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Fermions in gravity with local spin-base invariance

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We study a formulation of Dirac fermions in curved spacetime that respects general coordinate invariance as well as invariance under local spin-base transformations. The natural variables for this formulation are spacetime-dependent Dirac matrices subject to the Clifford-algebra constraint. In particular, a coframe, i.e. vierbein field is not required. The corresponding affine spin connection consists of a canonical part that is completely fixed in terms of the Dirac matrices and a free part that can be interpreted as spin torsion. A general variation of the Dirac matrices naturally induces a spinorial Lie derivative which coincides with the known Kosmann-Lie derivative in the absence of torsion. Using this formulation for building a field theory of quantized gravity and matter fields, we show that it suffices to quantize the metric and the matter fields. This observation is of particular relevance for field theory approaches to quantum gravity, as it can serve for a purely metric-based quantization scheme for gravity even in the presence of fermions.

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I. INTRODUCTION

Building a field theory of quantized gravity requires one to specify the fundamental degrees of freedom to be quantized. Unfortunately, the guidance from the corresponding classical theory, general relativity, is not particularly strong, as the variational principle applied to substantially different degrees of freedom can lead to the same equations of motion. Examples are given by (i) the conventional Einstein-Hilbert action in terms of metric degrees of freedom $g_{\mu\nu}$, or (ii) in terms of a vierbein $e_\mu{}^a$, or (iii) the first-order Hilbert Palatini action which in addition to the vierbein also depends on the spin connection $\omega_\mu{}^a{}_b$ (Einstein-Cartan theory). Many further variants along this line are known [1], all of which (in the absence of torsion or other deformations) have in common that they yield Einstein's equation on the classical level.

By contrast, if these various classically equivalent theories are quantized (by some appropriate method), the quantum versions should be expected to generically differ. This can be seen from the fact that, e.g., the relation between the metric and the vierbein,

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}, \tag{1}$$

implies that an appropriate measure for a functional integral over metrics $\mathcal{D}g$ is expected to differ from that over vierbeins $\mathcal{D}e$ by a nontrivial Jacobian. The resulting differences have explicitly been worked out recently in the asymptotic safety approach to quantum gravity [2,3], but should be expected to occur in any other field theory attempt at quantizing gravity as well. For instance, the renormalization group (RG) flow of metric-based quantum gravity [3] has been shown to differ from that of its vierbein-based counterpart [4,5] at least quantitatively. Qualitatively, new aspects arise from the Faddeev-Popov ghosts associated with local Lorentz invariance in the vierbein formulation [4]—a symmetry that is not present in the metric-based formulation. In the same spirit, quantizing Einstein-Cartan theory or its chiral variants leads to yet further sets of RG flows [6,7] even in the absence of any torsion.

As only one of these different quantum theories can be realized in Nature, criteria beyond pure mathematical consistency are required to distinguish between the different theories. As fermions occur in our universe, the use of a vierbein-based formalism seems mandatory, consequently giving preference to versions of quantum gravity where the corresponding fields are considered as fundamental or where at least vierbeins are formed prior to the metric in terms of even more fundamental degrees of freedom, see e.g., [8,9].

In the present work, we critically reexamine the seeming necessity of vierbein-based formulations in the presence of fermions on curved space. For this, we consider a more general formulation of fermions in gravity, where in addition to general coordinate invariance the symmetry under local spin-base transformations remains fully preserved [10,11]. With respect to our original motivation, it turns out that a purely metric-based quantization scheme appears much more natural, as the local spin-base fluctuations can be shown to represent a trivial factor of the measure. We emphasize that this observation does not invalidate a quantization of gravity in terms of vierbeins or other underlying degrees of freedom. Rather, the existence of fermions in the Universe does not provide an argument for ruling out metric-based quantization schemes of gravity. A similar conclusion has been drawn for the case that the observed fermions finally turn out to be Kähler fermions [5].

On our way to this central result, we will rederive and generalize the spin-base invariant formalism for fermions in curved space, following the work of Finster and Weldon [10,11]. In particular, we derive all details of the formalism as well as new results from very few underlying assumptions in a self-contained way. Among the new results, we show how the concept of spin torsion arises in this formalism and we discover a new simple relation between the general variation of Dirac matrices and the Kosmann-Lie derivative for spinors.

In order to contrast the spin-base invariant formalism discussed below with the standard vierbein formulation, let us briefly recall the elements of the standard construction for describing fermions in curved spacetime [12–15]: Once a suitable vierbein, satisfying Eq. (1) is introduced, the spin connection ${\omega_\mu}^a{}_b$ required to define fermionic dynamics is derived from the vierbein postulate as an algebraic equation

$$0 = \partial_\mu {e_\nu}^a - \Gamma^\kappa_{\mu\nu} {e_\kappa}^a + {\omega_\mu}^a{}_b {e_\nu}^b, \tag{2}$$

where $\Gamma^\kappa_{\mu\nu}$ is the—not necessarily symmetric—affine spacetime connection [14–16]. The Dirac matrices ${\gamma_{(e)}}_\mu$ within this vierbein formalism are given by

$${\gamma_{(e)}}_\mu = {e_\mu}^a {\gamma_{(\mathrm{f})}}_a, \tag{3}$$

where the ${\gamma_{(\mathrm{f})}}_a$ are fixed constant Dirac matrices satisfying the Clifford algebra for Minkowski space

$$\{{\gamma_{(\mathrm{f})}}_a, {\gamma_{(\mathrm{f})}}_b\} = 2\eta_{ab}\mathrm{I}, \tag{4}$$

where I is the unit matrix. In this way, the ${\gamma_{(e)}}_\mu$ are automatically compatible with the Clifford algebra

$$\{{\gamma_{(e)}}_\mu, {\gamma_{(e)}}_\nu\} = 2g_{\mu\nu}\mathrm{I}. \tag{5}$$

The covariant derivative for spinors ψ then reads

$${\nabla_{(e)}}_\mu \psi = \partial_\mu \psi + \frac{1}{8} {\omega_\mu}^{ab} [{\gamma_{(\mathrm{f})}}_a, {\gamma_{(\mathrm{f})}}_b]\psi. \tag{6}$$

For explicit calculations the Dirac operator $\nabla\!\!\!\!/_{(e)} = {\gamma_{(e)}}^\mu {\nabla_{(e)}}_\mu$ is often needed. In practical calculations, it can be more convenient to have the Dirac operator in a more adjusted basis concerning the actual choice of the Dirac matrices [17,18].

While this standard vierbein formalism is perfectly sufficient for a description of fermions in curved spacetime, several properties give rise to criticism at least from a conceptual (or aesthetic) viewpoint: the relevant objects for the physical system are the generally spacetime-dependent Dirac matrices γ_μ which have to satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}$. For a given metric, more solutions than only those parametrizable by a vierbein exist for the Dirac matrices [11]. This already indicates that the vierbein construction should be regarded as a special choice. On the other hand, it seems at odds with the principles of general relativity that a special inertial coframe ${e_\mu}^a$ has to be introduced in order to describe the fermions.

In addition, this choice introduces another symmetry, "physically" corresponding to the Lorentz symmetry of the tangential space related to the roman indices $a, b, \ldots$ (e.g., a local O(4) symmetry in a Euclidean formulation, which can be generalized to a $\mathrm{GL}(4,\mathbb{R})$ symmetry [19]). From the viewpoint of the Dirac matrices γ_μ, this symmetry seems artificial. By contrast, the relevant nontrivial symmetry of the Clifford algebra is the local spin-base symmetry $\mathrm{SL}(4,\mathbb{C})$ which is not fully reflected by the standard vierbein construction.

The spin-base invariant formalism [10,11] used and further developed in the present work does not require a coframe or vierbein construction. Still, in the absence of torsion it is completely compatible with the vierbein formalism in the sense that a vierbein construction can always be recovered as a special case. Important differences however arise in the presence of torsion, as discussed in Sec. III. The spin-base invariant formalism supports degrees of freedom within the affine spin connection, which can be interpreted as a spin torsion. This is in direct analogy to the affine spacetime connection which in general consists of a canonical part in terms of the Levi-Civita connection and a free part connected with spacetime torsion.

Following the principle of general covariance together with spin-base invariance, a field strength corresponding to a spin curvature can be constructed. The simplest action linear in this field strength defines a classical dynamical theory. The resulting equations of motion imply that the spin torsion vanishes in the absence of any sources. The metric-part of these equations of motion correspond to general relativity as expected.

This paper is organized as follows: in Sec. II, we specify all prerequisites and assumptions for constructing the spin-base invariant formalism. Section III is devoted to the analysis of the affine spin connection and the spin metric, which defines the relation between spinors and Dirac conjugated spinors. In Sec. IV a spinorial Lie derivative is constructed within the present framework which turns out to coincide with the Kosmann-Lie derivative known in the literature. The inclusion of an additional gauge symmetry is worked out in Sec. V. The field strength for spinors and the corresponding action linear in the field strength is derived in Sec. VI. We generalize our results, formulated for irreducible representations of the Dirac algebra, to reducible cases in Sec. VII. The implications of the spin-base invariant formalism for a possible quantized version of gravity and quantized matter is discussed in Sec. VIII on the level of a path integral approach. As a first hands-on application of the spin-base invariant formalism, we determine the response of several elements of the formalism

(Dirac matrices, spin connection, etc.) under a variation of the metric in Sec. IX. These results form elementary technical building blocks for generic quantum field theory computations. Conclusions are drawn in Sec. X. The uniqueness (up to a sign) of the spin metric is proven in Appendix A. In Appendix B we list several useful identities of the formalism for the simpler case of vanishing torsion, serving as a toolbox for a straightforward application of the formalism.

II. BASIC REQUIREMENTS FOR SPIN-BASE INVARIANCE

We aim at a generally covariant and spin-base invariant description of fermions without recourse to a vierbein construction. For this, only a few basic assumptions have to be made. We stress that these requirements are completely compatible with the vierbein formalism for torsion-free spacetimes.

First we fix the relation between the metric of a (pseudo-) Riemannian spacetime and the Dirac matrices γ^μ by demanding the Clifford algebra to hold locally,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathrm{I}, \qquad \gamma^\mu \in \mathbb{C}^{d_\gamma \times d_\gamma}. \tag{7}$$

Here, d_γ denotes the dimension of the Dirac matrices in the *irreducible* representation of the Clifford algebra, i.e., $d_\gamma = 2^{\lfloor d/2 \rfloor}$.

The Clifford algebra supports an $\mathrm{SL}(d_\gamma, \mathbb{C})$ symmetry.[1] We require this invariance under *spin-base transformations* with $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$ to hold locally for general action functionals involving the Dirac matrices and Dirac fermions ψ and their conjugate $\bar{\psi}$ obeying the transformation rules,

$$\begin{aligned} &\text{(i) } \gamma^\mu \to \mathcal{S}\gamma^\mu\mathcal{S}^{-1},\\ &\text{(ii) } \psi \to \mathcal{S}\psi,\\ &\text{(iii) } \bar{\psi} \to \bar{\psi}\mathcal{S}^{-1}. \end{aligned} \tag{8}$$

Dirac conjugation of a spinor ψ involves Hermitian conjugation and a *spin metric* h,

$$\bar{\psi} = \psi^\dagger h, \tag{9}$$

which is assumed to carry no scale,

$$|\det h| = 1. \tag{10}$$

Local spin-base invariance requires us to introduce a covariant derivative ∇_μ with the following standard properties,

$$\begin{aligned} &\text{(i) linearity:} && \nabla_\mu(\psi_1 + \psi_2) = \nabla_\mu\psi_1 + \nabla_\mu\psi_2,\\ &\text{(ii) product rule:} && \nabla_\mu(\psi\bar{\psi}) = (\nabla_\mu\psi)\bar{\psi} + \psi(\nabla_\mu\bar{\psi}),\\ &\text{(iii) covariance:} && \nabla_\mu\bar{\psi} = \overline{\nabla_\mu\psi}, \qquad \nabla_\mu\psi^\dagger = (\nabla_\mu\psi)^\dagger. \end{aligned} \tag{11}$$

Up to this point, local spin-base invariance is reminiscent of gauge invariance, however, with a noncompact gauge group. Now we make contact with general covariance by additionally demanding that ∇_μ has to coincide with the ordinary spacetime covariant derivative D_μ, if it acts on an object that is a scalar under spin-base transformations. A particularly important example is given by

$$\nabla_\mu(\bar{\psi}\gamma^\nu\psi) = D_\mu(\bar{\psi}\gamma^\nu\psi) = \partial_\mu(\bar{\psi}\gamma^\nu\psi) + \Gamma^\nu_{\mu\kappa}(\bar{\psi}\gamma^\kappa\psi), \tag{12}$$

where $\Gamma^\nu_{\mu\kappa}$ denotes the *metric compatible* affine spacetime connection

$$\Gamma^\nu_{\mu\kappa} = \left\{ \begin{matrix} \nu \\ \mu\kappa \end{matrix} \right\} + K^\nu{}_{\mu\kappa}. \tag{13}$$

Here $\left\{ \begin{matrix} \nu \\ \mu\kappa \end{matrix} \right\}$ is the Levi-Civita connection

$$\left\{ \begin{matrix} \nu \\ \mu\kappa \end{matrix} \right\} = \frac{1}{2} g^{\nu\lambda}(\partial_\mu g_{\lambda\kappa} + \partial_\kappa g_{\lambda\mu} - \partial_\lambda g_{\mu\kappa}), \tag{14}$$

and $K^\nu{}_{\mu\kappa}$ is the contorsion tensor. The contorsion $K^\nu{}_{\mu\kappa}$ and the torsion $C^\nu{}_{\mu\kappa}$ are related via

$$C^\nu{}_{\mu\kappa} = 2K^\nu{}_{[\mu\kappa]} \equiv K^\nu{}_{\mu\kappa} - K^\nu{}_{\kappa\mu}, \tag{15}$$

$$K^\nu{}_{\mu\kappa} = \frac{1}{2}(C^\nu{}_{\mu\kappa} + C_\kappa{}^\nu{}_\mu - C_{\mu\kappa}{}^\nu) \equiv -K_{\kappa\mu}{}^\nu, \tag{16}$$

where indices in square brackets […] are completely antisymmetrized.

The property (iii) of Eq. (11) can be interpreted as the spinorial analog to metric compatibility of general relativity with $\bar{\psi}$ corresponding to a covariant spin vector and ψ to a contravariant spin vector under spin-base transformations. From this viewpoint, the Hermitian conjugate spinor $\psi^\dagger$ should be considered as (the Hermitian conjugate of) a contravariant spin vector in contrast to the covariant $\bar{\psi}$. For instance, $\psi^\dagger$ transforms with $\mathcal{S}^\dagger$, whereas $\bar{\psi}$ transforms with $\mathcal{S}^{-1}$. Therefore, we need the additional *definition* $\nabla_\mu\psi^\dagger = (\nabla_\mu\psi)^\dagger$ in (iii) of Eq. (11) which reduces to an identity for the ordinary partial derivative in flat space.[2]

[1]In fact the Clifford algebra is invariant under a $\mathrm{GL}(d_\gamma, \mathbb{C})$ symmetry which locally factorizes into $\mathrm{SL}(d_\gamma, \mathbb{C}) \times \mathrm{U}(1) \times \mathbb{R}_+$. Here we first concentrate on the $\mathrm{SL}(d_\gamma, \mathbb{C})$ component, as the $\mathrm{U}(1) \times \mathbb{R}_+$ part does act trivially on the Dirac matrices. The inclusion of additional symmetry groups such as the U(1) factor is discussed in Sec. V.

[2]In fact this *definition* is not mandatory. If we dropped $\nabla_\mu\psi^\dagger = (\nabla_\mu\psi)^\dagger$, there would be no unique definition of the spin covariant derivative for $\psi^\dagger$ and h. These derivatives are, however, not necessary for calculational or conceptual concerns. With hindsight, only the second equality in Eq. (27) given below, $\partial_\mu h - h\Gamma_\mu - \Gamma_\mu^\dagger h = 0$, is needed which can be inferred from the property $\nabla_\mu\bar{\psi} = \overline{\nabla_\mu\psi}$ alone.

Finally, we require the action of a dynamical theory to be real, especially

$$\text{(i)} \quad (\bar{\psi}\psi)^* = \bar{\psi}\psi,$$
$$\text{(ii)} \quad \int_x (\bar{\psi}\nabla\!\!\!/\psi)^* = \int_x \bar{\psi}\nabla\!\!\!/\psi, \tag{17}$$

where $\int_x$ is a shorthand for $\int d^dx\sqrt{-g}$ with $g = \det g_{\mu\nu}$. Equations (17) also fix some of our conventions, i.e., in other conventions the reality conditions could look differently.

Based on these elementary requirements, the next section is devoted to the analysis of the spin connection that fully implements invariance under spin-base transformation. Apart from obvious conceptual advantages, this invariance might be of use in practical computations to choose a convenient set of Dirac matrices for a simplified construction of classical solutions [17,18]. For vanishing torsion, this spin connection can be made to agree with the standard spin connection constructed from the vierbeins $e_\mu{}^a$ if the Dirac matrices are spin-base transformed to those of the vierbein formalism.

III. PROPERTIES OF THE AFFINE SPIN CONNECTION

For the following analysis, we work in $d = 4$ spacetime dimensions, where $d_\gamma = 4$ holds for the dimension of the irreducible representation of the Clifford algebra. Generalizations to $d = 2$ and $d = 3$ can also be worked out straightforwardly [20]. Whenever suitable, we give the formulas for general d_γ to emphasize the generality of parts of the construction.

A cornerstone of the present construction is the Weldon theorem [11]. It states that an infinitesimal variation of the Dirac matrices which preserves the Clifford algebra can be decomposed into an infinitesimal variation of the inverse metric $\delta g^{\mu\nu}$ and an infinitesimal $\mathrm{SL}(4,\mathbb{C})_\gamma$ transformation $\delta\mathcal{S}_\gamma$:

$$\delta\gamma^\mu = \frac{1}{2}(\delta g^{\mu\nu})\gamma_\nu + [\delta\mathcal{S}_\gamma, \gamma^\mu]. \tag{18}$$

Here the notation distinguishes between $\mathrm{SL}(4,\mathbb{C})$ spin-base transformations $\mathcal{S}$ that simultaneously act on fermions as discussed above and constitute an invariance of the theory, and an (infinitesimal) $\mathrm{SL}(4,\mathbb{C})_\gamma$ transformation $\delta\mathcal{S}_\gamma$ that may only act on the Dirac matrices.[3] This theorem can straightforwardly be proved by using that every matrix in $\mathbb{C}^{4\times4}$ can uniquely be locally spanned by the explicit basis of the Clifford algebra I, γ_*, γ_α, $\gamma_*\gamma_\alpha$ and $[\gamma_\alpha, \gamma_\beta]$. Here,

$$\gamma_* = -\frac{\mathrm{i}}{4!}\tilde{\varepsilon}_{\mu_1\ldots\mu_4}\gamma^{\mu_1}\ldots\gamma^{\mu_4}, \qquad \tilde{\varepsilon}_{\mu_1\ldots\mu_4} = \sqrt{-g}\varepsilon_{\mu_1\ldots\mu_4} \tag{19}$$

is the generalized (generally spacetime-dependent) analog of $\gamma_5 = -\mathrm{i}\gamma_{(\mathrm{f})}{}^0\gamma_{(\mathrm{f})}{}^1\gamma_{(\mathrm{f})}{}^2\gamma_{(\mathrm{f})}{}^3$ in flat space. We have used the Levi-Civita symbol $\varepsilon_{\mu_1\ldots\mu_4}$ with $\varepsilon_{0123} = 1$ to define the Levi-Civita tensor $\tilde{\varepsilon}_{\mu_1\ldots\mu_4}$. Essential properties of γ_* are

$$\text{(i) } \{\gamma^\mu, \gamma_*\} = 0,$$
$$\text{(ii) } \mathrm{tr}\,\gamma_* = 0. \tag{20}$$

Now, let us assume that a spacetime is specified in terms of a set of Dirac matrices γ_μ, also defining the metric through the Clifford algebra Eq. (7), and in terms of contorsion $K^\nu{}_{\mu\kappa}$. Already from the Dirac matrices, we can determine a useful auxiliary Dirac-valued matrix $\hat{\Gamma}_\mu$ as a spacetime vector valued function of the γ^μ. It is defined by

$$D_{(\mathrm{LC})\mu}\gamma^\nu = \partial_\mu\gamma^\nu + \left\{ {\nu \atop \mu\kappa} \right\}\gamma^\kappa = -[\hat{\Gamma}_\mu, \gamma^\nu], \;\; \mathrm{tr}\,\hat{\Gamma}_\mu = 0, \tag{21}$$

where $D_{(\mathrm{LC})\mu}$ is the spacetime covariant derivative including the Levi-Civita connection, but disregarding any torsion. Equations (21) can be resolved explicitly in terms of the local Clifford basis:

$$\text{(i) } \hat{\Gamma}_\mu = p_\mu\gamma_* + v_\mu{}^\alpha\gamma_\alpha + a_\mu{}^\alpha\gamma_*\gamma_\alpha + t_\mu{}^{\alpha\beta}[\gamma_\alpha, \gamma_\beta],$$
$$\text{(ii) } p_\mu = \frac{1}{32}\mathrm{tr}(\gamma_*\gamma_\alpha\partial_\mu\gamma^\alpha),$$
$$\text{(iii) } v_\mu{}^\alpha = \frac{1}{48}\mathrm{tr}([\gamma^\alpha, \gamma_\beta]\partial_\mu\gamma^\beta),$$
$$\text{(iv) } a_\mu{}^\alpha = \frac{1}{8}\mathrm{tr}(\gamma_*\partial_\mu\gamma^\alpha),$$
$$\text{(v) } t_{\mu\alpha}{}^\beta = -\frac{1}{32}\mathrm{tr}(\gamma_\alpha\partial_\mu\gamma^\beta) - \frac{1}{8}\left\{ {\beta \atop \mu\alpha} \right\} \equiv -t_\mu{}^\beta{}_\alpha, \tag{22}$$

where all tensorial coefficients are functions of the Dirac matrices.

Next, we turn to the construction of the covariant derivative. From (ii) of Eq. (11) and $\nabla_\mu(\bar{\psi}\psi) = \partial_\mu(\bar{\psi}\psi)$ analogous to Eq. (12), we observe that the covariant derivative can be written as

$$\nabla_\mu\psi = \partial_\mu\psi + \Gamma_\mu\psi, \tag{23}$$

$$\nabla_\mu\bar{\psi} = \partial_\mu\bar{\psi} - \bar{\psi}\Gamma_\mu. \tag{24}$$

Here we have introduced the affine spin connection Γ_μ to be analyzed, which transforms as a vector under general

[3] In a slight abuse of language, we may call $\delta\mathcal{S}_\gamma$ a "spin-base fluctuation." Since it applies only to the Dirac matrices, it represents a physically relevant fluctuation of the spin basis for the Dirac matrices relative to that of the fermions. This is different from an infinitesimal version of the spin-base transformation $\mathcal{S} \in \mathrm{SL}(4,\mathbb{C})$ which are an invariance of the theory by definition.

coordinate transformations and inhomogeneously under spin-base transformations,

$$\Gamma_\mu \to \mathcal{S}\Gamma_\mu\mathcal{S}^{-1} - (\partial_\mu\mathcal{S})\mathcal{S}^{-1}. \tag{25}$$

From Eq. (21), it is immediate that $\hat{\Gamma}_\mu$ has the same transformation properties as Γ_μ both under general coordinate as well as spin-base transformations. Because Γ_μ is the connection for the spin-base transformations, it is composed from the generators of the symmetry group $\mathrm{SL}(d_\gamma,\mathbb{C})$, the traceless matrices. Therefore we can set the trace of Γ_μ to zero, $\mathrm{tr}\,\Gamma_\mu = 0$. In Sec. V below, we discuss generalizations including a trace part.

From the property of general covariance (12) together with the definition of Dirac conjugation in Eq. (9), we conclude that

$$\nabla_\mu\gamma^\nu = D_\mu\gamma^\nu + [\Gamma_\mu,\gamma^\nu] \equiv \left[\Gamma_\mu - \hat{\Gamma}_\mu - \frac{1}{8}K_{\rho\mu\lambda}[\gamma^\rho,\gamma^\lambda],\gamma^\nu\right], \tag{26}$$

$$\nabla_\mu h = \partial_\mu h - h\Gamma_\mu - \Gamma_\mu^\dagger h = 0, \tag{27}$$

where we have made use of the auxiliary matrix $\hat{\Gamma}_\mu$ defined in Eq. (21).

The challenging task is to find the maximum number of constraints on Γ_μ from the Dirac matrices and therefore from the metric and the actual choice of the spin base in order to identify its physical content. For this, we first consider the spin metric and notice with Eq. (9) that it transforms under spin-base transformations as

$$h \to \mathcal{S}^{\dagger^{-1}}h\mathcal{S}^{-1}. \tag{28}$$

Equation (i) of (17),

$$\psi^\dagger h\psi = \bar{\psi}\psi = (\bar{\psi}\psi)^* = \psi^{\mathrm{T}}h^*\psi^* = \psi^\dagger(-h^\dagger)\psi, \tag{29}$$

implies that the spin metric is anti-Hermitian

$$h^\dagger = -h. \tag{30}$$

Let us now define the Dirac conjugate of a matrix M as

$$\bar{M} = h^{-1}M^\dagger h \tag{31}$$

which implies

$$(\bar{\psi}M\psi)^* = \bar{\psi}\,\bar{M}\,\psi. \tag{32}$$

Using the standard relation

$$\partial_\mu\sqrt{-g} = \sqrt{-g}\,\Gamma^\kappa_{\mu\kappa} \equiv \sqrt{-g}\left\{\begin{matrix}\kappa\\ \mu\kappa\end{matrix}\right\}, \tag{33}$$

we can straightforwardly derive from Eq. (ii) of (17) together with Eqs. (21) and (27) that

$$\int_x \bar{\psi}\nabla\!\!\!/\psi = \int_x (\bar{\psi}\nabla\!\!\!/\psi)^* = \int_x \bar{\psi}(-\bar{\gamma}^\mu\nabla_\mu + [\bar{\Gamma}_\mu - \bar{\hat{\Gamma}}_\mu,\bar{\gamma}^\mu])\psi. \tag{34}$$

As this has to hold for arbitrary fermion fields, we deduce

$$\bar{\gamma}^\mu = -\gamma^\mu \tag{35}$$

and

$$0 = [\Delta\Gamma_\mu,\gamma^\mu], \qquad \Delta\Gamma_\mu = \Gamma_\mu - \hat{\Gamma}_\mu. \tag{36}$$

Since $\hat{\Gamma}_\mu$ is fully determined in terms of the Dirac matrices, Eq. (36) represents a first constraint on the components of the spin connection Γ_μ. This constraint can, of course, trivially be satisfied by identifying $\Gamma_\mu \overset{!}{=} \hat{\Gamma}_\mu$ and setting the difference to zero $\Delta\Gamma_\mu \overset{!}{=} 0$. From the present viewpoint, this is a perfectly legitimate choice, yielding one particular explicit realization of the spin connection being fully determined by the Dirac matrices. This choice has been advocated in [11], where it has also been shown that this spin-base invariant formalism contains the standard vierbein formalism as a subset: for Dirac matrices following the vierbein construction Eq. (3), the coefficients p_μ, $v_\mu{}^\alpha$, $a_\mu{}^\alpha$ all vanish, and the $t_\mu{}^{\alpha\beta}$ are given by

$$t_\mu{}^{\alpha\beta} = \frac{1}{8}\omega_\mu{}^{ab}e^\alpha{}_a e^\beta{}_b.$$

However, there is a priori no reason to single out this definition of the spin connection in terms of $\hat{\Gamma}_\mu$. Therefore, we investigate below the properties and possible further degrees of freedom contained in a possibly nonzero $\Delta\Gamma_\mu$.

Before we do so, let us extract an important consequence of Eq. (36): the covariant derivative of the Dirac matrices given in Eq. (26) reads after evaluating the Dirac matrix commutators,

$$\nabla_\mu\gamma^\nu = [\Delta\Gamma_\mu,\gamma^\nu] + K^\nu{}_{\mu\kappa}\gamma^\kappa. \tag{37}$$

Using the constraint (36), this implies

$$\nabla_\mu\gamma^\mu = K^\mu{}_{\mu\kappa}\gamma^\kappa. \tag{38}$$

In the presence of torsion with $K^\kappa{}_{\kappa\mu} \neq 0$, this result is incompatible with the vierbein postulate (2). In the present notation, the latter is equivalent to the vanishing covariant derivative of the Dirac matrices

$$\nabla_{(e)\mu}\gamma_{(e)}{}^\nu = 0, \tag{39}$$

where the $\gamma_{(e)}{}^\nu$ follow the vierbein construction Eq. (3). This discrepancy between our more general formalism and the conventional vierbein formalism is in line with the fact that the inclusion of torsion requires us to go beyond the conventional vierbein formalism such as, e.g., Einstein-Cartan theory. From the viewpoint of our spin-base invariant formalism, torsion can be accommodated in a straightforward manner on the basis of our requirements of Sec. II. [4]

In the remainder of this section, we concentrate on the analysis of the properties of the $\Delta\Gamma_\mu$ part of the spin connection. For this, it is useful to explicitly construct the spin metric h, as is done in Appendix A. For a given set of Dirac matrices, the spin metric turns out to be uniquely fixed up to a sign and can be parametrized by

$$h = \pm\, \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\varphi}{2}}\mathrm{e}^{\hat{M}}, \tag{40}$$

where φ and $\hat{M}$ are (up to a sign) implicitly defined by

$$\gamma_\mu^\dagger = -\mathrm{e}^{\hat{M}}\gamma_\mu\mathrm{e}^{-\hat{M}}, \qquad \mathrm{tr}\hat{M} = 0, \qquad \mathrm{e}^{\hat{M}^\dagger} = \mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{\hat{M}}. \tag{41}$$

The angle φ can only take discrete constant values,

$$\varphi \in \left\{ n\frac{2\pi}{d_\gamma} : n \in \{0, \ldots, d_\gamma - 1\} \right\}, \qquad \partial_\mu\varphi = 0. \tag{42}$$

These properties together with Eq. (27) imply another constraint for the spin connection (see Appendix A for details):

$$\Gamma_\mu + \bar{\Gamma}_\mu = \hat{\Gamma}_\mu + \bar{\hat{\Gamma}}_\mu = h^{-1}\partial_\mu h. \tag{43}$$

Even if we admit for a nonzero trace of the spin connection (cf. Sec. V), this constraint together with the properties of the spin metric imply that the trace part has to be purely imaginary

$$\mathrm{Re}\,\mathrm{tr}\,\Gamma_\mu = 0. \tag{44}$$

If we span $\Delta\Gamma_\mu$ also by the standard Clifford basis

$$\Delta\Gamma_\mu = \Delta p_\mu\gamma_* + \Delta v_\mu{}^\alpha\gamma_\alpha + \Delta a_\mu{}^\alpha\gamma_*\gamma_\alpha + \Delta t_\mu{}^{\alpha\beta}[\gamma_\alpha, \gamma_\beta] \tag{45}$$

and use the constraints (36) and (43), we conclude that

$$\begin{aligned} &\text{(i) } \Delta p_\mu = 0, \\ &\text{(ii) } \Delta v_{[\alpha\beta]} = 0,\ \Delta v_\mu{}^\alpha \in \mathbb{R}, \\ &\text{(iii) } \Delta a_\alpha{}^\alpha = 0,\ \Delta a_\mu{}^\alpha \in \mathbb{R}, \\ &\text{(iv) } \Delta t_\mu{}^{(\alpha\beta)} = 0,\ \Delta t_\beta{}^{\beta\alpha} = 0,\ \Delta t_\mu{}^{\alpha\beta} \in \mathbb{R}, \end{aligned} \tag{46}$$

where we use $(\ldots)$ to denote complete symmetrization of indices. In summary, this leaves us with 45 real parameters for $\Delta\Gamma_\mu$. It is important to note that the coefficient tensors in Eq. (46) do not change under spin-base transformations, since

$$\Delta\Gamma_\mu \to \mathcal{S}\Delta\Gamma_\mu\mathcal{S}^{-1}, \tag{47}$$

transforms homogeneously in contrast to Γ_μ and $\hat{\Gamma}_\mu$, cf. Eq. (25). Hence, spin-base transformations cannot be employed to transform any of these parameters to zero.

We interpret $\Delta\Gamma_\mu$ as a *spin torsion*. Similarly to general relativity where the torsion becomes visible in the antisymmetric part of $\Gamma^\lambda_{\mu\nu}$, also $\Delta\Gamma_\mu$ is contained in the *antisymmetric* part of the affine connection Γ_μ, cf. Eq. (43)

$$\frac{1}{2}(\Gamma_\mu - \bar{\Gamma}_\mu) = \frac{1}{2}(\hat{\Gamma}_\mu - \bar{\hat{\Gamma}}_\mu) + \Delta\Gamma_\mu, \tag{48}$$

where antisymmetrization is defined in terms of Dirac conjugation. Also the transformation behavior is reminiscent to that of torsion, since it transforms homogeneously under spin-base transformations and coordinate transformations.

In order to illustrate the physical meaning of $\Delta\Gamma_\mu$, let us consider the contribution of this spin torsion to the Dirac operator. Using the identities (valid for $d = d_\gamma = 4$),

$$\gamma_*[\gamma^\alpha, \gamma^\beta] = \frac{\mathrm{i}}{2}\tilde{\varepsilon}^{\alpha\beta\mu\nu}[\gamma_\mu, \gamma_\nu] \tag{49}$$

$$\{\gamma^\mu, [\gamma^\alpha, \gamma^\beta]\} = -4\mathrm{i}\tilde{\varepsilon}^{\mu\alpha\beta\nu}\gamma_*\gamma_\nu, \tag{50}$$

and taking the constraints (46) into account, we find

$$\bar{\psi}\gamma^\mu\Delta\Gamma_\mu\psi = \mathfrak{M}\bar{\psi}\psi - \mathfrak{A}_\mu\bar{\psi}\mathrm{i}\gamma_*\gamma^\mu\psi - \mathfrak{F}_{\mu\nu}\bar{\psi}\frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]\psi, \tag{51}$$

Here we have introduced the intuitive abbreviations

$$\mathfrak{M} = \Delta v_\alpha{}^\alpha \tag{52}$$

$$\mathfrak{A}^\nu = 2\Delta t_{\mu\alpha\beta}\tilde{\varepsilon}^{\mu\alpha\beta\nu} \tag{53}$$

$$\mathfrak{F}_{\mu\nu} = \Delta a^{[\alpha\beta]}\tilde{\varepsilon}_{\alpha\beta\mu\nu}, \tag{54}$$

for a scalar field (spacetime dependent mass) $\mathfrak{M}$, an axial vector field $\mathfrak{A}^\nu$, and an antisymmetric tensor field $\mathfrak{F}_{\mu\nu}$ all of

[4] It is worthwhile to note that Eqs. (37) and (38) actually do not intertwine torsion and the Dirac matrices. Since torsion is naturally contained in the full covariant derivative on the left-hand side, cf. Eqs. (12) and (13), the torsion terms naturally drop out of Eqs. (37) and (38). By contrast, torsion is trivially constrained to vanish in Eq. (39), if the covariant derivative on the left-hand side is assumed to also contain the antisymmetric part of the spacetime affine connection. Hence, it seems that the vierbein formalism could also accommodate torsion, if the vierbein postulate is generalized analogously to Eq. (37).

which have mass dimension one. We conclude that such fields can be accommodated in the spin torsion. They can obviously remain nonzero even in the limit of flat Minkowski space. It is interesting to observe that no pseudoscalar and no vector field, which would complete the possible bilinear fermion structures, occur in Eq. (51). As discussed in Sec. V, a vector field can straightforwardly be accommodated in the trace part of the spin connection.

Out of the 45 parameters of the spin torsion, the fields $\mathfrak{M}$, $\mathfrak{A}^\nu$, and $\mathfrak{F}_{\mu\nu}$ summarize 11 parameters. The remaining 34 can contribute to higher order operators, e.g., involving more covariant derivatives.

This comparatively large number of parameters of the spin torsion can, of course, be further constrained by additional symmetry requirements. For instance, in order to construct a chiral symmetry, we demand for a covariantly constant γ_* which facilitates the construction of covariantly constant chiral projectors,

$$0 = \nabla\!\!\!/\gamma_* = \gamma^\mu[\Delta\Gamma_\mu, \gamma_*] = 2\Delta v_\mu{}^\mu \gamma_* - \Delta a_{\mu\nu}[\gamma^\mu, \gamma^\nu], \tag{55}$$

which implies additional constraints for the spin torsion

$$\Delta v_\mu{}^\mu = 0, \qquad \Delta a_{[\mu\nu]} = 0, \tag{56}$$

leaving 38 free real parameters. In fact, this chiral-symmetry constraint requires only the scalar field $\mathfrak{M}$ and the antisymmetric tensor field $\mathfrak{F}_{\mu\nu}$ to vanish. The axial vector field $\mathfrak{A}^\nu$ (4 parameters) as well as the remaining 34 parameters possibly contributing to higher order operators are left untouched.

To summarize the present section, we now have a covariant derivative of Dirac fermions at our disposal which encodes a parallel transport of a Dirac spinor in curved spacetimes that respects general coordinate invariance as well as local spin-base invariance. More explicitly, given a spinor ψ which transforms as a scalar under coordinate transformations and a vector under spin-base transformations, its covariant derivative can be written as

$$\nabla_\mu \psi = \partial_\mu \psi + \hat{\Gamma}_\mu \psi + \Delta\Gamma_\mu \psi, \tag{57}$$

where $\hat{\Gamma}_\mu$ is fully determined in terms of spacetime dependent Dirac matrices also carrying metric information and $\Delta\Gamma_\mu$ denotes the spin torsion. This is in complete analogy to the covariant derivative of a spacetime vector which can be written in terms of the Levi-Civita connection (determined in terms of the metric) and the spacetime torsion. We would like to emphasize that the spin torsion and the spacetime torsion are mutually independent. They have to be fixed by corresponding external conditions or a corresponding dynamical theory.

IV. LIE DERIVATIVE

The standard Lie derivative $\mathcal{L}_v$ with respect to a vector field v^μ (considered as infinitesimal in the following) is defined by

$$\mathcal{L}_{v_1} v_2{}^\mu = v_1{}^\nu \partial_\nu v_2{}^\mu - v_2{}^\nu \partial_\nu v_1{}^\mu, \tag{58}$$

where $v_2{}^\mu$ is also some vector field. This geometrical structure can be used to implement the statement that Einstein's theory of general relativity is torsionfree. Demanding that the Lie derivative also equals the covariantized right-hand side,

$$\mathcal{L}_{v_1} v_2{}^\mu \overset{!}{=} v_1{}^\nu D_\nu v_2{}^\mu - v_2{}^\nu D_\nu v_1{}^\mu, \tag{59}$$

the torsion has to vanish.

This relation implies that $\Gamma^\lambda_{\mu\nu}$ has to be symmetric in $\mu \leftrightarrow \nu$ and therefore is equal to the Levi-Civita connection. If we wish to apply the same concept to the spin-base covariant derivative in order to exclude spin torsion, we first need a Lie derivative for spinors. In fact this has been a challenge of its own which has been extensively discussed in the literature [21–23].

In the following, we present an independent definition of a generalized Lie derivative for spinors $\tilde{\mathcal{L}}$ which is motivated by the Weldon theorem Eq. (18). Since the metric is encoded in the Dirac matrices in the present spin-base invariant formalism, it is natural to define the generalized Lie derivative in terms of its action on the Dirac matrices. From the Weldon theorem Eq. (18), we know that general Clifford-algebra compatible variations of the Dirac matrices can be decomposed into a metric variation $\delta g^{\mu\nu}$ and an infinitesimal spin-base transformation $\delta\mathcal{S}_\gamma$. As the Lie derivative can be related to infinitesimal diffeomorphisms, the variation of the metric occurring in the Weldon theorem Eq. (18) is given by the ordinary Lie derivative

$$\delta g^{\mu\nu} = \mathcal{L}_v g^{\mu\nu} = -g^{\mu\rho}\partial_\rho v^\nu - g^{\nu\rho}\partial_\rho v^\mu + v^\rho \partial_\rho g^{\mu\nu}. \tag{60}$$

However, in order to compare spinors under a variation of the metric without contributions from local spin-base variations, we keep the spin bases fixed. Hence, we define the generalized Lie derivative in terms of a variation of the Dirac matrices with $\delta\mathcal{S}_\gamma = 0$

$$\tilde{\mathcal{L}}_v \gamma^\mu = \frac{1}{2}(\mathcal{L}_v g^{\mu\nu})\gamma_\nu. \tag{61}$$

This way the Lie derivative gives us the variation of the spinors under diffeomorphisms with fixed spin bases, corresponding to a *comparability* of the in general different spin bases under *different* metrics. Of course we also demand the generalized Lie derivative $\tilde{\mathcal{L}}$ to fulfill a product rule and to coincide with the ordinary Lie derivative $\mathcal{L}$ if the considered object is a scalar under spin-base transformations,

$$\tilde{\mathcal{L}}_v \bar{\psi}\psi = \mathcal{L}_v \bar{\psi}\psi. \tag{62}$$

That justifies the general form of $\tilde{\mathcal{L}}$:

$$\begin{aligned}\tilde{\mathcal{L}}_v \psi &= \mathcal{L}_v \psi + \mathcal{Z}_v \psi,\\ \tilde{\mathcal{L}}_v \bar{\psi} &= \mathcal{L}_v \bar{\psi} - \bar{\psi}\mathcal{Z}_v,\\ \tilde{\mathcal{L}}_v \gamma^\mu &= \mathcal{L}_v \gamma^\mu + [\mathcal{Z}_v, \gamma^\mu]\end{aligned} \tag{63}$$

for some Clifford-algebra valued matrix $\mathcal{Z}_v$. We demand additionally for $\mathcal{Z}_v$ to be traceless,

$$\operatorname{tr}\mathcal{Z}_v = 0. \tag{64}$$

This condition is natural, as any nonzero trace part ∼I would not modify the Dirac matrices and hence leave also the geometry unaffected. Even if $\mathcal{Z}_v$ was not traceless, the trace part would act similar to that of the covariant derivative discussed in the next section and hence carry no independent information. The traceless part of $\mathcal{Z}_v$ can be calculated from Eq. (61), by a comparison with the ordinary Lie derivative of the Dirac matrices which can be derived straightforwardly,

$$\mathcal{L}_v \gamma^\mu = \frac{1}{2}(\mathcal{L}_v g^{\mu\nu})\gamma_\nu - \left[v^\rho \hat{\Gamma}_\rho + \frac{1}{8}(\partial_{[\rho} v_{\lambda]})[\gamma^\rho, \gamma^\lambda], \gamma^\mu\right]. \tag{65}$$

Hence, we can read off

$$\mathcal{Z}_v = v^\rho \hat{\Gamma}_\rho + \frac{1}{16}(\partial_\rho v_\lambda - \partial_\lambda v_\rho)[\gamma^\rho, \gamma^\lambda]. \tag{66}$$

This line of argument leads us to a generalized Lie derivative for Dirac spinors

$$\tilde{\mathcal{L}}_v \psi = v^\rho \partial_\rho \psi + v^\rho \hat{\Gamma}_\rho \psi + \frac{1}{8}(\partial_{[\rho} v_{\lambda]})[\gamma^\rho, \gamma^\lambda]\psi. \tag{67}$$

Now, the geometric argument for eliminating the spin torsion analogous to that of general relativity formulated by Eq. (59) can be completed: the analog requirement in spinor space is to demand our generalized Lie derivative to agree with a spinor-covariantized form:

$$\tilde{\mathcal{L}}_v \psi = v^\rho \nabla_\rho \psi + \frac{1}{8}(\partial_{[\rho} v_{\lambda]})[\gamma^\rho, \gamma^\lambda]\psi. \tag{68}$$

Then we can immediately conclude that

$$v^\rho \Delta\Gamma_\rho = 0, \tag{69}$$

for all (infinitesimal) vectors v^ρ. Therefore, relating the geometrical construction represented by the Lie derivative to the covariant derivative in spinor space implies that the spin torsion has to vanish.

In Eq. (68), we have only covariantized the spinorial part. Alternatively, we could also require the generalized Lie derivative to agree with its fully covariantized form leading to

$$\tilde{\mathcal{L}}_v \psi = v^\rho \nabla_\rho \psi + \frac{1}{8}(D_{[\rho} v_{\lambda]})[\gamma^\rho, \gamma^\lambda]\psi. \tag{70}$$

This requirement relates torsion and spin torsion,

$$\Delta\Gamma_\mu = \frac{1}{8} C_{\mu\rho\lambda}[\gamma^\rho, \gamma^\lambda], \qquad C^\sigma{}_{\sigma\mu} = 0. \tag{71}$$

Read together with Eqs. (v) of (22) and (45), this resembles the form of the spin connection $\Gamma_\mu = \hat{\Gamma}_\mu + \Delta\Gamma_\mu$, known from the vierbein formalism (with torsion replaced by contorsion), but with the additional constraint that the spacetime torsion needs to be traceless.

However, it is important to emphasize that this relation between spin torsion and torsion is only nontrivial, as long as we do not impose the condition (59) for Lie derivatives of vectors. In fact, treating spinors and vectors differently appears unnatural. Hence, imposing the covariantized form also for spacetime vectors, we have

$$\tilde{\mathcal{L}}_v(\bar{\psi}\gamma^\mu\psi) = \mathcal{L}_v(\bar{\psi}\gamma^\mu\psi) = v^\rho D_\rho(\bar{\psi}\gamma^\mu\psi) - (\bar{\psi}\gamma^\rho\psi) D_\rho v^\mu \tag{72}$$

which in combination with Eq. (70) immediately implies that both kinds of torsion have to vanish

$$\begin{aligned}&\text{(i) } C^\rho{}_{\mu\nu} = 0,\\ &\text{(ii) } \Delta\Gamma_\mu = 0.\end{aligned} \tag{73}$$

The fully covariantized form for our generalized Lie derivative in Eq. (70) is identical (up to torsion) to the Kosmann-Lie derivative discussed in the literature [21–23].

V. GAUGE FIELDS

In the preceding sections, we have set a possible trace part of the spin connection Γ_μ to zero, as such a trace part proportional to the identity in Dirac space ∼I does not transform the Dirac matrices nontrivially, cf. (i) of Eq. (8). If we allow for this generalization, the symmetry group can be extended to $\mathcal{G} \otimes \mathrm{SL}(d_\gamma, \mathbb{C})$, where $\mathcal{G}$ denotes the symmetry group of the trace part. The Clifford algebra is, of course, also invariant under this larger group, since the Dirac matrices and thus the geometry do not transform under $\mathfrak{g} \in \mathcal{G}$.

To construct a connection $\Gamma_{(\mathcal{G}\otimes\mathrm{SL})_\mu}$ for this larger group, we consider symmetry transformations $\mathfrak{g} \otimes \mathcal{S} \in \mathcal{G} \otimes \mathrm{SL}(d_\gamma, \mathbb{C})$ and find analogously to Eq. (25)

$$\begin{aligned}\Gamma_{(\mathcal{G}\otimes\mathrm{SL})_\mu} \to\ &\mathfrak{g} \otimes \mathcal{S}\Gamma_{(\mathcal{G}\otimes\mathrm{SL})_\mu}(\mathfrak{g} \otimes \mathcal{S})^{-1}\\ &- (\partial_\mu(\mathfrak{g} \otimes \mathcal{S}))(\mathfrak{g} \otimes \mathcal{S})^{-1}\end{aligned} \tag{74}$$

as the transformation property of the spin connection. Here we can use the product rule for the derivative and expand the inhomogeneous part,

$$(\partial_\mu(\mathfrak{g}\otimes\mathcal{S}))(\mathfrak{g}\otimes\mathcal{S})^{-1} = ((\partial_\mu\mathfrak{g})\mathfrak{g}^{-1})\otimes \mathrm{I} + \mathrm{I}_{(\mathcal{G})}\otimes((\partial_\mu\mathcal{S})\mathcal{S}^{-1}), \quad (75)$$

where $\mathrm{I}_{(\mathcal{G})}$ is the unit element of $\mathcal{G}$. Because of this behavior, it is sufficient to consider connections with the property

$$\Gamma_{(\mathcal{G}\otimes\mathrm{SL})_\mu} = \Gamma_{(\mathcal{G})_\mu}\otimes \mathrm{I} + \mathrm{I}_{(\mathcal{G})}\otimes\Gamma_\mu, \quad (76)$$

where $\Gamma_{(\mathcal{G})_\mu}$ is the connection for the group $\mathcal{G}$ and Γ_μ is the traceless connection for the $\mathrm{SL}(d_\gamma,\mathbb{C})$ part, i.e. the Γ_μ from the previous sections. Obviously, the Dirac trace part of $\Gamma_{(\mathcal{G}\otimes\mathrm{SL})_\mu}$ accommodates the connection for the group $\mathcal{G}$.

Similarly, a straightforward generalization of the spin metric suggests the form

$$h_{(\mathcal{G}\otimes\mathrm{SL})} = \mathrm{I}_{(\mathcal{G})}\otimes h, \quad (77)$$

with the corresponding transformation law

$$h_{(\mathcal{G}\otimes\mathrm{SL})}\to(\mathfrak{g}^\dagger\otimes\mathcal{S}^\dagger)^{-1}h_{(\mathcal{G}\otimes\mathrm{SL})}(\mathfrak{g}\otimes\mathcal{S})^{-1} \quad (78)$$

under a $\mathfrak{g}\otimes\mathcal{S}$ transformation. Requiring the transformation (78) to preserve Eq. (77), the elements of $\mathcal{G}$ need to be unitary

$$\mathfrak{g}^{-1} = \mathfrak{g}^\dagger. \quad (79)$$

If we now additionally demand for metric compatibility, Eq. (27), we get

$$\mathrm{I}_{(\mathcal{G})}\otimes(h^{-1}(\partial_\mu h)) = \Gamma_{(\mathcal{G}\otimes\mathrm{SL})_\mu} + \mathrm{I}_{(\mathcal{G})}\otimes h^{-1}\Gamma^\dagger_{(\mathcal{G}\otimes\mathrm{SL})_\mu}\mathrm{I}_{(\mathcal{G})}\otimes h, \quad (80)$$

from which we deduce with regard to Eq. (76) that the connection of $\mathcal{G}$ needs to be anti-Hermitian

$$\Gamma^\dagger_{(\mathcal{G})_\mu} = -\Gamma_{(\mathcal{G})_\mu}. \quad (81)$$

Here we also used Eq. (43). This justifies the introduction of the gauge field $\mathcal{A}_\mu$

$$\Gamma_{(\mathcal{G})_\mu} = \mathrm{i}\mathcal{A}_\mu \quad (82)$$

which is associated with the $\mathcal{G}$ symmetry. This field can in general be non-Abelian but is always Hermitian as is conventional in ordinary gauge field theory.

To summarize, the inclusion of a trace part in the spin connection Γ_μ can be viewed as an extension of the symmetry group from $\mathrm{SL}(d_\gamma,\mathbb{C})$ to $\mathcal{G}\otimes\mathrm{SL}(d_\gamma,\mathbb{C})$, with $\mathcal{G}$ being a unitary group. The spin connection can then be decomposed as

$$\Gamma_{(\mathcal{G}\otimes\mathrm{SL})_\mu} = \mathrm{i}\mathcal{A}_\mu\otimes\mathrm{I} + \mathrm{I}_{(\mathcal{G})}\otimes(\hat{\Gamma}_\mu + \Delta\Gamma_\mu), \quad (83)$$

or in short

$$\Gamma_\mu = \mathrm{i}\mathcal{A}_\mu + \hat{\Gamma}_\mu + \Delta\Gamma_\mu, \quad (84)$$

as it is understood and used in the following. Within the physical context of fermions in curved space, the $\mathrm{SL}(d_\gamma,\mathbb{C})$ part of the connection is always present in covariant derivatives of spinor fields, since it carries the information about how fermions evolve dynamically in a given curved space. By contrast, the gauge part of the connection may or may not be present depending on whether a fermion is charged under the group $\mathcal{G}$. Technically, the distinction among differently charged fermions may be parametrized by a charge matrix as a factor inside $\mathcal{A}_\mu$.

VI. SPIN CURVATURE

From our knowledge about the spinor covariant derivative and the associated spin connection, it is immediate to construct a curvature or field strength which we denote by spin curvature for short. Again, we motivate the definition for this spin curvature by analogy to the standard definition of the curvature tensor in general relativity (including torsion) [16],

$$R_{\mu\nu}{}^\lambda{}_\rho T^\rho = [D_\mu, D_\nu]T^\lambda + C^\sigma{}_{\mu\nu}D_\sigma T^\lambda, \quad \forall\, T^\rho \text{ tensor}. \quad (85)$$

This suggests the definition of the spin curvature $\Phi_{\mu\nu}$,

$$\Phi_{\mu\nu}\psi = [\nabla_\mu,\nabla_\nu]\psi + C^\sigma{}_{\mu\nu}\nabla_\sigma\psi. \quad (86)$$

More explicitly, it is given by

$$\Phi_{\mu\nu} = \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu + [\Gamma_\mu,\Gamma_\nu] \quad (87)$$

$$= \mathrm{i}\mathcal{F}_{\mu\nu} + \hat{\Phi}_{\mu\nu} + 2\partial_{[\mu}\Delta\Gamma_{\nu]} + 2[\hat{\Gamma}_{[\mu},\Delta\Gamma_{\nu]}] + [\Delta\Gamma_\mu,\Delta\Gamma_\nu], \quad (88)$$

where $\mathcal{F}_{\mu\nu}$ is the field strength tensor of the gauge field

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu,\mathcal{A}_\nu]. \quad (89)$$

The quantity $\hat{\Phi}_{\mu\nu}$ is the spin curvature induced by $\hat{\Gamma}_\mu$

$$\hat{\Phi}_{\mu\nu} = \partial_\mu\hat{\Gamma}_\nu - \partial_\nu\hat{\Gamma}_\mu + [\hat{\Gamma}_\mu,\hat{\Gamma}_\nu], \quad (90)$$

which can be related to the curvature tensor $R_{(\mathrm{LC})_{\mu\nu}}{}^\lambda{}_\rho$ defined in terms of the Levi-Civita connection $D_{(\mathrm{LC})_\mu}$ by the following observation:

$$R_{(\mathrm{LC})\mu\nu}{}^{\lambda}{}_{\rho}\gamma^{\rho} = [D_{(\mathrm{LC})\mu}, D_{(\mathrm{LC})\nu}]\gamma^{\lambda} = -[\hat{\Phi}_{\mu\nu}, \gamma^{\lambda}]. \tag{91}$$

This demonstrates the correspondence between the spin-torsion/torsionfree curvature expressions:

$$\hat{\Phi}_{\mu\nu} = \frac{1}{8} R_{(\mathrm{LC})\mu\nu\lambda\rho}[\gamma^{\lambda}, \gamma^{\rho}], \tag{92}$$

where

$$R_{(\mathrm{LC})\mu\nu}{}^{\lambda}{}_{\rho} = \partial_{\mu}\left\{ \begin{array}{c} \lambda \\ \nu\rho \end{array} \right\} - \partial_{\nu}\left\{ \begin{array}{c} \lambda \\ \mu\rho \end{array} \right\} + \left\{ \begin{array}{c} \lambda \\ \mu\sigma \end{array} \right\}\left\{ \begin{array}{c} \sigma \\ \nu\rho \end{array} \right\} - \left\{ \begin{array}{c} \lambda \\ \nu\sigma \end{array} \right\}\left\{ \begin{array}{c} \sigma \\ \mu\rho \end{array} \right\}. \tag{93}$$

These results together with the explicit representation (88) make it clear that the spin curvature does not carry information about the whole spacetime structure, since no spacetime-torsion dependent terms occur. Only spin torsion appears in Eq. (88) which is in line with the fact that also a covariant derivative acting on a Dirac spinor does not depend on spacetime torsion but only on spin torsion, cf. Eq. (57). A direct coupling between spinor degrees of freedom and spacetime torsion therefore requires ad-hoc higher-order coupling terms or higher-spin fields such as Rarita-Schwinger spinors, see below.

As a simple application of the spin curvature, let us construct the simplest classical field theory that can be formed out of the spin curvature. Since $\Phi_{\mu\nu}$ is Clifford-algebra valued, there exists already a spin base and diffeomorphism invariant quantity to linear order in the spin curvature. The simplest classical action thus is

$$S_{\Phi} = \frac{1}{16\pi G}\int_x \mathfrak{L}_{\Phi}, \qquad \mathfrak{L}_{\Phi} = \frac{2}{d_{\mathcal{A}} d_{\gamma}} \operatorname{tr}(\gamma^{\mu}\Phi_{\mu\nu}\gamma^{\nu}), \tag{94}$$

where G is a coupling constant, and $d_{\mathcal{A}}$ is the dimension of the representation of the gauge group $d_{\mathcal{A}} = \operatorname{tr} \mathrm{I}_{(\mathcal{G})}$. We set $d_{\mathcal{A}} = 1$ in the absence of any gauge group. The content of this field theory can be worked out more explicitly, using the identities

$$D_{(\mathrm{LC})\mu}\Delta\Gamma_{\nu} = (D_{(\mathrm{LC})\mu}\Delta v_{\nu}{}^{\alpha})\gamma_{\alpha} + (D_{(\mathrm{LC})\mu}\Delta a_{\nu}{}^{\alpha})\gamma_{*}\gamma_{\alpha} + (D_{(\mathrm{LC})\mu}\Delta t_{\nu}{}^{\alpha\beta})[\gamma_{\alpha}, \gamma_{\beta}] - [\hat{\Gamma}_{\mu}, \Delta\Gamma_{\mu}], \tag{95}$$

$$0 = \operatorname{tr}([\gamma^{\alpha}, \gamma^{\beta}]) = \operatorname{tr}(\gamma^{\mu}[\gamma^{\alpha}, \gamma^{\beta}]) = \operatorname{tr}(\gamma_{*}\gamma^{\mu}[\gamma^{\alpha}, \gamma^{\beta}]), \tag{96}$$

$$\operatorname{tr}([\gamma^{\mu}, \gamma^{\nu}][\gamma^{\alpha}, \gamma^{\beta}]) = 4d_{\gamma}(g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta}). \tag{97}$$

The Lagrangian reads in terms of the Levi-Civita curvature and the spin torsion coefficients

$$\mathfrak{L}_{\Phi} = R_{(\mathrm{LC})} + 4(\Delta v_{\mu}{}^{\mu})^2 - 4\Delta v_{\mu}{}^{\nu}\Delta v_{\nu}{}^{\mu} + 4\Delta a_{\mu}{}^{\nu}\Delta a_{\nu}{}^{\mu} + 64\Delta t_{\mu\nu\kappa}\Delta t^{\kappa\nu\mu}, \tag{98}$$

where $R_{(\mathrm{LC})} = R_{(\mathrm{LC})\mu\nu}{}^{\mu\nu}$. This action is rather similar to the (torsion-amended) Einstein-Hilbert action

$$S_R = \frac{1}{16\pi G}\int_x R,$$
$$R = R_{\mu\nu}{}^{\mu\nu},$$
$$R = R_{(\mathrm{LC})} + 2D_{(\mathrm{LC})\mu}K^{\mu}{}_{\nu}{}^{\nu} - K^{\rho}{}_{\nu}{}^{\nu}K_{\rho\mu}{}^{\mu} + K^{\rho\mu\nu}K_{\rho\nu\mu}, \tag{99}$$

which differs from Eq. (98) only in the torsion terms. Obviously, $\mathfrak{L}_{\Phi}$ cannot depend on spacetime torsion as Φ is blind to spacetime torsion as well.

This simple observation offers a speculative though interesting perspective: if classical GR was based on Eq. (94) (and possibly supplemented by higher order monomials of $\Phi_{\mu\nu}$) instead of Eq. (99), the absence of spacetime torsion in classical GR would be a natural self-evident consequence.

Instead, S_{Φ} confronts us with the presence of spin torsion terms in Eq. (98). In this simplest field theory, however, the spin torsion terms occur only algebraically, implying that the torsion fields remain nondynamical and satisfy particularly simple equations of motion.

Varying the action with respect to the fields $\Delta v_{\mu}{}^{\nu}$, $\Delta a_{\mu}{}^{\nu}$ and $\Delta t_{\mu}{}^{\alpha\beta}$, taking into account the constraints (46), we find

$$\delta\Delta v^{\mu\nu} = \frac{1}{2}(\delta^{\mu}_{\rho}\delta^{\nu}_{\lambda} + \delta^{\nu}_{\rho}\delta^{\mu}_{\lambda})\delta\Delta v^{\rho\lambda}, \tag{100}$$

$$\delta\Delta a^{\mu\nu} = \left[\delta^{\mu}_{\rho}\delta^{\nu}_{\lambda} - \frac{1}{4}g^{\mu\nu}g_{\rho\lambda}\right]\delta\Delta a^{\rho\lambda}, \tag{101}$$

$$\delta\Delta t^{\mu\alpha\beta} = \left[\delta^{\mu}_{\rho}\delta^{[\alpha}_{\lambda}\delta^{\beta]}_{\sigma} - \frac{1}{3}g_{\rho\lambda}g^{\mu[\alpha}\delta^{\beta]}_{\sigma} + \frac{1}{3}g_{\rho\sigma}g^{\mu[\alpha}\delta^{\beta]}_{\lambda}\right]\delta\Delta t^{\rho\lambda\sigma}. \tag{102}$$

Hence, the variations of the action yield

$$\frac{\delta S_{\Phi}}{\delta\Delta v^{\mu\nu}} = \frac{1}{2\pi G}((\Delta v_{\kappa}{}^{\kappa})g_{\mu\nu} - \Delta v_{\mu\nu}), \tag{103}$$

$$\frac{\delta S_{\Phi}}{\delta\Delta a^{\mu\nu}} = \frac{1}{2\pi G}\Delta a_{\mu\nu}, \tag{104}$$

$$\frac{\delta S_{\Phi}}{\delta\Delta t^{\mu\alpha\beta}} = \frac{8}{\pi G}\Delta t_{[\mu\alpha]\beta}. \tag{105}$$

Imposing an action principle $\delta S_{\Phi} = 0$ this requires the spin torsion to vanish in the absence of sources or boundary conditions for this simplest classical theory. The resulting theory is identical to classical general relativity.

We conclude this section with two additional remarks: First, a different definition of spin curvature would be suggested in the presence of Rarita-Schwinger spinors ψ^λ, being a first-rank tensor in spacetime as well as in Dirac space. In analogy to Eq. (85), we would define

$$\Phi_{\mu\nu}{}^{\lambda}{}_{\rho}\psi^{\rho} = [\nabla_\mu, \nabla_\nu]\psi^\lambda + C^{\sigma}{}_{\mu\nu}\nabla_\sigma\psi^\lambda. \tag{106}$$

This spin curvature can be decomposed into

$$\Phi_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho}\mathrm{I} + \Phi_{\mu\nu}g_{\lambda\rho}. \tag{107}$$

Here, the spacetime curvature tensor $R_{\mu\nu\lambda\rho}$ appears as the antisymmetric part of $\Phi_{\mu\nu\lambda\rho}$ in $\lambda\leftrightarrow\rho$ and the previous spin curvature $\Phi_{\mu\nu}g_{\lambda\rho}$ arises as the symmetric term. Forming suitable first order invariants of this spin curvature, we end up with actions of Einstein-Hilbert type including both spacetime and spin curvature. In the spirit of the speculative interpretation given above, the absence of spacetime torsion in our universe would fit well to a nonexistence of fundamental Rarita-Schwinger fields.

For our second remark, we disregard any torsion such that $\Phi_{\mu\nu} \to \mathrm{i}\mathcal{F}_{\mu\nu} + \hat{\Phi}_{\mu\nu}$. In this case, the second-order invariant of the spin curvature which is reminiscent to the kinetic term of a gauge theory reduces to

$$\frac{1}{d_{\mathcal{A}}d_\gamma}\,\mathrm{tr}\Phi_{\mu\nu}\Phi^{\mu\nu} \to -\frac{1}{d_{\mathcal{A}}}\,\mathrm{tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{8}R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}. \tag{108}$$

Naively, this seems to suggest that a gauge-gravity field theory links the coupling to the gauge fields to that of higher-order curvature terms. However, this connection can, of course, simply be broken explicitly by additional $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ terms in the action which are not part of a $\Phi_{\mu\nu}\Phi^{\mu\nu}$ term.

VII. REDUCIBLE REPRESENTATIONS

So far, our considerations have been based on the irreducible representation of the Clifford algebra characterized by $d_\gamma = 4$ in four spacetime dimensions. A generalization of our formalism to reducible representations is not completely trivial, since the construction of the spin connection makes explicit use of a particular complete basis of the Clifford algebra. The basis used above may not generalize straightforwardly to any reducible representation. Therefore, we confine ourselves to those reducible representations where the basis used so far is still sufficient.

Our construction leads to reducible representations with $d_\gamma = 4n$, for $n \in \mathbb{N}$. For this, we assume that the new Dirac matrices can be written as a tensor product of a possibly spacetime dependent matrix $A \in \mathbb{C}^{n\times n}$ of dimension n and the Dirac matrices $\gamma^\mu \in \mathbb{C}^{4\times 4}$ of the irreducible representation used above,

$$\gamma_{(d_\gamma)}{}^{\mu} = A \otimes \gamma^\mu, \tag{109}$$

obviously implying that $d_\gamma = 4n$. Of course, the set of $\gamma_{(d_\gamma)}{}^{\mu}$ shall also satisfy the Clifford algebra

$$\{\gamma_{(d_\gamma)}{}^{\mu}, \gamma_{(d_\gamma)}{}^{\nu}\} = 2g^{\mu\nu}\mathrm{I}_{(d_\gamma)} = 2g^{\mu\nu}\mathrm{I}_{(n)} \otimes \mathrm{I} \tag{110}$$

which tells us that A is idempotent.

$$A^2 = \mathrm{I}_{(n)} \tag{111}$$

has to hold at any spacetime point. Analogous to our previous construction, we need a covariant derivative $\nabla_{(d_\gamma)\mu}$ and a spin metric $h_{(d_\gamma)}$. We require the covariant derivative to factorize accordingly,

$$\nabla_{(d_\gamma)\mu}A \otimes \gamma^\nu = (\nabla_{(n)\mu}A) \otimes \gamma^\nu + A \otimes (\nabla_\mu\gamma^\nu), \tag{112}$$

where $\nabla_{(n)\mu}$ acts on the "A-part" and ∇_μ is identical to the covariant derivative in irreducible representation. Analogously to Eq. (26), we also demand for

$$\nabla_{(d_\gamma)\mu}A \otimes \gamma^\nu = D_\mu A \otimes \gamma^\nu + [\Gamma_{(d_\gamma)\mu}, A \otimes \gamma^\nu], \tag{113}$$

which tells us that the affine connection has to read

$$\Gamma_{(d_\gamma)\mu} = \Gamma_{(n)\mu} \otimes \mathrm{I} + \mathrm{I}_{(n)} \otimes \Gamma_\mu. \tag{114}$$

Because the irreducible component already carries all relevant structures for general covariance, the A-part in its simplest form should be covariantly constant,

$$0 = \nabla_{(n)\mu}A = \partial_\mu A + [\Gamma_{(n)\mu}, A]. \tag{115}$$

We can rewrite this into a condition for the connection $\Gamma_{(n)\mu}$ which has to satisfy

$$\Gamma_{(n)\mu} = A\Gamma_{(n)\mu}A^{-1} - (\partial_\mu A)A^{-1}. \tag{116}$$

For a given choice of A on a given spacetime, Eq. (116) may or may not have a solution in terms of a set of $\Gamma_{(n)\mu}$. If a solution exists, it completes the definition of the spin connection for this reducible representation. For the simpler case of constant matrices A, a solution is always given by $\Gamma_{(d_\gamma)\mu} = 0$.

The natural way to embed the spin-base transformations is given by the form

$$\mathcal{S}_{(d_\gamma)} = \mathrm{I}_{(n)} \otimes \mathcal{S}, \qquad \mathcal{S} \in \mathrm{SL}(4, \mathbb{C}). \tag{117}$$

The corresponding transformation law for the spin connection then reads

$$\Gamma_{(d_\gamma)\mu} \to \mathcal{S}_{(d_\gamma)}\Gamma_{(d_\gamma)\mu}\mathcal{S}_{(d_\gamma)}{}^{-1} - (\partial_\mu \mathcal{S}_{(d_\gamma)})\mathcal{S}_{(d_\gamma)}{}^{-1} \\ = \Gamma_{(d_\gamma)\mu} \otimes \mathrm{I} + \mathrm{I}_{(n)} \otimes (\mathcal{S}\Gamma_\mu \mathcal{S}^{-1} - (\partial_\mu \mathcal{S})\mathcal{S}^{-1}). \tag{118}$$

It is worthwhile to emphasize that the choice of the embedding (117) is not unique. Reducible representations of the Clifford algebra have a much larger symmetry of $\mathrm{SL}(d_\gamma = 4n, \mathbb{C})$, such that there are typically many more options of embedding $\mathrm{SL}(4, \mathbb{C})$ into $\mathrm{SL}(d_\gamma = 4n, \mathbb{C})$. The present choice is motivated by the similarity to the embedding of local Lorentz transformations that we would encounter in the corresponding vierbein formalism. Vierbeins transform under these Lorentz transformations as

$$e^\mu{}_a \to e^\mu{}_b \Lambda^b{}_a, \tag{119}$$

corresponding on the level of Dirac matrices to

$$\gamma_{(e)}{}^\mu \to \mathcal{S}_{\mathrm{Lor}} \gamma_{(e)}{}^\mu \mathcal{S}_{\mathrm{Lor}}{}^{-1}. \tag{120}$$

The matrix $\mathcal{S}_{\mathrm{Lor}}$ is given by

$$\mathcal{S}_{\mathrm{Lor}} = \exp\left(\frac{\eta_{ac}\omega^c{}_b}{8}[\gamma_{(\mathrm{f})}{}^a, \gamma_{(\mathrm{f})}{}^b]\right), \tag{121}$$

where the matrix $(\omega^a{}_b)$ is defined by

$$\Lambda^a{}_b = (\mathrm{e}^\omega)^a{}_b. \tag{122}$$

Promoting the (fixed) Dirac matrices to the reducible representation given above, the corresponding Lorentz transformation reads

$$\mathcal{S}_{\mathrm{Lor}(d_\gamma)} = \exp\left(\frac{\eta_{ac}\omega^c{}_b}{8}[A \otimes \gamma_{(\mathrm{f})}{}^a, A \otimes \gamma_{(\mathrm{f})}{}^b]\right) \equiv \mathrm{I}_n \otimes \mathcal{S}_{\mathrm{Lor}}, \tag{123}$$

which is structurally identical to our choice for the embedding of Eq. (117).

Finally, we also need the spin metric for the reducible representation, which has to satisfy

$$\gamma_{(d_\gamma)}{}^\dagger_\mu = -h_{(d_\gamma)}\gamma_{(d_\gamma)\mu} h_{(d_\gamma)}{}^{-1}. \tag{124}$$

It is obvious that this condition is satisfied by

$$h_{(d_\gamma)} = A \otimes h, \quad A^\dagger = A, \tag{125}$$

demanding that A is Hermitian in order to have $h_{(d_\gamma)}$ anti-Hermitian. Of course also the absolute value of the determinant is equal to one as required, since

$$|\det h_{(d_\gamma)}| = \sqrt{|\det A \otimes h|^2} = \sqrt{|\det \mathrm{I}_{(n)} \otimes h^2|} = 1. \tag{126}$$

This completes the construction of a generalization to particularly simple reducible representations of the Dirac algebra.

Again, the embedding (125) may not be unique. The present choice is intuitive, because in conventional choices for the flat spacetime Dirac matrices, the spin metric is simply given by $\gamma_{(\mathrm{f})0}$. In the corresponding reducible representation, the "new" $\gamma_{(d_\gamma)0}$ would read

$$\gamma_{(d_\gamma)0} = A \otimes \gamma_{(\mathrm{f})0}, \tag{127}$$

matching precisely with our extended spin metric.

Let us emphasize again that the straightforwardly induced symmetries of the present construction may not exhaust the full invariance of the reducible Clifford algebra. For instance, one can immediately verify that our construction is invariant under local $\mathrm{SU}(n) \otimes \mathrm{SL}(4, \mathbb{C})$ transformations, which is in general only a subgroup of the $\mathrm{SL}(d_\gamma, \mathbb{C})$ invariance of the Clifford algebra in reducible representation.

VIII. PATH INTEGRAL

As an application of the spin-base invariant formalism, let us discuss possible implications for quantizing gravity within a path integral framework. Of course, the question as to whether such a path integral exists is far from being settled. For the purpose of the following discussion, we simply assume that there is such a path integral possibly regularized in a symmetry-preserving way and possibly amended with a suitable gauge fixing procedure. For simplicity, we consider the case of vanishing spin torsion, spacetime torsion and gauge fields

$$\Delta\Gamma_\mu = 0, \quad C^\kappa{}_{\mu\nu} = 0, \quad \mathcal{A}_\mu = 0, \tag{128}$$

even though the following considerations will not interfere with any of these quantities. Also, we work manifestly in $d = 4$ where $d_\gamma = 4$.

So far, we took the viewpoint that the spacetime-dependent Dirac matrices γ_μ are the basic objects encoding the essential properties of the spacetime. In fact, given a set of Dirac matrices, we can compute the metric,

$$g_{\mu\nu} = \frac{1}{4}\mathrm{tr}(\gamma_\mu \gamma_\nu). \tag{129}$$

Also the spin metric necessary for including fermionic Dirac degrees of freedom is fixed (up to a sign) by the condition

$$h^\dagger = -h, \qquad \bar{\gamma}^\mu = -\gamma^\mu, \qquad |\det h| = 1, \tag{130}$$

see Appendix A and Eqs. (40)–(42). The Dirac matrices also determine the spin connection (up to spin torsion), cf. Eq. (22), and all these ingredients suffice to define a classical theory of gravity including dynamical fermions. One is hence tempted to base a quantized theory also on the Dirac matrices as the fundamental degree of freedom. This would be analogous to quantizing gravity in terms of a vierbein. Whereas this is certainly a valid and promising option, we show in the following that this Dirac matrix/vierbein quantization is actually not necessary.

Demanding that quantization preserves the local Clifford algebra constraint also off shell

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathrm{I}, \qquad \gamma^\mu \in \mathbb{C}^{4\times 4}, \tag{131}$$

(for a correspondingly off-shell metric), the Weldon theorem (18) already tells us that a fluctuation of the Dirac matrices can always be decomposed into a metric fluctuation and an $\mathrm{SL}(4,\mathbb{C})_\gamma$ fluctuation,

$$\delta\gamma^\mu = \frac{1}{2}(\delta g^{\mu\nu})\gamma_\nu + [\delta\mathcal{S}_\gamma, \gamma^\mu]. \tag{132}$$

Hence, we do not attempt to construct an integration measure for Dirac matrices "$\mathcal{D}\gamma$," satisfying the Dirac algebra constraint. Instead, it appears more natural to integrate over metrics and $\mathrm{SL}(4,\mathbb{C})_\gamma$ fluctuations. In the following, we show that the $\mathrm{SL}(4,\mathbb{C})_\gamma$ fluctuations factor out of the path integral because of spin-base invariance, such that a purely metric-based quantization scheme appears sufficient also in the presence of dynamical fermions.

The crucial starting point of our line of argument is the fact that all possible sets of Dirac matrices compatible with a given metric are connected with each other via $\mathrm{SL}(4,\mathbb{C})_\gamma$ transformations [24]. This means that we can cover the space of Dirac matrices by (i) choosing an arbitrary mapping $\tilde{\gamma}^\mu$ of the metric into the space of Dirac matrices satisfying the Clifford algebra

$$g_{\mu\nu} \to \tilde{\gamma}_\mu = \tilde{\gamma}_\mu(g), \tag{133}$$

and (ii) performing $\mathrm{SL}(4,\mathbb{C})_\gamma$ transformations $\mathrm{e}^{\mathcal{M}}$ of this mapping

$$\gamma_\mu(g) = \gamma_\mu(\tilde{\gamma}(g), \mathcal{M}(g)) = \mathrm{e}^{\mathcal{M}(g)}\tilde{\gamma}_\mu(g)\mathrm{e}^{-\mathcal{M}(g)} \tag{134}$$

where $\mathcal{M}$ is an arbitrary tracefree matrix which can be spanned by the generators of $\mathrm{SL}(4,\mathbb{C})_\gamma$ transformations. This matrix $\mathcal{M}$ may even depend on the metric if we demand $\gamma_\mu(g)$ to be a particular Dirac matrix compatible with the Clifford algebra independently of the choice of the representative Dirac matrix $\tilde{\gamma}_\mu$.

Equation (134) emphasizes the fact that every possible set of Dirac matrices yielding a given metric $g_{\mu\nu}$ can be constructed by this mapping.

The variation of the resulting Dirac matrices under an infinitesimal variation in terms of the metric $\delta g_{\mu\nu}$ can be represented analogously to the Weldon theorem:

$$\delta\gamma_\mu = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + [G^{\rho\lambda}\delta g_{\rho\lambda}, \gamma_\mu], \tag{135}$$

where the tensor $G_{\rho\lambda}$ is tracefree and depends on the actual choice of $\tilde{\gamma}_\mu(g)$ and $\mathcal{M}(g)$. $G^{\rho\lambda}$ can be calculated from

$$[[G^{\rho\lambda}, \gamma_\mu], \gamma^\mu] = \left[\frac{\partial\gamma_\mu(g)}{\partial g_{\rho\lambda}}, \gamma^\mu\right]. \tag{136}$$

The infinitesimal $\mathrm{SL}(4,\mathbb{C})_\gamma$ fluctuation $\delta\mathcal{S}_\gamma$ acting on the Dirac matrices, as it occurs in the Weldon theorem, is obviously given by

$$\delta\mathcal{S}_\gamma = G^{\rho\lambda}\delta g_{\rho\lambda}. \tag{137}$$

Now, the microscopic actions subject to quantization are considered to be functionals of the fermions and the Dirac matrices, $S[\psi, \bar{\psi}, \gamma]$. From our construction given above, the Dirac matrices arise from a representative Dirac matrix $\tilde{\gamma}^\mu(g)$ which is related to the metric by an arbitrary but fixed *bijection*, $g_{\mu\nu} \leftrightarrow \tilde{\gamma}^\mu$. The Dirac matrix γ^μ occurring in the action is then obtained via the $\mathrm{SL}(4,\mathbb{C})_\gamma$ transformation governed by $\mathcal{M}$, cf. Eq. (134). Therefore, it is useful to think of the action as a functional of the metric and of $\mathcal{M}$, $S[\psi, \bar{\psi}, g; \mathcal{M}]$. In particular, the freedom to choose $\mathcal{M}$ [or the corresponding $\mathrm{SL}(4,\mathbb{C})_\gamma$ group element] guarantees that the space of all possible Dirac matrices compatible with a given metric can be covered—for any choice of the representative $\tilde{\gamma}^\mu(g)$.

In addition to diffeomorphism invariance, we demand that the actions under consideration are invariant under spin-base transformations

$$S[\psi, \bar{\psi}, g; \mathcal{M}] \to S[\mathcal{S}\psi, \bar{\psi}\mathcal{S}^{-1}, g; \ln(\mathcal{S}\mathrm{e}^{\mathcal{M}})] \equiv S[\psi, \bar{\psi}, g; \mathcal{M}]. \tag{138}$$

Especially we may always choose

$$\mathcal{S} = \mathrm{e}^{-\mathcal{M}}, \tag{139}$$

such that

$$S[\psi, \bar{\psi}, g; \mathcal{M}] = S[\psi', \bar{\psi}', g; 0], \;\; \psi' = \mathrm{e}^{-\mathcal{M}}\psi, \;\; \bar{\psi}' = \bar{\psi}\mathrm{e}^{\mathcal{M}}. \tag{140}$$

The essential ingredient for a path integral quantization is the choice of the measure. As argued above, the present construction suggests, to integrate over metrics g and successively over $\mathcal{M}$ to cover the space of all Dirac matrices.

More specifically, let us study the expectation value of an operator $\hat{O}(\psi, \bar{\psi}, g; \mathcal{M})$ which is a scalar under spin-base transformations. For illustrative purposes, let us first consider only the functional integrations over the fermion and metric degrees of freedom:

$$O[\mathcal{M}] = \langle \hat{O}(\psi, \bar{\psi}, g; \mathcal{M}) \rangle \tag{141}$$

$$= \int \mathcal{D}g\mathcal{D}\psi\mathcal{D}\bar{\psi}\, \hat{O}(\psi, \bar{\psi}, g; \mathcal{M}) \mathrm{e}^{\mathrm{i}S[\psi,\bar{\psi},g;\mathcal{M}]}, \tag{142}$$

with suitable measures $\mathcal{D}g\mathcal{D}\psi\mathcal{D}\bar{\psi}$. The following argument only requires that the measure transforms in a standard manner under a *change of variables*

$$\mathcal{D}\psi = \mathcal{D}\psi' \left(\det \frac{\delta\psi'}{\delta\psi} \right)^{-1}. \tag{143}$$

As a consequence, $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ is invariant under spin-base transformations, since the Jacobians from $\mathcal{D}\psi$ and from $\mathcal{D}\bar{\psi}$ are inverse to each other

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \mathcal{D}(\mathcal{S}\psi)\mathcal{D}(\bar{\psi}\mathcal{S}^{-1}). \tag{144}$$

Because $\hat{O}$ is a scalar in Dirac space, it also needs to be invariant under spin-base transformations

$$\hat{O}(\psi, \bar{\psi}, g; \mathcal{M}) \to \hat{O}(\mathcal{S}\psi, \bar{\psi}\mathcal{S}^{-1}, g; \ln(\mathcal{S}\mathrm{e}^{\mathcal{M}})) \equiv \hat{O}(\psi, \bar{\psi}, g; \mathcal{M}). \tag{145}$$

Now it is easy to see that $O[\mathcal{M}]$ is actually independent of the choice of $\mathcal{M}(g)$

$$\begin{aligned} O[\mathcal{M}] &= \int \mathcal{D}g\mathcal{D}\psi\mathcal{D}\bar{\psi}\, \hat{O}(\psi, \bar{\psi}, g; \mathcal{M}) \mathrm{e}^{\mathrm{i}S[\psi,\bar{\psi},g;\mathcal{M}]} \\ &= \int \mathcal{D}g\mathcal{D}\psi'\mathcal{D}\bar{\psi}'\, \hat{O}(\psi', \bar{\psi}', g; 0) \mathrm{e}^{\mathrm{i}S[\psi',\bar{\psi}',g;0]} \\ &= O[0]. \end{aligned} \tag{146}$$

Therefore, every set of Dirac matrices compatible with a given metric contributes identically to such an expectation value. Hence, we may choose any convenient spin basis to simplify explicit computations. From another viewpoint, an additional functional integration over $\mathrm{SL}(4,\mathbb{C})_\gamma$ with a suitable measure $\mathcal{D}\mathcal{M}$ would have factored out of the path integral and thus can be included trivially in its normalization.

This concludes our argument that a quantization of interacting theories of fermions and gravity may be solely based on a quantization of the metric together with the fermions. The spin-base invariant formulation given here suggests that this quantization scheme is natural. A quantization in terms of vierbeins/Dirac matrices—though perhaps legitimate—is not mandatory.

In hindsight, our results rely crucially on the constraint that the fluctuations of the Dirac matrices satisfy the Clifford algebra Eq. (131) also off shell. If this assumption is relaxed, e.g., if the anticommutator of two Dirac matrices in the path integral is no longer bound to be proportional to the identity, a purely metric-based quantization scheme may no longer be possible.

IX. METRIC VARIATIONS IN THE SPIN-BASE INVARIANT FORMALISM

In this section, we discuss the response of several objects under variations of the metric, yielding a set of properties that may become relevant in concrete quantum gravity computations within the present formalism. The formalism has already been used successfully for theories with quantized fermions in curved spacetime [18,25].

For both perturbative as well as nonperturbative calculations, propagators are central objects. As they arise from two-point correlators, we study the response of several quantities up to second order in metric fluctuations in the following. Since field theory calculations generically need a spacetime "to stand on," we introduce a fiducial background metric $\bar{g}$ with respect to which variations are performed.

Let us first consider the variation of the Dirac matrices to second order in the fluctuations around this background,

$$\gamma_\mu(\bar{g} + \delta g) = \bar{\gamma}_\mu + \left.\frac{\partial\gamma_\mu(g)}{\partial g_{\rho\lambda}}\right|_{g=\bar{g}} \delta g_{\rho\lambda} + \frac{1}{2} \left.\frac{\partial^2\gamma^\mu(g)}{\partial g_{\alpha\beta}\partial g_{\rho\lambda}}\right|_{g=\bar{g}} \delta g_{\alpha\beta}\delta g_{\rho\lambda} + \mathcal{O}(\delta g^3), \tag{147}$$

where $\bar{\gamma}_\mu = \gamma_\mu(\bar{g})$. [5] From Eq. (135) we get

$$\frac{\partial\gamma_\mu(g)}{\partial g_{\rho\lambda}} = \frac{1}{2}\delta^{\rho\lambda}_{\mu\nu}\gamma^\nu(g) + [G^{\rho\lambda}(g), \gamma_\mu(g)], \tag{148}$$

with the *symmetrized* product of two deltas, $\delta^{\rho\lambda}_{\mu\nu} = \frac{1}{2}(\delta^\rho_\mu\delta^\lambda_\nu + \delta^\rho_\nu\delta^\lambda_\mu)$. We know that the first part on the right-hand side is obligatory and therefore cannot be eliminated by spin-base transformations. But the second term is only a variation of the spin base and can thus be transformed to zero, at least for the background field metric. Therefore we may demand

$$G^{\rho\lambda}(\bar{g}) = 0, \tag{149}$$

which corresponds to implicitly choosing part of the spin base.

Assuming that $\gamma_\mu(g)$ is a sufficiently smooth function of the metric, partial derivatives with respect to different metric components commute. This constrains the first

[5] Within the present section, the bar only refers to the background-field quantities and not the Dirac conjugation; $\bar{\gamma}_\mu$ here should thus not be confused with $h^{-1}\gamma_\mu^\dagger h$.

derivative of $G^{\rho\lambda}(g)$ which we call $G^{\alpha\beta\rho\lambda} = \frac{\partial G^{\rho\lambda}(g)}{\partial g_{\alpha\beta}}\Big|_{g=\bar{g}}$. It is useful to introduce the auxiliary tensor

$$\omega^{\rho\lambda\alpha\beta}{}_{\mu\nu} = \frac{1}{4}\delta^{\rho\lambda}_{\mu\kappa}\bar{g}^{\kappa\sigma}\delta^{\alpha\beta}_{\sigma\nu} = \omega^{\alpha\beta\rho\lambda}{}_{\nu\mu}, \tag{150}$$

which shows up in

$$\begin{aligned}\frac{\partial^2\gamma_\mu(g)}{\partial g_{\alpha\beta}\partial g_{\rho\lambda}}\Big|_{g=\bar{g}} &= -\omega^{\rho\lambda\alpha\beta}{}_{\mu\nu}\bar{\gamma}^\nu + [G^{\alpha\beta\rho\lambda},\bar{\gamma}_\mu] \\ &= -\omega^{\alpha\beta\rho\lambda}{}_{(\mu\nu)}\bar{\gamma}^\nu + \left[G^{\alpha\beta\rho\lambda} - \frac{1}{8}\omega^{\alpha\beta\rho\lambda}{}_{[\kappa\sigma]}[\bar{\gamma}^\kappa,\bar{\gamma}^\sigma],\bar{\gamma}_\mu\right].\end{aligned} \tag{151}$$

Again the first part is obligatory since it is symmetric in $(\alpha\beta)\leftrightarrow(\rho\lambda)$ and arises from the first term of Eq. (148). Therefore the simplest choice is

$$G^{\alpha\beta\rho\lambda} = \frac{1}{8}\omega^{\alpha\beta\rho\lambda}{}_{[\kappa\sigma]}[\bar{\gamma}^\kappa,\bar{\gamma}^\sigma], \tag{152}$$

leading to the simple variation to second order in δg,

$$\gamma_\mu(\bar{g}+\delta g) \simeq \bar{\gamma}_\mu + \frac{1}{2}\delta^{\rho\lambda}_{\mu\nu}\bar{\gamma}^\nu\delta g_{\rho\lambda} - \frac{1}{2}\omega^{\alpha\beta\rho\lambda}{}_{(\mu\nu)}\bar{\gamma}^\nu\delta g_{\alpha\beta}\delta g_{\rho\lambda}. \tag{153}$$

It is easy to check that the expansion Eq. (153) fulfills the Clifford algebra to order $\mathcal{O}(\delta g^2)$. Of course, if different conditions on $G^{\rho\lambda}$ or $G^{\alpha\beta\rho\lambda}$ are imposed, the variation of the Dirac matrices will have a different form.

With this result (or corresponding results for other conditions on $G^{\rho\lambda}$ or $G^{\alpha\beta\rho\lambda}$), variations of field monomials formulated in terms of the Dirac matrices with respect to the metric can be calculated straightforwardly. Immediate applications are the computation of the Hessian of a bare action, corresponding to the inverse bare graviton propagator, or a Hessian of an effective action, yielding the full propagator.

If further dynamical fermion fields are included, we also need the variations of the spin metric, etc., at least in principle. In practice, they turn out to be irrelevant at the two-point level, as demonstrated now: For example, the variation of the spin metric has to satisfy Eq. (35),

$$(\bar{\gamma}_\mu + \delta\gamma_\mu)^\dagger = -(\bar{h}+\delta h)(\bar{\gamma}_\mu+\delta\gamma_\mu)(\bar{h}+\delta h)^{-1}, \tag{154}$$

where $\bar{h}$ is the spin metric corresponding to $\bar{\gamma}_\mu$. The variations δh and $\delta\gamma_\mu$ parametrize the deviations of $h(\bar{g}+\delta g)$ and $\gamma_\mu(\bar{g}+\delta g)$ from the background-field quantities. For our choice Eq. (153), we have $\delta\gamma_\mu^\dagger = -\bar{h}(\delta\gamma_\mu)\bar{h}^{-1}$ neglecting terms with $\mathcal{O}(\delta g^3)$. This equation leads to

$$0 \simeq [(\bar{\gamma}_\mu+\delta\gamma_\mu)(\mathrm{I}-\bar{h}^{-1}\delta h),\bar{h}^{-1}\delta h]. \tag{155}$$

Here we have used, that δh is at least of order δg such that we only need to keep track of all terms up to order δg within the other terms, yielding

$$\begin{aligned}0 &\simeq [\bar{\gamma}_\mu + \delta\gamma_\mu, \bar{h}^{-1}\delta h] \\ &\simeq \left(\bar{g}_{\mu\rho} + \frac{1}{2}\delta g_{\mu\rho}\right)[\bar{\gamma}^\rho,\bar{h}^{-1}\delta h]\end{aligned} \tag{156}$$

by multiplying Eq. (155) from the right with $\mathrm{I}+\bar{h}^{-1}\delta h$. Multiplying by $\bar{g}^{\nu\mu} - \frac{1}{2}\bar{g}^{\nu\alpha}(\delta g_{\alpha\beta})\bar{g}^{\beta\mu}$, we find

$$\delta h = \varepsilon\bar{h} + \mathcal{O}(\delta g^3), \qquad \varepsilon\in\mathbb{R}, \tag{157}$$

for an arbitrary infinitesimal ε, which needs to be real because δh needs to be anti-Hermitian. But $h(\bar{g}+\delta g)$ still needs to have a determinant with absolute value equal to one, cf. Eq. (10), resulting in a constraint for ε

$$1 = |\det(\bar{h}+\delta h)| = |\det(\bar{h}(\mathrm{I}+\bar{h}^{-1}\delta h))| \simeq (1+\varepsilon)^4. \tag{158}$$

This equation only has two real solutions $\varepsilon_1 = -2$ and $\varepsilon_2 = 0$. Of course, ε_1 is not infinitesimal but corresponds to the discrete transformation $\bar{h}\to-\bar{h}$. This solution reflects the ambiguity in the choice of the sign of the spin metric and therefore is irrelevant. The relevant second solution shows that the spin metric is constant to second order in the metric variation

$$h(\bar{g}+\delta g) = \bar{h} + \mathcal{O}(\delta g^3). \tag{159}$$

Analogously, it can be derived from $\{\gamma_*,\gamma_\mu\} = 0$ and $\gamma_*^2 = \mathrm{I}$ that also γ_* is constant to second order

$$\gamma_*(\bar{g}+\delta g) = \gamma_*(\bar{g}) + \mathcal{O}(\delta g^3). \tag{160}$$

Finally, let us study the variation of the spin connection Γ_μ. For the spin torsion $\Delta\Gamma_\mu$ this is particularly simple, as it depends on the metric only through the base elements $\gamma_\mu, \gamma_*\gamma_\mu, [\gamma_\mu,\gamma_\nu]$ the variations of which are straightforward. The variation of the connection $\hat{\Gamma}_\mu$ can also be straightforwardly worked out using $D_{(\mathrm{LC})\mu}\gamma_\nu = -[\hat{\Gamma}_\mu,\gamma_\nu]$ and $\mathrm{tr}\,\hat{\Gamma}_\mu = 0$. We find

$$\hat{\Gamma}_\mu(\bar{g}+\delta g) = \hat{\Gamma}_\mu(\bar{g}) + \delta\hat{\Gamma}_\mu + \mathcal{O}(\delta g^3) \tag{161}$$

$$\begin{aligned}\delta\hat{\Gamma}_\mu = \frac{1}{8}[\bar{\gamma}^\kappa,\bar{\gamma}^\sigma]\Bigg[&\delta^{\alpha\beta}_{\mu[\kappa}\delta^\nu_{\sigma]} \\ &+\delta g_{\rho\lambda}\left(\omega^{\alpha\beta\rho\lambda}{}_{[\kappa\sigma]}\delta^\nu_\mu - 2\omega^{\alpha\beta\rho\lambda}{}_{\mu[\kappa}\delta^\nu_{\sigma]} - \frac{1}{2}\delta^{\alpha\beta}_{\mu[\kappa}\delta^{\rho\lambda}_{\sigma]\chi}\bar{g}^{\chi\nu}\right)\Bigg] \\ &\times\bar{D}_{(\mathrm{LC})_\nu}\delta g_{\alpha\beta}.\end{aligned} \tag{162}$$

In a certain sense, our conditions on $G^{\rho\lambda}$ or $G^{\alpha\beta\rho\lambda}$ represent a minimal choice as they minimize the number of terms present in the variation of the Dirac matrices to the corresponding order. Calculations should therefore simplify in comparison to other choices. Due to the direct relation of $G^{\rho\lambda}$ to spin base transformations, it is obvious that physical observables are independent of the choice of conditions.

It is interesting to note that the variation of the Dirac matrices Eq. (147) using the conditions Eq. (149) and (152) corresponds exactly to the result obtained within the vierbein formalism if the Lorentz symmetric gauge is used [26,27]. This gauge has already proved to be a useful choice within the vierbein formalism, and has for instance been used in a functional RG calculation of fermions in quantized gravity in [28]. Hence, our choice can be viewed as the direct generalization of the Lorentz symmetric gauge into the spin-base invariant formulation.

Let us finally comment on the differences between our metric-based quantization scheme and vierbein- (or Dirac matrix-)based schemes both of which are a priori legitimate strategies for quantization. An obvious difference occurs in the corresponding Hessians: given a bare or effective action $S[g]$, the second functional derivative with respect to the metric is different from that with respect to the vierbein, see [4,7] for explicit representations on the Einstein-Hilbert level. A second difference is more subtle: quantizing the vierbein requires further gauge fixing of the additional Lorentz symmetry. This gauge fixing goes along with additional Faddeev-Popov ghosts. Though they can be ignored in perturbation theory in the Lorentz symmetric gauge [26] as they are nonpropagating, they have been shown to contribute nonperturbatively in [4,7]. In our spin-base invariant formalism, there is no such artificial Lorentz symmetry and no corresponding ghosts. Instead we have a local spin-base invariance. As we have shown in the preceding section, the integral over spin bases factorizes in the functional integral in our metric-based quantization scheme such that observables can be computed in any desired spin base. Hence, we can just single out one spin base for the computation, e.g., by demanding Eqs. (149) and (152) to hold. Further ghosts could only appear if one wants to explicitly carry out the integral over spin bases with (symbolic) measure $\mathcal{DM}$ with a suitable spin-base gauge fixing. This is, however, simply not necessary in the present formalism. From another viewpoint, the choice of the spin basis as in Eqs. (149) and (152) plays the role of an external background field in our formalism rather than a "gauge"-fixed quantum field. Of course, other choices are equally legitimate as we have proved that spin-base invariant observables do not depend on this choice.

X. CONCLUSION AND OUTLOOK

In this paper we gave a first-principles approach to a local spin-base invariant approach to fermions in 4-dimensional curved spacetimes. While such a formalism already has been discussed and successfully used at several instances in the literature, our presentation carefully distinguishes between assumptions and consequences, paving the way for generalizations and possibly quantization.

One such generalization is the inclusion of torsion which we have worked out for the first time in this article. In addition to spacetime torsion, which can be included rather straightforwardly in the formalism, the spin connection admits further degrees of freedom which we interpret as spin torsion. Some of these degrees of freedom can be associated with a scalar, an axial vector, and an antisymmetric tensor field. For instance, the latter has a coupling to Dirac spinors in the form of a Pauli term. If the spin torsion contains such a contribution, its torquelike physical influence on the orientation of spin along a geodesic is obvious. Phenomenologically, such terms are similar to those discussed in standard model extensions due to Lorentz- or CPT-violation [29] and are typically tightly constrained, see, e.g., [30].

Further generalizations include the construction of spin curvature which can be used to define classical field theories of gravity (and fermions) in terms of the Dirac matrices (and Dirac spinors) as elementary degrees of freedom. We showed that the simplest possible field theory contains Einstein's theory of general relativity and predicts zero spacetime torsion and zero spin-torsion in absence of explicit sources or boundary conditions.

For vanishing spacetime and spin torsion, the spin-base invariant formalism can be mapped onto the conventional vierbein formalism which can be viewed as a "spin-base gauge-fixed" version of the invariant formalism.

As another generalization, the formalism suggests the definition of a generalized Lie derivative, which turns out to agree with the generalized Lie derivative proposed by Kosmann. In our formalism, this spinorial Lie derivative appears in a manner which can be given a geometrical meaning much in the same way as the Lie derivative for spacetime vectors can be associated with a geometrical interpretation.

As a main result, we used the formalism to show that a possible path integral quantization of gravity and fermionic matter fields can be solely based on an integration over metric and matter fluctuations. Despite the fact that the Dirac matrices appear to be the more fundamental degrees of freedom, their fluctuations can be parametrized by metric as well as spin-base fluctuations. We observe that the latter does not contribute to spin-base invariant observables and hence the spin-base fluctuations can be factored out of the quantum theory. In view of the increasing complexity of quantization schemes based on vierbeins and/or spin connections, the legitimation of a metric-based scheme (though still an open and frighteningly hard challenge) is good news.

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APPENDIX A: SPIN METRIC

For a given set of Dirac matrices encoding the spacetime metric via the Clifford-algebra constraint, also the spin metric h is fixed (up to a sign) by the requirements

$$\begin{aligned} &\text{(i) } \gamma_\mu^\dagger = -h\gamma_\mu h^{-1}, \\ &\text{(ii) } |\det h| = 1, \\ &\text{(iii) } h^\dagger = -h. \end{aligned} \tag{A1}$$

Let us first assume that there is at least one spin metric h_1, which satisfies all three conditions. Then we know, if there is another spin metric h_2, they must be related via

$$[h_2^{-1}h_1, \gamma_\mu] = 0, \tag{A2}$$

because both spin metrics have to fulfill

$$h_2\gamma_\mu h_2^{-1} = -\gamma_\mu^\dagger = h_1\gamma_\mu h_1^{-1}. \tag{A3}$$

Therefore, using Schur's Lemma [24],

$$h_2 = zh_1, \quad z \in \mathbb{C} \tag{A4}$$

has to hold. With (ii), it follows that

$$z = \mathrm{e}^{\mathrm{i}\arg z}. \tag{A5}$$

But if both spin metrics satisfy the condition (iii), then

$$\mathrm{e}^{-\mathrm{i}\arg z}h_1 = -\mathrm{e}^{-\mathrm{i}\arg z}h_1^\dagger = -h_2^\dagger = h_2 = \mathrm{e}^{\mathrm{i}\arg z}h_1 \tag{A6}$$

has to hold. Therefore both spin metrics have to be identical up to a sign,

$$h_2 = \pm h_1. \tag{A7}$$

This demonstrates the uniqueness (up to a sign) of the spin metric. Now we only need to prove the existence of one such spin metric h. For this, we first introduce the Matrix $\hat{M}$ satisfying

$$\gamma_\mu^\dagger = -\mathrm{e}^{\hat{M}}\gamma_\mu \mathrm{e}^{-\hat{M}}, \qquad \mathrm{tr}\,\hat{M} = 0. \tag{A8}$$

This equation implies

$$\gamma_\mu^\dagger = \mathrm{e}^{\hat{M}}\gamma_*\gamma_\mu(\mathrm{e}^{\hat{M}}\gamma_*)^{-1}. \tag{A9}$$

The matrices $\gamma_\mu^\dagger$ also satisfy the Clifford algebra. Since any two different sets of such matrices satisfying the Clifford algebra are connected by a similarity transformation [24], i.e. a spin-base transformation, $\mathrm{e}^{\hat{M}}\gamma_*$ must exist as it parametrizes this similarity transformation. Therefore also $\hat{M}$ must exist but may not be unique. The trace of $\hat{M}$ can always be set to zero, because the trace part commutes with all matrices and therefore drops out of Eq. (A8). The Hermitian conjugate of Eq. (A8) is

$$\gamma_\mu = -\mathrm{e}^{-\hat{M}^\dagger}\gamma_\mu^\dagger \mathrm{e}^{\hat{M}^\dagger}. \tag{A10}$$

Therefore, also

$$\mathrm{e}^{\hat{M}}\gamma_\mu \mathrm{e}^{-\hat{M}} = -\gamma_\mu^\dagger = \mathrm{e}^{\hat{M}^\dagger}\gamma_\mu \mathrm{e}^{-\hat{M}^\dagger} \tag{A11}$$

has to hold. Schur's Lemma again implies there exists a φ such that

$$\mathrm{e}^{\hat{M}^\dagger} = \mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{\hat{M}}, \qquad \varphi \in \mathbb{R}. \tag{A12}$$

This equation fixes $\mathrm{e}^{\mathrm{i}\varphi}$ once we have chosen a specific $\hat{M}$. Now we also know, that $\det \mathrm{e}^{\hat{M}} = 1$ and therefore the same has to hold for $\det \mathrm{e}^{\hat{M}^\dagger} = 1$. From this, we conclude that φ is limited to

$$\varphi \in \left\{ n\frac{2\pi}{d_\gamma}\colon\ n \in \{0, \ldots, d_\gamma - 1\} \right\}. \tag{A13}$$

The desired spin metric h is then given by

$$h = \mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\varphi}{2}}\mathrm{e}^{\hat{M}}. \tag{A14}$$

It is straightforward to show, that this metric satisfies (i)–(iii).

We continue with implementing the spin metric compatibility as expressed in Eq. (27). This tells us that

$$\Gamma_\mu + \bar{\Gamma}_\mu = h^{-1}\partial_\mu h \tag{A15}$$

has to hold. Taking into account that [cf. Eq. (21)]

$$-D_{(\mathrm{LC})_\mu}h\gamma^\nu h^{-1} = D_{(\mathrm{LC})_\mu}\gamma^{\nu\dagger} = (D_{(\mathrm{LC})_\mu}\gamma^\nu)^\dagger = -[\hat{\Gamma}_\mu, \gamma^\nu]^\dagger, \tag{A16}$$

we arrive at

$$[h^{-1}(\partial_\mu h) - \hat{\Gamma}_\mu - \bar{\hat{\Gamma}}_\mu, \gamma^\nu] = 0. \tag{A17}$$

Because $\mathrm{tr}\,\hat{\Gamma}_\mu = 0$, this implies

$$\hat{\Gamma}_\mu + \bar{\hat{\Gamma}}_\mu = h^{-1}\partial_\mu h - \frac{1}{d_\gamma}\mathrm{tr}(h^{-1}\partial_\mu h)\mathrm{I}. \tag{A18}$$

Now we use

$$\mathrm{tr}(\mathrm{e}^{-\hat{M}}\partial_\mu \mathrm{e}^{\hat{M}}) = \mathrm{tr}\left(\mathrm{e}^{-\hat{M}} \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{\hat{M}^k(\partial_\mu \hat{M})\hat{M}^{n-k-1}}{n!}\right) = \mathrm{tr}(\partial_\mu \hat{M}) = 0 \tag{A19}$$

to conclude

$$\frac{1}{d_\gamma}\mathrm{tr}(h^{-1}\partial_\mu h) = \frac{\mathrm{i}}{2}\partial_\mu \varphi. \tag{A20}$$

This leaves us with

$$\Gamma_\mu + \bar{\Gamma}_\mu = h^{-1}\partial_\mu h = \hat{\Gamma}_\mu + \bar{\hat{\Gamma}}_\mu + \frac{\mathrm{i}}{2}\partial_\mu \varphi \mathrm{I}, \tag{A21}$$

which implies that

$$\frac{\mathrm{i}}{2}\partial_\mu \varphi = \frac{1}{d_\gamma}\mathrm{tr}(\Gamma_\mu + \bar{\Gamma}_\mu) = \frac{2}{d_\gamma}\,\mathrm{Re}\,\mathrm{tr}\Gamma_\mu. \tag{A22}$$

Since the left-hand side is purely imaginary and the right-hand side is purely real both have to vanish. Because φ can only take discrete values, it must be a constant if we require it to be a sufficiently smooth function. This finally implies that

$$\mathrm{Re}\,\mathrm{tr}\,\Gamma_\mu = 0 \tag{A23}$$

and

$$\Gamma_\mu + \bar{\Gamma}_\mu = \hat{\Gamma}_\mu + \bar{\hat{\Gamma}}_\mu = h^{-1}\partial_\mu h \tag{A24}$$

have to hold. These two identities are used in Sec. III to constrain spin torsion.

APPENDIX B: TOOLBOX FOR THE SPIN-BASE INVARIANT FORMALISM

In this appendix, we summarize a set of commonly used formulas for the spin-base invariant formalism, which may serve as a toolbox for practical computations. For simplicity, we set spacetime torsion and spin torsion $\Delta\Gamma_\mu$ to zero.

Given a set of spacetime dependent Dirac matrices, the metric is encoded in the Clifford-algebra constraint and can straightforwardly be computed:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathrm{I}, \qquad g_{\mu\nu} = \frac{1}{d_\gamma}\mathrm{tr}(\gamma_\mu \gamma_\nu). \tag{B1}$$

The inclusion of fermion degrees of freedom requires a spin metric h for the definition of scalar products of spinors and conjugate spinors

$$\bar{\psi} = \psi^\dagger h. \tag{B2}$$

Though h can in principle be constructed explicitly, cf. Appendix A, only the algebraic relations that define h are typically needed in practical calculations,

$$\gamma^{\mu\dagger} = -h\gamma^\mu h^{-1}, \qquad |\det h| = 1, \qquad h^\dagger = -h. \tag{B3}$$

For covariant differentiation of spinors

$$\nabla_\mu \psi = \partial_\mu \psi + \Gamma_\mu \psi, \tag{B4}$$

the affine spin connection is needed, where in the absence of spin torsion ($\Delta\Gamma_\mu = 0$) $\Gamma_\mu = \hat{\Gamma}_\mu$ is implicitly given by

$$D_\mu \gamma^\nu = -[\Gamma_\mu, \gamma^\nu], \qquad \mathrm{tr}\,\Gamma_\mu = 0, \tag{B5}$$

and explicitly by

$$\begin{aligned}
&\text{(i)}\ \Gamma_\mu = p_\mu \gamma_* + v_\mu{}^\alpha \gamma_\alpha + a_\mu{}^\alpha \gamma_* \gamma_\alpha + t_\mu{}^{\alpha\beta}[\gamma_\alpha, \gamma_\beta],\\
&\text{(ii)}\ p_\mu = \frac{1}{32}\ \mathrm{tr}(\gamma_* \gamma_\alpha \partial_\mu \gamma^\alpha),\\
&\text{(iii)}\ v_\mu{}^\alpha = \frac{1}{48}\ \mathrm{tr}([\gamma^\alpha, \gamma_\beta]\partial_\mu \gamma^\beta),\\
&\text{(iv)}\ a_\mu{}^\alpha = \frac{1}{8}\ \mathrm{tr}(\gamma_* \partial_\mu \gamma^\alpha),\\
&\text{(v)}\ t_{\mu\alpha}{}^\beta = -\frac{1}{32}\mathrm{tr}(\gamma_\alpha \partial_\mu \gamma^\beta) - \frac{1}{8}\left\{ \begin{matrix} \beta \\ \mu\alpha \end{matrix} \right\} \equiv -t_\mu{}^\beta{}_\alpha.
\end{aligned} \tag{B6}$$

The covariant derivative satisfies the spin metric compatibility condition,

$$\nabla_\mu h = \partial_\mu h - h\Gamma_\mu - \Gamma_\mu^\dagger h = 0. \tag{B7}$$

The generalized Lie derivative $\tilde{\mathcal{L}}$ is given by

$$\tilde{\mathcal{L}}_v = \mathcal{L}_v \psi + \mathcal{Z}_v \psi, \tag{B8}$$

where $\mathcal{L}_v$ is the ordinary Lie derivative acting on ψ as on a spacetime scalar, and the matrix $\mathcal{Z}_v$ is implicitly given by

$$\tilde{\mathcal{L}}_v \gamma^\mu = \mathcal{L}_v \gamma^\mu + [\mathcal{Z}_v, \gamma^\mu] = \frac{1}{2}(\mathcal{L}_v g^{\mu\nu})\gamma_\nu, \ \ \mathrm{tr}\mathcal{Z}_v = 0 \tag{B9}$$

and explicitly by

$$\mathcal{Z}_v = v^\rho \Gamma_\rho + \frac{1}{16}(\partial_\rho v_\lambda - \partial_\lambda v_\rho)[\gamma^\rho, \gamma^\lambda]. \tag{B10}$$

For calculations in a quantized framework, the variations of the spinorial quantities with respect to metric fluctuations $\delta g_{\mu\nu}$ about a background metric $\bar{g}$ are needed. Choosing a suitable spin base, these variations acquire a minimal form (corresponding to the Lorentz symmetric gauge in the vierbein formalism). Up to second order, the minimal variations are given by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \tag{B11}$$

$$\gamma_\mu = \bar{\gamma}_\mu + \frac{1}{2}\delta g_{\mu\nu}\bar{\gamma}^\nu - \frac{1}{8}\delta g_{\mu\rho}\bar{g}^{\rho\lambda}\delta g_{\lambda\nu}\bar{\gamma}^\nu + \mathcal{O}(\delta g^3) \tag{B12}$$

$$h = \bar{h} + \mathcal{O}(\delta g^3) \tag{B13}$$

$$\gamma_* = \bar{\gamma}_* + \mathcal{O}(\delta g^3) \tag{B14}$$

$$\Gamma_\mu = \bar{\Gamma}_\mu + \frac{1}{8}[\bar{\gamma}^\kappa, \bar{\gamma}^\sigma]\bar{D}_\sigma \delta g_{\kappa\mu} + \frac{1}{8}[\bar{\gamma}^\kappa, \bar{\gamma}^\sigma]\delta g_{\sigma\rho}\bar{g}^{\rho\lambda} \times \left(\frac{1}{4}\delta^\nu_\mu \delta^{\alpha\beta}_{\kappa\lambda} + \delta^\alpha_\mu \delta^\nu_{[\kappa}\delta^\beta_{\lambda]}\right)\bar{D}_\nu \delta g_{\alpha\beta} + \mathcal{O}(\delta g^3), \tag{B15}$$

where barred quantities refer to the background.

The derivations of the identities of this toolbox as well as generalizations to nonzero torsion can be found in the main text.

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