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Holographic parity-violating charged fluid dual to Chern-Simons modified gravity

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We discuss the $(2+1)$-dimensional parity-violating charged fluid on a finite cutoff surface Σ_c, dual to the nondynamical and dynamical Chern-Simons (CS) modified gravities. Using the nonrelativistic long-wavelength expansion method, the field equations are solved up to $\mathcal{O}(\epsilon^2)$ in the nondynamical model. It is shown that there exists nonvortical dual fluid with shear viscosity η and Hall viscosity η_A on the cutoff surface Σ_c. The ratio of shear viscosity over entropy density η/s of the fluid takes the universal value $1/4\pi$, while the ratio of Hall viscosity over entropy density η_A/s depends on the Σ_c and black brane charge q. Moreover, the nonvortical dual fluid obeys the magnetohydrodynamic (MHD) equation. However, these kinematic viscosities ν and ν_A related to η and η_A do not appear in this MHD equation due to the constraint condition $\tilde{\partial}^2\beta_j = 0$ for the $(2+1)$-dimensional dual fluid. Then, we extend our discussion to the dynamical CS modified gravity and show that the dual vortical fluid possesses another so-called Curl viscosity ζ_A, whose ratio to entropy density ζ_A/s also depends on the Σ_c and q. Moreover, the value of η/s still equals $1/4\pi$ and the result of η_A/s agrees with the previous result under the probe limit of the pseudoscalar field at the infinite boundary in the charged black brane background for the dynamical CS modified gravity. This vortical dual fluid corresponds to the MHD turbulence equation in plasma physics.

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I. INTRODUCTION

Recently, there have been a lot of studies on the fluid/gravity dualities [1–3], which are considered special applications of the AdS/CFT correspondence [4–6]. It was argued that the dual field theory at the anti–de Sitter (AdS) boundary can be described by hydrodynamics in the long-wavelength limit. The ability to derive hydrodynamic equations and transport coefficients from this duality provides fresh perspectives in understanding holography. A remarkable description of the fluid/gravity duality was further set up on a finite cutoff surface Σ_c outside the horizon [7]. The discussions have been extended to different models in Einstein relativity [8–11] and modified gravity models with higher-order curvatures corrections [12–14]. Imposing the Petrov-like condition on the $\Sigma_c(r = r_c)$ in the near horizon limit, the incompressible Navier-Stokes equations (or modified equations) for a fluid living on the flat (or spatially curved) spacetime with one fewer dimensions have been demonstrated in Refs. [15–21]. The physics on a finite cutoff surface Σ_c with finite energy scale is appealing since it could be reached by experiments. The study of holography on the finite surface Σ_c may be helpful to understand the microscopic origin of gravity. Other recent works on the fluid/gravity correspondence can be found in Refs. [22–28].

Besides the shear viscosity η and bulk viscosity ζ appearing in the usual hydrodynamic system, we know that in the parity-violating hydrodynamic system, there exists other important transport coefficients, the Hall viscosity η_A and Curl viscosity ζ_A, which are often studied in condensed matter physics. The Hall viscosity, which is a nondissipative viscosity coefficient, does not contribute to the entropy production of the fluid and has been frequently investigated in the field theory approach [29–36]. In the quantum Hall fluids, at zero temperature, the usual dissipative shear and bulk viscosities vanish, while the nondissipative Hall viscosity can be nonzero, provided that the quantum Hall fluid has an energy gap and broken time-reversal symmetry [37]. How can we study this Hall viscosity from holography? This is an interesting question to pursue.

Recently, the fluid/gravity duality was explored in a system with parity violation. The Chern-Simons (CS) modified gravity generally possesses parity-violating gravitational term, even including electromagnetic CS term in the action [38,39], which is considered a simple model to realize the holographic description of a $(2+1)$-dimensional isotropic fluid with broken spatial parity [40–43]. It is expected that in this gravity model, the dual fluid may possess a nonzero Hall viscosity at the AdS boundary. Since the Hall viscosity is related to the presence of a nontrivial background scalar field, it is natural to anticipate that it encodes the parity violation in CS gravity. The Hall viscosity of the dual fluid can be affected by the electromagnetic field if one considers the influence by the electromagnetic CS term on the phase transition of holographic superconductors in four dimensions [44,45]. In addition, the vorticity of holographic

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fluid is another interesting property, and the holographic boundary vortical fluids of analogue gravity systems have been explored in Refs. [46–50].

Using the nonrelativistic fluid expansion method, Cai *et al.* [51] investigated the $(2+1)$-dimensional parity-violating hydrodynamics, dual to the dynamical CS modified gravity on a finite cutoff surface Σ_c outside the uncharged black brane horizon. They presented the dual hydrodynamics with Hall viscosity and Curl viscosity obeying the incompressible Navier-Stokes equations. Note that the CS gravity model has two frameworks, the dynamical one and the nondynamical one, which are classified by whether or not there is a kinetic term for the scalar field in the action [52]. In this paper, we will extend the study to discuss the holographic hydrodynamics dual to nondynamical and dynamical CS modified gravity, respectively. Besides the gravitational CS term, we will include the electromagnetic CS term in our discussion. We will show that the holographic fluid/gravity duality can be realized both in the nondynamical and the dynamical CS gravities. In the nondynamical model, the dual nonvortical fluid possesses the shear viscosity η and Hall viscosity η_A and obeys the magnetohydrodynamic (MHD) equation. However, these kinematic viscosities ν and ν_A related to η and η_A do not appear in this MHD equation due to the constraint condition $\tilde{\partial}^2\beta_j = 0$. Here, the ratio η/s of the fluid equals $1/4\pi$, while the ratio η_A/s depends on the cutoff surface Σ_c and black brane charge q. As to the dynamical model, the dual vortical fluid obeys the MHD turbulence equation in plasma physics. Besides the shear and Hall viscosities, the dual fluid possesses another so-called Curl viscosity ζ_A, whose ratio to entropy density ζ_A/s depends on the Σ_c and black brane charge q.

The outline of this paper is as follows. In Sec. II, we adopt two finite diffeomorphism transformations and make nonrelativistic hydrodynamic expansion to a general black brane metric, the pseudoscalar and electromagnetic fields. By applying this formalism to nondynamical CS modified gravity coupled to the electromagnetic field, we calculate the stress-energy tensor of the dual fluid through the Brown-York tensor and analyze the properties of the dual fluid on the cutoff surface Σ_c. In Sec. III, we extend the above investigation to the dynamical CS modified gravity. We finally summarize our results in Sec. IV.

II. DUAL FLUID TO NONDYNAMICAL CS MODIFIED GRAVITY

With the electromagnetic CS term $\theta\tilde{F}F$, the action of nondynamical CS modified gravity model reads [53]

$$\mathcal{I}_G = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda - 4\pi G F_{\mu\nu}F^{\mu\nu}) + \frac{1}{4}\int d^4x\sqrt{-g}(\lambda_1\theta\tilde{R}R + \lambda_2\theta\tilde{F}F), \quad (1)$$

where λ_1 and λ_2 are the coupling constants, $\theta\tilde{R}R$ and $\theta\tilde{F}F$ are gravitational and electromagnetic CS terms with

$$\tilde{R}_{\mu\nu}{}^{\rho\tau} = \frac{1}{2}\epsilon^{\rho\tau\chi\phi}R_{\mu\nu\chi\phi}, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\tau}F_{\rho\tau},$$
$$\tilde{R}R = \tilde{R}^{\mu\nu\rho\tau}R_{\nu\mu\rho\tau}, \qquad \tilde{F}F = \tilde{F}^{\mu\nu}F_{\mu\nu}.$$

Here, $\epsilon^{\mu\nu\rho\tau}$ is the four-dimensional Levi-Cività tensor in the bulk with the convention $\epsilon^{r\tau xy} = 1/\sqrt{-g}$. The strengths of the gravitational and electromagnetic CS corrections are controlled by the pseudoscalar field θ. Usually, the pseudoscalar field θ is not a constant, but a function of spacetime, thus, serving as a deformation function. If $\theta = \text{const}$, CS modified gravity reduces to the Einstein gravity. The negative cosmological constant Λ equals $-3/l^2$, where l is the AdS radius. We take $l = 1$ in what follows, for convenience.

As usual, we obtain the field equations by varying the action with respect to the metric, electromagnetic, and pseudoscalar fields, respectively, yielding

$$W_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + 16\pi G\lambda_1 C_{\mu\nu} + 8\pi G T^{(A)}_{\mu\nu} = 0, \quad (2)$$

$$W^{\nu}_{(A)} = \nabla_\mu F^{\mu\nu} - \lambda_2\partial_\mu\theta\tilde{F}^{\mu\nu} = 0, \quad (3)$$

$$W_{(\theta)} = \lambda_1\tilde{R}R + \lambda_2\tilde{F}F = 0, \quad (4)$$

where the stress-energy tensor of the electromagnetic field $T^{(A)}_{\mu\nu}$ and the so-called Cotton tensor $C_{\mu\nu}$ are

$$T^{(A)}_{\mu\nu} = \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F_\mu{}^\alpha F_{\nu\alpha},$$
$$C_{\mu\nu} = \theta_{,\sigma}\epsilon^{\alpha\beta\sigma}{}_{(\mu}R_{\nu)\beta;\alpha} + \theta_{;\sigma\tau}\tilde{R}^{\sigma}{}_{(\mu}{}^{\tau}{}_{\nu)}.$$

It is interesting to take the covariant derivative of the equations of motion (EOM) Eq. (2),

$$\nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right) + 16\pi G\lambda_1\nabla^\mu C_{\mu\nu} + 8\pi G\nabla^\mu T^{(A)}_{\mu\nu} = 0. \quad (5)$$

As we know, the Bianchi identity enforces $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}) = 0$. The covariant derivatives of the Cotten tensor $C_{\mu\nu}$ and the stress-energy tensor of electromagnetic $T^{(A)}_{\mu\nu}$ satisfy [53,54]

$$\nabla^\mu C_{\mu\nu} = -\frac{1}{8}\partial_\nu\theta\tilde{R}R, \qquad \nabla^\mu T^{(A)}_{\mu\nu} = -\frac{\lambda_2}{4}\partial_\nu\theta\tilde{F}F. \quad (6)$$

Since the pseudoscalar field is spacetime coordinate dependent, which leads to $\partial_\nu\theta \neq 0$, then Eq. (5) reduces to

$$\lambda_1 \tilde{R}R + \lambda_2 \tilde{F}F = 0, \tag{7}$$

which is exactly the pseudoscalar field equation $W_{(\theta)} = 0$. Hence, the pseudoscalar field equation is not independent of the EOM (2).

Considering the traceless properties of $C_{\mu\nu}$ and stress-energy tensor $T^{(A)}_{\mu\nu}$, we have

$$W_{\mu\nu} = E_{\mu\nu} + 16\lambda_1 \pi G C_{\mu\nu} = 0, \tag{8}$$

where $E_{\mu\nu} = R_{\mu\nu} - \Lambda g_{\mu\nu} + 8\pi G T^{(A)}_{\mu\nu}$, and we have used the trace of EOM $R = 4\Lambda$.

To study the dynamics of the dual fluid in $(2+1)$-dimensional flat spacetime, we assume the general $(3+1)$-dimensional black brane metric [40]

$$ds^2 = -f(r)d\tau^2 + 2H(r)drd\tau + r^2 dx_i dx^i, \quad i = 1, 2. \tag{9}$$

Then, the induced metric on the cutoff surface $\Sigma_c(r = r_c)$ outside the horizon r_h with the intrinsic coordinates $\tilde{x}^a \sim (\tilde{\tau} = \sqrt{f(r_c)}\tau, \tilde{x}^i = r_c x^i)$ is

$$\begin{aligned} ds^2_{2+1} &= \gamma_{ab} dx^a dx^b = -f(r_c)d\tau^2 + r_c^2 dx_i dx^i \\ &= -d\tilde{\tau}^2 + \delta_{ij} d\tilde{x}^i d\tilde{x}^j. \end{aligned} \tag{10}$$

We require the metric Eq. (10) to be flat when perturbing the bulk metric Eq. (9) and will investigate the dual fluid living on the $\Sigma_c(r = r_c)$.

Substituting the metric Eq. (9) into field equation (8), we find that the Cotten tensor $C_{\mu\nu}$ automatically vanishes, and $R_{\mu\nu} - \Lambda g_{\mu\nu}$ only depends on r. This leads the electromagnetic tensor $T^{(A)}_{\mu\nu}$ to be only r dependent. In this paper, we only consider the electric field for the static background solution. Hence, the vector potential A only depends on r, which ensures that $T^{(A)}_{\mu\nu}$ is only r dependent. We set $A_\mu dx^\mu = A(r, q)d\tau$, where q is related to the charge of the black hole and then $A'(r, q)$ is obviously nonzero. As to the pseudoscalar field θ for the static and stable background configuration, we consider the pseudoscalar field θ to be spatially dependent first without loss of generality. From the electromagnetic field equation (3), there exists three components of the electromagnetic field equation:

$$\begin{aligned} W^{x_1}{}_{(A)} &= -\frac{\lambda_2 A'(r,q)}{r^2 H(r)} \frac{\partial\theta(r, x_1, x_2)}{\partial x_1} = 0, \\ W^{x_2}{}_{(A)} &= \frac{\lambda_2 A'(r,q)}{r^2 H(r)} \frac{\partial\theta(r, x_1, x_2)}{\partial x_2} = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} W^{\tau}{}_{(A)} = \frac{1}{rH^3(r)} [&rH'(r)A'(r,q) - 2A'(r)H(r) \\ &- rA''(r,q)H(r)] = 0. \end{aligned} \tag{12}$$

Therefore, the pseudoscalar field θ only depends on r and is independent of coordinates x_1 and x_2 to keep Eq. (3) satisfied for $A'(r, q) \neq 0$. In addition, it is worth noting that the pseudoscalar field $\theta(r)$ for the background solution usually has been employed in some other discussion on the holographic models for fluid/gravity duality [40–42].

As to the bulk metric Eq. (9), following Ref. [9], we can introduce two types of diffeomorphism transformations: (i) a Lorentz boost with constant boost parameter β_i and (ii) a transformation of r and associated rescalings of τ and x^i. Taking the nonrelativistic hydrodynamic long-wavelength expansion parametrized by $\epsilon \to 0$, we have

$$\partial_\tau \sim \epsilon^2, \qquad \partial_i \sim \epsilon, \qquad \partial_r \sim \epsilon^0. \tag{13}$$

Together with $\beta^i = \frac{r_c}{\sqrt{f(r_c)}} v^i$, $v_i = v_i(\tau, x^i)$, $P = P(\tau, x^i)$, and scaling $v_i \sim \epsilon$ and $P \sim \epsilon^2$, we can express the transformed bulk metric up to $\mathcal{O}(\epsilon^2)$ in the form [51]

$$\begin{aligned} ds^2 = {}& -f(r)d\tau^2 + 2H(r)drd\tau + r^2 dx_i dx^i - 2r^2\left(1 - \frac{r_c^2 f(r)}{r^2 f(r_c)}\right) v_i dx^i d\tau - \frac{2r_c^2 H(r)}{f(r_c)} v_i dx^i dr + r^2\left(1 - \frac{r_c^2 f(r)}{r^2 f(r_c)}\right) \\ &\times \left(v^2 d\tau^2 + \frac{r_c^2 v_i v_j}{f(r_c)} dx^i dx^j\right) + f(r)\left(\frac{rf'(r)}{f(r)} - \frac{r_c f'(r_c)}{f(r_c)}\right) P d\tau^2 + \frac{r_c^2 H(r)}{f(r_c)} v^2 dr d\tau \\ &+ \left(\frac{r_c f'(r_c)H(r)}{f(r_c)} - 2H(r) - 2rH'(r)\right) P dr d\tau + \mathcal{O}(\epsilon^3), \end{aligned} \tag{14}$$

where the terms in the last two lines are all of $\mathcal{O}(\epsilon^2)$.

Under these two types of diffeomorphism transformations, both $\theta(r)$ and $A_\mu dx^\mu$ will be expanded. After promoting v^i and P to be (τ, x^i) dependent and adopting the scaling $v_i \sim \epsilon$ and $P \sim \epsilon^2$, we have

$$A(r,q)d\tau \to A(r,q)\left[d\tau - \frac{r_c^2 v_i(\tau, x^i)}{f(r_c)} dx^i + \frac{r_c^2 v(\tau, x^i)^2}{2f(r_c)} d\tau + \left(\frac{r_c f'(r_c)}{2f(r_c)} - \frac{A'(r,q)r}{A(r,q)}\right) P(\tau, x^i) d\tau\right] + \mathcal{O}(\epsilon^3), \tag{15}$$

$$\theta(r) \to \theta(r) - r\theta'(r)P(\tau, x^i). \quad (16)$$

In this charged configuration, we only focus on the electromagnetic degrees of freedom (d.o.f.) induced by the above two kinds of diffeomorphisms, which can be roughly regraded as gravitational and do not turn on the independent electromagnetic d.o.f. [13]. Note that the same approach recently has been adopted to set up a new magnetohydrodynamic/gravity correspondence in higher-dimensional flat Minkowski space for the independent electromagnetic d.o.f. [11].

Now substituting the perturbed black brane metric Eq. (14), electromagnetic field Eq. (15), and pseudoscalar field Eq. (16) into field equations (2)–(4), we have the EOM at $\mathcal{O}(\epsilon^0)$:

$$C^{(0)}_{rr} = C^{(0)}_{\tau\tau} = C^{(0)}_{ii} = 0, \qquad E^{(0)}_{rr} = \frac{2H'(r)}{rH(r)} = 0,$$
$$E^{(0)}_{\tau\tau} = f(r)\left[-3 - \frac{f'(r)H'(r)}{2H^3(r)} + \frac{f''(r)}{2H^2(r)} + \frac{f'(r)}{rH^2(r)} - \frac{4\pi G A'^2(r,q)}{H^2(r)}\right] = 0,$$
$$E^{(0)}_{ii} = 3r^2 + \frac{rf(r)H'(r)}{H^3(r)} - \frac{f(r)}{H^2(r)} - \frac{rf'(r)}{H^2(r)} - \frac{4\pi G r^2 A'^2(r,q)}{H^2(r)} = 0, \qquad (i = 1, 2). \quad (17)$$

The pseudoscalar field equation automatically reaches $W^{(0)}_{(\theta)} = 0$ and the electromagnetic field equation $W^{\tau(0)}_{(A)}$ takes the same form as Eq. (12). From $E^{(0)}_{rr}$, the function $H(r)$ should equal a constant and here we take $H(r) = 1$ for simplicity. Then, $f(r)$ and $A(r, q)$ can be obtained in the forms

$$f(r) = r^2 - \frac{m}{r} + \frac{q^2}{r^2}, \qquad A_\mu dx^\mu = \frac{1}{\sqrt{4\pi G}}\frac{q}{r}d\tau. \quad (18)$$

The integral constants m and q here are related to the gravitational mass $M = \frac{mV_2}{8\pi G}$ and the total charge $Q^2 = \frac{4\pi q^2}{G}$, respectively. Moreover, m in terms of the real root of $f(r_h) = 0$ is $m = r_h^3 + \frac{q^2}{r_h}$. Then, the Hawking temperature T_h of the black brane is obtained:

$$T_h = \frac{f'(r_h)}{4\pi} = \frac{1}{4\pi}\left(3r_h - \frac{q^2}{r_h^3}\right). \quad (19)$$

The condition $3r_h^4 \geq q^2$ should be satisfied for $T_h \geq 0$. Moreover, the perturbed metric Eq. (14) also solves these field equations (2)–(4) at $\mathcal{O}(\epsilon)$.

At $\mathcal{O}(\epsilon^2)$, a correction term

$$ds_c^2 = r^2(F(r)\sigma_{ij} + F_A(r)\sigma^A_{ij})dx^i dx^j \quad (20)$$

needs to be added to the perturbed metric Eq. (14) to cancel the terms of the tensor sector and pseudoscalar sector due to the spatial $SO(2)$ rotation symmetry of the black brane background [51]. Here, σ_{ij} and σ^A_{ij} take $\partial_{(i}v_{j)} - \frac{1}{2}\delta_{ij}\partial_k v^k$ and $\frac{1}{2}(\epsilon_{ik}\sigma_j{}^k + \epsilon_{jk}\sigma_i{}^k)$, respectively. The gauges $F(r_c) = 0$ and $F_A(r_c) = 0$ are chosen to keep the induced metric γ_{ab} invariant. Then, the equations of the pseudoscalar and electromagnetic fields at $\mathcal{O}(\epsilon^2)$ are obtained:

$$W^{(2)}_{(\theta)} = \left[\frac{\lambda_1 f^2(r)}{r^3H^3(r)}\left(\frac{f'(r)}{f(r)} - \frac{2}{r}\right)^2 + \frac{\lambda_2(A^2(r,q))'}{2r^2H(r)}\right]\frac{r_c^2\Omega}{f(r_c)} = 0, \quad (21)$$

$$W^{r(2)}_{(A)} = \frac{A'(r,q)}{H^2(r)}\partial_i v^i = 0, \quad (22)$$

$$W^{\tau(2)}_{(A)} = -\frac{F'(r)A'(r,q)}{H^2(r)}\partial_i v^i + \lambda_2\frac{r_c^2 A(r,q)\theta'(r)}{r^2 f(r_c)}\Omega = 0, \quad (23)$$

where Eq. (22) leads to the incompressible condition $\partial_i v^i = 0$ of the dual fluid on the Σ_c for $A'(r,q) \neq 0$. Then, we can also obtain the so-called nonvortical condition $\Omega \equiv \epsilon^{ij}\partial_i v_j = 0$ for Eqs. (21) and (23) with solutions (18). With these incompressible and nonvortical conditions, a new constraint condition at $\mathcal{O}(\epsilon^3)$ reads as

$$\partial^2 v_i = 0, \qquad i = 1, 2. \quad (24)$$

From EOM $W^{(2)}_{\mu\nu} = 0$, the Cotten tensors $C^{(2)}_{\mu\nu}$ are given by

$$C^{(2)}_{rr} = -\frac{r_c^2 H(r)}{2r^4 f(r_c)}\frac{d}{dr}\left[\frac{r^2 f(r)}{H^2(r)}\left(\frac{f'(r)}{f(r)} - \frac{2}{r}\right)\theta'(r)\right]\Omega,$$
$$C^{(2)}_{\tau\tau} = -\frac{r_c^2}{2r^3 H(r)f(r_c)}\frac{d}{dr}\left[\frac{rf^{3/2}(r)}{H^2(r)}\left(\frac{f'(r)}{f(r)} - \frac{2}{r}\right)\theta'(r)\right]\Omega,$$
$$C^{(2)}_{xx} + C^{(2)}_{yy} = -\frac{r_c^2}{2H(r)f(r_c)}\frac{d}{dr}\left[\frac{f^2(r)}{H^2(r)}\left(\frac{f'(r)}{f(r)} - \frac{2}{r}\right)\theta'(r)\right]\Omega, \quad (25)$$

and also vanish with $\Omega = 0$. With the help of the incompressible condition $\partial_i v^i = 0$, $E^{(2)}_{\mu\nu}$ disappears when imposing the requirement

$$\frac{d}{dr}\left[r^2\left(\frac{f(r)}{H(r)}F'(r) + 1\right)\right]\tilde{\sigma}_{ij} + \frac{d}{dr}\left[r^2\left(\frac{f(r)}{H(r)}F'_A(r) + \frac{\lambda_1 f(r)\theta'(r)}{2H^2(r)}\left(\frac{f'(r)}{f(r)} - \frac{2}{r}\right)\right)\right]\tilde{\sigma}^A_{ij} = 0. \quad (26)$$

Notice that $\tilde{\sigma}_{ij}$ and $\tilde{\sigma}^A_{ij}$ have different tensor structures, which leads two second-order differential equations, which can be solved separately as

$$F'(r) = \frac{H(r)}{f(r)}\left(\frac{c_F}{r^2} - 1\right),$$
$$F_A'(r) = \frac{1}{f(r)}\left(\frac{H(r)}{r^2}c_{F_A} - \frac{\lambda_1 f'(r)\theta'(r)}{2H(r)}\right) + \frac{\lambda_1\theta'(r)}{rH(r)}. \quad (27)$$

The integration constants c_F and c_{F_A} are determined by keeping the functions $F(r)$ and $F_A(r)$ regular at the horizon r_h. It is easy to find that the constants c_F and c_{F_A} are

$$c_F = r_h^2, \qquad c_{F_A} = \frac{\lambda_1 r_h^2 f'(r_h)\theta'(r_h)}{2H^2(r_h)}. \quad (28)$$

According to the fluid/gravity duality, the Brown-York tensor on the Σ_c can be identified as the energy-momentum tensor of the dual fluid. Since the existence of the gravitational CS term, there are three possible contributions that need to be explained: the usual Gibbons-Hawking boundary term, a term arising from the variation of the gravitational CS term, and the boundary counterterm. The Brown-York tensor T_{ab}^{BY} on the Σ_c can be derived from

$$T_{ab}^{\mathrm{BY}} = \frac{1}{8\pi G}(K\gamma_{ab} - K_{ab} - T_{ab}^{cs} + \mathcal{C}\gamma_{ab}), \quad (29)$$

where $\gamma_{ab} = g_{ab} - n_a n_b$ is an induced metric on the Σ_c, and K is the trace of the extrinsic curvature tensor K_{ab} of Σ_c which is defined by $K_{ab} = \gamma^{\delta}{}_{a}\nabla_{\delta}n_b$. As shown in Ref. [55], the contribution T_{ab}^{cs} from $\theta\tilde{R}R$ does not contribute to the Brown-York tensor. $\mathcal{C}$ is an unfixed constant which can bring a finite result when the cutoff surface goes to the AdS boundary, as determined below.

Plugging the perturbed metric Eqs. (14) and (20) into Eq. (29), the Brown-York tensor T_{ab}^{BY} of the dual fluid in the $\tilde{x}^a \sim (\tilde{\tau}, \tilde{x}^i)$ coordinates can be described as

$$\tilde{T}_{ab}^{\mathrm{BY}} = \tilde{T}_{ab}^{(0)} + \tilde{T}_{ab}^{(1)} + \tilde{T}_{ab}^{(2)} + \mathcal{O}(\epsilon^3), \quad (30)$$

where

$$8\pi G\tilde{T}_{ab}^{(0)}d\tilde{x}^a d\tilde{x}^b = -\left(\frac{2\sqrt{f(r_c)}}{r_c} + \mathcal{C}\right)d\tilde{\tau}^2 + \frac{1}{\sqrt{f(r_c)}}\left(\frac{f'(r_c)}{2} + \frac{f(r_c)}{r_c} + \mathcal{C}\right)d\tilde{x}_i d\tilde{x}^i,$$
$$8\pi G\tilde{T}_{ab}^{(1)}d\tilde{x}^a d\tilde{x}^b = -\left(\frac{f(r)}{r^2}\right)'_c \frac{r_c^2\beta_i}{\sqrt{f(r_c)}}d\tilde{x}^i d\tilde{\tau},$$
$$8\pi G\tilde{T}_{ab}^{(2)}d\tilde{x}^a d\tilde{x}^b = \left(\frac{f(r)}{r^2}\right)'_c \frac{r_c^2}{2\sqrt{f(r_c)}}[(2P + \beta^2)d\tilde{\tau}^2 + (\beta_i\beta_j + \kappa P\delta_{ij})d\tilde{x}^i d\tilde{x}^j]$$
$$- [(1 + f(r_c)F'(r_c))\tilde{\sigma}_{ij} + f(r_c)F_A'(r_c)\tilde{\sigma}_{ij}^A]d\tilde{x}^i d\tilde{x}^j + \mathcal{O}(\epsilon^3), \quad (31)$$

with $\kappa = \frac{r_c^3}{2f(r_c)}(\frac{f(r)}{r^2})'_c - 3 - r_c(\frac{f(r)}{r^2})''_c/(\frac{f(r)}{r^2})'_c$. Here, the trace of the stress-energy tensor $\tilde{T}_{ab}$ in the $\tilde{x}^a \sim (\tilde{\tau}, \tilde{x}^i)$ coordinates can be computed up to $\mathcal{O}(\epsilon^2)$ with $\tilde{T}_c = \tilde{T}_{ab}^{\mathrm{BY}}\tilde{\gamma}^{ab} = \frac{2K+3\mathcal{C}}{8\pi G}$. For the boundary at infinity, we can take the corresponding factor $\mathcal{C} = -2$ to remove the divergence in the energy-momentum tensor.

In these $(2+1)$-dimensional parity-violating hydrodynamic systems, the energy-momentum tensor of the fluid with the first-order gradient expansion usually takes the following form:

$$\tilde{T}^{ab} = \rho\tilde{u}^a\tilde{u}^b + p\tilde{P}^{ab} - 2\eta\tilde{\sigma}^{ab} - \zeta\tilde{\Theta}\tilde{P}^{ab} - 2\eta_A\tilde{\sigma}_A^{ab} - \zeta_A\tilde{\Omega}\tilde{P}^{ab}, \quad (32)$$

where $\tilde{P}_{ab} = \tilde{\gamma}_{ab} + \tilde{u}_a\tilde{u}_b$. The shear viscosity η and the bulk viscosity ζ are canonical transport coefficients, while the Hall viscosity η_A and curl viscosity ζ_A arise from the parity-violating effect. Here, $\tilde{u}^a = \frac{(1,\beta^i)}{\sqrt{1-\beta^2}}$, ρ is the energy density, p is the pressure, $\tilde{\sigma}_{ab}$ is the shear, and $\tilde{\Theta} = \tilde{\partial}_a\tilde{u}^a$ describes the expansion.

Under the nonrelativistic long-wavelength expansion, in the above stress-energy tensor we have $\tilde{\Theta} = 0$ by using the incompressible condition $\tilde{\partial}_a\tilde{u}^a \sim \tilde{\partial}_i\beta^i = 0$ at the order ϵ^2, which results in the vanishing of the term $\zeta\tilde{\Theta}\tilde{P}_{ab}$. With the incompressible and nonvortical conditions ($\tilde{\Theta} = 0$ and $\tilde{\Omega} = 0$), up to $\mathcal{O}(\epsilon^2)$, the components of the energy-momentum tensor in the nonrelativistic limit are given by

$$\tilde{T}_{\tau\tau} = \rho + (p+\rho)\beta^2, \qquad \tilde{T}_{\tau i} = -(p+\rho)\beta_i,$$
$$\tilde{T}_{ij} = (p+\rho)\beta_i\beta_j + p\delta_{ij} - 2\eta\tilde{\sigma}_{ij} - 2\eta_A\tilde{\sigma}_{ij}^A. \quad (33)$$

The energy density ρ_0 and pressure p_0 of the dual fluid at $\mathcal{O}(\epsilon^0)$ take the following form:

$$\rho_0 = -\frac{\sqrt{f(r_c)}}{4\pi G r_c} - \frac{\mathcal{C}}{8\pi G},$$
$$p_0 = \frac{1}{8\pi G\sqrt{f(r_c)}}\left(\frac{f'(r_c)}{2} + \frac{f(r_c)}{r_c}\right) + \frac{\mathcal{C}}{8\pi G},$$
$$\omega = \rho_0 + p_0 = \frac{r_c^2}{16\pi G\sqrt{f(r_c)}}\left(\frac{f(r)}{r^2}\right)'_c. \quad (34)$$

Up to $\mathcal{O}(\epsilon^2)$, the energy density ρ_c and the pressure p_c are corrected to be

$$\rho_c = \rho_0 + 2\omega P, \qquad p_c = p_0 + \omega\kappa P, \tag{35}$$

and the transport coefficients such as the shear viscosity η and Hall viscosity η_A of the dual fluid are given by

$$\eta = \frac{1 + f(r_c)F'(r_c)}{16\pi G}, \qquad \eta_A = \frac{f(r_c)F'_A(r_c)}{16\pi G}. \tag{36}$$

Based on Eq. (28), the shear viscosity η and Hall viscosity η_A are obtained:

$$\begin{aligned} \eta &= \frac{1}{16\pi G}\frac{r_h^2}{r_c^2}, \\ \eta_A &= \frac{\lambda_1}{32\pi G r_c^2}(r_h^2\theta'(r_h)f'(r_h) - r_c^2 f'(r_c)\theta'(r_c) \\ &\quad + 2r_c f(r_c)\theta'(r_c)). \end{aligned} \tag{37}$$

It is worth noting that the Hall viscosity η_A depends on the gravitational CS term $\lambda_1\theta\tilde{R}R$. If taking the vanishing of $\lambda_1\theta\tilde{R}R$ for the parameter $\lambda_1 = 0$, we have $\eta_A = 0$.

From the metric (9), we consider a quotient under shift of x^i, $x^i \sim x^i + n^i$ with $n^i \in Z$. The spatial R^2 on the Σ_c turns out to be a two-tours T^2 with r_c-dependent volume $V_2(r_c) = r_c^2$. Then, the entropy density s_c on the Σ_c is described by $S/V_2(r_c)$ in the form $\frac{1}{4G}\frac{r_h^2}{r_c^2}$ [7]. So, the ratios of the shear viscosity and Hall viscosity to entropy density read as

$$\begin{aligned} \frac{\eta}{s_c} &= \frac{1}{4\pi}, \\ \frac{\eta_A}{s_c} &= \frac{\lambda_1}{8\pi}\left[\left(3r_h - \frac{q^2}{r_h^3}\right)\theta'(r_h) - \left(3r_h + \frac{3q^2}{r_h^3} - \frac{4q^2}{r_h^2 r_c}\right)\theta'(r_c)\right]. \end{aligned} \tag{38}$$

Apparently, the ratio η/s_c is independent of r_c and does not receive any influence from the gravitational and electromagnetic CS terms. However, the ratio η_A/s_c is cutoff dependent and background dependent. If we take the cutoff surface to approach the black brane horizon, $r_c \to r_h$, η_A/s_c vanishes. In the infinite boundary limit $r_c \to \infty$, if we take the following assumptions $\theta(r_c) \to 0$, η_A/s_c becomes $\frac{\eta_A}{s_c} = \frac{\lambda_1\theta'(r_h)}{8\pi}(3r_h - \frac{q^2}{r_h^3})$, which is non-negative for $T_h \geq 0$.

The local temperature T_c on the Σ_c is identified as the temperature of the dual fluid. With the Tolman relation, we get the local temperature T_c:

$$T_c = \frac{T_h}{\sqrt{f(r_c)}} = \frac{1}{4\pi\sqrt{f(r_c)}}\left(3r_h - \frac{q^2}{r_h^3}\right). \tag{39}$$

With $T_c = 0$, the ratio η_A/s_c of the dual fluid disappears in the infinite boundary. It implies that the parity-violating dual fluid corresponds to quantum Hall fluid with time-reversal symmetry.

In addition, we define the chemical potential μ_c as $\mu_c = \frac{1}{4\pi G\sqrt{f(r_c)}}(\frac{q}{r_h} - \frac{q}{r_c})$ and the charge density $q_c = \frac{q}{V_2(r_c)}$ with $\frac{q}{r_c^2}$ on the Σ_c. Then, the thermodynamic relation can be verified,

$$\omega - s_c T_c = q_c\mu_c. \tag{40}$$

The conservation equations of the Brown-York tensor on the Σ_c, the so-called momentum constraint, can be deduced from EOM (2),

$$\begin{aligned} &-(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu})n^\mu\gamma^\nu{}_b \\ &\quad = (16\lambda_1\pi G C_{\mu\nu} + 8\pi G T^{(A)}_{\mu\nu})n^\mu\gamma^\nu{}_b \\ &\quad \Rightarrow \tilde{\partial}^a\tilde{T}^{BY}_{ab} = T^{(A)}_{\mu b}n^\mu, \end{aligned} \tag{41}$$

where n^μ is the unit normal vector of Σ_c, and the Cotton tensor $C_{\mu\nu}$ vanishes since it has no contribution to the source terms of the momentum constraint up to $\mathcal{O}(\epsilon^3)$. Taking the index $b = \tau$, the temporal component of the momentum constraint at $\mathcal{O}(\epsilon^2)$ reads as

$$\begin{aligned} \tilde{\partial}^a\tilde{T}^{BY}_{a\tau} &= T^{(A)}_{\mu\tau}n^\mu = \frac{1}{\sqrt{f(r_c)}}T^r{}_\tau = 0, \\ &\Rightarrow -\left(\frac{f(r)}{r^2}\right)'_c\frac{r_c^2}{\sqrt{f(r_c)}}\tilde{\partial}_i\beta^i = 0, \end{aligned} \tag{42}$$

which leads to the incompressible condition of the dual fluid $\tilde{\partial}_i\beta^i = 0$.

Taking the index $b = j$, the spatial component of the momentum constraint at $\mathcal{O}(\epsilon^3)$ is given by

$$\begin{aligned} \tilde{\partial}^a\tilde{T}^{BY}_{aj} &= T^{(A)}_{\mu j}n^\mu = F_{ja}J^a, \\ &\Rightarrow \left(\frac{f(r)}{r^2}\right)'_c\frac{r_c^2}{2\sqrt{f(r_c)}}(\tilde{\partial}_\tau\beta_j + \beta^i\tilde{\partial}_i\beta_j + \kappa\tilde{\partial}_j P) \\ &\qquad - [(1 + f(r_c)F'(r_c))\tilde{\partial}^2\beta_j + f(r_c)F'_A(r_c)\epsilon^{ij}\tilde{\partial}^2\beta_i] \\ &= F_{ja}J^a. \end{aligned} \tag{43}$$

With Eq. (24), the vanishing of $\partial^2 v_j$ implies $\tilde{\partial}^2\beta_j = 0$ at $\mathcal{O}(\epsilon^3)$. Then, the momentum constraint reduces to

$$\tilde{\partial}_\tau\beta_j + \beta^i\tilde{\partial}_i\beta_j + \tilde{\partial}_j P_r = f_j, \qquad (j = 1, 2), \tag{44}$$

$$\tilde{\partial}_i\beta^i = 0, \qquad \tilde{\Omega} \equiv \epsilon^{ij}\tilde{\partial}_i\beta_j = 0, \tag{45}$$

which corresponds to the MHD equation [56,57]. Note that the nonvortical dual fluid possesses the shear viscosity η

and Hall viscosity η_A, but these kinematic viscosities ν and ν_A related to η and η_A do not appear in Eq. (44). It is special for the $(2+1)$-dimensional dual fluid. Here, the pressure density and external force density read as

$$f_j = \frac{F_{ja}J^a}{r_c\omega}, \qquad P_r = \frac{\tilde{p}_c - \tilde{p}_0}{\tilde{\rho}_0 + \tilde{p}_0} = \frac{\tilde{p}_c - \tilde{p}_0}{\omega} = \kappa P. \tag{46}$$

For this external force density f_j, the term $F_{ja}J^a$ consists of $F_{ji}J^i$ and $F_{j\tau}J^\tau$, where $F_{j\tau}J^\tau$ arises from the background electric field, while $F_{ji}J^i$ corresponds to the Lorentz force due to the magnetic field arising from the perturbation of the background electric field. Moreover, the current J^a dual to the bulk electromagnetic field is obtained by $J^a = -n_\mu F^{\mu a}$ on the Σ_c. We have $J^\tau = -n_r F^{r\tau}$ at $\mathcal{O}(\epsilon^0)$ and $J^i = -n_r F^{ri}$ at $\mathcal{O}(\epsilon)$. The partial derivative for the boundary current J^a satisfies $\partial_a J^a \sim \epsilon^2$. With the electromagnetic field equation (3) and the transformation of $\theta(r)$ Eq. (15), there exists a current conservation law $\partial_a J^a = 0$ at $\mathcal{O}(\epsilon^2)$, which is not affected by the pseudoscalar field. This shows that the conservation law of the boundary current J^a coincides with the incompressible condition $\tilde{\partial}_i\beta^i = 0$ for the constant dual charge density.

On the other hand, the MHD equation (44) can be expanded in the form

$$\tilde{\partial}_1\tilde{\partial}_\tau\beta_2 + \tilde{\partial}_1(\beta^1\tilde{\partial}_1\beta_2) + \tilde{\partial}_1(\beta^2\tilde{\partial}_2\beta_2) + \tilde{\partial}_1\tilde{\partial}_2 P_r = \partial_1 f_2, \tag{47}$$

$$\tilde{\partial}_2\tilde{\partial}_\tau\beta_1 + \tilde{\partial}_2(\beta^1\tilde{\partial}_1\beta_1) + \tilde{\partial}_2(\beta^2\tilde{\partial}_2\beta_1) + \tilde{\partial}_2\tilde{\partial}_1 P_r = \partial_2 f_1. \tag{48}$$

Considering Eqs. (47) and (48), we can obtain

$$\tilde{\partial}_\tau(\tilde{\partial}_1\beta_2 - \tilde{\partial}_2\beta_1) + \tilde{\partial}_1(\beta^1\tilde{\partial}_1\beta_2) - \tilde{\partial}_2(\beta^2\tilde{\partial}_2\beta_1) + \tilde{\partial}_1(\beta^2\tilde{\partial}_2\beta_2) - \tilde{\partial}_2(\beta^1\tilde{\partial}_1\beta_1) = \partial_1 f_2 - \partial_2 f_1. \tag{49}$$

Using the nonvortical condition $\tilde{\Omega} \equiv \epsilon^{ij}\tilde{\partial}_i\beta_j = 0$ and the incompressible condition $\partial_i\beta^i = 0$, the above equation leads to

$$\tilde{\partial}_\tau\Omega + \beta^j\tilde{\partial}_j\Omega = \epsilon^{ij}\partial_i f_j = 0. \tag{50}$$

In the nondynamic case, there is a constraint condition $\epsilon^{ij}\partial_i f_j = 0$ for the external force, but this constraint of the external force is of the order ϵ^4, while the MHD equation we focused on is of the order ϵ^3. The constraint conditions we considered in the paper, such as the nonvortical and incompressible conditions, are of the order not higher than ϵ^3. In the order of ϵ^3, the above MHD equation for the dual fluid is kept.

We can also try to set up the holographic duality between the nondynamical CS gravity and $(2+1)$-dimensional vortical fluid in the cutoff flat surface Σ_c, namely, $\Omega \neq 0$. Notice that the expressions for $C^{(2)}_{\mu\nu}$ [Eq. (25)] do not disappear for black brane solutions $f(r)$ and $H(r)$ [Eq. (18)]. As in Ref. [51], we introduce some correction terms in the perturbed metric to cancel the residual curl scalar Ω at $\mathcal{O}(\epsilon^2)$,

$$ds_s^2 = (-f(r)k(r)d\tau^2 + 2H(r)h(r)drd\tau + r^2 g(r)dx_i dx^i)\Omega, \tag{51}$$

and then the overall perturbed metric with Eqs. (14) and (20) is given by

$$ds_o^2 = ds^2 + ds_c^2 + ds_s^2. \tag{52}$$

Inserting this overall perturbed metric into the field equations (2)–(4), we find that the pseudoscalar field equation still takes the form of Eq. (21), and the electromagnetic field equations at $\mathcal{O}(\epsilon^2)$ are changed to

$$W^{r(2)}_{(A)} = \frac{A'(r,q)}{H^2(r)}\partial_i v^i + \frac{3h(r)f(r)A'(r,q)}{2H^3(r)} \times \left[\frac{f'(r)}{f(r)} - \frac{H'(r)}{H(r)} + \frac{h'(r)}{3h(r)}\right]\Omega = 0, \tag{53}$$

$$W^{\tau(2)}_{(A)} = -\frac{F'(r)A'(r,q)}{H^2(r)}\partial_i v^i + \left[g'(r)A'(r,q) + \lambda_2\frac{r_c^2 A(r,q)\theta'(r)}{r^2 f(r_c)}\right]\Omega = 0. \tag{54}$$

Consider $A'(r,q) \neq 0$, $\Omega \neq 0$, and different structures of $\partial_i v^i$ and Ω in these electromagnetic field equations, Eqs. (53) and (54) lead to the incompressible condition $\partial_i v^i = 0$ and

$$\frac{f'(r)}{f(r)} - \frac{H'(r)}{H(r)} + \frac{h'(r)}{3h(r)} = 0, \tag{55}$$

$$g'(r)A'(r,q) + \lambda_2\frac{r_c^2 A(r,q)\theta'(r)}{r^2 f(r_c)} = 0. \tag{56}$$

In addition, in order to satisfy the pseudoscalar field equation (21) with $\Omega \neq 0$, we have

$$\frac{\lambda_1 f^2(r)}{r^3 H^3(r)}\left(\frac{f'(r)}{f(r)} - \frac{2}{r}\right)^2 + \frac{\lambda_2 (A^2(r,q))'}{2r^2 H(r)} = 0. \tag{57}$$

One can see that Eq. (57) does not vanish with $H(r) = 1$ and the black brane solution $f(r)$ Eq. (18), while it disappears in the trivial case for the locally pure AdS spacetimes with $f(r) = r^2$ and $A(r,q) = 0$. Therefore, this new fluid/gravity duality does not set up in the nondynamical CS modified gravity.

It is interesting to note that the action of dynamical CS modified gravity is related to the kinetic term for the pseudoscalar field θ [39]; it is expected that this dynamical CS modified gravity can help to overcome the difficulty. We will discuss this possibility in the next section.

III. DUAL FLUID TO DYNAMICAL CS MODIFIED GRAVITY

The action of dynamical CS modified gravity coupled to the electromagnetic CS term is [41]

$$\mathcal{I}_G=\frac{1}{16\pi G}\int d^4x\sqrt{-g}(R-2\Lambda-4\pi GF_{\mu\nu}F^{\mu\nu}) + \int d^4x\sqrt{-g}\left(\frac{\lambda_1}{4}\theta\tilde{R}R+\frac{\lambda_2}{4}\theta\tilde{F}F-\frac{1}{2}\partial_\mu\theta\partial^\mu\theta-V(\theta)\right). \tag{58}$$

The corresponding new equations of motion (NEOM), the electromagnetic and scalar fields equations read

$$\hat{W}_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}+16\lambda_1\pi GC_{\mu\nu} + 8\pi GT^{(\theta)}_{\mu\nu}+8\pi GT^{(A)}_{\mu\nu}=0, \tag{59}$$

$$\hat{W}^{\nu}_{(A)}=\nabla_\mu F^{\mu\nu}-\lambda_2\partial_\mu\theta\tilde{F}^{\mu\nu}=0, \tag{60}$$

$$\hat{W}_{(\theta)}=\frac{\lambda_1}{4}\tilde{R}R+\frac{\lambda_2}{4}\tilde{F}F+\Box\theta-\frac{dV}{d\theta}=0, \tag{61}$$

with the stress-energy tensor of the pseudoscalar field

$$T^{(\theta)}_{\mu\nu}=-\partial_\mu\theta\partial_\nu\theta+\frac{1}{2}g_{\mu\nu}(\partial\theta)^2+g_{\mu\nu}V(\theta).$$

Obviously, the new electromagnetic field equation $\hat{W}^{\nu}_{(A)}$ is not influenced by the newly added dynamical terms of the pseudoscalar field and still takes the expression as Eq. (3). Similarly, the new pseudoscalar field equation can be derived also by the covariant derivative of NEOM, and the NEOM can also be rewritten as

$$\hat{W}_{\mu\nu}=\hat{E}_{\mu\nu}+16\lambda_1\pi GC_{\mu\nu}=0, \tag{62}$$

with $\hat{E}_{\mu\nu}=E_{\mu\nu}-8\pi G(\partial_\mu\theta\partial_\nu\theta+g_{\mu\nu}V(\theta))$.

Similar to the analysis for the case in the nondynamical CS modified gravity, we also consider that the vector potential A only relates to r for the background configuration in this case, and then the pseudoscalar field θ should be related only to r from the electromagnetic field equation (60). Then, substituting the overall perturbed black brane metric Eq. (52), perturbed pseudoscalar field Eq. (16), and electromagnetic field Eq. (15) into the NEOM $\hat{W}_{\mu\nu}$ and the new pseudoscalar field equation $\hat{W}_{(\theta)}$, respectively, the background equations for the NEOM at $\mathcal{O}(\epsilon^0)$ are obtained:

$$\hat{E}^{(0)}_{rr}=E^{(0)}_{rr}-8\pi G\theta'^2(r)=0,$$
$$\hat{E}^{(0)}_{\tau\tau}=E^{(0)}_{\tau\tau}+8\pi Gf(r)V(\theta)=0,$$
$$\hat{E}^{(0)}_{ii}=E^{(0)}_{ii}-8\pi Gr^2V(\theta)=0,\quad C^{(0)}_{rr}=C^{(0)}_{\tau\tau}=C^{(0)}_{ii}=0. \tag{63}$$

The new pseudoscalar field equation at $\mathcal{O}(\epsilon^0)$ can be worked out as

$$\hat{W}^{(0)}_{(\theta)}=-\frac{dV(\theta)}{d\theta}+\frac{\theta'(r)f(r)}{H^2(r)}\left(\frac{2}{r}-\frac{H'(r)}{H(r)}+\frac{f'(r)}{f(r)}\right) +\frac{\theta''(r)f(r)}{H^2(r)}=0, \tag{64}$$

and the new electromagnetic field has only a τ component and takes the same expression for Eq. (12) in nondynamical CS modified gravity.

Unfortunately, getting the analytic solutions for functions $f(r)$, $H(r)$, and $A(r,q)$ from field equations (12), (63), and (64) is hard work. We can expand the functions $f(r)$, $H(r)$, $A(r,q)$, and $\theta(r)$ with a small parameter ξ,

$$\begin{aligned} f(r)&=f_0(r)+\xi f_1(r)+\cdots,\\ H(r)&=H_0(r)+\xi H_1(r)+\cdots,\\ A(r,q)&=A_0(r,q)+\xi A_1(r,q)+\cdots,\\ \theta(r)&\to\xi\theta(r),\\ V(\theta)&\to\xi^2V(\theta), \end{aligned} \tag{65}$$

where $f_0(r)$, $H_0(r)$, and $A_0(r,q)$ can be obtained by solving these field equations (63) at $\mathcal{O}(\xi^0)$, which reads as Eq. (18),

$$f_0(r)=r^2-\frac{m}{r}+\frac{q^2}{r^2},\qquad H_0(r)=1,$$
$$A_0(r,q)=\frac{1}{\sqrt{4\pi G}}\frac{q}{r}. \tag{66}$$

Substituting Eq. (65) into field equations (12), (63), and (64), at $\mathcal{O}(\xi)$, we have

$$f_1(r)=-\frac{4\sqrt{\pi G}qC_1}{r^2},\qquad H_1(r)=0,$$
$$A_1(r,q)=-\frac{C_2}{r}+C_3, \tag{67}$$

and the pseudoscalar θ obeys the following equation:

$$\theta''(r)+\left(\frac{f_0'(r)}{f_0(r)}+\frac{2}{r}\right)\theta'(r)-\frac{dV(\theta)/d\theta}{f_0(r)}=0, \tag{68}$$

where C_1, C_2, and C_3 are the integral constants. This scalar field equation for the pseudoscalar field is the same as Eq. (17) in Refs. [41,42]. Specifying the form of the potential, the solutions of this scalar field equation and asymptotical behaviors of the pseudoscalar field have been also discussed in Refs. [41,42]. With these background equations $\hat{W}^{(0)}_{\mu\nu}$, $W^{\tau(0)}_{(A)}$, and $\hat{W}^{(0)}_{(\theta)}$, we find that the perturbed metric together with the perturbed electromagnetic and pseudoscalar fields automatically satisfy the field equations (59)–(61) at $\mathcal{O}(\epsilon)$.

At $\mathcal{O}(\epsilon^2)$, the new electromagnetic field equations (60) still take the forms of Eqs. (53) and (54). Then, we can also obtain the incompressible condition $\partial_i v^i = 0$ with $\Omega \neq 0$, $A'(r,q) \neq 0$, and different structures of Ω and $\partial_i v^i$. The new pseudoscalar field equation $\hat{W}^{(2)}_{(\theta)}$ reads as

$$\hat{W}^{(2)}_{(\theta)} = \left[\frac{\theta'(r)f(r)f^{1/2}(r_c)}{H^2(r)}(k'(r)-h'(r)+g'(r)) + f^{1/2}(r_c)(k(r)-2h(r))\frac{dV(\theta)}{d\theta} + \frac{\lambda_1 r_c^2 f^2(r)}{r^3 H^3(r) f(r_c)}\left(\frac{f'(r)}{f(r)}-\frac{2}{r}\right)^2 + \frac{\lambda_2 r_c^2 (A^2(r,q))'}{2r^2 H(r) f(r_c)}\right]\Omega. \tag{69}$$

Although the existence of the pseudoscalar field makes it hard to get the solution, it can avoid the trivial solution we met in the uncharged case. Under the incompressible condition $\partial_i v^i = 0$, the nonvanishing components of $\hat{E}^{(2)}_{\mu\nu}$ are expressed as

$$\begin{aligned}
\hat{E}^{(2)}_{rr} &= \left[\frac{2h'(r)}{r} + \left(\frac{H'(r)}{H(r)} - \frac{2}{r}\right)g'(r) - g''(r)\right]\Omega,\\
\hat{E}^{(2)}_{\tau\tau} &= \left[(g'(r)-h'(r))f'(r) - k'(r)f(r)\left(\frac{H'(r)}{2H(r)} - \frac{7f'(r)}{4f(r)} - \frac{2}{r}\right) + f(r)k''(r)\right.\\
&\quad \left. - (k(r)-2h(r))f(r)/r\left(\frac{H'(r)}{H(r)} - \frac{3f'(r)}{f(r)} - \frac{2}{r}\right) + 8\pi G A'^2(r,q)(k(r)-h(r))\right]\frac{f(r)\Omega}{2H^2(r)},\\
\hat{E}^{(2)}_{xx} + \hat{E}^{(2)}_{yy} &= \left[2(h'(r)-k'(r)) + rg'(r)\left(\frac{H'(r)}{H(r)} - \frac{f'(r)}{f(r)} - \frac{4}{r}\right) - rg''(r)\right.\\
&\quad \left. + 2(k(r)-2h(r))\left(\frac{H'(r)}{H(r)} - \frac{f'(r)}{f(r)} - \frac{1}{r}\right) + 8\pi G r f(r) A'^2(r,q)(g(r)-h(r))\right]\frac{rf(r)\Omega}{H^2(r)},
\end{aligned} \tag{70}$$

with the requirement of Eq. (26). The components of $C^{(2)}_{\mu\nu}$ are shown in Eq. (25). With the Dirichlet boundary condition $k(r_c) = 0$ and $g(r_c) = 0$, the equations of motion (25) and (70), pseudoscalar field equation (69), and electromagnetic field equations (53) and (54) at $\mathcal{O}(\epsilon^2)$ are expected to be formally solved with the background equations, but here we will not concentrate on finding these solutions.

Plugging the overall perturbed metric Eq. (52) into the Brown-York tensor T^{BY}_{ab} Eq. (29) of the dual fluid, T^{BY}_{ab} in the $\tilde{x}^a \sim (\tilde{\tau}, \tilde{x}^i)$ coordinates can be described as

$$\tilde{T}^{\mathrm{BY}}_{ab} = \tilde{T}^{(0)}_{ab} + \tilde{T}^{(1)}_{ab} + \tilde{T}^{(2)}_{ab} + \mathcal{O}(\epsilon^3), \tag{71}$$

where

$$\begin{aligned}
8\pi G \tilde{T}^{(0)}_{ab} d\tilde{x}^a d\tilde{x}^b &= -\left[\frac{2\sqrt{f(r_c)}}{r_c H(r_c)} + \mathcal{C}\right] d\tilde{\tau}^2 + \left[\frac{f'(r_c) r_c + 2f(r_c)}{2r_c H(r_c)\sqrt{f(r_c)}} + \mathcal{C}\right] d\tilde{x}_i d\tilde{x}^i,\\
8\pi G \tilde{T}^{(1)}_{ab} d\tilde{x}^a d\tilde{x}^b &= -\left(\frac{f(r)}{r^2}\right)'_c \frac{r_c^2 \beta_i}{H(r_c)\sqrt{f(r_c)}} d\tilde{x}^i d\tilde{\tau},\\
8\pi G \tilde{T}^{(2)}_{ab} d\tilde{x}^a d\tilde{x}^b &= \left(\frac{f(r)}{r^2}\right)'_c \frac{r_c^2}{2H(r_c)\sqrt{f(r_c)}}\left[\left(\left(2 - \frac{4f(r_c)H'(r_c)}{r_c^2 H(r_c)}\left(\frac{f(r)}{r^2}\right)'^{-1}_c\right)P\right.\right.\\
&\quad \left. + \beta^2 - \frac{2f(r_c)^{3/2}(h(r_c) - r_c g'(r_c))}{r_c^3}\left(\frac{f(r)}{r^2}\right)'^{-1}_c \tilde{\Omega}\right) d\tilde{\tau}^2 + (\beta_i\beta_j + \kappa(r_c) P \delta_{ij}) d\tilde{x}^i d\tilde{x}^j\Big]\\
&\quad - \left[\left(1 + \frac{f(r_c)}{H(r_c)}F'(r_c)\right)\tilde{\sigma}_{ij} + \frac{f(r_c)}{H(r_c)}F'_A(r_c)\tilde{\sigma}^A_{ij} + \frac{\varpi(r_c)}{2H(r_c)}\tilde{\Omega}\delta_{ij}\right] d\tilde{x}^i d\tilde{x}^j + \mathcal{O}(\epsilon^3),
\end{aligned} \tag{72}$$

with

$$\kappa(r_c) = \frac{r_c^3}{2f(r_c)}\left(\frac{f(r)}{r^2}\right)_c' - 3 - r_c\left(\frac{f(r)}{r^2}\right)_c' / \left(\frac{f(r)}{r^2}\right)_c'' + \frac{(r^2 f(r))_c'}{r_c^3}\frac{H'(r_c)}{H(r_c)},$$
$$\varpi(r_c) = (f'(r_c) + 2f(r_c)/r_c)h(r_c) - 2f(r_c)(g'(r_c) + k'(r_c)).$$

Here we have used the Dirichlet boundary condition $k(r_c) = 0$ and $g(r_c) = 0$. We can choose the Landau frame to make the term of $\tilde{\Omega}$ disappear in Eq. (72), which is necessary to match the stress-energy tensor of the fluid, as we will discuss below.

Now, the components of the stress-energy tensor in the nonrelativistic limit with the incompressible conditions ($\tilde{\Theta} = 0$) up to $\mathcal{O}(\epsilon^2)$ are

$$\tilde{T}_{\tau\tau} = \rho + (p+\rho)\beta^2, \qquad \tilde{T}_{\tau i} = -(p+\rho)\beta_i,$$
$$\tilde{T}_{ij} = (p+\rho)\beta_i\beta_j + p\delta_{ij} - 2\eta\tilde{\sigma}_{ij} - 2\eta_A\tilde{\sigma}^A_{ij} - \zeta_A\tilde{\Omega}\delta_{ij}. \quad (73)$$

Then, the energy density ρ_0 and pressure p_0 of the dual fluid at $\mathcal{O}(\epsilon^0)$ satisfy

$$\omega = \rho_0 + p_0 = \frac{r_c^2}{16\pi G H(r_c)\sqrt{f(r_c)}}\left(\frac{f(r)}{r^2}\right)_c'. \quad (74)$$

The transformed energy density ρ_c and the pressure p_c up to $\mathcal{O}(\epsilon^2)$ are corrected to be

$$\rho_c = \rho_0 + \left(2 - \frac{4f(r_c)H'(r_c)}{r_c^2 H(r_c)}\left(\frac{f(r)}{r^2}\right)_c^{\prime -1}\right)\omega P,$$
$$p_c = p_0 + \omega\kappa(r_c)P, \quad (75)$$

and the transport coefficients, such as the shear viscosity η, Hall viscosity η_A, and Curl viscosity ζ_A of the dual fluid, are given by

$$\eta = \frac{1}{16\pi G}\left(1 + \frac{f(r_c)}{H(r_c)}F'(r_c)\right),$$
$$\eta_A = \frac{1}{16\pi G}\frac{f(r_c)F_A'(r_c)}{H(r_c)}, \qquad \zeta_A = \frac{\varpi(r_c)}{16\pi G H(r_c)}. \quad (76)$$

With Eq. (28), the shear viscosity η takes the same value in the nondynamical CS case, and its ratio η/s_c equals $1/4\pi$; the Hall viscosity η_A and its ratio are obtained

$$\eta_A = \frac{\lambda_1}{16\pi G}\left(\frac{r_h^2\theta'(r_h)f'(r_h)}{2r_c^2 H^2(r_h)} - \frac{f'(r_c)\theta'(r_c)}{2H^2(r_c)} + \frac{f(r_c)\theta'(r_c)}{r_c H^2(r_c)}\right),$$
$$\frac{\eta_A}{s_c} = \frac{\lambda_1}{4\pi}\left(\frac{\theta'(r_h)f'(r_h)}{2H^2(r_h)} - \frac{r_c^2 f'(r_c)\theta'(r_c)}{2r_h^2 H^2(r_c)} + \frac{r_c f(r_c)\theta'(r_c)}{r_h^2 H^2(r_c)}\right), \quad (77)$$

and the ratio of the Curl viscosity reads as $\zeta_A/s_c = \frac{r_c^2\varpi(r_c)}{4\pi r_h^2 H(r_c)}$. However, the other two ratios η_A/s_c and ζ_A/s_c are cutoff dependent and background dependent.

If we take the cutoff surface to approach the black brane horizon, $r_c \to r_h$, η_A/s_c vanishes while ζ_A/s_c arrives at a finite value $\frac{f'(r_h)h(r_h)}{4\pi H(r_h)}$. In the infinite boundary limit $r_c \to \infty$; if we take the following assumptions,

$$\frac{f(r_c)}{r_c^2} \to 1 - \mathcal{O}(r_h^3/r_c^3), \qquad H(r_c) \to 1,$$
$$\theta(r_c) \to \mathcal{O}(r_h^m/r_c^m), \quad (78)$$

η_A/s_c becomes

$$\frac{\eta_A}{s_c} = \frac{\lambda_1}{8\pi}\frac{\theta'(r_h)f'(r_h)}{H^2(r_h)} \quad (79)$$

when $m > 3$. Note that the ratio η_A/s_c in our charged black brane background takes the same form as in the neutral black brane background [41], but in our case, the electromagnetic field influence is imprinted in the metric function. Our result agrees with the previous result by using the probe limit of the pseudoscalar field based on the same action by including the electromagnetic CS terms [42].

The conservation equations of the Brown-York tensor on the Σ_c, the so-called momentum constraint, can be deduced from NEOM (59),

$$-\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right)n^\mu\gamma^\nu{}_b$$
$$= (16\lambda_1\pi G C_{\mu\nu} + 8\pi G\gamma T^{(\theta)}_{\mu\nu} + 8\pi G T^{(A)}_{\mu\nu})n^\mu\gamma^\nu{}_b$$
$$\Rrightarrow \tilde{\partial}^a\tilde{T}^{\rm BY}_{ab} = (T^{(\theta)}_{\mu b} + T^{(A)}_{\mu b})n^\mu. \quad (80)$$

Taking the index $b = \tau$, the temporal component of the momentum constraint at $\mathcal{O}(\epsilon^2)$ reads as

$$\tilde{\partial}^a\tilde{T}^{\rm BY}_{a\tau} = (T^{(\theta)}_{\mu\tau} + T^{(A)}_{\mu\tau})n^\mu = 0,$$
$$\Rightarrow -\left(\frac{f(r)}{r^2}\right)_c'\frac{r_c^2}{H(r_c)\sqrt{f(r_c)}}\tilde{\partial}_i\beta^i = 0, \quad (81)$$

which leads to the incompressible condition of the dual fluid $\tilde{\partial}_i\beta^i = 0$.

Taking the index $b = j$, the spatial component of the momentum constraint at $\mathcal{O}(\epsilon^3)$ is given by

$$\tilde{\partial}^a\tilde{T}^{\rm BY}_{aj} = (T^{(\theta)}_{\mu j} + T^{(A)}_{\mu j})n^\mu, \quad (82)$$

$$\Rightarrow \left(\frac{f(r)}{r^2}\right)'_c \frac{r_c^2}{2H(r_c)\sqrt{f(r_c)}}(\tilde{\partial}_\tau\beta_j + \beta^i\tilde{\partial}_i\beta_j + \kappa(r_c)\tilde{\partial}_j P) - \left[\left(1+\frac{f(r_c)}{H(r_c)}F'(r_c)\right)\tilde{\partial}^2\beta_j + \frac{f(r_c)}{H(r_c)}F'_A(r_c)\epsilon^{ij}\partial^2\beta_i + \frac{\varpi(r_c)}{2H(r_c)}\epsilon^{ik}\partial_i\partial_j\beta_k\right] = \frac{r_c\sqrt{f(r_c)}\theta'^2(r_c)}{H^2(r_c)}\partial_j P + F_{ja}J^a. \tag{83}$$

With the momentum constraint, we can obtain the incompressible Navier-Stokes equations,

$$\tilde{\partial}_\tau\beta_j + \beta^i\tilde{\partial}_i\beta_j + \tilde{\partial}_j P_r - \nu\tilde{\partial}^2\beta_j - \nu_A\epsilon^{ij}\partial^2\beta_i - \xi_A\epsilon^{ik}\partial_i\partial_j\beta_k = f_j, \qquad \tilde{\partial}_i\beta^i = 0, \tag{84}$$

which correspond to the MHD turbulence equation with viscosity in plasma physics [58]. Here, the external force density reads $f_j = \frac{\sqrt{f(r_c)}\theta'^2(r_c)}{H^2(r_c)\omega}\partial_j P + \frac{F_{ja}J^a}{r_c\omega}$. Besides the Lorentz force due to the magnetic field, which arises from the perturbation of the electric field and electric force for the electric field, it is worth noting that the forcing term f_j is also affected by the pseudoscalar field θ with $\frac{\sqrt{f(r_c)}\theta'^2(r_c)}{H^2(r_c)\omega}\partial_j P$. The pressure density P_r equals $\kappa(r_c)P$ and these kinematic viscosities ν, ν_A, and ξ_A are defined by $\nu = \eta/\omega$, $\nu_A = \eta_A/\omega$, and $\xi_A = \zeta_A/\omega$.

IV. CLOSING REMARKS

Based on the static black brane metric, we applied the two finite diffeomorphism transformations and nonrelativistic long-wavelength expansion to derive the bulk equations of motion up to $\mathcal{O}(\epsilon^2)$ at an arbitrary cutoff surface Σ_c outside the horizon in the nondynamical and dynamical CS modified gravities. In this nondynamical model, the dual nonvortical fluid possesses the shear viscosity η and Hall viscosity η_A. According to the momentum constraint from the conservation equations of the Brown-York tensor, the dual nonvortical fluid obeys the MHD equation. However, these kinematic viscosities ν and ν_A related to η and η_A do not appear in this MHD equation, which is special for the $(2+1)$-dimensional dual fluid. The ratio η/s_c equals the universal value $1/4\pi$, while the ratio η_A/s depends on the r_c and black brane charge q. In the dynamical framework, besides the shear viscosity η and Hall viscosity η_A, the dual fluid possesses another so-called Curl viscosity ζ_A, whose ratio to entropy density ζ_A/s also depends on the Σ_c and black brane charge q. Moreover, the dual vortical fluid obeys the MHD turbulence equation with external force density influenced by the electromagnetic and pseudoscalar fields. At the infinite boundary, the ratio η_A/s_c agrees with the previous result by using the probe limit of the pseudoscalar field in the charged black brane background. In addition, even though the electromagnetic field is related to the pseudoscalar field, there exists the current conservation law $\partial_a J^a = 0$ at the order ϵ^2 in both cases, which is not affected by the pseudoscalar field.

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