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Self-dual road to noncommutative gravity with twist: A new analysis

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The field equations of noncommutative gravity can be obtained by replacing all exterior products by twist-deformed exterior products in the action functional of general relativity and are here studied by requiring that the torsion 2-form should vanish and that the Lorentz-Lie-algebra-valued part of the full connection 1-form should be self-dual. Two other conditions, expressing self-duality of a pair 2-forms occurring in the full curvature 2-form, are also imposed. This leads to a systematic solution strategy, here displayed for the first time, where all parts of the connection 1-form are first evaluated, and hence the full curvature 2-form, and eventually all parts of the tetrad 1-form, when expanded on the basis of γ matrices. By assuming asymptotic expansions which hold up to first order in the noncommutativity matrix in the neighborhood of the vanishing value for noncommutativity, we find a family of self-dual solutions of the field equations. This is generated by solving first an inhomogeneous wave equation on 1-forms in a classical curved spacetime (which is itself self-dual and solves the vacuum Einstein equations), subject to the Lorenz gauge condition. In particular, when the classical undeformed geometry is Kasner spacetime, the above scheme is fully computable out of solutions of the scalar wave equation in such a Kasner model.

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I. INTRODUCTION

The tetrad formalism of Cartan [1] has proved useful in both gravity (see, e.g., Sec. 10.1 of Ref. [2]) and supergravity [3], as well as in gravitational instanton theory [4] and in modern approaches to the Hamiltonian formulation of general relativity [5,6]. Within this framework, one assumes that a set of local Lorentz frames exists, whose global existence is ensured if the classical spacetime manifold is parallelizable. The covariant components $g_{\mu\nu}$ of the metric tensor can then be reexpressed through tetrad covectors $e^a{}_\mu$ in the form

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab} = e_{b\mu} e^b{}_\nu, \tag{1.1}$$

so that, on defining the tetrad 1-forms

$$e^a \equiv e^a{}_\mu \mathrm{d}x^\mu, \tag{1.2}$$

the spacetime metric reads eventually

$$g = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = e^a \otimes e^b \eta_{ab}. \tag{1.3}$$

Of course, the "dual" description in terms of tetrad vectors $e^\mu{}_a$ is also possible. One then finds, from the contravariant metric components

$$g^{\mu\nu} = e^\mu{}_a e^\nu{}_b \eta^{ab} = e^{\mu b} e^\nu{}_b, \tag{1.4}$$

jointly with the tetrad vector fields [2]

$$e_a \equiv e^\mu{}_a \frac{\partial}{\partial x^\mu}, \tag{1.5}$$

the other useful formula

$$g = g^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = e_a \otimes e_b \eta^{ab}. \tag{1.6}$$

On the other hand, several investigations of noncommutative gravity have exploited the tetrad and spin connection as well, but by replacing the ordinary exterior products of forms with deformed exterior products [7]. This would be fair enough, with no need for extra mathematical machinery, if it were not for the fact that attempts of providing a rigorous definition of noncommutative gravity equations *jointly with their solution* had been unsuccessful, at least within the framework of twist differential geometry (see Appendix A) in the version considered in Refs. [8,9]. One of the basic aspects of tetrad formalism for noncommutative gravity is to expand the tetrad on the basis of γ

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matrices, so that one deals actually with the space of tetrad 1-forms with components given by 4×4 matrices, i.e. (see Appendix B)

$$V_j{}^k = (V_\mu)_j{}^k \mathrm{d}x^\mu, \tag{1.7}$$

where

$$(V_\mu)_j{}^k = V^a{}_\mu (\gamma_a)_j{}^k + \tilde{V}^a{}_\mu (\gamma_a \gamma_5)_j{}^k. \tag{1.8}$$

Hereafter, following Ref. [7], the noncommutativity we consider is given by the Moyal-Weyl $\star$ product associated with a constant antisymmetric matrix $\theta^{\rho\sigma}$ in the generic coordinates x^μ, which obey the "deformed" commutation law

$$x^\rho \star x^\sigma - x^\sigma \star x^\rho \equiv \mathrm{i}\theta^{\rho\sigma}. \tag{1.9}$$

Moreover, following again Ref. [7], the additive structures are not modified, while multiplicative structures get deformed. Thus, *the notion of tensors that we consider remains the same as in the classical commutative setting, while the tensor product and the exterior product (and only these structures) are deformed*. For a deeper look at the mathematical foundations, we refer the reader to Appendix A and to the work in Ref. [10].

Both $V^a{}_\mu$ and $\tilde{V}^a{}_\mu$ depend on the noncommutativity matrix $\theta^{\rho\sigma} = \theta^{[\rho\sigma]}$ and are approximated by means of even and odd [7] asymptotic expansions in the neighborhood of $\theta^{\rho\sigma} = 0$ of the form

$$V^a{}_\mu(\theta) = V^a{}_\mu(-\theta) \sim e^a{}_\mu + \mathrm{O}(\theta^2), \tag{1.10}$$

$$\tilde{V}^a{}_\mu(\theta) = -\tilde{V}^a{}_\mu(-\theta) \sim \theta^{\rho\sigma} P^a{}_{\mu[\rho\sigma]} + \mathrm{O}(\theta^3). \tag{1.11}$$

(Notice, however, that $\tilde{V}$ has no commutative analogue; it is generated by the requirement that the action functional introduced in the next section be fully invariant under $\star$-gauge transformations. We refer to Appendix B for details.)

This is indeed a crucial point of all our analysis. The full theory, at the nonperturbative level, is nonlocal, but there is not yet any experimental evidence of finite effects resulting from finite values of $\theta^{\rho\sigma}$. Thus, we limit ourselves to studying the behavior of noncommutative gravity in the neighborhood of $\theta^{\rho\sigma} = 0$. The existence of even and odd parts of the tetrad is an exact property [7], but we focus on their asymptotics (1.10) and (1.11).

The full connection 1-form is also a 4×4 matrix of 1-forms [7] expandable as (see Appendix B)

$$\Omega_j{}^k = (\Omega_\mu)_j{}^k \mathrm{d}x^\mu = [\omega_\mu^{ab} (\Gamma_{ab})_j{}^k + \mathrm{i}\omega_\mu \delta_j{}^k + \tilde{\omega}_\mu (\gamma_5)_j{}^k] \mathrm{d}x^\mu, \tag{1.12}$$

with

$$\Gamma_{ab} = \frac{1}{4}\gamma_{ab} = \frac{1}{8}(\gamma_a\gamma_b - \gamma_b\gamma_a) = \frac{1}{4}\gamma_{[a}\gamma_{b]}. \tag{1.13}$$

The components ω_μ^{ab} take values in the Lie algebra of the Lorentz group and hence carry Lorentz-frame indices. One deals therefore with a 1-form $\omega^{ab} = \omega_\mu^{ab} \mathrm{d}x^\mu$, which yields the usual spin connection in the commutative limit, while ω_μ and $\tilde{\omega}_\mu$ are components of purely noncommutative 1-forms $\omega = \omega_\mu \mathrm{d}x^\mu$ and $\tilde{\omega} = \tilde{\omega}_\mu \mathrm{d}x^\mu$, introduced, as $\tilde{V}$, to fulfill the $\star$-gauge invariance of the theory (see again Appendix B).

The full curvature 2-form is defined by

$$\mathcal{R} \equiv \mathrm{d}\Omega - \Omega \wedge_\star \Omega \tag{1.14}$$

and, by writing explicitly all matrix, coordinate, and Lorentz-frame indices, reads as

$$\mathcal{R}_j{}^k = \frac{1}{2} (\mathcal{R}_{\mu\nu})_j{}^k \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{1.15}$$

where [7]

$$(\mathcal{R}_{\mu\nu})_j{}^k = R_{\mu\nu}^{ab} (\Gamma_{ab})_j{}^k + \mathrm{i} r_{\mu\nu} \delta_j{}^k + \tilde{r}_{\mu\nu} (\gamma_5)_j{}^k. \tag{1.16}$$

With this notation, $R_{\mu\nu}^{ab}$ are the components of the Lorentz-Lie-algebra-valued 2-form [for more precise language, see what we write after Eq. (B4) of Appendix B]

$$R^{ab} = \frac{1}{2} R_{\mu\nu}^{ab} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu,$$

while $r_{\mu\nu}$ and $\tilde{r}_{\mu\nu}$ are the components of the 2-forms

$$r = \frac{1}{2} r_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \quad \text{and} \quad \tilde{r} = \frac{1}{2} \tilde{r}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

By virtue of (1.12) and (1.14)–(1.16), and exploiting the definition of Hodge dual (both ω^{ab} and R^{ab} can be treated as 2-forms with respect to Lorentz-frame indices, as discussed in Appendix C)

$$^{(*)}\omega^{ab} \equiv \frac{1}{2} \varepsilon^{ab}{}_{cd} \omega^{cd}, \tag{1.17}$$

$$^{(*)}R^{ab} \equiv \frac{1}{2} \varepsilon^{ab}{}_{cd} R^{cd}, \tag{1.18}$$

and the twist-deformed exterior product of 1-forms $\alpha_\mu \mathrm{d}x^\mu$ and $\beta_\nu \mathrm{d}x^\nu$,

$$\alpha \wedge_\star \beta = \alpha_{[\mu} \star \beta_{\nu]} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{1.19}$$

with

$$\alpha_\mu \star \beta_\nu \sim \alpha_\mu \beta_\nu + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho \alpha_\mu)(\partial_\sigma \beta_\nu) + O(\theta^2) \quad \text{as } \theta^{\rho\sigma} \to 0, \tag{1.20}$$

$$\alpha_{[\mu} \star \beta_{\nu]} \equiv \frac{1}{2}(\alpha_\mu \star \beta_\nu - \alpha_\nu \star \beta_\mu), \tag{1.21}$$

one finds eventually the components (cf. [7])

$$\begin{aligned} R^{ab}_{\mu\nu} &= 2\partial_{[\mu}\omega^{ab}_{\nu]} + (\omega^{b}_{c[\mu} \star \omega^{ca}_{\nu]} - \omega^{a}_{c[\mu} \star \omega^{cb}_{\nu]}) \\ &\quad - 2i(\omega^{ab}_{[\mu} \star \omega_{\nu]} + \omega_{[\mu} \star \omega^{ab}_{\nu]}) \\ &\quad - 2i[{}^{(*)}\omega^{ab}_{[\mu} \star \tilde{\omega}_{\nu]} + \tilde{\omega}_{[\mu} \star {}^{(*)}\omega^{ab}_{\nu]}], \end{aligned} \tag{1.22}$$

$$r_{\mu\nu} = 2\partial_{[\mu}\omega_{\nu]} - \frac{i}{4}\omega_{cd[\mu} \star \omega^{cd}_{\nu]} - 2i(\omega_{[\mu} \star \omega_{\nu]} - \tilde{\omega}_{[\mu} \star \tilde{\omega}_{\nu]}), \tag{1.23}$$

$$\tilde{r}_{\mu\nu} = 2\partial_{[\mu}\tilde{\omega}_{\nu]} + \frac{i}{4}{}^{(*)}\omega_{cd[\mu} \star \omega^{cd}_{\nu]} - 2i(\omega_{[\mu} \star \tilde{\omega}_{\nu]} + \tilde{\omega}_{[\mu} \star \omega_{\nu]}). \tag{1.24}$$

The use of Hodge duals is suggested by what one finds in simpler circumstances. For example, in general relativity, self-duality (respectively, anti–self-duality) of the spin-connection 1-form, i.e.

$${}^{(*)}\omega^{ab} = \pm i\omega^{ab}, \tag{1.25}$$

is a sufficient condition for self-duality (respectively, anti–self-duality) of the curvature 2-form:

$${}^{(*)}R^{ab} = \pm i R^{ab}. \tag{1.26}$$

More precisely, Eq. (1.26) is important because, from the condition of the vanishing torsion 2-form, i.e.

$$T \equiv de - \omega \wedge e = 0, \tag{1.27}$$

one finds a solution of the vacuum Einstein equations, since the latter read as

$${}^{(*)}R \wedge e = 0, \tag{1.28}$$

while from $dT = 0$ and from $R = d\omega - \omega\wedge\omega$ and associativity of the exterior product one finds

$$R \wedge e = 0, \tag{1.29}$$

which coincides with Eq. (1.28) upon imposing self-duality or anti–self-duality: ${}^{(*)}R = \pm iR$ according to Eq. (1.26).

The plan of our paper is hence as follows. In Sec. II, we write the torsion-free field equations of noncommutative gravity, without any coupling to other fields, and in Sec. III, we consider their self-dual form. In Sec. IV, we study the self-duality conditions on ω^{ab}, while Sec. V expresses self-duality of the 2-forms r and $\tilde{r}$. The remaining self-dual equations are reexpressed in Sec. VI. In Secs. VII–XI, we study, to first order in noncommutativity, the resulting set of equations for the 1-form ω^{ab}, jointly with the 1-forms $\omega, \tilde{\omega}$ and the components V^a_μ and $\tilde{V}^a_\mu$ of the tetrad 1-form. Our results and the open problems are described in Sec. XII, while Appendixes A–C describe in detail the foundations of the concepts we have been using and the operations we have been performing.

II. TORSION-FREE FIELD EQUATIONS

The basic assumption of quantum theory [11] is that every isolated dynamical system can be described by a characteristic action functional S. Our paper does not deal with quantum theory but prepares the ground for it by studying the Euler-Lagrange equations for a given choice of action functional. By relying upon Ref. [7], the starting point is an action for gravity where all exterior products are replaced by twist-deformed exterior products (see Appendix A for a definition), i.e.

$$S = \int \mathrm{Tr}(i\mathcal{R} \wedge_\star V \wedge_\star V \gamma_5). \tag{2.1}$$

This action is invariant under ordinary diffeomorphisms as well as $\star$ diffeomorphisms (we refer to Sec. III of Appendix A for details), and it is also invariant under $\star$-gauge transformations, as described in Appendix B. It leads to the field equations

$$\mathrm{Tr}\left[\gamma_c \gamma_5 (V \wedge_\star \mathcal{R} + \mathcal{R} \wedge_\star V)\right] = 0, \tag{2.2}$$

$$\mathrm{Tr}\left[\gamma_c (V \wedge_\star \mathcal{R} + \mathcal{R} \wedge_\star V)\right] = 0. \tag{2.3}$$

Bearing in mind what we said in the introduction, we now consider the torsion 2-form in the noncommutative setting [7], i.e.

$$T \equiv dV - \Omega \wedge_\star V - V \wedge_\star \Omega, \tag{2.4}$$

and investigate the consequence of requiring it to vanish. This is not mandatory but certainly legitimate. Indeed, from $T = 0$ one finds

$$dV = \Omega \wedge_\star V + V \wedge_\star \Omega, \tag{2.5}$$

while from $dT = 0$ one obtains

$$
\begin{aligned}
0 &= -\mathrm{d}(\Omega\wedge_\star V) - \mathrm{d}(V\wedge_\star\Omega) \\
&= -(\mathrm{d}\Omega)\wedge_\star V + \Omega\wedge_\star \mathrm{d}V - \mathrm{d}V\wedge_\star\Omega + V\wedge_\star \mathrm{d}\Omega \\
&= -(\mathcal{R} + \Omega\wedge_\star\Omega)\wedge_\star V + \Omega\wedge_\star(\Omega\wedge_\star V + V\wedge_\star\Omega) \\
&\quad - (\Omega\wedge_\star V + V\wedge_\star\Omega)\wedge_\star\Omega + V\wedge_\star(\mathcal{R} + \Omega\wedge_\star\Omega) \\
&= -\mathcal{R}\wedge_\star V + V\wedge_\star\mathcal{R}, \qquad (2.6)
\end{aligned}
$$

because also the twist-deformed exterior product is associative [7]. Thus, we find the simple but nontrivial property

$$\mathcal{R}\wedge_\star V = V\wedge_\star\mathcal{R},$$

and the torsion-free versions of the field equations (2.2) and (2.3) become

$$\mathrm{Tr}[\gamma_c\gamma_5 V\wedge_\star\mathcal{R}] = 0, \qquad (2.7)$$

$$\mathrm{Tr}[\gamma_c V\wedge_\star\mathcal{R}] = 0. \qquad (2.8)$$

An equivalent result would be obtained by defining the torsion 2-form according to

$$T \equiv \mathrm{d}V - \Omega\wedge_\star V,$$

because Eq. (2.6) would then reduce to $\mathcal{R}\wedge_\star V = 0$, which again turns Eqs. (2.2) and (2.3) into the torsion-free form (2.7) and (2.8).

From the expansion (1.8), we note that

$$\gamma_5 V = -\tilde{V}^a\gamma_a - V^a\gamma_a\gamma_5, \qquad (2.9)$$

and hence it is *a priori* clear that the two sets of equations are obtained one from the other by interchanging V^a_μ with $\tilde{V}^a_\mu$, as was pointed out at a later stage in Ref. [8]. Now, from the decompositions (1.8) and (1.16), the traces in Eqs. (2.7) and (2.8) are found to be (see Appendix B)

$$
\begin{aligned}
0 &= \mathrm{Tr}[\gamma_c\gamma_5 \mathrm{i}V\wedge_\star\mathcal{R}] \\
&= -\varepsilon_{abcd}V^d\wedge_\star R^{ab} - 4\mathrm{i}\eta_{cd}V^d\wedge_\star\tilde{r} \\
&\quad - \mathrm{i}(\eta_{bc}\eta_{ad} - \eta_{ac}\eta_{bd})\tilde{V}^d\wedge_\star R^{ab} \\
&\quad + 4\eta_{cd}\tilde{V}^d\wedge_\star r, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mathrm{Tr}[\gamma_c \mathrm{i}V\wedge_\star\mathcal{R}] \\
&= \varepsilon_{abcd}\tilde{V}^d\wedge_\star R^{ab} + 4\mathrm{i}\eta_{cd}\tilde{V}^d\wedge_\star\tilde{r} \\
&\quad + \mathrm{i}(\eta_{bc}\eta_{ad} - \eta_{ac}\eta_{bd})V^d\wedge_\star R^{ab} \\
&\quad - 4\eta_{cd}V^d\wedge_\star r. \qquad (2.11)
\end{aligned}
$$

Now we exploit the definition (1.18) of the Hodge dual, jointly with the identity

$$(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd})R^{ab} = R_{dc} - R_{cd} = 2R_{dc},$$

to write Eqs. (2.7) and (2.8) in the form

$$V^d\wedge_\star {}^{(*)}R_{cd} + \tilde{V}^d\wedge_\star(-\mathrm{i}R_{cd}) + 2(\mathrm{i}V_c\wedge_\star\tilde{r} - \tilde{V}_c\wedge_\star r) = 0, \qquad (2.12)$$

$$\tilde{V}^d\wedge_\star {}^{(*)}R_{cd} + V^d\wedge_\star(-\mathrm{i}R_{cd}) + 2(\mathrm{i}\tilde{V}_c\wedge_\star\tilde{r} - V_c\wedge_\star r) = 0. \qquad (2.13)$$

Recall now that any 2-form F can be decomposed into its self-dual (F^+) and anti–self-dual part (F^-) according to $F = F^+ + F^-$, where (the imaginary unit occurs because of the Lorentzian signature)

$$
\begin{aligned}
F^+ &\equiv \frac{1}{2}(F - \mathrm{i}^{(*)}F), \\
F^- &\equiv \frac{1}{2}(F + \mathrm{i}^{(*)}F).
\end{aligned}
$$

We apply this decomposition to the Lorentz-Lie-algebra-valued 2-form R_{cd} in Eqs. (2.12) and (2.13) and also to the 2-forms r and $\tilde{r}$ therein. We further multiply both equations by $-\mathrm{i}$ and hence get

$$
\begin{aligned}
&V^d\wedge_\star(R^-_{cd} - R^+_{cd}) + \tilde{V}^d\wedge_\star(R^-_{cd} + R^+_{cd}) \\
&\quad - 2V_c\wedge_\star(\tilde{r}^- + \tilde{r}^+) - 2\mathrm{i}\tilde{V}_c\wedge_\star(r^- + r^+) = 0, \qquad (2.14)
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{V}^d\wedge_\star(R^-_{cd} - R^+_{cd}) + V^d\wedge_\star(R^-_{cd} + R^+_{cd}) \\
&\quad - 2\tilde{V}_c\wedge_\star(\tilde{r}^- + \tilde{r}^+) - 2\mathrm{i}V_c\wedge_\star(r^- + r^+) = 0. \qquad (2.15)
\end{aligned}
$$

III. SELF-DUAL TORSION-FREE EQUATIONS

In the self-dual case, one sets to 0 all anti–self-dual parts of the curvature, i.e.

$$R^-_{cd} = 0, \qquad r^- = 0, \qquad \tilde{r}^- = 0, \qquad (3.1)$$

where the Hodge dual of r and $\tilde{r}$, *provided one works to linear order in* θ as we are doing, can be reexpressed through the undeformed Levi-Civita symbol with coordinate indices for curved spacetime, i.e. (see the detailed discussion in Appendix C)

$$
\begin{aligned}
{}^{(*)}r_{\mu\nu} &\sim \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}r_{\rho\sigma} + \mathrm{O}(\theta^2), \\
{}^{(*)}\tilde{r}_{\mu\nu} &\sim \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\tilde{r}_{\rho\sigma} + \mathrm{O}(\theta^2), \quad \text{as } \theta \to 0. \qquad (3.2)
\end{aligned}
$$

We now insert (3.1) into (2.14) and (2.15) and add up the two resulting equations to find the set of "self-dual equations"

$$(\tilde{V}^d - V^d)\wedge_\star R^+_{cd} - 2V_c\wedge_\star\tilde{r}^+ - 2\mathrm{i}\tilde{V}_c\wedge_\star r^+ = 0, \qquad (3.3)$$

$$(\tilde{V}_c + V_c)\wedge_\star(\tilde{r}^+ + \mathrm{i}r^+) = 0. \qquad (3.4)$$

In the anti–self-dual case, one sets instead to 0 the self-dual parts of the curvature, i.e.

$$R^{+}_{cd} = 0, \qquad r^{+} = 0, \qquad \tilde{r}^{+} = 0. \tag{3.5}$$

By inserting (3.5) into (2.14) and (2.15) and subtracting the two resulting equations, we find the "anti–self-dual equations"

$$(\tilde{V}^d + V^d)\wedge_\star R^{-}_{cd} - 2V_c\wedge_\star \tilde{r}^{-} - 2\mathrm{i}\tilde{V}_c\wedge_\star r^{-} = 0, \tag{3.6}$$

$$(V_c - \tilde{V}_c)\wedge_\star(\tilde{r}^{-} - \mathrm{i}r^{-}) = 0. \tag{3.7}$$

IV. SELF-DUALITY OF ω^{ab}

The conditions $R^{-}_{cd} = 0$ or $R^{+}_{cd} = 0$ considered in Sec. III may still lead to complicated equations, as is clear from (1.22). However, Eq. (1.22) tells us something more helpful: if the 1-form ω^{ab} is self-dual or anti–self-dual, according to (1.25), this is a sufficient condition for self-duality or anti–self-duality of R^{ab} itself. Indeed one finds, by virtue of (1.25),

$$\begin{aligned} &{}^{(*)}[\mathrm{d}\omega^{ab} - \mathrm{i}(\omega^{ab}\wedge_\star\omega + \omega\wedge_\star\omega^{ab})] \\ &\quad = \pm\mathrm{i}[\mathrm{d}\omega^{ab} - \mathrm{i}(\omega^{ab}\wedge_\star\omega + \omega\wedge_\star\omega^{ab})], \end{aligned} \tag{4.1}$$

$$\begin{aligned} &{}^{(*)}[-\mathrm{i}({}^{(*)}\omega^{ab}\wedge_\star\tilde{\omega} + \tilde{\omega}\wedge_\star{}^{(*)}\omega^{ab})] \\ &\quad = -[-\mathrm{i}(\omega^{ab}\wedge_\star\tilde{\omega} + \tilde{\omega}\wedge_\star\omega^{ab})] \\ &\quad = \pm\mathrm{i}[-\mathrm{i}({}^{(*)}\omega^{ab}\wedge_\star\tilde{\omega} + \tilde{\omega}\wedge_\star{}^{(*)}\omega^{ab})]. \end{aligned} \tag{4.2}$$

Moreover, by virtue of the identity [8]

$$\begin{aligned} \varepsilon^{abcd}\varepsilon_{defg} &= \delta^a_d(\delta^b_e\delta^c_f - \delta^c_e\delta^b_f) + \delta^b_d(\delta^c_e\delta^a_f - \delta^a_e\delta^c_f) \\ &\quad + \delta^c_d(\delta^a_e\delta^b_f - \delta^b_e\delta^a_f), \end{aligned} \tag{4.3}$$

one finds

$$\begin{aligned} {}^{(*)}(\omega^a_c\wedge_\star\omega^{cb}) &= \frac{1}{2}\varepsilon^{ab}{}_{ef}\,\omega^e_{c[\mu}\star\omega^{cf}_{\nu]}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= \mp\frac{\mathrm{i}}{4}\varepsilon^{ab}{}_{ef}\,\varepsilon^e_{cpq}\,\omega^{pq}_{[\mu}\star\omega^{cf}_{\nu]}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= \mp\frac{\mathrm{i}}{2}(\omega^{cb}_{[\mu}\star\omega^a_{c\nu]} + \omega^{ac}_{[\mu}\star\omega^b_{c\nu]})\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= \pm\mathrm{i}\,\omega^a_{c[\mu}\star\omega^{cb}_{\nu]}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu, \end{aligned} \tag{4.4}$$

and an analogous procedure holds for the Hodge dual of $\omega^b_c\wedge_\star\omega^{ca}$. Hence one finds

$$\begin{aligned} {}^{(*)}R^{ab} &= {}^{(*)}\Big[\mathrm{d}\omega^{ab} - \frac{1}{2}\omega^a_c\wedge_\star\omega^{cb} + \frac{1}{2}\omega^b_c\wedge_\star\omega^{ca} \\ &\qquad - \mathrm{i}(\omega^{ab}\wedge_\star\omega + \omega\wedge_\star\omega^{ab}) \\ &\qquad - \mathrm{i}({}^{(*)}\omega^{ab}\wedge_\star\tilde{\omega} + \tilde{\omega}\wedge_\star{}^{(*)}\omega^{ab})\Big] \\ &= \pm\mathrm{i}R^{ab}, \end{aligned} \tag{4.5}$$

provided that

$$\varepsilon^{ab}{}_{cd}\,\omega^{cd}_\mu = 2\mathrm{i}\,\omega^{ab}_\mu. \tag{4.6}$$

V. SELF-DUALITY OF THE 2-FORMS r AND $\tilde{r}$

Self-duality of the 2-forms r and $\tilde{r}$ means setting to 0 their anti–self-dual parts, which implies that

$$r_{\mu\nu} = -\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}r_{\rho\sigma}, \tag{5.1}$$

$$\tilde{r}_{\mu\nu} = -\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}\tilde{r}_{\rho\sigma}. \tag{5.2}$$

By virtue of (1.23)–(1.25), this leads to the equations

$$\begin{aligned} &2\partial_{[\mu}\omega_{\nu]} + 2\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}\partial_{[\rho}\omega_{\sigma]} - \frac{\mathrm{i}}{4}\omega_{cd[\mu}\star\omega^{cd}_{\nu]} + \frac{1}{4}\varepsilon_{\mu\nu}{}^{\rho\sigma}\omega_{cd[\rho}\star\omega^{cd}_{\sigma]} \\ &\quad - 2\mathrm{i}(\omega_{[\mu}\star\omega_{\nu]} - \tilde{\omega}_{[\mu}\star\tilde{\omega}_{\nu]}) + 2\varepsilon_{\mu\nu}{}^{\rho\sigma}(\omega_{[\rho}\star\omega_{\sigma]} - \tilde{\omega}_{[\rho}\star\tilde{\omega}_{\sigma]}) \\ &= 0, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &2\partial_{[\mu}\tilde{\omega}_{\nu]} + 2\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}\partial_{[\rho}\tilde{\omega}_{\sigma]} - \frac{1}{4}\omega_{cd[\mu}\star\omega^{cd}_{\nu]} - \frac{\mathrm{i}}{4}\varepsilon_{\mu\nu}{}^{\rho\sigma}\omega_{cd[\rho}\star\omega^{cd}_{\sigma]} \\ &\quad - 2\mathrm{i}(\omega_{[\mu}\star\tilde{\omega}_{\nu]} + \tilde{\omega}_{[\mu}\star\omega_{\nu]}) + 2\varepsilon_{\mu\nu}{}^{\rho\sigma}(\omega_{[\rho}\star\tilde{\omega}_{\sigma]} + \tilde{\omega}_{[\rho}\star\omega_{\sigma]}) \\ &= 0. \end{aligned} \tag{5.4}$$

VI. REMAINING SELF-DUAL EQUATIONS

If Eqs. (4.6), (5.3), and (5.4) are fulfilled, we can omit the $^{+}$ superscript for the curvatures $R_{cd}, r, \tilde{r}$ in (3.3) and (3.4). Moreover, the form of (3.3) and (3.4) suggests defining

$$U^a_\mu \equiv \tilde{V}^a_\mu + V^a_\mu, \tag{6.1}$$

$$W^a_\mu \equiv \tilde{V}^a_\mu - V^a_\mu, \tag{6.2}$$

after which one can use the explicit form of the twist-deformed exterior product of a 1-form $\alpha = \alpha_\lambda \mathrm{d}x^\lambda$ with a 2-form $\gamma = \frac{1}{2}\gamma_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$, i.e.

$$\alpha\wedge_\star\gamma = \frac{1}{6}(\alpha_\lambda\star\gamma_{\mu\nu} + \alpha_\mu\star\gamma_{\nu\lambda} + \alpha_\nu\star\gamma_{\lambda\mu})\mathrm{d}x^\lambda\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu. \tag{6.3}$$

Hence one finds that Eq. (3.4) reads as

$$[U_{c\lambda}\star(\tilde{r}_{\mu\nu}+\mathrm{i}r_{\mu\nu})+U_{c\mu}\star(\tilde{r}_{\nu\lambda}+\mathrm{i}r_{\nu\lambda}) + U_{c\nu}\star(\tilde{r}_{\lambda\mu}+\mathrm{i}r_{\lambda\mu})]\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu} = 0. \tag{6.4}$$

Remarkably, this is also part of Eq. (3.3) in the unknowns U^a_λ and W^a_λ, which therefore reduces to an equation in the unknown W^a_λ, i.e.

$$[(W^d_\lambda\star R_{cd\mu\nu}+W^d_\mu\star R_{cd\nu\lambda}+W^d_\nu\star R_{cd\lambda\mu}) + (W_{c\lambda}\star(\tilde{r}_{\mu\nu}-\mathrm{i}r_{\mu\nu})+W_{c\mu}\star(\tilde{r}_{\nu\lambda}-\mathrm{i}r_{\nu\lambda}) + W_{c\nu}\star(\tilde{r}_{\lambda\mu}-\mathrm{i}r_{\lambda\mu}))]\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu} = 0. \tag{6.5}$$

VII. SELF-DUAL EQUATIONS TO FIRST ORDER IN $\theta^{\mu\nu}$

At this stage we can study the full set of self-dual equations (4.6), (5.3), (5.4), (6.4), and (6.5) to first order in $\theta^{\rho\sigma}$ as $\theta^{\rho\sigma}$ approaches 0, since, as we said already after (1.11), there is no observational evidence so far of a regime where noncommutativity produces (even just) finite effects, which would justify, in turn, the consideration of higher orders in $\theta^{\rho\sigma}$. Hence we assume the existence, in the neighborhood of $\theta^{\rho\sigma}=0$, of the asymptotic expansion

$$\omega^{ab}_\mu \sim {}^{(0)}\omega^{ab}_\mu + \theta^{\rho\sigma}C^{ab}_{\mu[\rho\sigma]} + \mathrm{O}(\theta^2), \tag{7.1}$$

where ${}^{(0)}\omega^{ab}_\mu$ is the classical spin connection from the classical tetrad [2],

$${}^{(0)}\omega^{ab}_\mu = \frac{1}{2}e^{a\nu}(e^b_{\nu,\mu}-e^b_{\mu,\nu}) - \frac{1}{2}e^{b\nu}(e^a_{\nu,\mu}-e^a_{\mu,\nu}) + \frac{1}{2}e^{a\nu}e^{b\sigma}(e^c_{\nu,\sigma}-e^c_{\sigma,\nu})e_{c\mu}, \tag{7.2}$$

jointly with (cf. [7])

$$\omega_\mu(\theta) = -\omega_\mu(-\theta) \sim \theta^{\rho\sigma}A_{\mu[\rho\sigma]} + \mathrm{O}(\theta^3) \text{ as } \theta^{\rho\sigma}\to 0, \tag{7.3}$$

$$\tilde{\omega}_\mu(\theta) = -\tilde{\omega}_\mu(-\theta) \sim \theta^{\rho\sigma}B_{\mu[\rho\sigma]} + \mathrm{O}(\theta^3) \text{ as } \theta^{\rho\sigma}\to 0, \tag{7.4}$$

while the asymptotic expansion of V^a_μ and $\tilde{V}^a_\mu$ is described by (1.10) and (1.11).

The resulting solution strategy is therefore as follows. First, solve Eq. (4.6) to first order in $\theta^{\rho\sigma}$, by insertion of (7.1). Then solve Eqs. (5.3) and (5.4) for ω_μ and $\tilde{\omega}_\nu$ to first order in $\theta^{\rho\sigma}$, bearing in mind their limiting form in such a case, i.e.

$$2\partial_{[\mu}\omega_{\nu]} + 2\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}\partial_{[\rho}\omega_{\sigma]} + \mathrm{i}\left(-\frac{1}{4}\omega_{cd[\mu}\star\omega^{cd}_{\nu]} - \frac{\mathrm{i}}{4}\varepsilon_{\mu\nu}{}^{\rho\sigma}\omega_{cd[\rho}\star\omega^{cd}_{\sigma]}\right) + \mathrm{O}(\theta^2) = 0, \tag{7.5}$$

$$2\partial_{[\mu}\tilde{\omega}_{\nu]} + 2\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}\partial_{[\rho}\tilde{\omega}_{\sigma]} + \left(-\frac{1}{4}\omega_{cd[\mu}\star\omega^{cd}_{\nu]} - \frac{\mathrm{i}}{4}\varepsilon_{\mu\nu}{}^{\rho\sigma}\omega_{cd[\rho}\star\omega^{cd}_{\sigma]}\right) + \mathrm{O}(\theta^2) = 0. \tag{7.6}$$

At that stage, $R^{ab}_{\mu\nu}$ can be evaluated from (1.22) to first order in $\theta^{\rho\sigma}$, as well as $\tilde{r}_{\mu\nu}\pm\mathrm{i}r_{\mu\nu}$ from (1.23) and (1.24), i.e.

$$\tilde{r}_{\mu\nu}+\mathrm{i}r_{\mu\nu} \sim 2[\partial_{[\mu}\tilde{\omega}_{\nu]} + \mathrm{i}\partial_{[\mu}\omega_{\nu]}] + \mathrm{O}(\theta^2) \quad \text{as } \theta^{\rho\sigma}\to 0, \tag{7.7}$$

$$\tilde{r}_{\mu\nu}-\mathrm{i}r_{\mu\nu} \sim 2[\partial_{[\mu}\tilde{\omega}_{\nu]} - \mathrm{i}\partial_{[\mu}\omega_{\nu]}] - \frac{1}{2}\omega_{cd[\mu}\star\omega^{cd}_{\nu]} + \mathrm{O}(\theta^2) \quad \text{as } \theta^{\rho\sigma}\to 0. \tag{7.8}$$

Thus, one can solve Eq. (6.4) for U^a_μ and then Eq. (6.5) for W^a_μ and eventually obtain V^a_μ and $\tilde{V}^a_\mu$ from (6.1) and (6.2), i.e.

$$V^a_\mu = \frac{1}{2}(U^a_\mu - W^a_\mu), \qquad \tilde{V}^a_\mu = \frac{1}{2}(U^a_\mu + W^a_\mu).$$

VIII. SOLUTION OF EQ. (4.6) TO FIRST ORDER

The insertion of the asymptotic expansion (7.1) into the self-duality condition (4.6) for the 1-form ω^{ab} yields the equation

$$[\varepsilon^{ab}{}_{cd}{}^{(0)}\omega^{cd}_\mu - 2\mathrm{i}{}^{(0)}\omega^{ab}_\mu] = \theta^{\rho\sigma}[2\mathrm{i}C^{ab}{}_{\mu[\rho\sigma]} - \varepsilon^{ab}{}_{cd}C^{cd}{}_{\mu[\rho\sigma]}]. \tag{8.1}$$

This condition should be identically satisfied, and we notice that the left-hand side is independent of $\theta^{\rho\sigma}$, while the right-hand side does depend on it. Thus, we should set them to 0 separately, and a sufficient condition is fulfillment of the following self-duality conditions (with respect to Lorentz-frame indices):

$$\varepsilon^{ab}{}_{cd}{}^{(0)}\omega^{cd}_\mu = 2\mathrm{i}{}^{(0)}\omega^{ab}_\mu, \tag{8.2}$$

$$\varepsilon^{ab}{}_{cd}C^{cd}_{\mu[\rho\sigma]} = 2\mathrm{i}C^{ab}_{\mu[\rho\sigma]}. \tag{8.3}$$

Interestingly, Eq. (8.2) implies that the curvature 2-form ${}^{(0)}R^{ab}$ of the classical background is itself self-dual. Moreover, Eq. (8.2) also implies that, for a given choice of solution of the self-duality condition for the classical spin connection, there exists a two-parameter family of solutions of Eq. (8.3) reading as

$$C^{ab}_{\mu[\rho\sigma]} = \psi_1 {}^{(0)}\omega^{ab}_{\mu} F_{\rho\sigma} + \psi_2 W_{\mu} {}^{(0)}R^{ab}_{\rho\sigma}, \tag{8.4}$$

where ψ_1 and ψ_2 are parameters, $F_{\rho\sigma} = -F_{\sigma\rho} = F_{[\rho\sigma]}$ are the components of a generic 2-form $F = \frac{1}{2}F_{\lambda\mu}\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\mu}$, and W_{μ} are the components of a 1-form $W_{\mu}\mathrm{d}x^{\mu}$.

IX. COMPONENTS OF THE CURVATURE FORM

We can begin by studying Eq. (7.6), here reexpressed explicitly in the form

$$\partial_{\mu}\tilde{\omega}_{\nu} - \partial_{\nu}\tilde{\omega}_{\mu} + 2\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}\partial_{\rho}\tilde{\omega}_{\sigma} - \frac{1}{8}(\omega_{cd\mu}\star\omega^{cd}_{\nu} - \omega_{cd\nu}\star\omega^{cd}_{\mu} + 2\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}\omega_{cd\rho}\star\omega^{cd}_{\sigma}) + \mathrm{O}(\theta^2) = 0. \tag{9.1}$$

At this stage, we exploit the asymptotic expansion (1.20) of the twist-deformed product of components of 1-forms and the asymptotic expansion (7.1) of the 1-form ω^{ab}. Hence we find the first-order expansion

$$\omega_{cd\mu}\star\omega^{cd}_{\nu} \sim {}^{(0)}\omega_{cd\mu}{}^{(0)}\omega^{cd}_{\nu} + {}^{(0)}\omega_{cd\mu}\theta^{\alpha\beta}C^{cd}_{\nu[\alpha\beta]} + {}^{(0)}\omega^{cd}_{\nu}\theta^{\alpha\beta}C_{cd\mu[\alpha\beta]} + \frac{\mathrm{i}}{2}\theta^{\rho\sigma}({}^{(0)}\omega_{cd\mu,\rho})({}^{(0)}\omega^{cd}_{\nu,\sigma}) + \mathrm{O}(\theta^2). \tag{9.2}$$

Interestingly, this first-order asymptotic expansion leads to exact cancellation of the four terms involving $C^{ab}_{\mu[\rho\sigma]}$ in the course of evaluating, in Eq. (9.1), the difference between the first two terms within the round bracket which is multiplied by $-\frac{1}{8}$. Thus, by virtue of the asymptotic expansion (7.4), Eq. (9.1) becomes the following partial differential equation in the unknown $B_{\mu[\rho\sigma]}$:

$$\theta^{\alpha\beta}[B_{\nu[\alpha\beta],\mu} - B_{\mu[\alpha\beta],\nu} + 2\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}B_{\sigma[\alpha\beta],\rho}] = \frac{\mathrm{i}}{8}\theta^{\alpha\beta}({}^{(0)}\omega_{cd\mu,\alpha})({}^{(0)}\omega^{cd}_{\nu,\beta}) + \frac{\mathrm{i}}{4}\varepsilon_{\mu\nu}{}^{\rho\sigma}\left[{}^{(0)}\omega_{cd\rho}{}^{(0)}\omega^{cd}_{\sigma} + \frac{\mathrm{i}}{2}\theta^{\alpha\beta}({}^{(0)}\omega_{cd\rho,\alpha})({}^{(0)}\omega^{cd}_{\sigma,\beta})\right]. \tag{9.3}$$

Bearing in mind that the Levi-Civita tensor is fully antisymmetric, and also the identity, [see Eq. (10.20)]

$${}^{(0)}\omega_{cd[\rho}{}^{(0)}\omega^{cd}_{\sigma]} = 0,$$

the term independent of $\theta^{\alpha\beta}$ in Eq. (9.3) is found to vanish, so that this equation reduces to

$$\theta^{\alpha\beta}[B_{\nu[\alpha\beta],\mu} - B_{\mu[\alpha\beta],\nu} + 2\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}B_{\sigma[\alpha\beta],\rho}] = \frac{\mathrm{i}}{8}\theta^{\alpha\beta}[({}^{(0)}\omega_{cd\mu,\alpha})({}^{(0)}\omega^{cd}_{\nu,\beta}) + \mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}({}^{(0)}\omega_{cd\rho,\alpha})({}^{(0)}\omega^{cd}_{\sigma,\beta})] \equiv U_{\mu\nu}. \tag{9.4}$$

Now we define, from the right-hand side of (7.3) and (7.4),

$$\mathcal{A}_{\mu} \equiv \theta^{\rho\sigma}A_{\mu[\rho\sigma]}, \qquad \mathcal{B}_{\mu} \equiv \theta^{\rho\sigma}B_{\mu[\rho\sigma]}, \tag{9.5}$$

and we exploit the constancy of $\theta^{\alpha\beta}$ to define the skew-symmetric "field strengths"

$$G_{\mu\nu} \equiv \partial_{\mu}\mathcal{A}_{\nu} - \partial_{\nu}\mathcal{A}_{\mu}, \qquad H_{\mu\nu} \equiv \partial_{\mu}\mathcal{B}_{\nu} - \partial_{\nu}\mathcal{B}_{\mu}. \tag{9.6}$$

Hence we find, from (7.7) and (7.8), the asymptotic expansions in the self-dual case:

$$\tilde{r}_{\mu\nu} + \mathrm{i}r_{\mu\nu} \sim H_{\mu\nu} + \mathrm{i}G_{\mu\nu} + \mathrm{O}(\theta^2) \quad \text{as } \theta^{\alpha\beta} \to 0, \tag{9.7}$$

$$\tilde{r}_{\mu\nu} - \mathrm{i}r_{\mu\nu} \sim H_{\mu\nu} - \mathrm{i}G_{\mu\nu} - \frac{\mathrm{i}}{4}\theta^{\alpha\beta}({}^{(0)}\omega_{cd\mu,\alpha})({}^{(0)}\omega^{cd}_{\nu,\beta}) + \mathrm{O}(\theta^2) \quad \text{as } \theta^{\alpha\beta} \to 0. \tag{9.8}$$

In these formulas, $G_{\mu\nu}$ and $H_{\mu\nu}$ are found by solving equations like (9.4), as is shown in Sec. X.

Last, but not least, we have to evaluate the curvature components $R^{ab}_{\mu\nu}$ from Eq. (1.22) to first order in $\theta^{\alpha\beta}$. By virtue of the self-duality assumption (4.6) and of the asymptotic expansions used so far, we find

$$R^{ab}_{\mu\nu} \sim 2[\partial_{[\mu}{}^{(0)}\omega^{ab}_{\nu]} + \theta^{\alpha\beta}\partial_{[\mu}C^{ab}_{\nu][\alpha\beta]}] + {}^{(0)}\omega^{b}{}_{c[\mu}{}^{(0)}\omega^{ca}_{\nu]} - {}^{(0)}\omega^{a}{}_{c[\mu}{}^{(0)}\omega^{cb}_{\nu]} + \frac{\mathrm{i}}{2}\theta^{\alpha\beta}[(\partial_{\alpha}{}^{(0)}\omega^{b}{}_{c[\mu})(\partial_{\beta}{}^{(0)}\omega^{ca}_{\nu]}) - (\partial_{\alpha}{}^{(0)}\omega^{a}{}_{c[\mu})(\partial_{\beta}{}^{(0)}\omega^{cb}_{\nu]})] + 2\theta^{\alpha\beta}[{}^{(0)}\omega^{cb}_{[\mu}C^{a}{}_{|c|\nu][\alpha\beta]} + {}^{(0)}\omega^{ca}_{[\nu}C^{b}{}_{|c|\mu][\alpha\beta]}] + 2[{}^{(0)}\omega^{ab}_{[\mu}(\mathcal{B} - \mathrm{i}\mathcal{A})_{\nu]} + (\mathcal{B} - \mathrm{i}\mathcal{A})_{[\mu}{}^{(0)}\omega^{ab}_{\nu]}] + \mathrm{O}(\theta^2) \quad \text{as } \theta^{\alpha\beta} \to 0, \tag{9.9}$$

where Eq. (8.4) can be used to express $C^{ab}_{\mu[\alpha\beta]}$. In the next section, we are going to solve Eq. (9.4) and the associated equation for $\mathcal{A}_\mu$. This makes it possible to compute all asymptotic expansions of the curvature forms, and the first-order part of the tetrad can be found eventually from the equation derived in Sec. XI.

X. WAVE EQUATIONS FOR $\mathcal{A}_\mu$ AND $\mathcal{B}_\mu$

Our self-dual solution scheme is fully computable provided that one is able to obtain the general solution of first-order partial differential equations like (9.4). For this purpose, we begin by remarking that Eq. (9.4) can be written in the form

$$H_{\mu\nu} + 2\mathrm{i}\,{}^{(*)}H_{\mu\nu} = U_{\mu\nu}, \tag{10.1}$$

while, from Eq. (7.5), one finds

$$G_{\mu\nu} + 2\mathrm{i}\,{}^{(*)}G_{\mu\nu} = -\mathrm{i}U_{\mu\nu}. \tag{10.2}$$

Note now that the Hodge dual of Eq. (10.1) yields

$${}^{(*)}H_{\mu\nu} - 2\mathrm{i}H_{\mu\nu} = {}^{(*)}U_{\mu\nu}, \tag{10.3}$$

and hence Eqs. (10.1) and (10.3) lead to

$$H_{\mu\nu} = \frac{1}{3}[2\mathrm{i}\,{}^{(*)}U_{\mu\nu} - U_{\mu\nu}]. \tag{10.4}$$

Moreover, since the right-hand side of Eq. (10.2) is $-\mathrm{i}$ times the right-hand side of Eq. (10.1), we find also

$$G_{\mu\nu} = -\frac{\mathrm{i}}{3}[2\mathrm{i}\,{}^{(*)}U_{\mu\nu} - U_{\mu\nu}]. \tag{10.5}$$

Note now that, from the point of view of partial differential equations, Eq. (10.4) can be written explicitly as

$$\partial_\mu\mathcal{B}_\nu - \partial_\nu\mathcal{B}_\mu = \nabla_\mu\mathcal{B}_\nu - \nabla_\nu\mathcal{B}_\mu = \frac{1}{3}[2\mathrm{i}\,{}^{(*)}U_{\mu\nu} - U_{\mu\nu}], \tag{10.6}$$

where ∇_μ is a torsion-free metric compatible covariant derivative of the classical background endowed with classical tetrad covectors $e^a{}_\mu$. We need the transition from ∂_μ to ∇_μ, because the latter makes it possible to act with an appropriate derivative operator on both sides of the tensor equation (10.6), i.e.

$$\nabla^\mu\nabla_\mu\mathcal{B}_\nu - \nabla^\mu\nabla_\nu\mathcal{B}_\mu = -Q_\nu, \tag{10.7}$$

having defined

$$Q_\nu \equiv -\frac{1}{3}\nabla^\mu[2\mathrm{i}\,{}^{(*)}U_{\mu\nu} - U_{\mu\nu}]. \tag{10.8}$$

Eventually, this reads as

$$(-\delta^\mu_\nu\Box + R^\mu_\nu)\mathcal{B}_\mu + \nabla_\nu(\mathrm{div}\mathcal{B}) = Q_\nu, \tag{10.9}$$

where $-\delta^\mu_\nu\Box + R^\mu_\nu$ is the wave operator in curved spacetime acting on (co)vectors. It maps elements of $T_p(M)$ into elements of $T_p(M)$, and elements of $T^*_p(M)$ into elements of $T^*_p(M)$; in the language of differential forms, it reads as $\mathrm{d}\delta + \delta\mathrm{d}$, δ being the codifferential. Upon imposing the Lorenz gauge condition

$$\mathrm{div}\mathcal{B} = \nabla^\mu\mathcal{B}_\mu = 0, \tag{10.10}$$

Eq. (10.9) becomes the familiar inhomogeneous wave equation in curved spacetime, for which existence theorems for the solution are available, since the pioneering work of Leray [12] on the existence of Green functions of hyperbolic operators in curved spacetime [13].

Interestingly, we have therefore found that solutions of the equation

$$(-\delta^\mu_\nu\Box + R^\mu_\nu)\mathcal{B}_\mu = Q_\nu, \tag{10.11}$$

with $\mathcal{B}_\mu$ satisfying the Lorenz gauge, generate solutions of a family of self-dual noncommutative gravity field equations, in the way made precise by Sec. IX and the following section. In particular, on considering classical backgrounds which solve the vacuum Einstein equations in four dimensions, the Ricci term in Eq. (10.11) vanishes, and our wave operator takes the simple form

$$P^\mu_\nu \equiv -\delta^\mu_\nu\Box, \tag{10.12}$$

and $\mathcal{B}_\mu$ reads as

$$\mathcal{B}_\mu = b_\mu + \tilde{\mathcal{B}}_\mu, \tag{10.13}$$

where b_μ is the general solution of the homogeneous equation

$$\Box b_\mu = 0, \tag{10.14}$$

while $\tilde{\mathcal{B}}_\mu$ is a particular solution of the inhomogeneous equation (10.11) with a vanishing Ricci term. In terms of the Green function $G_{\lambda\nu'} \equiv G_{\lambda\nu}(x, x')$ of the operator P^λ_μ, which solves by definition the equation [14]

$$P^\lambda_\mu G^{\nu'}_\lambda = \delta^\nu_\mu\frac{\delta(x, x')}{\sqrt{-g}}, \tag{10.15}$$

one finds

$$\tilde{\mathcal{B}}_\mu = \int G^{\nu'}_\mu Q_\nu(x')\sqrt{-g(x')}\mathrm{d}^4x'. \tag{10.16}$$

To obtain an explicit example, we may consider the classical self-dual spin connection of a Kasner spacetime

[15], which belongs to the class of Bianchi models. In such a case, the metric reads as

$$g = -\mathrm{d}t \otimes \mathrm{d}t + t^{2p_1}\mathrm{d}x \otimes \mathrm{d}x + t^{2p_2}\mathrm{d}y \otimes \mathrm{d}y + t^{2p_3}\mathrm{d}z \otimes \mathrm{d}z, \tag{10.17}$$

where the p_i are constants satisfying the conditions

$$\sum_{i=1}^{3} p_i = 1, \tag{10.18}$$

$$\sum_{i=1}^{3} p_i^2 = 1, \tag{10.19}$$

called the Kasner plane and Kasner 2-sphere condition, respectively. Each $t = \text{const}$ hypersurface of this cosmological model, which solves the vacuum Einstein equations, is a flat three-dimensional space, and the worldlines of constant x, y, z are timelike geodesics along which galaxies or other matter, viewed as test particles, can be imagined to move [16]. This model represents an expanding universe, since the volume element is constantly increasing, but the expansion is anisotropic. The distances parallel to the x axis expand at a rate proportional to t^{p_1}, while those along the y axis can expand at a rate proportional to t^{p_2}. Moreover, along one of the axes, distances contract rather than expand. Thus, if blackbody radiation were emitted at one time t in a Kasner universe and never subsequently scattered, later observers would see blueshifts near one pair of antipodes on the sky and redshifts in most other directions [16]. Despite these features not vindicated by observations, the model remains of interest, both in the analysis of classical cosmological singularities [17] and for our purposes, since we have no *a priori* reasons for selecting a particular self-dual solution of the vacuum Einstein equations, but we rather try to build their noncommutative counterpart with the help of geometric and analytic techniques.

In a Kasner spacetime, the spin connection satisfies the self-duality condition (8.2), and its nonvanishing components are given by [we use the general formula (7.2), and our coordinate indices μ range from 0 through 3]

$${}^{(0)}\omega_i^{ab} = -(\delta^{a0}\delta_i^b - \delta^{b0}\delta_i^a)p_i t^{p_i-1}, \quad \forall\ i = 1,2,3. \tag{10.20}$$

Thus, the tensor $U_{\mu\nu}$ given by the right-hand side of Eq. (9.4) is found to vanish [because the term in square brackets on the right-hand side of (9.4) vanishes if $\alpha \neq \beta$ and $\mu \neq \nu$], which implies in turn that the field strengths $H_{\mu\nu}$ and $G_{\mu\nu}$ vanish, by virtue of the general formulas (10.4) and (10.5). Hence both $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$ *can* be expressed as the gradient of one and the same scalar function ϕ, i.e.

$$\mathcal{A}_\mu = \mathcal{B}_\mu = \nabla_\mu \phi, \tag{10.21}$$

and the Lorenz gauge condition upon them leads to the scalar wave equation for ϕ, i.e.

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\phi = 0. \tag{10.22}$$

In the Kasner coordinates of Eq. (10.17), this reads as

$$\left[-\frac{\partial^2}{\partial t^2} - \frac{1}{t}\frac{\partial}{\partial t} + t^{2p_1}\frac{\partial^2}{\partial x^2} + t^{2p_2}\frac{\partial^2}{\partial y^2} + t^{2p_3}\frac{\partial^2}{\partial z^2}\right]\phi = 0. \tag{10.23}$$

The work in Ref. [18] suggests looking for solutions in the form

$$\phi(t,x,y,z) = \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 A(k,t) \mathrm{e}^{\mathrm{i}(k_1 x + k_2 y + k_3 z)}, \tag{10.24}$$

where k is a concise notation for the triple (k_1, k_2, k_3) and $A(k,t)$ solves, from (10.23), the partial differential equation

$$\left[-\frac{\partial^2}{\partial t^2} - \frac{1}{t}\frac{\partial}{\partial t} + U(k,t)\right]A(k,t) = 0, \tag{10.25}$$

having defined

$$U(k,t) \equiv \sum_{i=1}^{3} t^{2p_i}k_i^2. \tag{10.26}$$

One can turn Eq. (10.25) into a simpler equation, where the coefficient of the first derivative vanishes, by setting

$$A(k,t) = t^\alpha W(k,t). \tag{10.27}$$

This yields the equation

$$\left[-\frac{\alpha^2}{t^2} - \frac{(2\alpha+1)}{t}\frac{\dot{W}}{W} - \frac{\ddot{W}}{W} + U(k,t)\right]A(k,t) = 0, \tag{10.28}$$

where our goal is achieved by setting $\alpha = -\frac{1}{2}$. Hence we find that W should solve the equation

$$\left[\frac{\partial^2}{\partial t^2} + \frac{1}{4t^2} - U(k,t)\right]W(k,t) = 0. \tag{10.29}$$

To get an understanding of some features of the possible solutions, we may consider the particular case $p_1 = 1$, $p_2 = p_3 = 0$, which is consistent with the Kasner conditions (10.18) and (10.19). Hence we arrive at the equation

$$\left[\frac{\partial^2}{\partial t^2}+\frac{1}{4t^2}-t^2k_1^2-k_2^2-k_3^2\right]W(k,t)=0. \qquad (10.30)$$

At this stage it is clear that we cannot find solutions by means of finitely many positive or negative powers of t. We rather have to consider W as a function admitting a Laurent expansion as $t\in]0,\infty[$, i.e.

$$W(k,t)=\sum_{n=-\infty}^{\infty}W_n(k)t^n. \qquad (10.31)$$

By virtue of (10.31), Eq. (10.30) takes the form

$$\sum_{m=-\infty}^{\infty}f_m(k)t^m=0, \qquad (10.32)$$

where

$$f_m(k)\equiv\left(m+\frac{3}{2}\right)^2W_{m+2}(k)-(k_2^2+k_3^2)W_m(k)-k_1^2W_{m-2}(k). \qquad (10.33)$$

Thus, having to set $f_m(k)=0$ for all m, we obtain a countable infinity of three-term recurrence relations for the evaluation of $W(k,t)$ and hence of $A(k,t)=\frac{1}{\sqrt{t}}W(k,t)$, which yields in turn $\phi(t,x,y,z)$ from (10.24). We also notice that the particular solutions of Eq. (10.23) with slow spatial variation are "harmful" in that they have a logarithmic dependence on time and hence blow up in the neighborhood of $t=0$.

The wave equation in Kasner had been studied in Ref. [19] for a quantum scalar field with mass and a conformal coupling term to gravity, with application to the regularized and renormalized energy-momentum tensor. Moreover, the classical wave equation in a Kasner spacetime had been studied in Ref. [20] for the electromagnetic potential, where the author obtained plane-wave solutions such that the temporal component of the electromagnetic potential vanishes, jointly with two of the spatial components. The work in Ref. [21] had instead evaluated directly the electric and magnetic field in Bianchi models, including a Kasner universe.

XI. EQUATIONS FOR THE TETRAD AND THEIR SOLUTION

First, by relabeling dummy indices and exploiting the skew symmetry of $R_{cd\mu\nu}$, $r_{\mu\nu}$, $\tilde{r}_{\mu\nu}$ and of the exterior product $\mathrm{d}x^\lambda\wedge\mathrm{d}x^\mu$, we find that the three terms on the first line of Eq. (6.5) are equal, and the same holds for the three terms on the second and third line of Eq. (6.5). Thus, upon defining

$$Z_{ca\mu\nu}\equiv R_{ca\mu\nu}+\eta_{ca}(\tilde{r}_{\mu\nu}-\mathrm{i}r_{\mu\nu}), \qquad (11.1)$$

we find that Eq. (6.5) can be expressed in the form

$$W_\lambda^a\star Z_{ca\mu\nu}\mathrm{d}x^\lambda\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu=0, \qquad (11.2)$$

where, in light of (1.10), (1.11), and (6.2), W_λ^a has the asymptotic expansion

$$W_\lambda^a\sim-e_\lambda^a+\theta^{\alpha\beta}P^a_{\lambda[\alpha\beta]}+\mathrm{O}(\theta^2)\quad\text{as }\theta^{\alpha\beta}\to0, \qquad (11.3)$$

while, in light of (9.8), (9.9), and (11.1), we write

$$Z_{ca\mu\nu}\sim{}^{(0)}R_{ca\mu\nu}+\theta^{\alpha\beta}Z_{ca\mu\nu[\alpha\beta]}+\mathrm{O}(\theta^2)\quad\text{as }\theta^{\alpha\beta}\to0, \qquad (11.4)$$

where ${}^{(0)}R_{ca\mu\nu}$ is the θ-independent part of the asymptotics (9.9), while $\theta^{\alpha\beta}Z_{ca\mu\nu[\alpha\beta]}$ is the sum of the parts linear in θ in the asymptotic expansions (9.8) and (9.9), i.e.

$$\theta^{\alpha\beta}Z_{ca\mu\nu[\alpha\beta]}\equiv\theta^{\alpha\beta}\Big\{2\partial_{[\mu}C_{|ca|\nu][\alpha\beta]}+\frac{\mathrm{i}}{2}[(\partial_\alpha{}^{(0)}\omega_{ad[\mu})(\partial_\beta{}^{(0)}\omega^d_{|c|\nu]})-(\partial_\alpha{}^{(0)}\omega_{cd[\mu})(\partial_\beta{}^{(0)}\omega^d_{|a|\nu]})]$$
$$+2[{}^{(0)}\omega^d_{a[\mu}C_{|cd|\nu][\alpha\beta]}+{}^{(0)}\omega^d_{c[\nu}C_{|ad|\mu][\alpha\beta]}]\Big\}+2[{}^{(0)}\omega_{ca[\mu}(\mathcal{B}-\mathrm{i}\mathcal{A})_{\nu]}+(\mathcal{B}-\mathrm{i}\mathcal{A})_{[\mu}{}^{(0)}\omega_{|ca|\nu]}]$$
$$+\eta_{ca}\left[H_{\mu\nu}-\mathrm{i}G_{\mu\nu}-\frac{\mathrm{i}}{4}\theta^{\alpha\beta}({}^{(0)}\omega_{pq\mu,\alpha})({}^{(0)}\omega^{pq}_{\nu,\beta})\right]. \qquad (11.5)$$

Thus, the term $W_\lambda^a\star Z_{ca\mu\nu}$ in Eq. (11.2) is found to have, in the neighborhood of $\theta^{\alpha\beta}=0$, the asymptotic expansion

$$W_\lambda^a\star Z_{ca\mu\nu}\sim-e_\lambda^a{}^{(0)}R_{ca\mu\nu}+\theta^{\alpha\beta}\left[-e_\lambda^aZ_{ca\mu\nu[\alpha\beta]}+P^a_{\lambda[\alpha\beta]}{}^{(0)}R_{ca\mu\nu}-\frac{\mathrm{i}}{2}e^a_{\lambda,\alpha}({}^{(0)}R_{ca\mu\nu,\beta})\right]+\mathrm{O}(\theta^2), \qquad (11.6)$$

where the term independent of $\theta^{\alpha\beta}$ on the right-hand side of (11.6) gives a vanishing contribution to Eq. (11.2), if the classical background is taken to solve the vacuum Einstein equations as we have done in Sec. X. Thus, Eq. (11.2) yields the following "solution" for $P^a_{\lambda[\alpha\beta]}$, which expresses the odd part of the tetrad in the asymptotic expansion (1.11):

$$P^a_{\lambda[\alpha\beta]}\,{}^{(0)}R_{ca\mu\nu} = e^a_\lambda Z_{ca\mu\nu[\alpha\beta]} + \frac{\mathrm{i}}{2} e^a_{\lambda,[\alpha}\,{}^{(0)}R_{|ca\mu\nu|,\beta]}. \quad (11.7)$$

This equation should be studied jointly with Eq. (6.4), where the three terms are equal, so that it reads

$$U_{c\lambda}\star(\tilde{r}_{\mu\nu} + \mathrm{i}r_{\mu\nu})\mathrm{d}x^\lambda\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu = 0. \quad (11.8)$$

By working to first order in $\theta^{\alpha\beta}$, and introducing the 2-forms G and H corresponding to the field strengths $G_{\mu\nu}$ and $H_{\mu\nu}$, i.e.

$$G \equiv \frac{1}{2}G_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu, \qquad H \equiv \frac{1}{2}H_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu, \quad (11.9)$$

Eq. (6.4) leads to the nontrivial restriction

$$e^c\wedge(H + \mathrm{i}G) = 0. \quad (11.10)$$

As far as we can see, this means that we should choose the solutions of the wave equations for $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$ in such a way that the resulting 2-forms G and H fulfill Eq. (11.10). After having checked this, the task remains of solving Eq. (11.7).

In the case of a classical background of the Kasner type, as considered in the end of Sec. X, both G and H vanish, and hence Eq. (11.10) is identically satisfied, whereas Eq. (11.7) takes a simplified form, obtained by setting $G_{\mu\nu} = H_{\mu\nu} = 0$ and $\mathcal{A}_\mu = \mathcal{B}_\mu$ in the formula (11.5). Moreover, in a Kasner background, (11.5) is further simplified by the vanishing of contributions built from partial derivatives of the classical spin connection, while (1.22) and (10.20) lead to the following formulas for nonvanishing components of the classical curvature 2-form:

$$\begin{aligned} {}^{(0)}R^{ab}_{0i} &= -(\delta^{a0}\delta^b_i - \delta^{b0}\delta^a_i)p_i(p_i - 1)t^{p_i-2}, \\ &\forall\ i = 1,2,3, \end{aligned} \quad (11.11)$$

$${}^{(0)}R^{ab}_{ij} = (\delta^a_i\delta^b_j - \delta^a_j\delta^b_i)p_ip_jt^{p_i+p_j-2}, \quad \forall\ i,j = 1,2,3. \quad (11.12)$$

These formulas, bearing also in mind that the tetrad covectors for the metric (10.17) read as (with no summation over i on the right-hand side)

$$e^a{}_\lambda = \delta^a_0\delta_{\lambda 0} + \delta^a_i t^{p_i}\delta_{\lambda i}, \quad (11.13)$$

imply that the skew symmetrization of partial derivatives on the right-hand side of Eq. (11.7) vanishes, because the product of such partial derivatives therein is always proportional to the symmetric term $\delta_{\alpha 0}\delta_{\beta 0}$. Thus, Eq. (11.7) reduces to

$$P^a_{\lambda[\alpha\beta]}\,{}^{(0)}R_{ca\mu\nu} = e^a_\lambda Z_{ca\mu\nu[\alpha\beta]}, \quad (11.14)$$

where, on the left-hand side, we read off components of the classical curvature 2-form from (11.11) and (11.12), while on the right-hand side we use (11.13) for the classical tetrad and read off from (11.5) the nonvanishing terms in $Z_{ca\mu\nu[\alpha\beta]}$, i.e.

$$\begin{aligned} Z_{ca\mu\nu[\alpha\beta]} &= 2\partial_{[\mu}C_{|ca|\nu][\alpha\beta]} \\ &\quad + 2[{}^{(0)}\omega^d_{a[\mu}C_{|cd|\nu][\alpha\beta]} + {}^{(0)}\omega^d_{c[\nu}C_{|ad|\mu][\alpha\beta]}] \\ &\quad + 2(1-\mathrm{i})[{}^{(0)}\omega_{ca[\mu}B_{\nu][\alpha\beta]} + B_{[\mu|[\alpha\beta]|}\,{}^{(0)}\omega_{|ca|\nu]}]. \end{aligned} \quad (11.15)$$

In this expression of $Z_{ca\mu\nu[\alpha\beta]}$ in the Kasner case, we can set, bearing in mind the definition (9.5),

$$B_{\mu[\alpha\beta]} = \frac{\varepsilon_{\alpha\beta}\mathcal{B}_\mu}{\theta^{\rho\sigma}\varepsilon_{\rho\sigma}}, \quad (11.16)$$

where $\mathcal{B}_\mu$ is obtained from the gradient of the function (10.24), while the tensor $C^{ab}_{\mu[\rho\sigma]}$ admits the general decomposition displayed in (8.4). We cannot make our solution more explicit than this. For each choice of F and W in (8.4), we have a form of $Z_{ca\mu\nu[\alpha\beta]}$, and hence Eqs. (11.11)–(11.16) yield algebraic equations for the components $P^a_{\lambda[\alpha\beta]}$, i.e. the odd part of the tetrad in the asymptotic expansion (1.11). Our solution task is hence fully accomplished to first order in $\theta^{\rho\sigma}$.

Note also that, when $\theta^{\rho\sigma}$ has such an orientation to the three preferred Kasner axes for which only θ^{yt} and θ^{zx} are nonvanishing and equal to Θ_1 and Θ_2, respectively (the theorem on the reduction to canonical form [22] of $\theta^{\rho\sigma}$ ensures this is always possible), its effect reduces to obtaining the following formula for our $B_{\mu[\alpha\beta]}$:

$$B_{\mu[\alpha\beta]} = \frac{\varepsilon_{\alpha\beta}\mathcal{B}_\mu}{2(\Theta_1 + \Theta_2)}. \quad (11.17)$$

XII. RESULTS AND OPEN PROBLEMS

In this paper, we have tried to develop a powerful "calculus" to find solutions of the field equations of noncommutative gravity, motivated by the unsuccessful attempt of applying the Seiberg-Witten map [8,9] to this task when the action functional is built from twist-deformed exterior products. As far as we know, our analysis is original, and its results can be summarized as follows.

(i) On assuming that the spacetime manifold is parallelizable, so that tetrads can be introduced, the torsion-free equations resulting from the action (2.1) take the index-free form (2.7) and (2.8) or, with Lorentz-frame indices made manifest, the form (2.12) and (2.13).

(ii) In the self-dual (respectively, anti–self-dual) case, such equations reduce to (3.3) and (3.4) [respectively, (3.6) and (3.7)]. Self-duality (respectively, anti–self-duality) of the 1-form ω^{ab} [see (4.6)] is a sufficient condition for self-duality (respectively, anti–self-duality) of the Lorentz-Lie-algebra-valued part of the full curvature 2-form. The remaining parts of the curvature 2-form are self-dual if Eqs. (5.3) and (5.4) are satisfied. The full set of self-dual equations consists of (4.6), (5.3), (5.4), (6.4), and (6.5).

(iii) The self-dual equations can be solved by assuming that the tetrad and connection admit an asymptotic expansion (not of Poincaré type; see Appendix D and examples in Ref. [23]) to first order in noncommutativity in the neighborhood of $\theta^{\rho\sigma}$. This assumption does not exploit the full potentialities of the twist-deformed exterior product but might be appropriate after all, since no experimental evidence is available as yet of finite (let alone "large") effects resulting from noncommutativity.

(iv) Furthermore, all our field equations can be explicitly solved provided that one is able to integrate the first-order partial differential equation (9.4), which turns out to be equivalent to a inhomogeneous wave equation on 1-form fields, subject to a Lorenz gauge condition.

(v) To first order in noncommutativity, the tetrad should fulfill Eq. (11.7), provided the consistency condition (11.10) is satisfied.

(vi) The whole scheme has been tested when the classical background is Kasner spacetime, which is a Bianchi model solving the vacuum Einstein equations with a self-dual spin connection. In such a case, the solution of the scalar wave equation (10.23) is the desired "generator" of a solution for the tetrad form and connection form, to first order in noncommutativity.

We find it encouraging that the self-dual option can be pursued to the extent shown in our paper, without any use of the Seiberg-Witten map or yet other techniques applied in the previous literature [24–37], and the nontrivial cancellations of terms encountered at some stages provide further evidence in favor of a new level of internal consistency of gravity being in sight for the first time. Nevertheless, the mathematical potentialities of noncommutative gravity remain largely unexplored, especially the field equations and their solutions at finite values of $\theta^{\alpha\beta}$.

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APPENDIX A: TWIST DIFFERENTIAL GEOMETRY

1. Twist

In this section, we review the concept of twist, together with some of the noncommutative geometry associated with it. The presentation is based on Refs. [7,38,39].

Let Ξ be the linear space of smooth vector fields on a smooth manifold M, and $U\Xi$ its universal enveloping algebra [if G is a connected Lie group whose Lie algebra $\mathcal{G}$ is spanned by the vector fields $\{L_\alpha\}$, the universal enveloping algebra $U(\mathcal{G})$ is defined to be the algebra generated by the L_α's and the identity, with relations given by the Lie brackets [40]]. Given the commutative algebra of functions on M, denoted by $\mathrm{Fun}(M) \equiv A$, many associative noncommutative products may be obtained from the usual pointwise product $\mu(f \otimes g) = fg$ via the action of a twist operator $\mathcal{F} \in U\Xi \otimes U\Xi$:

$$f \star g = \mu\{\mathcal{F}^{-1}(f \otimes g)\}. \tag{A1}$$

We denote the deformed algebra of functions by $A_\star$. The associativity of the product is a consequence of the defining properties of the twist (an element of $U\Xi \otimes U\Xi$ is said to be a twist if it is invertible, is properly normalized, and satisfies a cocycle condition). On using the standard notation

$$\mathcal{F} = \mathcal{F}^\alpha \otimes \mathcal{F}_\alpha, \qquad \mathcal{F}^{-1} = \bar{\mathcal{F}}^\alpha \otimes \bar{\mathcal{F}}_\alpha, \tag{A2}$$

with $\mathcal{F}^\alpha, \mathcal{F}_\alpha, \bar{\mathcal{F}}^\alpha, \bar{\mathcal{F}}_\alpha$ elements of $U\Xi$, the star product acquires the form

$$f \star g = \bar{\mathcal{F}}^\alpha(f) \bar{\mathcal{F}}_\alpha(g), \tag{A3}$$

where the elements of $U\Xi$ act on functions as Lie derivatives. They are sums of products of vector fields: the Lie derivative with respect to products of vector fields is thus extended by means of

$$\mathcal{L}_{vw\ldots} = \mathcal{L}_v \mathcal{L}_w \ldots. \tag{A4}$$

The class of $\star$ products which can be obtained by a twist is quite rich. Among them, a wide class is given by the so-called Abelian twists:

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ab} X_a \otimes X_b}, \tag{A5}$$

with X_a mutually commuting vector fields and θ^{ab} a constant antisymmetric matrix. The Moyal twist is a particularly simple instance of such a family with $X_a = \partial_a$, the infinitesimal generators of translations, globally defined on $\mathbf{R}^d$.

We also introduce the universal $\mathcal{R}$ matrix

$$\mathcal{R} := \mathcal{F}_{21} \mathcal{F}^{-1}, \tag{A6}$$

where by definition $\mathcal{F}_{21} := \mathcal{F}_\alpha \otimes \mathcal{F}^\alpha$. Hereafter we use the notation

$$\mathcal{R} = R^\alpha \otimes R_\alpha, \qquad \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha. \tag{A7}$$

The $\mathcal{R}$ matrix measures the noncommutativity of the $\star$ product. Indeed, it is easy to see that

$$h \star g = \bar{R}^\alpha(g) \star \bar{R}_\alpha(h). \tag{A8}$$

The permutation group in noncommutative space is naturally represented by $\mathcal{R}$. Formula (A8) says that the $\star$ product is $\mathcal{R}$ commutative in the sense that, if we permute (exchange) two functions by using the $\mathcal{R}$-matrix action, then the result does not change.

2. Vector and tensor fields

We now use the twist to deform the spacetime commutative geometry into a noncommutative one. The guiding principle is the one used to deform the product of functions into the $\star$ product of functions. Every time we have a bilinear map

$$\mu\colon\ X \times Y \to Z, \tag{A9}$$

where X, Y, Z are vector spaces, with an action of $\mathcal{F}^{-1}$ on X and Y, we can combine this map with the action of the twist. In this way we obtain a deformed version $\mu_\star$ of the initial bilinear map μ:

$$\begin{aligned} &\mu_\star := \mu \circ \mathcal{F}^{-1},\\ &\mu_\star\colon\ X \times Y \to Z\\ &(\mathsf{x}, \mathsf{y}) \mapsto \mu_\star(\mathsf{x}, \mathsf{y}) = \mu(\bar{\mathcal{F}}^\alpha(\mathsf{x}), \bar{\mathcal{F}}_\alpha(\mathsf{y})). \end{aligned} \tag{A10}$$

The $\star$ product on the space of functions is recovered by setting $X = Y = A = \mathrm{Fun}(M)$. We now study the case of vector fields, 1-forms, and tensor fields.

Vector fields $\Xi_\star$.—We deform the A-module structure of vector fields, that is, the product $\mu\colon A \otimes \Xi \to \Xi$ between the space of functions on the spacetime M and vector fields. According to the general prescription Eq. (A10), the product $\mu\colon A \otimes \Xi \to \Xi$ is deformed into the product

$$h \star v = \bar{\mathcal{F}}^\alpha(h) \bar{\mathcal{F}}_\alpha(v). \tag{A11}$$

The action of $\bar{\mathcal{F}}^\alpha \in U\Xi$ on vector fields is given by repeated use of the Lie derivative as in (A4). This definition is compatible with the $\star$ product in A. We denote the space of vector fields with this $\star$ multiplication by $\Xi_\star$. As vector spaces $\Xi = \Xi_\star$, but Ξ is an A module while $\Xi_\star$ is an $\mathcal{A}_\star$ module.

1-forms $\Omega_\star$.—Analogously, we deform the product $\mu\colon A \otimes \Omega \to \Omega$ between the space A of functions on spacetime M and 1-forms. As for vector fields, we have

$$h \star \rho = \bar{\mathcal{F}}^\alpha(h) \bar{\mathcal{F}}_\alpha(\rho). \tag{A12}$$

The action of $\bar{\mathcal{F}}_\alpha$ on forms is given by iterating the Lie derivative action of vector fields on forms, as a trivial generalization of Eq. (A4). Forms can be multiplied by functions from the left or from the right (they are an A bimodule). If we deform the multiplication from the right, we obtain the new product

$$\rho \star h = \bar{\mathcal{F}}^\alpha(\rho) \bar{\mathcal{F}}_\alpha(h), \tag{A13}$$

and we move h to the left with the help of the $\mathcal{R}$ matrix,

$$\rho \star h = \bar{R}^\alpha(h) \star \bar{R}_\alpha(\rho). \tag{A14}$$

We have therefore defined the $A_\star$ bimodule of 1-forms.

Tensor fields $\mathcal{T}_\star$.—Tensor fields form an algebra with the tensor product $\otimes$ (over the algebra of functions). We define $\mathcal{T}_\star$ to be the noncommutative algebra of tensor fields. As vector spaces $\mathcal{T} = \mathcal{T}_\star$. The noncommutative and associative tensor product is obtained by applying (A10):

$$\tau \otimes_\star \tau' := \bar{\mathcal{F}}^\alpha(\tau) \otimes \bar{\mathcal{F}}_\alpha(\tau'). \tag{A15}$$

Here again the action of the twist on tensors is via the Lie derivative. Use of the Leibniz rule gives the action of the Lie derivative on a generic tensor.

There is a natural action of the permutation group on undeformed arbitrary tensor fields:

$$\tau \otimes \tau' \xrightarrow{\sigma} \tau' \otimes \tau. \tag{A16}$$

In the deformed case, it is the $\mathcal{R}$ matrix that provides a representation of the permutation group on $\star$-tensor fields:

$$\tau \otimes_\star \tau' \xrightarrow{\sigma_\mathcal{R}} \bar{R}^\alpha(\tau') \otimes_\star \bar{R}_\alpha(\tau). \tag{A17}$$

It is easy to check that, consistently with $\sigma_\mathcal{R}$ being a representation of the permutation group, we have $(\sigma_\mathcal{R})^2 = \mathrm{id}$.

Exterior forms $\Omega_\star^\circ = \oplus_p \Omega_\star^p$.—Exterior forms form an algebra with product $\wedge\colon \Omega^\circ \times \Omega^\circ \to \Omega^\circ$. According to the general prescription (A10), we $\star$ deform the wedge product

$$\theta \wedge_\star \theta' = \bar{\mathcal{F}}^\alpha(\theta) \wedge \bar{\mathcal{F}}_\alpha(\theta'). \tag{A18}$$

As a particular instance of the tensor product above, the exterior product is associative and $\bar{\mathcal{F}}^\alpha, \bar{\mathcal{F}}_\alpha$ act as Lie derivatives. Therefore, the exterior derivative d, commuting with the Lie derivative, is undeformed and satisfies the standard graded Leibniz rule

$$\mathrm{d}(\theta \wedge_\star \theta') = \mathrm{d}\theta \wedge_\star \theta' + (-1)^{\deg(\theta)} \theta \wedge_\star \mathrm{d}\theta'. \tag{A19}$$

For Abelian twists constructed with globally defined vector fields, the ordinary integral of forms verifies the graded cyclicity property, that is, up to boundary terms,

$$\int \theta\wedge_\star\theta' = \int \theta\wedge\theta' = (-1)^{\deg(\theta)\deg(\theta')}\int \theta'\wedge\theta$$
$$= (-1)^{\deg(\theta)\deg(\theta')}\int \theta'\wedge_\star\theta, \qquad (A20)$$

with $\theta\wedge\theta'$ a form of maximal rank on the spacetime manifold. It is possible to show that the graded cyclicity holds for more general twists.

As for complex conjugation, we have, for Abelian twists defined in terms of real fields X_a,

$$(\theta\wedge_\star\theta')^* = (-1)^{\deg(\theta)\deg(\theta')}\theta'^*\wedge_\star\theta^*, \qquad (A21)$$

which holds in particular for functions.

3. Infinitesimal ⋆ diffeomorphisms

We have mentioned in Sec. II that the gravity action Eq. (2.1) is invariant under standard diffeomorphisms, which are generated by vector fields, that act on forms through the Lie derivative. Indeed we have

$$\mathcal{L}_v \int 4-\text{form} = \int \mathrm{d}(i_v 4-\text{form}), \qquad (A22)$$

which yields a boundary term. Interestingly, the ⋆ action in (2.1) is also invariant with respect to ⋆ diffeomorphisms. Let us describe the ⋆-Lie algebra structure of their infinitesimal generators.

Following the general prescription (A10), we may combine the usual Lie derivative on functions $\mathcal{L}_u h = u(h)$ with the twist $\mathcal{F}$:

$$\mathcal{L}^\star_u(h) := \bar{\mathcal{F}}^\alpha(u)(\bar{\mathcal{F}}_\alpha(h)). \qquad (A23)$$

We obtain in this way the ⋆-Lie derivative on the algebra of functions $A_\star$. The differential operator $\mathcal{L}^\star_u$ satisfies the deformed Leibniz rule

$$\mathcal{L}^\star_u(h\star g) = \mathcal{L}^\star_u(h)\star g + \bar{R}^\alpha(h)\star\mathcal{L}^\star_{\bar{R}_\alpha(u)}(g). \qquad (A24)$$

This deformed Leibniz rule is intuitive: in the second addend we have exchanged the order of u and h, and this is achieved by the action of the $\mathcal{R}$ matrix, which provides a representation of the permutation group. In the commutative case, the commutator of two vector fields is again a vector field; we have the Lie algebra of vector fields. In this ⋆-deformed case, we have a similar situation. It is possible to verify that

$$\mathcal{L}^\star_u\mathcal{L}^\star_v - \mathcal{L}^\star_{\bar{R}^\alpha(v)}\mathcal{L}^\star_{\bar{R}_\alpha(u)} = \mathcal{L}^\star_{[u,v]_\star}, \qquad (A25)$$

where we have defined the new vector field

$$[u,v]_\star := [\bar{\mathcal{F}}^\alpha(u), \bar{\mathcal{F}}_\alpha(v)]; \qquad (A26)$$

again as in (A10), the deformed bracket is obtained from the undeformed one via composition with the twist:

$$[,]_\star = [,]\circ\mathcal{F}^{-1}. \qquad (A27)$$

Therefore, in the presence of twisted noncommutativity, we associate to the usual Lie algebra of vector fields, Ξ, $\Xi_\star$, the algebra of vector fields equipped with the ⋆ bracket (A26) or equivalently (A27). The map $[,]_\star\colon \Xi_\star\times\Xi_\star \to \Xi_\star$ is a bilinear map and verifies the ⋆ antisymmetry and the ⋆-Jacoby identity

$$[u,v]_\star = -[\bar{R}^\alpha(v), \bar{R}_\alpha(u)]_\star, \qquad (A28)$$

$$[u,[v,z]_\star]_\star = [[u,v]_\star, z]_\star + [\bar{R}^\alpha(v), [\bar{R}_\alpha(u), z]_\star]_\star. \qquad (A29)$$

We have constructed the deformed Lie algebra of vector fields $\Xi_\star$. As vector spaces $\Xi = \Xi_\star$, but $\Xi_\star$ is a ⋆-Lie algebra. We stress that a ⋆-Lie algebra is not a generic name for a deformation of a Lie algebra. Rather, it is a quantum Lie algebra of a quantum (symmetry) group [41].

Equation (A24) makes vector fields into ⋆ derivations of $\mathcal{A}_\star$. Moreover, it is compatible with the ⋆ multiplication on the left by elements of $\mathcal{A}_\star$, making $\Xi_\star$ into a left $\mathcal{A}_\star$ module.

⋆-vector fields are the infinitesimal generators of ⋆ diffeomorphisms. It is not difficult to verify that the action (2.1) is invariant. We have indeed

$$\mathcal{L}^\star_v \int 4-\text{form} = \int \mathcal{L}^\star_v(4-\text{form})$$
$$= \int \mathcal{L}_{\bar{\mathcal{F}}^\alpha(v)}\bar{\mathcal{F}}_\alpha(4-\text{form}). \qquad (A30)$$

On using Eq. (A4) to compute $\mathcal{L}_{\bar{\mathcal{F}}^\alpha(v)}$ and observing that $\bar{\mathcal{F}}_\alpha$ itself acts on forms as a Lie derivative, we end up with the integral of the external derivative of a top form, as in Eq. (A22), which yields again a boundary term.

APPENDIX B: ⋆-GAUGE TRANSFORMATIONS AND TRACES IN THE FIELD EQUATIONS

The form of the expansions (1.8), (1.12), and (1.16) can be understood by making the following considerations [7]. If two infinitesimal gauge transformations τ and τ' are given, reading as

$$\tau = I + \delta\varepsilon, \qquad \tau' = I + \delta\varepsilon', \qquad (B1)$$

where $\varepsilon = \varepsilon^A T^A$ and T^A are the generators of the algebra of the group under consideration, the deformed commutator of τ and τ' can be expressed in the form

$$[\tau, \tau']_\star = [\delta\varepsilon, \delta\varepsilon']_\star = \frac{1}{2}\{\delta\varepsilon^A, \delta\varepsilon'^B\}_\star [T^A, T^B] + \frac{1}{2}[\delta\varepsilon^A, \delta\varepsilon'^B]_\star \{T^A, T^B\}. \quad \text{(B2)}$$

Thus, since the deformed commutator of infinitesimal gauge parameters does not vanish, for a generic Lie algebra it is necessary to perform an extension so as to include also the anticommutators of generators, hence considering all their possible products.

In the specific case of the spinor representation of the Lorentz group, the expansion of the noncommutative gauge parameter ε contains also contributions proportional to the identity matrix I and to γ_5, i.e.

$$\varepsilon = \epsilon^{ab}\Gamma_{ab} + \mathrm{i}\hat{\varepsilon} I + \tilde{\varepsilon}\gamma_5, \quad \text{(B3)}$$

where the new parameters $\hat{\varepsilon}$ and $\tilde{\varepsilon}$, absent in the commutative setting, can be chosen to be real as the remaining ones, which is equivalent to imposing the Hermiticity condition

$$-\gamma_0 \varepsilon \gamma_0 = \varepsilon^\dagger. \quad \text{(B4)}$$

Thus, to achieve closure of noncommutative gauge transformations, the original Lorentz group $SO(3,1)$ of commutative theory has been extended to the group $SO(3,1) \times U(1) \times \mathbf{R}^+$, where the matrices $\mathrm{i}I$ and γ_5 are the generators of the compact component $U(1)$ and noncompact component $\mathbf{R}^+$, respectively. More precisely, the original gauge group $SL(2,C)$ has been therefore extended to the $\star$-gauge group $GL(2,C)$.

Since, under infinitesimal $\star$-gauge transformations, the full connection form Ω (called spin connection) undergoes the variation

$$\delta_\varepsilon \Omega = \delta\varepsilon - [\Omega, \varepsilon]_\star, \quad \text{(B5)}$$

it takes values, jointly with the curvature 2-form, in the $GL(2,C)$ Lie algebra given by even products of γ matrices, according to the expansions (1.12) and (1.16), respectively. The reality conditions for the 1-forms ω, $\tilde{\omega}$, and the 2-forms r, $\tilde{r}$, can be summarized through the Hermiticity conditions

$$-\gamma_0 \Omega \gamma_0 = \Omega^\dagger, \qquad -\gamma_0 \mathcal{R} \gamma_0 = \mathcal{R}^\dagger. \quad \text{(B6)}$$

Moreover, the infinitesimal $\star$-gauge transformations for tetrads read as

$$\delta_\varepsilon V = -[V, \varepsilon]_\star, \quad \text{(B7)}$$

and they "close" in the linear space generated by odd γ matrices, i.e. both γ_a and $\gamma_a\gamma_5$, the latter resulting from the anticommutator $\{\gamma_{ab}, \gamma_c\}$. Hence one arrives at the expansion (1.8).

On using Eqs. (B5) and (B7), it can be easily verified that the variation of the action (2.1) with respect to $\star$-gauge transformations vanishes [7]; that is, the model is $\star$-gauge invariant, with gauge group $GL(2,C)$.

We also find it helpful for the general reader to evaluate the six traces which contribute to the field equation (2.10), i.e. [7]

$$\tau_1 \equiv \mathrm{Tr}\left\{\frac{\mathrm{i}}{4}\gamma_c\gamma_5(V^d\gamma_d \wedge_\star R^{ab}\gamma_{ab})\right\} = -\frac{\mathrm{i}}{4}\mathrm{Tr}(\gamma_{ab}\gamma_c\gamma_d\gamma_5)V^d \wedge_\star R^{ab} = -\varepsilon_{abcd}V^d \wedge_\star R^{ab}, \quad \text{(B8)}$$

$$\tau_2 \equiv \mathrm{Tr}\{-\gamma_c\gamma_5 V^d\gamma_d \wedge_\star r\} = \mathrm{Tr}(\gamma_c\gamma_d\gamma_5)V^d \wedge_\star r = 0, \quad \text{(B9)}$$

$$\tau_3 \equiv \mathrm{Tr}\{\mathrm{i}\gamma_c\gamma_5 V^d\gamma_d \wedge_\star \tilde{r}\gamma_5\} = -\mathrm{i}\mathrm{Tr}(\gamma_c\gamma_d)V^d \wedge_\star \tilde{r} = -4\mathrm{i}V_c \wedge_\star \tilde{r}, \quad \text{(B10)}$$

$$\tau_4 \equiv \mathrm{Tr}\left\{\frac{\mathrm{i}}{4}\gamma_c\gamma_5\tilde{V}^d\gamma_d\gamma_5 \wedge_\star R^{ab}\gamma_{ab}\right\} = -\frac{\mathrm{i}}{4}\mathrm{Tr}(\gamma_c\gamma_d\gamma_{ab})\tilde{V}^d \wedge_\star R^{ab} = -\mathrm{i}(\eta_{bc}\eta_{ad} - \eta_{ac}\eta_{bd})\tilde{V}^d \wedge_\star R^{ab}, \quad \text{(B11)}$$

$$\tau_5 \equiv \mathrm{Tr}\{-\gamma_c\gamma_5\tilde{V}^d\gamma_d\gamma_5 \wedge_\star r\} = \mathrm{Tr}(\gamma_c\gamma_d)\tilde{V}^d \wedge_\star r = 4\tilde{V}_c \wedge_\star r, \quad \text{(B12)}$$

$$\tau_6 \equiv \{\mathrm{i}\gamma_c\gamma_5\tilde{V}^d\gamma_d\gamma_5 \wedge_\star \tilde{r}\gamma_5\} = 0. \quad \text{(B13)}$$

APPENDIX C: THE HODGE DUAL

Our definitions of Hodge duals (1.17) and (1.18) are inspired by earlier work in the literature. For example, the work in Ref. [4] used precisely the definition (1.17) to derive self-dual solutions of Euclidean gravity, i.e. the asymptotically locally Euclidean Eguchi-Hanson instanton. What is nontrivial in these definitions is that ω^{ab} is a 1-form $\omega^{ab}_\mu \mathrm{d}x^\mu$ but, being Lie algebra valued and skew symmetric: $\omega^{ab} = -\omega^{ba}$, makes it possible to define a Hodge dual (1.17) with respect to Lorentz-frame indices. The same holds for R^{ab}, which is a 2-form written after Eq. (1.16). Our Levi-Civita symbol with frame indices, ε_{abcd}, is precisely the one used in Ref. [7], i.e. the standard undeformed Levi-Civita symbol with frame indices, obtainable from flat-space γ matrices according to

$$\varepsilon_{abcd} = \frac{\mathrm{i}}{4}\mathrm{Tr}(\gamma_{ab}\gamma_c\gamma_d\gamma_5). \quad \text{(C1)}$$

By contrast, the 2-forms r and $\tilde{r}$ introduced in Sec. I are Lorentz scalars or, in other words, 0-forms from the point of view of frame indices, and hence for them we have to generalize the definition of the Hodge dual in curved spacetime. Indeed, in Riemannian geometry, the Hodge dual of a 2-form α (β being another 2-form) admits the intrinsic definition

$$^{(*)}\alpha\wedge\beta = (\alpha,\beta)V_4, \tag{C2}$$

where (α,β) is the interior product of α with β and V_4 is the volume 4-form. With index notation, one then writes [42]

$$(^{(*)}\alpha)_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\alpha_{\rho\sigma}, \tag{C3}$$

where $\varepsilon_{\mu\nu\rho\sigma} \equiv \sqrt{\det g}\epsilon_{\mu\nu\rho\sigma}$, where $\epsilon_{\mu\nu\rho\sigma}$ is equal to 1 (respectively, -1) for even (respectively, odd) permutation of the indices and equal to 0 otherwise. The "curved" Levi-Civita symbol $\varepsilon_{\mu\nu\rho\sigma}$ is a covariant tensor density of weight -1, whereas $\varepsilon^{\mu\nu\rho\sigma}$ is a contravariant tensor density of weight $+1$. Last, but not least, the Levi-Civita symbol with a pair of covariant and a pair of contravariant indices is a tensor of type (2,2), skew symmetric in both pairs of indices.

Both definitions recalled so far make it quite clear that, to define the Hodge dual, one needs a metric. In Lorentzian geometry, the metric has signature 2 in dimension 4, and hence the Hodge dual becomes a complex structure, its square being equal to minus the identity.

Within the framework of twist differential geometry applied to gravity, we know that the tensor product gets deformed according to our prescription (A15). Thus, we deform the tensor product $e^a \otimes e^b$ in Eq. (1.3), after pointing out that the expansion (1.8) can be written in the form

$$V_\mu = E^a{}_\mu\gamma_a, \tag{C4}$$

where

$$E^a{}_\mu \equiv V^a{}_\mu I - \tilde{V}^a{}_\mu\gamma_5, \tag{C5}$$

or, with matrix indices made explicit,

$$(E^a{}_\mu)_j{}^l \equiv V^a{}_\mu\delta_j{}^l - \tilde{V}^a{}_\mu(\gamma_5)_j{}^l. \tag{C6}$$

Thus, what corresponds to the tetrad 1-forms e^a of Eq. (1.2) is the matrix of 1-forms $E^a = E^a{}_\mu dx^\mu$, and the previous considerations suggest considering the following definition of metric (the factor $\frac{1}{4}$ is introduced to compensate for $\mathrm{Tr}I = 4$):

$$g \equiv \frac{1}{4}\mathrm{Tr}(E^a \otimes_\star E^b)\eta_{ab} = g_{\mu\nu}(\theta)dx^\mu \otimes dx^\nu, \tag{C7}$$

where

$$g_{\mu\nu}(\theta) = \frac{1}{4}\mathrm{Tr}(E^a{}_\mu\star E^b_\nu)\eta_{ab} = (V^a{}_\mu\star V^b{}_\nu + \tilde{V}^a{}_\mu\star\tilde{V}^b{}_\nu)\eta_{ab}, \tag{C8}$$

which reduces to (1.1) as $\theta \to 0$. Furthermore, we note that, similarly to the way in which the undeformed tetrad $e^a{}_\mu$ turns the Levi-Civita symbol (C1) into its curved spacetime counterpart according to

$$\epsilon_{\mu\nu\rho\sigma} = \varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho e^d{}_\sigma, \tag{C9}$$

we can now define, with the help of $E^a{}_\mu$,

$$\begin{aligned} E_{\mu\nu\rho\sigma} &\equiv \mathrm{Tr}[\varepsilon_{abcd}(E^a{}_\mu\star E^b{}_\nu\star E^c{}_\rho\star E^d{}_\sigma)] \\ &= \varepsilon_{abcd}(E^a{}_\mu)_j{}^l\star(E^b{}_\nu)_l{}^m\star(E^c{}_\rho)_m{}^p\star(E^d{}_\sigma)_p{}^j. \end{aligned} \tag{C10}$$

To raise and lower indices of $E_{\mu\nu\rho\sigma}$, some equally legitimate (but different) prescriptions are available, i.e.

$$\begin{aligned} E_{\mu\nu}{}^{\rho\sigma} \equiv (g^{\rho\alpha}\star g^{\sigma\beta}\star E_{\mu\nu\alpha\beta}) \quad \text{or} \\ (g^{\rho\alpha}\star E_{\mu\nu\alpha\beta}\star g^{\beta\sigma}) \quad \text{or} \quad (E_{\mu\nu\alpha\beta}\star g^{\alpha\rho}\star g^{\beta\sigma}), \end{aligned} \tag{C11}$$

as well as other prescriptions differing for the relative order of indices of metric components. Some freedom is also available in the definition of contravariant metric $g^{\mu\nu}$, i.e.

$$g_{\mu\nu}\star g^{\nu\lambda} = \delta_\mu{}^\lambda \quad \text{or} \quad g^{\mu\nu}\star g_{\nu\lambda} = \delta^\mu{}_\lambda. \tag{C12}$$

Last, to define the Hodge dual of a 2-form when curved spacetime is deformed according to the prescriptions of twist differential geometry, we consider (cf. Ref. [42])

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma} \equiv \sqrt{\det g(\theta)}\star E_{\mu\nu}{}^{\rho\sigma}, \tag{C13}$$

where we define, inspired by matrix calculus,

$$\det g(\theta) \equiv E^{i_1\cdots i_4}\star g_{1i_1}\star\cdots\star g_{4i_4}, \tag{C14}$$

with the understanding that $E^{i_1\cdots i_4}$ is defined according to (C10) and the metric components are defined according to (C8). We propose therefore the following definition of the Hodge dual of a 2-form $\alpha = \frac{1}{2}\alpha_{\mu\nu}dx^\mu\wedge dx^\nu$:

$$(^{(*)}\alpha)_{\mu\nu} \equiv \frac{1}{2}\mathcal{E}_{\mu\nu}{}^{\rho\sigma}\star\alpha_{\rho\sigma} \sim \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\alpha_{\rho\sigma} + (\mathrm{O}(\theta)\alpha)_{\mu\nu}, \tag{C15}$$

where $\varepsilon_{\mu\nu}{}^{\rho\sigma}$ includes $\sqrt{\det g(\theta = 0)}$.

The transformation properties of our geometric objects are defined when one considers their behavior under infinitesimal $\star$ diffeomorphisms studied in Sec. III of Appendix A. This involves studying the $\star$-Lie derivative of deformed products according to our Eq. (A24). In our application to gravity, we shall therefore write Eq. (A24) in the form

$$\mathrm{Tr}[\mathcal{L}_u^\star(E^a{}_\mu \star E^b{}_\nu)] = \mathrm{Tr}[(\mathcal{L}_u^\star E^a{}_\mu)\star E^b{}_\nu + \bar{R}^\alpha(E^a{}_\mu)\star \mathcal{L}^\star_{\bar{R}_\alpha(u)} E^b{}_\nu]. \qquad \text{(C16)}$$

The passage to some sort of "exponentiation" to obtain the full set of finite $\star$ diffeomorphisms is a challenging open problem, as far as we know.

Interestingly, our definition expressed by (C15) leads to our asymptotic expansions (3.2), which tell us that, to first order in θ, since both $r_{\mu\nu}$ and $\tilde{r}_{\mu\nu}$ are odd functions of θ, one can keep using the Levi-Civita symbol with coordinate indices of the undeformed curved spacetime. In general, *for the purpose of studying linear effects of θ, the various conceivable definitions of the Hodge dual of a 2- form lead always to the asymptotic expansions (3.2).*

APPENDIX D: ASYMPTOTIC EXPANSIONS

Following Ref. [23], we find it appropriate to stress that the notion of asymptotic expansion has nothing to do with the notion of series, despite the confusing use of the term "asymptotic series" in the literature. A series has infinitely many terms, whereas, by definition (see below), *an asymptotic expansion has only finitely many terms*. Talking about convergence (or lack of) of an asymptotic expansion is therefore meaningless. The confusion arises because, in several cases, the Taylor expansion in the neighborhood of a real point x_0 of the function under consideration can be extended arbitrarily far away from x_0, and one can then try to understand whether the Taylor series converges and what is the relation between its sum and the function one started from. This problem, however, has no relation whatsoever with the study of the behavior of the given function in the neighborhood of x_0.

The existence of asymptotic expansions with a large number of terms is a very special phenomenon. For example, the function

$$x \to x^2 + x \sin x$$

has an asymptotic expansion with one term only, i.e. $x^2 + o(x^2)$, in the neighborhood of $+\infty$. Another example is provided by the number $\pi(x)$ of prime numbers smaller than or equal to x, for which

$$\pi(x) \sim \int_2^x \frac{\mathrm{d}t}{\log t}.$$

In general, one starts by considering the set $\mathcal{E}$ of functions of the form [23]

$$g\colon\ x \to g(x) \equiv x^\alpha (\log x)^\beta \mathrm{e}^{P(x)}, \qquad \text{(D1)}$$

where α, β are real nonvanishing constants, and

$$P(x) \equiv \sum_{j=1}^k c_j x^{\gamma_j}, \qquad \text{(D2)}$$

where the c_j are real constants of arbitrary sign, while

$$\gamma_1 > \gamma_2 > \cdots > \gamma_k > 0. \qquad \text{(D3)}$$

By definition, given a function f, its asymptotic expansion with k terms with respect to the set $\mathcal{E}$ is meant to be the sum [23]

$$\Sigma_k \equiv \sum_{j=1}^k b_j g_j, \qquad \text{(D4)}$$

where the b_j are nonvanishing constants and the g_j are functions belonging to the set $\mathcal{E}$ such that

$$g_{j+1} = o(g_j), \quad \forall\ j\colon\ 1 \le j \le k-1. \qquad \text{(D5)}$$

One then writes

$$f = \sum_{j=1}^k b_j g_j + o(g_k). \qquad \text{(D6)}$$

The difference $f - \Sigma_k$ is called the *remainder* of the asymptotic expansion [23]. In our paper, we write this last formula with the equality symbol replaced by the $\sim$ symbol, which is more commonly used in the physics-oriented literature, although less consistent with our source [23].

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