How does Casimir energy fall? IV. Gravitational interaction of regularized quantum vacuum energy

K. A. Milton*

Laboratoire Kastler Brossel, Université Pierre et Marie Curie, 4, Place Jussieu Case 74, F-75252 Paris Cedex 05, France

K. V. Shajesh

Department of Physics, Southern Illinois University-Carbondale, Carbondale, Illinois 62901, USA

S. A. Fulling

Departments of Mathematics and Physics, Texas A&M University, College Station, Texas 77843, USA

Prachi Parashar

H. L. Dodge Department of Physics and Astronomy, University of Oklahoma, Norman, Oklahoma 73019, USA

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Several years ago we demonstrated that the Casimir energy for perfectly reflecting and imperfectly reflecting parallel plates gravitated normally, that is, obeyed the equivalence principle. At that time the divergences in the theory were treated only formally, without proper regularization, and the coupling to gravity was limited to the canonical energy-momentum-stress tensor. Here we strengthen the result by removing both of those limitations. We consider, as a toy model, massless scalar fields interacting with semitransparent (δ-function) potentials defining parallel plates, which become Dirichlet plates for strong coupling. We insert space and time point-split regulation parameters, and obtain well-defined contributions to the self-energy of each plate, and the interaction energy between the plates. (This self-energy does not vanish even in the conformally coupled, strong-coupled limit.) We also compute the local energy density, which requires regularization near the plates. In general, the energy density includes a surface energy that resides precisely on the boundaries. This energy is also regulated. The gravitational interaction of this well-defined system is then investigated, and it is verified that the equivalence principle is satisfied.

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I. INTRODUCTION

The subject of quantum vacuum energy (the Casimir effect) dates from the same year as the discovery of renormalized quantum electrodynamics, 1948 [1]. It puts the lie to the presumption that zero-point energy is not observable [2–4]. On the other hand, because of the severe divergence structure of the theory, controversy has surrounded it from the beginning. Here we will deal with divergences carefully, by using point splitting in space and time.

The volume divergence, sometimes called the bulk term, is rather easily isolated, and apparently has no physical consequences, since it does not refer to anything but the properties of empty space. Once bodies are introduced, additional divergences appear. Sharp boundaries and even soft ones give rise to divergences in the local energy density near the surface [5–7]. Curvature introduces additional divergences, and if the surfaces possess discontinuities such as corners, there will be additional divergent terms associated with these. These divergences may make it impossible to extract meaningful self-energies of single objects, the cancellations for the electromagnetic field at perfectly conducting planes [1] and spheres [8] being accidental [5]. How can something finite be meaningfully extracted from this wealth of infinities (which are actually finite, but very large, if a physically reasonable microscopic cutoff is inserted)? These objections have been most forcefully presented by Graham *et al.* [9,10], and by Barton [11,12], but they date back to Deutsch and Candelas [5].

In fact, it has appeared for some time that these surface divergences can be dealt with successfully in a process of renormalization (see for example, Refs. [13,14]) and that finite self-energies, in a generalization of the sense of Boyer [8], may be extracted. So in this paper we will consider not only the universally recognized unambiguous Casimir interaction energies, but also the divergent, but regulated, self-energies of the separate bodies, here planar objects. It is critical to do this, because gravity couples to the local energy-momentum tensor, and such surface divergences and self-energies promise serious difficulties. How is the

*Permanent address: H. L. Dodge Department of Physics and Astronomy, University of Oklahoma, Norman, OK 73019, USA. milton@nhn.ou.edu

completely finite Casimir interaction energy of a pair of parallel conducting plates, as well as the divergent self-energies of nonideal plates, accelerated by gravity? We must also address the issue of the renormalization of Einstein's equations resulting from singular Casimir surface energy densities [15,16]. The resolution of these questions turns out to be surprisingly less straightforward than the reader might suspect.

In the remainder of the Introduction we shall recapitulate the previous papers in this series [17–19]. We use natural units (in particular, $c = 1$), so that energy is identified with mass, and acceleration has the units of inverse length.

A. Gravitational coupling to an ideal Casimir apparatus

Brown and Maclay [20] showed that, for parallel perfectly conducting plates separated by a distance a in the z direction, the electromagnetic stress tensor acquires the vacuum expectation value between the plates

$$\langle T^{\mu\nu}\rangle = \frac{\mathcal{E}_C}{a}\mathrm{diag}(1,-1,-1,3), \qquad \mathcal{E}_C = -\frac{\pi^2}{720a^3}, \quad (1.1)$$

$\mathcal{E}_C$ being Casimir's energy per unit area. Outside the plates the value of $\langle T^{\mu\nu}\rangle$ is 0. What is the gravitational interaction of this Casimir apparatus? As shown in Ref. [17], this question can be most simply addressed through use of the gravitational definition of the energy-momentum tensor,

$$\delta W_m \equiv -\frac{1}{2}\int (dx)\sqrt{-g}\delta g^{\mu\nu}T_{\mu\nu} = \frac{1}{2}\int (dx)\sqrt{-g}\delta g_{\mu\nu}T^{\mu\nu}. \quad (1.2)$$

For a weak field,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}, \quad (1.3)$$

so if we think of turning on the gravitational field as a small perturbation, we can ignore $\sqrt{-g}$. The gravitational energy, for a static situation, is therefore given by ($\delta W = -\int dt\delta E$),

$$E_g = -\int (d\mathbf{x})h_{\mu\nu}T^{\mu\nu}. \quad (1.4)$$

The Fermi metric locally describes an inertial coordinate system:

$$h_{00} = -gz, \qquad h_{0i} = h_{ij} = 0, \quad (1.5)$$

which is appropriate for describing a constant gravitational field. Let us consider a Casimir apparatus of parallel plates separated by a distance a, with transverse dimensions $L \gg a$. Let the apparatus be oriented at an angle α with respect to the direction of gravity. The Cartesian coordinate system attached to the Earth is (x, y, z), where z is the direction of $-\mathbf{g}$. See Fig. 1. Now we calculate the gravitational energy

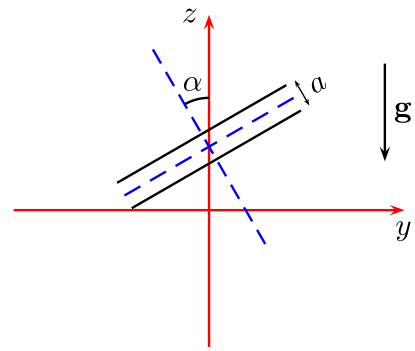

FIG. 1 (color online). A Casimir apparatus of two parallel plates, the normal to which makes an angle α with respect to the direction of gravity, the negative z axis. The parallel plates are indicated by the heavy lines.

$$E_g = \int (d\mathbf{x})gzT^{00} = g\mathcal{E}_C L^2 z_0 + K, \quad (1.6)$$

where K is a constant, independent of the center z_0 of the apparatus. Thus, the gravitational force per area $A = L^2$ on the apparatus is independent of orientation:

$$\mathcal{F} \equiv \frac{F}{A} = -\frac{\partial E_g}{A\partial z_0} = -g\mathcal{E}_C, \quad (1.7)$$

a small upward push. Therefore, $\mathcal{E}_C$ just adds to the mass energy of the plates, precisely in accordance with the equivalence principle.

B. Uniform acceleration, semitransparent plates

A more exact relativistic calculation is based on the use of Rindler coordinates to describe constant acceleration [18]. In the balance of this paper, for simplicity, we will consider scalar fields interacting with δ-function (semitransparent) plates. Relativistically, uniform (but necessarily ξ-dependent) acceleration is described by hyperbolic motion,

$$t = \xi \sinh \tau, \qquad z = \xi \cosh \tau, \quad (1.8)$$

which induces the metric

$$dt^2 - dz^2 - d\mathbf{r}_\perp^2 = \xi^2 d\tau^2 - d\xi^2 - d\mathbf{r}_\perp^2. \quad (1.9)$$

The d'Alembertian operator has cylindrical form

$$\partial^2 = -\left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 + \nabla_\perp^2$$
$$= -\frac{1}{\xi^2}\left(\frac{\partial}{\partial \tau}\right)^2 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial}{\partial \xi}\right) + \nabla_\perp^2. \quad (1.10)$$

For two semitransparent (δ-function) [6,21] plates at ξ_1 and ξ_2, the Green's function can be written as

$$G(x,x') = \int \frac{d\omega}{2\pi}\frac{(d\mathbf{k}_\perp)}{(2\pi)^2} e^{-i\omega(\tau-\tau')} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp} g(\xi,\xi'), \tag{1.11}$$

where the reduced Green's function satisfies

$$\left[-\frac{\omega^2}{\xi^2} - \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + k_\perp^2 + \lambda_1\delta(\xi-\xi_1) + \lambda_2\delta(\xi-\xi_2)\right] \times g(\xi,\xi') = \frac{1}{\xi}\delta(\xi-\xi'), \tag{1.12}$$

which we recognize as just the problem of two concentric semitransparent cylinders [22] with the replacements $m \to \zeta = -i\omega$ and $\kappa \to k$. The explicit solution for the reduced Green's function g is given in Ref. [18] in terms of modified Bessel functions, $I_\zeta(k_\perp\xi)$, $K_\zeta(k_\perp\xi)$.

The *canonical* energy-momentum tensor for a scalar field is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}, \qquad \mathcal{L} = -\frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - \frac{1}{2}V\phi^2, \tag{1.13}$$

where the Lagrange density includes the δ-function potential,

$$V = \lambda_1\delta(\xi-\xi_1) + \lambda_2\delta(\xi-\xi_2). \tag{1.14}$$

Using the equation of motion,

$$(-\partial^2 + V)\phi = 0, \tag{1.15}$$

we find the energy density to be

$$T_{00} = \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}\phi\frac{\partial^2}{\partial\tau^2}\phi + \frac{\xi}{2}\frac{\partial}{\partial\xi}\left(\phi\xi\frac{\partial}{\partial\xi}\phi\right) + \frac{\xi^2}{2}\nabla_\perp\cdot(\phi\nabla_\perp\phi). \tag{1.16}$$

We obtain the vacuum expectation value of the stress tensor, $\langle T_{\mu\nu}\rangle$, from the replacement

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{i}G(x,y). \tag{1.17}$$

The (gravitational) force density is given by [23]

$$f_\lambda = -\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}T^\nu{}_\lambda) + \frac{1}{2}T^{\mu\nu}\partial_\lambda g_{\mu\nu}, \tag{1.18}$$

so the gravitational force per unit area on the system is, upon integration by parts,

$$\frac{1}{g}\mathcal{F} = \int d\xi\xi f_\xi = -\int\frac{d\xi}{\xi^2}T_{00} = \int d\xi\xi \times \int\frac{d\hat\zeta(d\mathbf{k}_\perp)}{(2\pi)^3}\hat\zeta^2 g(\xi,\xi), \qquad (\zeta = \hat\zeta\xi). \tag{1.19}$$

This is the change of momentum per unit Rindler coordinate time τ, which when multiplied by the gravitational acceleration at ξ_0, namely, $g = 1/\xi_0$, is the gravitational force/area $\mathcal{F}$ on the Casimir energy in an apparatus centered at Rindler position ξ_0. The reader is referred to Ref. [18] for details. For the purposes here, all we need is the weak acceleration limit. This is the limit in which ξ, ξ', ξ_1, and ξ_2 all tend to infinity, but expanded about ξ_0 so that differences such as $\xi - \xi'$ are finite. Likewise, we rescale $\zeta = \xi_0\hat\zeta$, and regard $\hat\zeta$ and $\kappa^2 = k_\perp^2 + \hat\zeta^2$ as finite. Then the Green's function reduces to exactly the expected result, for example, between the plates, $\xi_1 < \xi$, $\xi' < \xi_2$ $(a = \xi_2 - \xi_1)$,

$$\xi_0 g(\xi,\xi') \to \frac{1}{2\kappa}e^{-\kappa|\xi-\xi'|} + \frac{1}{2\kappa\Delta}\left[\frac{\lambda_1\lambda_2}{(2\kappa)^2}2\cosh\kappa(\xi-\xi') - \frac{\lambda_1}{2\kappa}\left(1+\frac{\lambda_2}{2\kappa}\right)e^{-\kappa(\xi+\xi'-2\xi_2)} - \frac{\lambda_2}{2\kappa}\left(1+\frac{\lambda_1}{2\kappa}\right)e^{\kappa(\xi+\xi'-2\xi_1)}\right], \tag{1.20}$$

where the denominator (which has a simple interpretation in terms of multiple reflections) is

$$\Delta = \left(1+\frac{\lambda_1}{2\kappa}\right)\left(1+\frac{\lambda_2}{2\kappa}\right)e^{2\kappa a} - \frac{\lambda_1\lambda_2}{4\kappa^2}. \tag{1.21}$$

The flat-space limit also holds outside the plates.

From this follows the explicit force per unit area on the two-plate apparatus as

$$\mathcal{F} = \frac{g}{96\pi^2a^3}\int_0^\infty dy y^3\frac{1+\frac{1}{y+\lambda_1 a}+\frac{1}{y+\lambda_2 a}}{(\frac{y}{\lambda_1 a}+1)(\frac{y}{\lambda_2 a}+1)e^y - 1} - \frac{g}{96\pi^2a^3}\int_0^\infty dy y^2\left[\frac{1}{\frac{y}{\lambda_1 a}+1} + \frac{1}{\frac{y}{\lambda_2 a}+1}\right] = -g(\mathcal{E}_C + \mathcal{E}_{S_1} + \mathcal{E}_{S_2}), \tag{1.22}$$

which is just $-g$ times the Casimir energy/area of the two semitransparent plates, including the divergent parts associated with each plate. Note that the divergent parts are independent of the separation between the plates. The divergent terms are self-energies $\mathcal{E}_{S_{1,2}}$ which simply renormalize the mass/area of each plate:

$$\mathcal{E}_{\text{total}} = m_1 + m_2 + \mathcal{E}_{S_1} + \mathcal{E}_{S_2} + \mathcal{E}_C = M_1 + M_2 + \mathcal{E}_C, \tag{1.23}$$

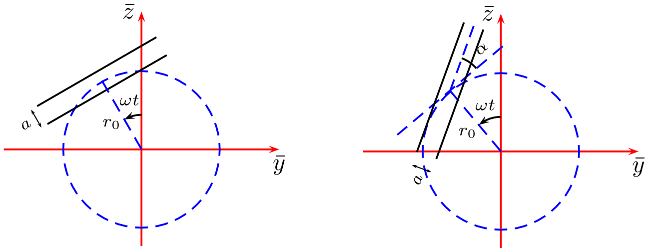

FIG. 2 (color online). Casimir apparatus undergoing circular motion. The Casimir energy contributes in the usual manner to the inertial mass of the system, and the divergent contributions to the energy renormalize the masses of the two Casimir plates. The first panel shows the normal of the apparatus in the radial direction, the second with that axis making an arbitrary angle α with respect to the radius.

and thus the gravitational force on the entire apparatus obeys the equivalence principle

$$\mathcal{F} = -g(M_1 + M_2 + \mathcal{E}_C). \tag{1.24}$$

This calculation has been implicitly carried out in the vacuum state of the field quantized in the Rindler coordinate system. For completeness one should also consider the presence of Unruh radiation (or Hawking-Hartle radiation, in the case of a Schwarzschild gravitational source). This complication is left for later investigation.

A third paper in this series [19] considered a Casimir apparatus undergoing centripetal acceleration as shown in Fig. 2. The centripetal force on the apparatus rotating with angular speed ω, $\omega r \ll 1$ is

$$\mathbf{F} = -\omega^2 \int (d\mathbf{r})\mathbf{r}T_{00}(\mathbf{r}) = -\omega^2 \mathbf{r}_{\rm CM}(m_1 + m_2 + E_{S_1} + E_{S_2} + E_C), \tag{1.25}$$

where $\mathbf{r}_{\rm CM}$ is the position vector of the center of energy. Again, the self-energies correctly renormalize the mass of the plates.

Other work demonstrating that Casimir energy possesses the correct Einstein inertia includes Ref. [24].

II. REGULATED CALCULATION OF CASIMIR ENERGY OF PARALLEL SEMITRANSPARENT PLATES

A. Fundamental formulas

In this and the following two sections we will consider Minkowski spacetime. We will also be freely using the same symbols to represent operators and functions illustrated by the Green's function

$$G(x, x') = \langle x|G|x'\rangle. \tag{2.1}$$

The imaginary frequency is represented by ζ.

The fundamental formula for the Casimir energy can be taken to be the famous trace-log formula,[1]

$$E = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \mathrm{Tr}\ln \mathcal{G}. \tag{2.2}$$

From this, by formal integration by parts, one obtains another commonly used form

$$E = -\int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\zeta^2 \mathrm{Tr}\mathcal{G}. \tag{2.3}$$

But one might rightly be suspicious of this because the integrals are not well defined. We will properly define the regulated versions of these integrals in the following subsection.

The analysis sketched in the Introduction may be equally well criticized for not dealing with divergences properly, and including manipulations with divergent integrals. In this paper, we will remedy this situation. We also wish to include arbitrary values of the conformal coupling parameter, because these correspond to more general couplings to gravity, and include the conformally coupled case which may have special virtues [25]. We will consider two semitransparent plates interacting with a massless scalar field, with the potential

$$V = \lambda_1\delta(z) + \lambda_2\delta(z - a). \tag{2.4}$$

The time-Fourier transformed Green's function has the form

$$\mathcal{G}(\mathbf{r}, \mathbf{r}'; \omega) = \int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp} g(z, z'), \tag{2.5}$$

where, between the plates, $0 < z, z' < a$, the reduced Green's function has precisely the form given in Eq. (1.20) with ξ and ξ' replaced by z and z', $\xi_1 = 0$, $\xi_2 = a$, and $\hat{\zeta} \to \zeta$. In particular, $\kappa = \sqrt{k_\perp^2 + \zeta^2}$. In the region outside the plates, the reduced Green's function has the form

$$z, z' < 0\colon\ g(z, z') = \frac{e^{-\kappa|z-z'|}}{2\kappa} - \frac{e^{\kappa(z+z')}}{2\kappa\Delta}\left[\frac{\lambda_2}{2\kappa}\left(1 - \frac{\lambda_1}{2\kappa}\right) + \frac{\lambda_1}{2\kappa}\left(1 + \frac{\lambda_2}{2\kappa}\right)e^{2\kappa a}\right], \tag{2.6a}$$

$$z, z' > a\colon\ g(z, z') = \frac{e^{-\kappa|z-z'|}}{2\kappa} - \frac{e^{\kappa(2a-z-z')}}{2\kappa\Delta}\left[\frac{\lambda_1}{2\kappa}\left(1 - \frac{\lambda_2}{2\kappa}\right) + \frac{\lambda_2}{2\kappa}\left(1 + \frac{\lambda_1}{2\kappa}\right)e^{2\kappa a}\right], \tag{2.6b}$$

where Δ is given by Eq. (1.21).

[1] A convincing argument for using this as a starting point is that then the correct free energy emerges upon replacing the imaginary frequency integral by the sum over Matsubara frequencies.

B. Point-split regularization

To define the integrals, we adopt point splitting in the time and the transverse directions (but not in the z direction, so as not to complicate the boundary conditions):

$$\tau = t_E - t'_E \to 0, \qquad \mathbf{R}_\perp = (\mathbf{r} - \mathbf{r}')_\perp \to 0. \tag{2.7}$$

Here we have made a Euclidean rotation,

$$\omega \to i\zeta, \qquad t \to it_E, \quad \text{so } \omega t \to -\zeta t_E. \tag{2.8}$$

Effectively $\mathcal{G}$ is now the cylinder kernel in the sense of Refs. [15,16,26]. For our transversely translationally invariant system, if we insert Eq. (2.5) into our fundamental form for the energy (2.2), and use the above regulator factors, we obtain for the energy per unit area

$$\mathcal{E} = -\frac{1}{2}\int \frac{d\zeta}{2\pi}\int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\delta}} \mathrm{Tr}\ln g, \tag{2.9}$$

in terms of the reduced Green's function. Here we have united frequency and transverse momentum as $\boldsymbol{\kappa} = (\zeta, \mathbf{k}_\perp)$, and similarly united the time and transverse spatial splittings as $\boldsymbol{\delta} = (\tau, \mathbf{R}_\perp)$. Let γ be the angle between $\boldsymbol{\delta}$ and the time axis. Thus, $\gamma = 0$ corresponds to time-splitting regularization, $\gamma = \pi/2$ to transverse space splitting. The latter splitting is in the neutral direction, as defined in Ref. [16], that is, not involved in the definition of the relevant stress-tensor component, or in the geometrically relevant direction. In that case, the integration by parts in passing to the regulated form of the energy (2.3) is legitimate, because the cutoff function does not depend on ζ. In general, when integrating over the spherical angles for $\boldsymbol{\kappa}$, α, and β, we encounter

$$\begin{aligned} f(\gamma) &= \int_{-1}^{1} d\cos\alpha \int_0^{2\pi} d\beta \cos^2\alpha\, e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\delta}} \to \frac{4\pi}{3}, \\ w(\gamma) &= \int_{-1}^{1} d\cos\alpha \int_0^{2\pi} d\beta\, e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\delta}} \to 4\pi. \end{aligned} \tag{2.10}$$

The limits are as $|\boldsymbol{\delta}| = \delta \to 0$. For transverse space splitting, the explicit forms of the cutoff functions are

$$\begin{aligned} f(\pi/2) &= 4\pi\left(-\frac{\cos\kappa\delta}{(\kappa\delta)^2} + \frac{\sin\kappa\delta}{(\kappa\delta)^3}\right), \\ w(\pi/2) &= 4\pi\frac{\sin\kappa\delta}{\kappa\delta}, \end{aligned} \tag{2.11}$$

and as a result

$$\mathcal{E}(\gamma = \pi/2) = -\frac{1}{(2\pi)^3}\int_0^\infty d\kappa\kappa^4 f(\pi/2)\mathrm{Tr}g. \tag{2.12}$$

For time splitting, the corresponding forms of the cutoff functions are obtained from

$$f(0) = \frac{d}{d\delta}[\delta f(\pi/2)], \qquad w(0) = w(\pi/2). \tag{2.13}$$

However, in this case the integration by parts leading to the regulated form of Eq. (2.3) proceeds as follows:

$$\begin{aligned} \mathcal{E}(\gamma = 0) &= -\frac{1}{2}\int \frac{d\zeta}{2\pi}\frac{(d\mathbf{k}_\perp)}{(2\pi)^2} e^{i\zeta\tau}\mathrm{Tr}\ln g = -\int_0^\infty \frac{d\kappa\kappa^2}{(2\pi)^3} \\ &\quad \times \int_{-1}^{1} d\cos\alpha \int_0^{2\pi} d\beta\kappa\cos\alpha \frac{1}{i\delta}(e^{i\kappa\delta\cos\alpha} - 1)\mathrm{Tr}g, \end{aligned} \tag{2.14}$$

which uses the indefinite integral

$$\int d\zeta e^{i\zeta\tau} = \frac{1}{i\tau}(e^{i\zeta\tau} - 1), \tag{2.15}$$

and, in view of the realization of g^{-1} as a differential operator,

$$g^{-1} = \zeta^2 + k_\perp^2 - \frac{d^2}{dz^2} + V, \tag{2.16}$$

we have

$$g^{-1}\frac{\partial}{\partial\zeta}g = -\frac{\partial}{\partial\zeta}g^{-1}g = -2\zeta g. \tag{2.17}$$

Now the integral over the angles in Eq. (2.14) is

$$\int_{-1}^{1} d\cos\alpha \int_0^{2\pi} d\beta\kappa\cos\alpha\frac{1}{i\delta}(e^{i\kappa\delta\cos\alpha} - 1) = \kappa^2 f(\pi/2), \tag{2.18}$$

that is, there is no difference between time and space splitting. The pressure anomaly [16] apparently affects only Eq. (2.3), not Eq. (2.2). Thus,

$$\mathcal{E}(\gamma = 0) = \mathcal{E}(\gamma = \pi/2) = -\frac{1}{(2\pi)^3}\int_0^\infty d\kappa\kappa^4 f(\pi/2)\mathrm{Tr}g. \tag{2.19}$$

(We have not examined other values of γ.)

Hence, if we insert the reduced Green's function given above in Eqs. (1.20) and (2.6), we find the energy/area to be given by (L_z is the extent of the system in the z direction)

$$\begin{aligned} \mathcal{E} = -\int_0^\infty \frac{d\kappa\kappa^2}{(2\pi)^3}\kappa^2 f(\pi/2)\Bigg\{\frac{L_z}{2\kappa} + \frac{1}{4\kappa^2\Delta}\Bigg[4(\kappa a + 1)\frac{\lambda_1\lambda_2}{(2\kappa)^2} \\ - 2e^{2\kappa a}\left(\frac{\lambda_1 + \lambda_2}{2\kappa} + 2\frac{\lambda_1\lambda_2}{(2\kappa)^2}\right)\Bigg]\Bigg\}. \end{aligned} \tag{2.20}$$

First we look at the Weyl, or bulk, term, that would be present with no boundaries, corresponding to the term in Eq. (2.20) proportional to L_z:

$$E_W(\gamma=\pi/2) = -\frac{V}{8\pi^3}\int_0^\infty d\kappa\kappa^2 f(\pi/2)\frac{1}{2}\kappa = -\frac{V}{2\pi^2\delta^4}, \tag{2.21}$$

just as expected. If we had replaced $f(\pi/2)$ by $f(0)$ to obtain the corresponding time-split divergence we would have obtained using Eq. (2.13)

$$E_W(\gamma=0) = \frac{3}{2\pi^2}\frac{V}{\delta^4}, \tag{2.22}$$

as is familiar, but as we have seen, if we regard Eq. (2.2) rather than Eq. (2.3) as fundamental, this is not legitimate, and Eq. (2.21) is the bulk energy for either type of regularization.

C. Self- and interaction energies

It is then straightforward to calculate the balance of the energy/area ($y = 2\kappa a$):

$$\mathcal{E} - \mathcal{E}_W = \frac{1}{128\pi^3 a^3}\int_0^\infty dy y^2 f(\pi/2)\left(\frac{1}{\frac{y}{\lambda_1 a}+1} + \frac{1}{\frac{y}{\lambda_2 a}+1}\right) - \frac{1}{96\pi^2 a^3}\int_0^\infty dy y^3 \frac{1+\frac{1}{y+\lambda_1 a}+\frac{1}{y+\lambda_2 a}}{(\frac{y}{\lambda_1 a}+1)(\frac{y}{\lambda_2 a}+1)e^y - 1}. \tag{2.23}$$

We have set the cutoff to zero in the second, finite term. That term is the same as given in Eq. (1.22) for the Casimir interaction energy $\mathcal{E}_C$ of parallel semitransparent plates.

The divergent term, which agrees with that in Eq. (1.22), $\mathcal{E}_{S_{1,2}}$, when the formal replacement in Eq. (2.10) is made, is the sum of contributions from each plate separately, which are unaware of the other plate. The self-energy of a single plate is, as $\delta \to 0$,

$$\mathcal{E}_{S_i} = \frac{\lambda_i}{8\pi^2}\left[\frac{1}{\delta^2} - \frac{\lambda_i}{8}\frac{\pi}{\delta} - \frac{\lambda_i^2}{12}\left(\ln\lambda_i\delta/2 + \gamma - \frac{4}{3}\right)\right] + O(\delta), \quad i = 1, 2. \tag{2.24}$$

This is for finite λ_i, $\lambda_i\delta \ll 1$. (Here γ is Euler's constant.) This expansion can be found from the heat kernel expansion given, for example, in Ref. [27]; to compare with our spatial-splitting results, we convert the heat kernel expansion to the cylinder kernel expansion using the formulas in Ref. [26]. See also Refs. [13,14]. In the Dirichlet limit, $\lambda_i \to \infty$, the self-energy is more divergent:

$$\mathcal{E}_{S_i} = \frac{1}{8\pi}\frac{1}{\delta^3}. \tag{2.25}$$

This also corresponds to the known surface term in the heat kernel expansion. The total energy thus has four components:

$$\mathcal{E} = \mathcal{E}_W + \mathcal{E}_{S_1} + \mathcal{E}_{S_2} + \mathcal{E}_C. \tag{2.26}$$

The interpretation of this result is straightforward: the Weyl term $\mathcal{E}_W$ is the unobservable vacuum energy of empty space, the self-energies $\mathcal{E}_{S_{1,2}}$ renormalize masses of the plates, and only the interaction term $\mathcal{E}_C$ is the observable Casimir energy.

III. LOCAL ENERGY DENSITY

A. Forms of stress tensor

To answer the question of how Casimir energy interacts with gravity, we must look at local quantities. The stress tensor, now including the conformal term, for a massless scalar field is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}(\partial_\lambda\phi\partial^\lambda\phi + V\phi^2) - \eta(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)\phi^2, \tag{3.1}$$

where η is the conformal parameter; $\eta = 1/6$ is the choice that makes conformal invariance manifest. Then, the Fourier-transformed expectation value of the stress tensor $t^{\mu\nu}$ given by

$$\langle T^{\mu\nu}\rangle = \int \frac{d\zeta}{2\pi}\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}e^{i\zeta\tau}e^{i\mathbf{k}_\perp\cdot\mathbf{R}_\perp}t^{\mu\nu}(z,z')|_{z'\to z} \tag{3.2}$$

is obtained with the quantum-mechanical replacement (1.17). In particular, the energy density is

$$T^{00} = \frac{1}{2}(\partial^0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}V\phi^2 - \eta\nabla^2\phi^2. \tag{3.3a}$$

Equation (3.3a) is the form obtained directly by variation of the Lagrangian with respect to g_{00}, but it can be rewritten using the equation of motion (1.15), including the potential (2.4), without changing the numerical values of T^{00}. For example, the equation of motion can be used to eliminate V entirely:

$$T^{00} = \frac{1}{2}(\partial^0\phi)^2 - \frac{1}{2}\phi(\partial^0)^2\phi + \frac{1}{4}(1-4\eta)\nabla^2\phi^2, \tag{3.3b}$$

which generalizes the flat-space analog of Eq. (1.16). If we take the vacuum expectation value of this, use the transform (3.2), and integrate over all space, we immediately obtain, for the spatial regulator, the energy (2.19).

The form (3.3b) does not mean that the energy density is free of δ functions, however; from Eq. (1.15) it is clear that the singularities in V must be compensated by singularities in the second-order z derivatives of ϕ, hence ultimately of the reduced Green's function g. Note that such terms are absent from the part of Eq. (3.3b) that survives when $4\eta = 1$. A case can be made for using the equation of motion in the reverse direction in the remaining term, replacing it as

$$T^{00} = \frac{1}{2}(\partial^0\phi)^2 - \frac{1}{2}\phi(\partial^0)^2\phi + \frac{1-4\eta}{2}[(\nabla\phi)^2 + \phi(\partial_0)^2\phi + V\phi^2]. \tag{3.3c}$$

In this form the energy density that resides exactly on the surface is exhibited explicitly by the δ functions in V, because (as will be verified) the time derivatives and first-order space derivatives are benign. In particular, the surface energy arises only when $4\eta \neq 1$.

To forestall confusion we must belabor two elementary distinctions. First, in the remainder of this section we will see that the largest parts of the bulk energy are concentrated close to the plates; such terms have also sometimes been called "surface energy," but here we will reserve that term for energy density that resides exactly on the surface. Second, because $\nabla^2\phi^2$ is a divergence, its integral over the region between the plates (or the region to either side) can be reduced to a surface integral over the plates; but that is merely a mathematical representation of energy that physically resides in the bulk. However, as we will see in Sec. IV, this surface integral is another way of describing the surface energy that resides on the plates.

B. Energy density in bulk

For purposes of calculation, we may use any of the forms of the energy density given above, Eqs. (3.3a), (3.3b), or (3.3c), which directly lead to the following alternative expressions for the energy density in "reduced form":

$$t^{00}(z,z) = \frac{1}{2}(-\zeta^2 + k_\perp^2 + \partial_z\partial_{z'})g(z,z')|_{z'\to z} - \eta\partial_z^2 g(z,z) + \frac{1}{2}V(z)g(z,z) \tag{3.4a}$$

$$= -\zeta^2 g(z,z) + \frac{1}{2}(1-4\eta)(\partial_z^2 + \partial_z\partial_{z'})g(z,z')|_{z'\to z} \tag{3.4b}$$

$$= -\zeta^2 g(z,z) + \frac{1}{2}(1-4\eta)(\kappa^2 + \partial_z\partial_{z'} + V)g(z,z')|_{z'\to z}. \tag{3.4c}$$

Deferring close examination of the surface terms to Sec. IV, we now study the energy density in the regions excluding the plates themselves ($z \neq 0$, a):

$$u(z) = \langle T^{00}\rangle = u_{\rm int}(z) + u_1(z) + u_2(z), \tag{3.5}$$

where, excluding the Weyl term,

$$u_{\rm int}(z<0) = \frac{\eta - 1/6}{\pi^2}\int_0^\infty d\kappa\kappa^3\lambda_2\left[\frac{1}{2\kappa+\lambda_1}\frac{1}{\Delta} - \frac{e^{-2\kappa a}}{2\kappa+\lambda_2}\right]e^{2\kappa z}, \tag{3.6a}$$

$$u_{\rm int}(0<z<a) = \frac{1}{\pi^2}\int_0^\infty d\kappa\kappa^3\frac{\lambda_1\lambda_2}{(2\kappa)^2}\frac{1}{\Delta}\left\{-\frac{1}{6} + \left(\eta - \frac{1}{6}\right)\times\left[\frac{1}{1+2\kappa/\lambda_1}e^{-2\kappa z} + \frac{1}{1+2\kappa/\lambda_2}e^{-2\kappa(a-z)}\right]\right\}, \tag{3.6b}$$

$$u_{\rm int}(z>a) = \frac{\eta - 1/6}{\pi^2}\int_0^\infty d\kappa\kappa^3\lambda_1\times\left[\frac{1}{2\kappa+\lambda_2}\frac{1}{\Delta} - \frac{e^{-2\kappa a}}{2\kappa+\lambda_1}\right]e^{2\kappa(a-z)}, \tag{3.6c}$$

while the parts referring to each plate separately are

$$u_1(z) = -\frac{1}{16\pi^3}\int_0^\infty d\kappa\kappa^3[(1-4\eta)w(\pi/2) - f(\pi/2)]\times\frac{1}{1+2\kappa/\lambda_1}e^{-2\kappa|z|}, \tag{3.7}$$

while $u_2(z)$ is obtained from $u_1(z)$ by replacing λ_1 by λ_2 and z by $a-z$. (This is a symmetry of the energy density.) In these equations Δ is still given by Eq. (1.21) and the cutoff functions w and f are given in Eq. (2.10). When we are not too close to the plates, we can replace the cutoff function as follows:

$$(1-4\eta)w(\pi/2) - f(\pi/2) \to -16\pi(\eta - 1/6). \tag{3.8}$$

The replacement is valid for $|z| \gg \delta$ or $|z-a| \gg \delta$, where the integral over κ are absolutely convergent.

C. Strong coupling

The integrals over κ can be carried out explicitly in the case of strong coupling, $\lambda_{1,2} \to \infty$. In this Dirichlet limit, u_2 and $u_{\rm int}$ cancel for $z<0$, and only u_1 contributes there, while for $z>a$ only u_2 survives. (We will also see in Sec. IV that the surface energy vanishes in strong coupling.) For example, in strong coupling, the energy density below the plate is everywhere for a spatial cutoff ($\gamma = \pi/2$),

$$u(z<0) = \frac{3}{8\pi^2}\frac{(\eta - 1/6)}{(z^2+\delta^2/4)^2} + \frac{1}{32\pi^2}(1-4\eta)\frac{\delta^2}{(z^2+\delta^2/4)^3}, \tag{3.9}$$

which is finite as $z \to 0$, and reduces to the familiar result

$$u(z<0) = \frac{3}{8\pi^2}\frac{\eta - 1/6}{z^4}, \tag{3.10}$$

if $|z| \gg \delta$. Equation (3.9) agrees with the result given in Ref. [16] found for a single plate for the special case $\eta = 1/4$. Above the top plate, the energy density is given by the same expression (3.9) with $z \to a - z$. And in between, one finds

$$u(0<z<a) = -\frac{\pi^2}{1440a^4} + \frac{3(\eta-1/6)}{8\pi^2a^4}[\zeta(4,1+z/a) + \zeta(4,2-z/a)] + u(z<0) + u(z>a), \tag{3.11}$$

where the last two terms mean that the divergent terms (as $\delta\to 0$) are the same on both sides of the plates. Here we have used the definition of the Hurwitz zeta function,

$$\zeta(s,x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^s}, \tag{3.12}$$

which has the property

$$\zeta(s,x) = \frac{1}{x^s} + \zeta(s,x+1). \tag{3.13}$$

Now when we integrate over all space, the Hurwitz zeta functions telescope, and we obtain the energy per unit area

$$\mathcal{E}-\mathcal{E}_W = -\frac{\pi^2}{1440a^3} + \frac{\eta-1/6}{4\pi^2a^3} + \left[\int_{-\infty}^{\infty} + \int_{-a}^{a}\right] \times dz\left\{\frac{3}{8\pi^2}\frac{\eta-1/6}{(z^2+\delta^2/4)^2} + \frac{\delta^2}{32\pi^2}\frac{1-4\eta}{(z^2+\delta^2/4)^3}\right\}. \tag{3.14}$$

The integrals occurring here, for $\delta/a\to 0$, are

$$\int_{-a}^{a}dz\frac{1}{(z^2+\delta^2/4)^2} = \frac{4\pi}{\delta^3} - \frac{2}{3a^3},$$
$$\int_{-a}^{a}dz\frac{1}{(z^2+\delta^2/4)^3} = \frac{12\pi}{\delta^5} - \frac{2}{5a^5}. \tag{3.15}$$

Thus the terms in Eq. (3.14) proportional to $\eta-1/6$ cancel, including the term coming from $1-4\eta = 1/3-4(\eta-1/6)$, and we are left with

$$\mathcal{E}-\mathcal{E}_W = -\frac{\pi^2}{1440a^3} + \frac{1}{4\pi\delta^3}. \tag{3.16}$$

This gives us the Casimir interaction energy plus the self-energy of both plates, twice Eq. (2.25).

If we do the temporal splitting, $\gamma=0$, the surface divergences are slightly modified. Thus, for example,

$$u(z<0) = \frac{3}{8\pi^2}\frac{\eta-1/6}{(z^2+\delta^2/4)^2} - \frac{\delta^2}{8\pi}\frac{\eta}{(z^2+\delta^2/4)^3}. \tag{3.17}$$

So now when we integrate the energy density over all three regions the $(1-4\eta)$ term instead has the factor -4η, so the self-energy term in Eq. (3.16) changes to

$$\mathcal{E}_{S_1+S_2} = -\frac{1}{2\pi\delta^3}. \tag{3.18}$$

This is exactly what is required by the recipe (2.13) for passing from space splitting to time splitting, since the total energy only depends on the cutoff function f. Starting from the local energy density, the total energy would therefore seem to be given by the regulated version of Eq. (2.3), namely

$$\mathcal{E}(\gamma) = -\frac{1}{(2\pi)^3}\int_0^\infty d\kappa\kappa^4 f(\gamma)\mathrm{Tr}g, \tag{3.19}$$

which is different for $\gamma=0$ from that given in Eq. (2.19). Consistency, the perhaps dubious requirement that the regulated energy have the same form, suggests, therefore, that the spatial splitting $\gamma=\pi/2$ is preferred, the point being that calculating the energy from the energy density leads to Eq. (2.3), not the more stable Eq. (2.2).

D. Numerical results

The energy density for the Dirichlet limit and with *spatial* splitting is shown in Fig. 3. It is seen that in each case, the energy density is concentrated near the surfaces, and that when integrated, the rather different energy densities correspond to a unique energy per unit area equal to that given by Eq. (3.16). In comparison, the interaction energy density is negligible. This makes precise what we mean by saying that the surface divergences are without consequence, giving rise to a self-energy of each plate, which can be considered as renormalizing the mass of the

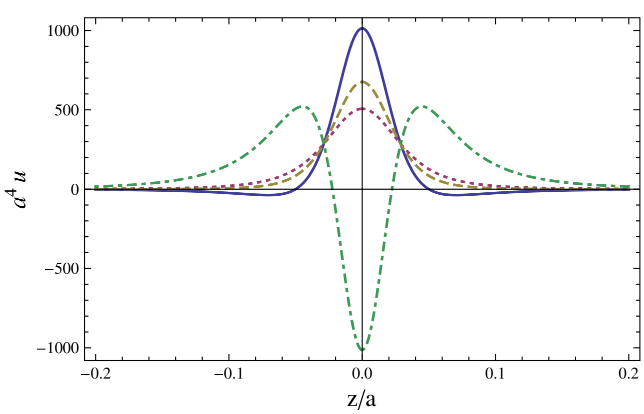

FIG. 3 (color online). The energy density (in units of a^4) near one of the Dirichlet plates. Note that the energy density is very small except close to the plate. The curves are all for the cutoff $\delta = 0.1a$. The curves are all for the spatial-splitting regularization, but for different values of the conformal parameter η. The solid (blue) curve is for the canonical case, $\eta=0$; the short-dashed (red) curve is for $\eta=1/4$, where the surface energy is zero; the long-dashed (yellow) curve is for $\eta=1/6$, the conformal value, and the dot-dashed, green curve is for $\eta=1$. The energy per unit area corresponding to any of these energy densities is the same, $\mathcal{E}=79.5/a^3$. This is the value given by Eq. (2.25), since the interaction energy density is negligible compared to the self-energy densities.

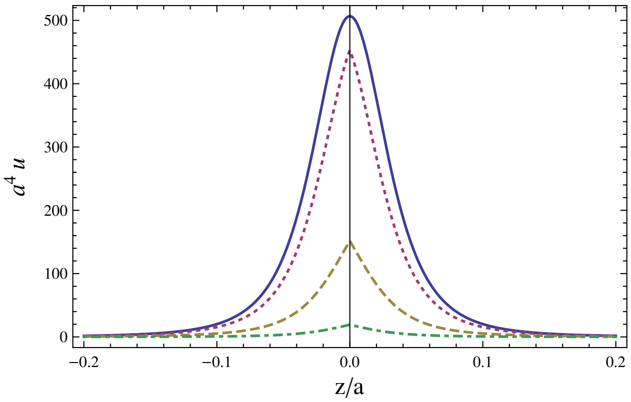

FIG. 4 (color online). The energy density (in units of a^4) near one of the plates for various values of the coupling λ. Again the energy density is very small except close to the plate. The curves are all for the cutoff $\delta = 0.1a$. They are all for the case where the surface term is zero, $\eta = 1/4$. The solid (blue) curve is for the Dirichlet limit, $\lambda \to \infty$. The short-dashed (red) curve is for $\lambda = 100$; the long-dashed (yellow) curve is for $\lambda = 10$, and the dot-dashed (green) curve is for $\lambda = 1$. In all cases, the interaction energy density is negligible compared to the self-energy density. Consequently, the integrated energy in each case agrees with that found from Eq. (2.24).

plates. Note that these self-energy densities do not vanish when $\eta = 1/6$, a fact that is completely overlooked by a naive calculation without cutoff [see Eq. (3.10)].

For finite coupling we must proceed numerically. In Fig. 4 we similarly plot the Casimir energy density for finite λ for the case when there is no surface term (as we shall see in the next section), that is, when $\eta = 1/4$ so the $\nabla^2\phi^2$ term in Eq. (3.3b) vanishes. The energy density localized near the surfaces, corresponding to the cutoff-dependent terms u_1 and u_2, Eq. (3.7), vastly dominate over the interaction energy density. As the coupling $\lambda \to \infty$, the Dirichlet limiting form is rapidly approached.

IV. SURFACE TERMS

In the previous section we only considered points not on the plates at $z = 0$ and $z = a$. But there are surface terms residing exactly on the plates that need to be included to get the total energy. If we naively only included the integrated local energy density in each region, and just dropped the divergent terms, we would get

$$\int_{-\infty}^{\infty} dz u(z) = -\frac{1}{96\pi^3 a^3}\int_0^\infty dy y^3 \frac{1 + \frac{12(\eta - 1/6)}{y+\lambda_1 a} + \frac{12(\eta-1/6)}{y+\lambda_2 a}}{(\frac{y}{\lambda_1 a}+1)(\frac{y}{\lambda_2 a}+1)e^y - 1}, \tag{4.1}$$

which disagrees with the correct interaction energy contained in Eq. (2.23) except for $\eta = 1/4$ or for $\lambda \to \infty$, where in either case the surface contribution to the energy vanishes.

The local surface energy density can be most easily found by using the energy density in the form (3.3c). It is only the potential term (which of course vanishes off the plates) that gives the surface energy: That due to the lower plate is therefore

$$\Delta u_1(z) = \frac{1-4\eta}{2}\lambda_1 \delta(z)\int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\int\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}e^{i\kappa\cdot\delta}g(0,0), \tag{4.2}$$

where we can take $g(0,0)$ to be given by Eq. (2.6a) in the limit as $z = z' \to 0$, since the Green's function is continuous. This gives immediately

$$\Delta u_1(z) = \delta(z)\frac{(1-4\eta)}{32\pi^3}\int_0^\infty d\kappa\kappa w(\gamma)\frac{\lambda_1}{1+\lambda_1/2\kappa}\left[1-\frac{\lambda_2}{2\kappa}\frac{1}{\Delta}\right], \tag{4.3}$$

and the energy density residing on the upper plate Δu_2 is given by a similar expression obtained by interchanging λ_1 and λ_2 and replacing z by $a - z$. Note that the first term in Δu_i is a contribution to the self-energy, while the second term contributes to the interaction energy. Then the total energy density is, rather than that given in Eq. (3.5),

$$u(z) = u_{\rm int} + u_1(z) + \Delta u_1(z) + u_2(z) + \Delta u_2(z), \tag{4.4}$$

where $u_{\rm int}$ is given by Eq. (3.6), and $u_1(z)$ by Eq. (3.7). Integrating, we straightforwardly recover the total energy:

$$\int_{-\infty}^{\infty} dz u(z) = \mathcal{E} - \mathcal{E}_W. \tag{4.5}$$

This is exactly the result (2.23) obtained directly.

It appears that there is a self-energy contribution to the surface energy in the strong-coupling (Dirichlet) limit,

$$\lambda \to \infty:\ \Delta u_1(z) = \delta(z)\frac{1-4\eta}{16\pi^3}\int_0^\infty d\kappa\kappa^2 w(\gamma). \tag{4.6}$$

However, using the expression for the cutoff function $w(0) = w(\pi/2)$ given in Eq. (2.11), we see that the integral here is zero:

$$\int_0^\infty d\kappa\kappa^2 w(\pi/2) = 4\pi\frac{1}{\delta}\frac{d}{d\delta}\int_0^\infty d\kappa \cos\kappa\delta = 0, \tag{4.7}$$

since the last integral vanishes in a distributional sense—see Ref. [7]. It is familiar that there should be no surface term for Dirichlet boundaries.

There is another, equivalent approach to the surface energy, which is applicable to surfaces that are not described by potentials, such as Robin boundaries. It is known that, except in the Dirichlet (or Neumann) limit, one must include a term that resides exactly on the boundary

[6,26,28–33]. This comes simply from integrating Eq. (3.3b) over some arbitrary volume V with boundary ∂V,

$$\int_V (d\mathbf{r})\langle T^{00}\rangle = -\int_V (d\mathbf{r})\int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\zeta^2\mathcal{G}(\mathbf{r},\mathbf{r}) + \frac{1-4\eta}{2} \times \int_{\partial V} d\mathbf{S}\cdot\nabla\int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\mathcal{G}(\mathbf{r},\mathbf{r}')|_{\mathbf{r}'\to\mathbf{r}}. \quad (4.8)$$

The first term on the right is the total energy; the last term is the negative of the boundary energy. [If we were to integrate over all space, including the plates, the interior surface terms would disappear, and we would recover the result (4.5).] We can now apply this identity as follows. Let the volume integral over the energy density be only over the three regions outside the potentials, that is for $z < 0$, $0 < z < a$, and $a < z$. The surfaces at $z = 0$ and $z = a$ are outside the region of the volume integration. Thus ∂V are surfaces just above and below the $z = 0$ and $z = a$ planes. Because the Green's function is continuous, the total energy term is insensitive to the surfaces, which have measure zero. On the other hand, the boundary terms do not cancel, because the first derivatives are discontinuous, so they give an additional contribution to the energy. If we call the boundary term $-\Delta\mathcal{E}$, we have

$$\mathcal{E} = \int_V (d\mathbf{r})\langle T^{00}\rangle + \Delta\mathcal{E}. \quad (4.9)$$

The integral over the energy density in the bulk (i.e., excluding the plates) must be supplemented by the surface energy $\Delta\mathcal{E}$. Combining the contributions coming from above and below the two surfaces, we get here an additional contribution to the energy that resides exactly on the surface:

$$\Delta\mathcal{E} = -\frac{1-4\eta}{2}\int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}e^{i\kappa\cdot\delta}\sum_{\text{plates}}\mathbf{n}\cdot\nabla g(z,z')|_{z'\to z}$$
$$= -\frac{1-4\eta}{32\pi^3}\int_0^\infty d\kappa\kappa w(\gamma)\left[-\frac{\lambda_1}{1+\lambda_1/2\kappa} - \frac{\lambda_2}{1+\lambda_2/2\kappa} + \frac{1}{\Delta}\frac{\lambda_1\lambda_2}{2\kappa}\left(\frac{1}{1+\lambda_1/2\kappa} + \frac{1}{1+\lambda_2/2\kappa}\right)\right]. \quad (4.10)$$

The sum over $\mathbf{n}\cdot\nabla$ on each plate signifies the outward normal gradients from each region, $\mathbf{n}\cdot\nabla = \pm\partial/\partial z$, with the $+$ sign corresponding to the boundary of the $z < 0$ region at $z = 0$, the $+$ and $-$ signs referring to the boundaries of the $0 < z < a$ region at $z = a$ and $z = 0$, respectively, and $-$ sign for the boundary of the $z > a$ region at $z = a$. Note that the surface term depends only on the regulator function w and not on f, so it has the same value for both temporal and spatial splitting. Not surprisingly, this agrees with our previous calculation,

$$\Delta\mathcal{E} = \int_{-\infty}^{\infty} dz(\Delta u_1 + \Delta u_2). \quad (4.11)$$

V. HOW DOES SURFACE ENERGY FALL?

Now we see that the arguments sketched in the Introduction continue to hold. Either by looking in flat (Minkowski) space at the interaction of the Casimir apparatus with a weak (Newtonian) gravitational field, or by working in Rindler coordinates and looking at the limit of small acceleration, we see that the integral of the local energy density occurs, which gives the total energy. There are divergences in the local energy density as the surfaces are approached, and there are divergent contributions to the surface energy that live entirely on the plates of the Casimir apparatus. But we have regulated the integrals with spatial and temporal cutoffs, and obtained therefore unique finite values for the total energy. (The local energy density depends on the conformal parameter.) The terms divergent as the cutoff goes to zero are contained in self-energies serving to renormalize the masses of the plates, so are unobservable. Both the finite, cutoff-independent, Casimir interaction energy, and the divergent, cutoff-dependent, self-energies gravitate normally, that is, they obey the equivalence principle.

To reiterate, we have found an extremely simple answer to the question of how Casimir energy gravitates: just like any other form of energy,

$$\mathcal{F} = -g\mathcal{E}_C. \quad (5.1)$$

This result is independent of the orientation of the Casimir apparatus relative to the gravitational field. This refutes the claim sometimes attributed to Feynman that virtual photons do not gravitate. After a period of confusion, other authors agree with our conclusion [34]. However, the previous arguments were formal, in that divergent self-energies were not properly defined. We have now regulated everything consistently, for both the global and local descriptions. We have also considered arbitrary conformal coupling parameter for the scalar field. These calculations show, quite generally, that the total Casimir energy, including the divergent parts, which renormalize the masses of the plates, possesses the gravitational mass demanded by the equivalence principle. Similar conclusions were drawn by Saharian *et al.* [35] for the finite interactions between Dirichlet, Neumann, and conducting plates. What is new in the present work is the explicit recognition that there is a surface energy density residing on the Casimir plates, which has been well defined through point-splitting regularization. When that term is included, the integrated energy density equals the total energy. Of course, if we considered only smooth potentials, the surface energy would become continuously distributed throughout the region of the potential.

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