\nu}\Omega^{;\mu\nu} - R(\partial\Omega)^2 + 4R_{\mu\nu}\Omega^{;\mu}\Omega^{;\nu}) \\ & + 8\Omega^{-3}((\square\Omega)(\partial\Omega)^2 - 4\Omega_{;\mu\nu}\Omega^{;\mu}\Omega^{;\nu}) + 24\Omega^{-4}(\partial\Omega)^2], \end{aligned} \quad (\text{B18})$$

and the transformed Gauss-Bonnet term reads

$$\begin{aligned} \bar{\mathcal{G}} = & \Omega^{-4}[\mathcal{G} + 4\Omega^{-1}(2R_{\mu\nu}\Omega^{;\mu\nu} - R\square\Omega) + 2\Omega^{-2}(4((\square\Omega)^2 - \Omega_{;\mu\nu}\Omega^{;\mu\nu}) - 8R_{\mu\nu}\Omega^{;\mu}\Omega^{;\nu} + 2R(\partial\Omega)^2) \\ & + 8\Omega^{-3}(4\Omega_{;\mu\nu}\Omega^{;\mu}\Omega^{;\nu} - (\square\Omega)(\partial\Omega)^2) - 24\Omega^{-4}(\partial\Omega)^2]. \end{aligned} \quad (\text{B19})$$

The GB action becomes, after integrating by parts [e.g. terms $\nabla^\mu(\Omega^{-7}\Omega_{;\mu}\Omega_{;\nu}\Omega^{;\nu})$ or $\nabla^\mu(\Omega^{-1}R_{\mu\nu}\Omega^{;\nu})$],

$$\int d^4x\sqrt{\bar{g}}\bar{\mathcal{G}} = \int d^4x\sqrt{g}(\mathcal{G} + \Delta\mathcal{L}_H), \quad (\text{B20})$$

where the additional terms $\Delta\mathcal{L}_H$ can be expressed in Horndeski form (3)–(6) with

$$\Delta G_2(\phi, X) = -176\Omega^{-4}X_\Omega, \quad \Delta G_3(\phi, X) = -48\Omega^{-3}X_\Omega, \quad \Delta G_4(\phi, X) = 8\Omega^{-2}X_\Omega, \quad \Delta G_5(\phi, X) = -8\Omega^{-1}, \quad (\text{B21})$$

and $\Omega = \Omega(\phi)$, $X_\Omega = -\frac{1}{2}(\partial\Omega)^2$. Therefore, one concludes that adding the Gauss-Bonnet term to the Horndeski action does not change the structure of the theory under a purely conformal transformation; it merely changes the functions $G_i(\Omega(\phi), X)$.

b. Normalized pure disformal relation

We will now compute the transformation rules for the Gauss-Bonnet term under a map given by a normalized, pure disformal relation (B1). The \bar{R}^2 term follows from (B5), while the other terms read

$$\begin{aligned} \bar{R}_{\mu\nu}\bar{R}^{\mu\nu} = & [R_{\mu\nu}^2] - 2\gamma^2(\langle R_{\mu\nu}^2 \rangle - \langle R_{\mu\nu}R^{\alpha\mu\beta\nu} \rangle) + \gamma^4(\langle\langle R_{\alpha\mu\beta\nu}R^{\gamma\mu\delta\nu} \rangle\rangle + \langle R_{\mu\nu} \rangle^2) + 2\gamma^2\{[\Pi][R_{\mu\nu}\Pi] - [\Pi R_{\mu\nu}\Pi] \\ & + \gamma^2(\langle\langle R_{\mu\alpha\nu\beta}\Pi^{\beta\gamma}\Pi_\gamma^\beta \rangle\rangle) + \langle\Pi R_{\mu\nu}\Pi\rangle + 2\langle R\Pi^2 \rangle - \langle R_{\alpha\mu\beta\nu}\Pi^{\alpha\beta} \rangle[\Pi] - 2\langle R_{\mu\nu}\Pi \rangle[\Pi] - \langle\Pi \rangle[R_{\mu\nu}\Pi] + \gamma^4(\langle\langle\Pi^{\mu\alpha}R_{\lambda\alpha\sigma\beta}\Pi^{\beta\nu} \rangle\rangle) \\ & - \langle R_{\alpha\mu\beta\nu}\Pi^{\alpha\beta} \rangle\langle\Pi \rangle + \langle R_{\mu\nu} \rangle(\langle\Pi^2 \rangle - \langle\Pi \rangle[\Pi])\} + \gamma^4\{[\Pi^4] - 2[\Pi][\Pi^3] + [\Pi]^2[\Pi^2] + 2\gamma^2([\Pi^3]\langle\Pi \rangle - [\Pi]^2\langle\Pi^2 \rangle) \\ & - [\Pi][\Pi^2]\langle\Pi \rangle + 3[\Pi]\langle\Pi^3 \rangle - 2\langle\Pi^4 \rangle) + \gamma^4(\langle\Pi \rangle^2([\Pi]^2 + [\Pi^2]) - 2\langle\Pi \rangle\langle\Pi^3 \rangle - 2[\Pi]\langle\Pi \rangle\langle\Pi^2 \rangle + 2\langle\Pi^2 \rangle^2)\}, \end{aligned} \quad (\text{B22})$$

$$\begin{aligned} \bar{R}_{\alpha\beta\mu\nu}\bar{R}^{\alpha\beta\mu\nu} = & [[R_{\alpha\beta\mu\nu}^2]] - 4\gamma^2\langle R_{\mu\alpha\beta\gamma}R^{\nu\alpha\beta\gamma} \rangle + 4\gamma^4\langle\langle R_{\alpha\mu\beta\nu}R^{\alpha\lambda\beta\sigma} \rangle\rangle \\ & + 4\gamma^2(\langle[\Pi^{\alpha\gamma}R_{\alpha\beta\gamma\delta}\Pi^{\beta\delta}] \rangle + 4\gamma^2\langle\Pi^{\alpha\beta}R_{\mu\alpha\beta\gamma}\Pi^{\gamma\nu} \rangle + 2\gamma^4\{\langle\Pi \rangle\langle R_{\alpha\mu\beta\nu}\Pi^{\alpha\beta} \rangle - \langle\Pi^{\alpha\lambda}R_{\mu\alpha\nu\beta}\Pi^{\beta\sigma} \rangle\}) \\ & + \gamma^4\{2([\Pi^2]^2 - [\Pi^4]) - 8\gamma^2(\langle\Pi^2 \rangle[\Pi^2] - \langle\Pi^4 \rangle) + 4\gamma^4(\langle\Pi^2 \rangle^2 - 2\langle\Pi^3 \rangle\langle\Pi \rangle + \langle\Pi \rangle^2[\Pi^2])\}. \end{aligned} \quad (\text{B23})$$

The total result is

$$\begin{aligned}
\bar{G} = & R^2 - 4[R_{\alpha\beta}^2] + [[R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}] - 4\gamma^2\{\langle R_{\alpha\beta\gamma\mu}R^{\alpha\beta\gamma\nu} \rangle - 2\langle R_{\alpha\mu}R^{\alpha\nu} \rangle - 2\langle R_{\alpha\mu\beta\nu}R^{\alpha\beta} \rangle + R\langle R_{\mu\nu} \rangle\} \\
& + \gamma^2\{2R[\Pi]^2 - 2R[\Pi^2] + 8[\Pi R_{\alpha\beta}\Pi] + 4[[\Pi^{\alpha\gamma}R_{\alpha\beta\gamma\delta}\Pi^{\beta\delta}] - 8[\Pi][R_{\alpha\beta}\Pi]\} \\
& - 4\gamma^4\{2\langle R_{\mu\alpha\nu\beta}\Pi^{\alpha\gamma}\Pi_{\gamma}^{\beta} \rangle - 4\langle \Pi^{\alpha\beta}R_{\mu\alpha\beta\gamma}\Pi^{\gamma\nu} \rangle + 4\langle R_{\alpha\beta} \cdot \Pi^2 \rangle - R\langle \Pi^2 \rangle + 2\langle \Pi R_{\alpha\beta}\Pi \rangle \\
& - 2[\Pi]\langle R_{\alpha\mu\beta\nu}\Pi^{\alpha\beta} \rangle - 4[\Pi]\langle R_{\mu\alpha}\Pi \rangle + R[\Pi]\langle \Pi \rangle + \langle R_{\mu\nu} \rangle([\Pi]^2 - [\Pi^2]) - 2\langle \Pi \rangle[R_{\alpha\beta}\Pi]\} \\
& + \gamma^4\{[\Pi]^4 - 6[\Pi]^2[\Pi^2] + 3[\Pi^2]^2 + 8[\Pi][\Pi^3] - 6[\Pi^4]\} \\
& + 4\gamma^6\{6\langle \Pi^4 \rangle - 6\langle \Pi^3 \rangle[\Pi] + 3\langle \Pi^2 \rangle[\Pi]^2 - \langle \Pi \rangle[\Pi]^3 - 3\langle \Pi^2 \rangle[\Pi^2] + 3\langle \Pi \rangle[\Pi][\Pi^2] - 2\langle \Pi \rangle[\Pi^3]\}. \tag{B24}
\end{aligned}$$

Here the terms arising from $R \cdot R$, $R \cdot \Pi$ and $\Pi \cdot \Pi$ correspond to the lines 1, 2–4, 5–6. The first three terms are just the Gauss-Bonnet term of the unbarred metric.

A theory whose Lagrangian density includes a $\int d^4x \sqrt{-\bar{g}} \bar{G}$ term of the above form does not belong to the Horndeski Lagrangian. This follows from the presence of terms with up to four contractions of the second derivatives of the scalar field π in the transformed

Gauss-Bonnet term (B24). However, we conjecture that the equations of motion for such a theory will be second order through the existence of implicit constraints (cf. Sec. IV), as the variation with respect to the metric would involve the Jacobian determinant and the higher-order terms introduced by the disformal transformation would not be present in the original frame. The effects of the Gauss-Bonnet term in disformally coupled theories will be analyzed elsewhere.

-
- [1] S. Perlmutter *et al.* (Supernova Cosmology Project), *Astrophys. J.* **517**, 565 (1999).
- [2] A. G. Riess *et al.* (Supernova Search Team), *Astron. J.* **116**, 1009 (1998).
- [3] G. Hinshaw *et al.* (WMAP Collaboration), *Astrophys. J. Suppl. Ser.* **208**, 19 (2013).
- [4] L. Anderson *et al.* (SDSS-III Collaboration), *Mon. Not. R. Astron. Soc.* **427**, 3435 (2013).
- [5] P. Ade *et al.* (Planck Collaboration), arXiv:1303.5076.
- [6] A. H. Guth, *Phys. Rev. D* **23**, 347 (1981).
- [7] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989).
- [8] J. Martin, *C.R. Phys.* **13**, 566 (2012).
- [9] E. J. Copeland, M. Sami, and S. Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006).
- [10] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, *Phys. Rep.* **513**, 1 (2012).
- [11] M. Ostrogradski, *Mem. Ac. St. Petersburg VI*, **4**, 385 (1850).
- [12] R. P. Woodard, *Lect. Notes Phys.* **720**, 403 (2007).
- [13] D. Lovelock, *J. Math. Phys. (N.Y.)* **12**, 498 (1971).
- [14] J. Z. Simon, *Phys. Rev. D* **41**, 3720 (1990).
- [15] G. W. Horndeski, *Int. J. Theor. Phys.* **10**, 363 (1974).
- [16] C. Deffayet, S. Deser, and G. Esposito-Farese, *Phys. Rev. D* **80**, 064015 (2009).
- [17] C. Deffayet, X. Gao, D. Steer, and G. Zahariade, *Phys. Rev. D* **84**, 064039 (2011).
- [18] M. Zumalacarregui, T. S. Koivisto, and D. F. Mota, *Phys. Rev. D* **87**, 083010 (2013).
- [19] A. De Felice, T. Kobayashi, and S. Tsujikawa, *Phys. Lett. B* **706**, 123 (2011).
- [20] G. Leon and E. N. Saridakis, *J. Cosmol. Astropart. Phys.* **03** (2013) 025.
- [21] T. Baker, P. G. Ferreira, and C. Skordis, *Phys. Rev. D* **87**, 024015 (2013).
- [22] L. Amendola, M. Kunz, M. Motta, I. Saltas, and I. Sawicki, *Phys. Rev. D* **87**, 023501 (2013).
- [23] M. Motta, I. Sawicki, I. D. Saltas, L. Amendola, and M. Kunz, *Phys. Rev. D* **88**, 124035 (2013).
- [24] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, *Prog. Theor. Phys.* **126**, 511 (2011).
- [25] K. Koyama, G. Niz, and G. Tasinato, *Phys. Rev. D* **88**, 021502 (2013).
- [26] R. Kase and S. Tsujikawa, *J. Cosmol. Astropart. Phys.* **08** (2013) 054.
- [27] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, *Phys. Rev. Lett.* **108**, 051101 (2012).
- [28] C. Brans and R. Dicke, *Phys. Rev.* **124**, 925 (1961).
- [29] E. E. Flanagan, *Classical Quantum Gravity* **21**, 3817 (2004).
- [30] N. Deruelle and M. Sasaki, *Springer Proc. Phys.* **137**, 247 (2011).
- [31] T. Chiba and M. Yamaguchi, *J. Cosmol. Astropart. Phys.* **10** (2013) 040.
- [32] C. de Rham and A. J. Tolley, *J. Cosmol. Astropart. Phys.* **05** (2010) 015.
- [33] D. Bettoni and S. Liberati, *Phys. Rev. D* **88**, 084020 (2013).
- [34] T. S. Koivisto, D. F. Mota, and M. Zumalacarregui, *Phys. Rev. Lett.* **109**, 241102 (2012).
- [35] A. Vainshtein, *Phys. Lett.* **39B**, 393 (1972).
- [36] J. Garcia-Bellido, J. Rubio, M. Shaposhnikov, and D. Zenhausern, *Phys. Rev. D* **84**, 123504 (2011).
- [37] S. Ferrara, R. Kallosh, A. Linde, A. Marrani, and A. Van Proeyen, *Phys. Rev. D* **82**, 045003 (2010).
- [38] S. Ferrara, R. Kallosh, A. Linde, A. Marrani, and A. Van Proeyen, *Phys. Rev. D* **83**, 025008 (2011).
- [39] A. Nicolis, R. Rattazzi, and E. Trincherini, *Phys. Rev. D* **79**, 064036 (2009).

- [40] A. Barreira, B. Li, C. Baugh, and S. Pascoli, *Phys. Rev. D* **86**, 124016 (2012).
- [41] A. Barreira, B. Li, A. Sanchez, C. M. Baugh, and S. Pascoli, *Phys. Rev. D* **87**, 103511 (2013).
- [42] J. D. Bekenstein, *Phys. Rev. D* **48**, 3641 (1993).
- [43] N. Kaloper, *Phys. Lett. B* **583**, 1 (2004).
- [44] M. Clayton and J. Moffat, *Phys. Lett. B* **460**, 263 (1999).
- [45] J. Magueijo, *Rep. Prog. Phys.* **66**, 2025 (2003).
- [46] B. A. Bassett, S. Liberati, C. Molina-Paris, and M. Visser, *Phys. Rev. D* **62**, 103518 (2000).
- [47] J. D. Bekenstein and R. H. Sanders, *Astrophys. J.* **429**, 480 (1994).
- [48] J. D. Bekenstein, *Phys. Rev. D* **70**, 083509 (2004).
- [49] M. Milgrom, *Phys. Rev. D* **80**, 123536 (2009).
- [50] J. Noller, *J. Cosmol. Astropart. Phys.* **07** (2012) 013.
- [51] P. Brax, *Phys. Lett. B* **712**, 155 (2012).
- [52] C. de Rham and G. Gabadadze, *Phys. Rev. D* **82**, 044020 (2010).
- [53] C. de Rham, G. Gabadadze, and A. J. Tolley, *Phys. Rev. Lett.* **106**, 231101 (2011).
- [54] T. S. Koivisto, [arXiv:0811.1957](https://arxiv.org/abs/0811.1957).
- [55] M. Zumalacarregui, T. Koivisto, D. Mota, and P. Ruiz-Lapuente, *J. Cosmol. Astropart. Phys.* **05** (2010) 038.
- [56] J. Cembranos, A. Dobado, and A. L. Maroto, *Phys. Rev. D* **68**, 103505 (2003).
- [57] J. Cembranos, A. Dobado, and A. L. Maroto, *Phys. Rev. Lett.* **90**, 241301 (2003).
- [58] T. Koivisto, D. Wills, and I. Zavala, [arXiv:1312.2597](https://arxiv.org/abs/1312.2597).
- [59] T. S. Koivisto and D. E. Wills, *Int. J. Mod. Phys. D* **22**, 1342024 (2013).
- [60] D. Bettoni, S. Liberati, and L. Sindoni, *J. Cosmol. Astropart. Phys.* **11** (2011) 007.
- [61] D. Bettoni, V. Pettorino, S. Liberati, and C. Baccigalupi, *J. Cosmol. Astropart. Phys.* **07** (2012) 027.
- [62] G. J. Olmo, H. Sanchis-Alepuz, and S. Tripathi, *Phys. Rev. D* **80**, 024013 (2009).
- [63] E. Goulart and F. Falciano, *Classical Quantum Gravity* **30**, 155020 (2013).
- [64] S. Mukohyama and J.-P. Uzan, *Phys. Rev. D* **87**, 065020 (2013).
- [65] J. Magueijo, M. Rodriguez-Vazquez, H. Westman, and T. Zlosnik, [arXiv:1311.4481](https://arxiv.org/abs/1311.4481).
- [66] S. Mukohyama, *Phys. Rev. D* **87**, 085030 (2013).
- [67] P. Brax, C. Burrage, and A.-C. Davis, *J. Cosmol. Astropart. Phys.* **10** (2012) 016.
- [68] C. van de Bruck and G. Sculthorpe, *Phys. Rev. D* **87**, 044004 (2013).
- [69] C. van de Bruck, J. Morrice, and S. Vu, *Phys. Rev. Lett.* **111**, 161302 (2013).
- [70] P. Brax, C. Burrage, A.-C. Davis, and G. Gubitosi, *J. Cosmol. Astropart. Phys.* **11** (2013) 001.
- [71] L. Amendola, K. Enqvist, and T. Koivisto, *Phys. Rev. D* **83**, 044016 (2011).
- [72] J.-P. Bruneton and G. Esposito-Farese, *Phys. Rev. D* **76**, 124012 (2007).
- [73] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
- [74] P. Pani, T. P. Sotiriou, and D. Vernieri, *Phys. Rev. D* **88**, 121502 (2013).
- [75] M. Spivak, *Calculus on Manifolds. A Modern Approach to Classical Theorems of Advanced Calculus* (Benjamin, New York, 1965).
- [76] K. Fujikawa, *Phys. Rev. Lett.* **44**, 1733 (1980).
- [77] Y. Fujii and T. Nishioka, *Phys. Rev. D* **42**, 361 (1990).
- [78] F. Albareti, J. Cembranos, A. de la Cruz-Dombriz, and A. Dobado, *J. Cosmol. Astropart. Phys.* **07** (2013) 009.
- [79] S. Renaux-Petel, S. Mizuno, and K. Koyama, *J. Cosmol. Astropart. Phys.* **11** (2011) 042.
- [80] J. York, W. James, *Phys. Rev. Lett.* **28**, 1082 (1972).
- [81] G. Gibbons and S. Hawking, *Phys. Rev. D* **15**, 2752 (1977).
- [82] I. D. Saltas and M. Hindmarsh, *Classical Quantum Gravity* **28**, 035002 (2011).
- [83] T.-j. Chen, M. Fasiello, E. A. Lim, and A. J. Tolley, *J. Cosmol. Astropart. Phys.* **02** (2013) 042.
- [84] J. D. Barrow and S. Cotsakis, *Phys. Lett. B* **214**, 515 (1988).
- [85] K.-i. Maeda, *Phys. Rev. D* **39**, 3159 (1989).
- [86] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, *J. Cosmol. Astropart. Phys.* **08** (2013) 025.
- [87] G. Gabadadze, K. Hinterbichler, J. Khoury, D. Pirtskhalava, and M. Trodden, *Phys. Rev. D* **86**, 124004 (2012).
- [88] T. Kobayashi, N. Tanahashi, and M. Yamaguchi, *Phys. Rev. D* **88**, 083504 (2013).
- [89] A. Padilla and V. Sivanesan, *J. High Energy Phys.* **04** (2013) 032.
- [90] V. Sivanesan, [arXiv:1307.8081](https://arxiv.org/abs/1307.8081).
- [91] T. Biswas, E. Gerwick, T. Koivisto, and A. Mazumdar, *Phys. Rev. Lett.* **108**, 031101 (2012).
- [92] S. A. Appleby, A. De Felice, and E. V. Linder, *J. Cosmol. Astropart. Phys.* **10** (2012) 060.
- [93] J. B. Jimenez, A. L. D. Froes, and D. F. Mota, *Phys. Lett. B* **725**, 212 (2013).
- [94] E. Babichev, C. Deffayet, and G. Esposito-Farese, *Phys. Rev. D* **84**, 061502 (2011).
- [95] D. Brizuela, J. M. Martín-García, and G. A. Mena Marugan, *Gen. Relativ. Gravit.* **41**, 2415 (2009).
- [96] J. M. Martín-García, <http://www.xact.es/>.
- [97] S. C. Davis, *Phys. Rev. D* **67**, 024030 (2003).