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Half-integral conservative post-Newtonian approximations in the redshift factor of black hole binaries

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Recent perturbative self-force computations [Shah *et al.*, Phys. Rev. D **89**, DM11229 (2014)], both numerical and analytical, have determined that half-integral post-Newtonian terms arise in the conservative dynamics of black hole binaries moving on exactly circular orbits. We look at the possible origin of these terms within the post-Newtonian approximation, find that they essentially originate from nonlinear "tail-of-tail" integrals and show that, as demonstrated in the previous paper, their first occurrence is at the 5.5PN order. The post-Newtonian method we use is based on a multipolar–post-Minkowskian treatment of the field outside a general matter source, which is reexpanded in the near zone and extended inside the source thanks to a matching argument. Applying the formula obtained for generic sources to compact binaries, we obtain the redshift factor of circular black hole binaries (without spins) at 5.5PN order in the extreme mass ratio limit. Our result fully agrees with the determination of the 5.5PN coefficient by means of perturbative self-force computations reported in the previously cited paper.

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I. INTRODUCTION

Post-Newtonian (PN) approximations (see Ref. [1] for a review) are well suited to describe the inspiraling phase of compact binary systems, when the post-Newtonian parameter $\epsilon \sim v/c$ is small independently of the mass ratio $q = m_1/m_2$ between the compact bodies. On the other hand, self-force (SF) analyses, based on black hole perturbation theory [2–5] (see Refs. [6–8] for reviews), give an accurate description of extreme mass ratio binaries for which $q \ll 1$, even in the strong field regime. The problem of the comparison between these two powerful methods in their common domain of validity, that of the slow-motion and weak-field regime of an extreme mass ratio compact binary system, has received a great deal of attention recently [9–12].

These efforts rely on the identification of a suitable gauge invariant quantity derived by means of the very separate and distinct SF and PN calculations. The results can be usefully compared, regardless of their manner of computation. At the heart of all previous comparisons lies a quantity that has come to be known as the redshift factor or observable, and was identified and first shown by Detweiler [9] to be gauge invariant for circular orbits. It can be characterized as the redshift a photon would experience in escaping from the small compact object to infinity along the orbital axis. It is directly related to the particle's Killing energy that is associated with the helical Killing symmetry. The redshift factor will be at the basis of the comparison we pursue here.

In the most recent PN-SF comparison (see the companion paper [12]), it was found that the post-Newtonian expansion of the redshift factor for extreme mass ratio compact binaries contains half-integral PN terms starting at the 5.5PN order. This result had previously been unexpected, because one may naively think that half-integral PN terms are associated with gravitational radiation reaction damping. However here they actually describe the *conservative* part of the dynamics, since the compact binary moves on an exactly circular orbit, and dissipative radiation reaction effects are explicitly neglected.

The goal of the present paper is to explain this fact using post-Newtonian theory, and to directly compute, using PN methods, the dominant half-integral 5.5PN coefficient for comparison with the SF result, obtained both numerically and analytically in Ref. [12]. We shall find perfect agreement with that result, given by Eq. (20) in [12], showing again strong internal consistency between analytical PN and numerical/analytical SF methods, and their joint effectiveness in describing the dynamics of compact binary systems.

We shall compute here the redshift factor introduced in Ref. [9], for a particle moving on an exact circular orbit around a Schwarzschild black hole. The ensuing space-time is helically symmetric, with a helical Killing vector K^α such that its value K_1^α at the location of the particle (labeled 1) is tangent to the particle's four-velocity u_1^α, defined as usual with unit timelike norm. The redshift factor u_1^T is then defined geometrically by

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$$u_1^\alpha = u_1^T K_1^\alpha. \tag{1.1}$$

Adopting a coordinate system in which the helical Killing vector reads $K^\alpha\partial_\alpha = \partial_t + \Omega\partial_\varphi$ (which defines its normalization), where Ω is the orbital frequency of the circular motion, the redshift factor reduces to the t component $u_1^T = u_1^t \equiv \mathrm{d}t/\mathrm{d}\tau_1$ of the particle's four-velocity (where $\mathrm{d}\tau_1$ is the particle's proper time), namely

$$u_1^T = \left[-g_{\alpha\beta}(y_1)\frac{v_1^\alpha v_1^\beta}{c^2}\right]^{-1/2}. \tag{1.2}$$

Here $g_{\alpha\beta}(y_1)$ denotes the metric evaluated at the particle's location $y_1^\alpha = (ct, y_1^i)$ by means of an appropriate self-field regularization (in principle dimensional regularization [13,14]), and $v_1^\alpha \equiv \mathrm{d}y_1^\alpha/\mathrm{d}t = (c, v_1^i)$ is the ordinary coordinate velocity.

Our strategy will be to obtain first the metric $g_{\alpha\beta}$ in the exterior of a general matter system by means of a multipolar–post-Minkowskian expansion [15], and to extend next the validity of the solution inside the source using a matching argument. More precisely, we consider in a first stage a general smooth matter distribution with compact support and slow internal velocities (post-Newtonian source). The field outside the post-Newtonian source is a solution of the vacuum field equations, which is reexpanded in the exterior part of the source's near zone. The matching argument we use is based on a variant of the method of matched asymptotic expansions which has been developed to connect the exterior near-zone field to the inner field of a post-Newtonian source (see e.g. Ref. [1]). At the dominant level, we will deal with a homogeneous solution of the wave equation which, being of the type retarded minus advanced, is regular all over the near zone of the source, and thus can directly be extended by matching inside the source.

Eventually, in a second stage, the source will be specialized to a binary point-particle system and the metric will be evaluated at the location of one of the particles. In principle, our PN calculations are valid for any mass ratio, but it turns out that the multipole interactions needed at 5.5PN order are rather involved for arbitrary mass ratios. In the extreme mass ratio (SF) limit, we shall essentially find that only one simple multipole interaction is required, namely the interaction between two mass monopoles and the mass quadrupole moment (consistently with an observation made in Ref. [12]), known in the literature as a "tail-of-tail" [16].

In the context of general relativity, tails are nonlinear effects physically due to the backscattering of linear waves from the space-time curvature generated by the total mass of the source. They are nonlinear in the usual language of the PN approximation (which expands flat space-time retarded wave operators), because they are associated with the nonlinear coupling between radiative multipole moments and the source's mass monopole. The tails imply a nonlocality in time since they involve an integral depending on the history of the source from the remote past to the current time. They are also appropriately referred to as "hereditary" contributions [17], in contrast to the "instantaneous" contributions which depend on the dynamics of the source only at the current time. In this paper we shall prove that half-integral conservative post-Newtonian terms are due to hereditary effects. In the process, we shall shed all unnecessary instantaneous terms and focus primarily on the relevant hereditary contributions.

The plan of this article is as follows. In Sec. II we use dimensionality arguments to discuss the first occurrence of half-integral conservative PN terms. In Sec. III we present the source terms for the so-called tail-of-tail hereditary integrals that are responsible for the 5.5PN effect in the extreme mass ratio limit. Section IV is devoted to basic formulas enabling us to obtain the near-zone expansion of a retarded integral, given that of its source. Finally, in Sec. V we obtain the piece of the metric (in two different gauges) corresponding to the tail-of-tail at 5.5PN order and compute the redshift factor. A most crucial but technical proof is relegated to the Appendix.

II. DIMENSIONALITY ARGUMENTS

We look at the dominant occurrence of terms at *half-integral* PN orders, i.e. at $\frac{n}{2}$PN orders where n is an *odd* integer, that arise in the conservative dynamics of binary point-particles systems, moving on exactly circular orbits. Such terms cannot stem from nontail/nonhereditary sources, and may be expected to occur first at rather high PN order. Indeed, any instantaneous (nontail) term at any half-integral PN order will be zero for circular orbits, as can be shown by a simple dimensional argument. To see this, let us look at the general structure of instantaneous terms in the redshift factor (1.2). We assume that the expression of u_1^T is given in the frame of the center of mass, and has been consistently order reduced, i.e. that all accelerations have been replaced by the lower-order equations of motion—the normal practice in PN theory. We have (in order of magnitude)

$$(u_1^T)_{\text{inst}} \sim \sum_{j,k,p,q} \nu^j \left(\frac{Gm}{r_{12}c^2}\right)^k \left(\frac{\boldsymbol{v}_{12}^2}{c^2}\right)^p \left(\frac{\boldsymbol{n}_{12}\cdot\boldsymbol{v}_{12}}{c}\right)^q, \tag{2.1}$$

where $m = m_1 + m_2$ is the sum of the two masses, $\nu = m_1 m_2/m^2$ the symmetric mass ratio, $r_{12} = |\boldsymbol{y}_1 - \boldsymbol{y}_2|$ the relative distance between particles and $\boldsymbol{n}_{12} = (\boldsymbol{y}_1 - \boldsymbol{y}_2)/r_{12}$ the relative direction. Furthermore $\boldsymbol{v}_{12}^2 = \dot{r}_{12}^2 + r_{12}^2\Omega^2$ is the squared Euclidean norm of the relative velocity between the two particles, and $\boldsymbol{n}_{12}\cdot\boldsymbol{v}_{12} = \dot{r}_{12}$ is the Euclidean scalar product between the unit separation vector and the relative velocity. In Eq. (2.1) we have assumed that we take the expansion when the mass ratio $\nu \to 0$. For comparison with the SF based calculation in linear perturbation theory, we can limit ourselves to terms linear in ν.

The simple counting of the powers of $1/c$ shows that the post-Newtonian order of the generic term in Eq. (2.1) is given by $\frac{n}{2}$PN where

$$n = 2k + 2p + q - 2. \tag{2.2}$$

If n is an odd integer, then q is also an odd integer; hence Eq. (2.1) contains at least one factor $\boldsymbol{n}_{12} \cdot \boldsymbol{v}_{12}$ and vanishes for circular orbits. The crucial point in this argument is that we are dealing with *instantaneous* (nonhereditary) terms, so that the velocity $\boldsymbol{v}_{12}$ and unit direction $\boldsymbol{n}_{12}$ are taken at the same time, which is the current time t at which we are evaluating those quantities. Thus there is no integration over some intermediate time extending from the infinite past up to t, which would allow a coupling between these vectors at different times. In conclusion, half-integral conservative post-Newtonian terms that are instantaneous give zero, and only truly hereditary integrals can contribute.

It is known that the first hereditary integral in the near-zone metric is the tail occurring at the 4PN order [18,19]. This tail is associated with the mass quadrupole moment, and produces both conservative and dissipative effects. Higher-order tails are associated with higher multipole moments (mass octupole, current quadrupole, etc.) and arise at higher but still integral PN orders (5PN, 6PN, etc.). The conservative part of these tail effects is responsible for the appearance of logarithmic terms in the redshift factor as well as the ADM mass and angular momentum of the binary system, which have previously been computed at 4PN [11,20] and 5PN [11,21] orders.

Interestingly, as we shall see now, the next complicated hereditary integrals called tails-of-tails [16] do occur at half-integral PN orders, and give a first contribution at precisely the 5.5PN order. We first simplify the problem by noticing that, for comparison with linear SF results, any product between two or more mass or current multipole moments I_P and J_P other than the mass M can be discarded, since each multipole carries in front a mass ratio ν, and we want to compute (1.2) at linear order in ν. The only moment that does not carry a factor ν is the mass monopole or total ADM mass M of the source. We thus consider only multipole interactions of type $M \times \cdots \times M \times I_P$ or $M \times \cdots \times M \times J_P$.

We shall prove below that, at the dominant level, the relevant piece of the metric is a homogeneous solution of the wave equation of the type retarded minus advanced which is regular all over the near zone of the source. The near-zone expansion of such a homogeneous solution, when $r = |\mathbf{x}| \to 0$, is of the type $\hat{n}_L r^{\ell+2i}$ with i being a positive integer and $\hat{n}_L$ the symmetric trace-free (STF) angular factor. The general structure of this term in the "gothic" metric deviation, corresponding to the interaction $M \times \cdots \times M \times I_P$, is[1]

$$h^{\alpha\beta}_{M\times\cdots\times M\times I_P} \sim \sum_{k,p,\ell,i} \frac{G^k M^{k-1}}{c^{3k+p}} \hat{n}_L \left(\frac{r}{c}\right)^{\ell+2i} \times \int_{-\infty}^{+\infty} \mathrm{d}u \kappa^{\alpha\beta}_{LP}(t,u) I^{(a)}_P(u). \tag{2.3}$$

Here k is the number of moments in the particular interaction we are considering; it is thus made of $k-1$ mass monopoles M and one nonstatic multipole I_P. The tensorial function $\kappa^{\alpha\beta}_{LP}(t,u)$ denotes a certain dimensionless hereditary kernel (typically a logarithmic kernel as we shall demonstrate below). The number of time derivatives on the moment is $a = k + p + \ell + 2i + 1$. Counting the powers of $1/c$ we find that the PN order of the generic term in Eq. (2.3) is $n = 3k + p + \ell + 2i + s - 2$, where s is the number of spatial indices among $\alpha\beta$, i.e. $s = 0$, 1, 2 according to whether $\alpha\beta = 00$, $0i$, or ij. Now we have the inequality $|\ell - p| \leq s$ ("law of addition of angular momenta") which states that the indices on the STF tensors $\hat{n}_L$ and $I^{(a)}_P$ must be either some free spatial indices coming from $\alpha\beta = 0i$ or ij, or be contracted with each other. In fact we have $\ell = p$ when $s = 0$, $\ell = p - 1$ or $p + 1$ when $s = 1$, and $\ell = p + 2$, p or $p - 2$ when $s = 2$. Notice that s has always the same parity as $\ell - p$. From this we can write the PN order as

$$n = 3k + 2p + 2j - 2, \tag{2.4}$$

where j is a positive integer. For a half-integral PN order we must have k odd; hence $k = 3, 5, \ldots$, since we eliminate $k = 1$ which corresponds to a linear term deprived of tail. For $k \geq 5$, recalling that we have at least $p \geq 2$ for evolving mass moments, we see that the PN order (2.4) satisfies $n \geq 17$, which means at least 8.5PN order. We can thus restrict ourselves to the case of cubic interactions $k = 3$ for the structure portrayed in Eq. (2.3). In this

[1] Our notation is as follows: the gothic metric deviation is $h^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta}$ where g and $g^{\alpha\beta}$ are, respectively, the determinant and the inverse of $g_{\alpha\beta}$, and $\eta^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$; $L = i_1 \cdots i_\ell$ or $P = i_1 \cdots i_p$ denote multi-indices composed of ℓ or p spatial indices (ranging from 1 to 3); $\partial_L = \partial_{i_1} \cdots \partial_{i_\ell}$ is the product of ℓ partial derivatives $\partial_i \equiv \partial/\partial x^i$; $x_L = x_{i_1} \cdots x_{i_\ell}$ is the product of ℓ spatial positions x_i; similarly $n_L = n_{i_1} \cdots n_{i_\ell}$ is the product of ℓ unit vectors $n_i = x_i/r$; the symmetric-trace-free (STF) projection is indicated with a hat, i.e. $\hat{x}_L \equiv \mathrm{STF}[x_L]$, $\hat{n}_L \equiv \mathrm{STF}[n_L]$, $\hat{\partial}_L \equiv \mathrm{STF}[\partial_L]$. The mass and current multipole moments I_P and J_P are STF, $I_P = \hat{I}_P$ and $J_P = \hat{J}_P$. In the case of summed-up (dummy) multi-indices L or P, we do not write the ℓ or p summations from 1 to 3 over the dummy indices. Symmetrization over indices is denoted by $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$. Time derivatives of the moments are indicated by superscripts (n).

case we have $n = 7 + 2p + 2j$ corresponding to terms 5.5PN, 6.5PN, … for the mass quadrupole $p = 2$, to terms 6.5PN, 7.5PN, … for the mass octupole $p = 3$, and so on. The first occurrences of half-integral orders for the current moments can be deduced from the previous discussion by noticing that the polar tensor $aaa\varepsilon_{ija}J_{aP-1}$, with $aaa\varepsilon_{ija}$ denoting the three-dimensional Levi-Civita tensor, has the same physical dimension as $\mathrm{d}I_P/\mathrm{d}t$ and is endowed with one extra index. Therefore, the relations between a, n and k, p, $\ell + 2i$ follow formally from those obtained for I_P by making the substitutions $a \to a - 1$ and $p \to p + 1$. This yields terms 6.5PN, 7.5PN, … for the current quadrupole $p = 2$, terms 7.5PN, 8.5PN, … for the current octupole $p = 3$, and so on.

We find in the end that the minimal order for which we have an occurrence of half-integral PN hereditary terms (at linear order in the mass ratio ν) is 5.5PN. It corresponds to the cubic interaction $M \times M \times I_{ij}$, between two mass monopoles and the mass quadrupole moment, or tail-of-tail. Nevertheless, it should be noted that the structure $\sim\hat{n}_L r^{\ell+2i}$ assumed in Eq. (2.3) for terms of half-integral PN order is only the starting point of a PN iteration. It should generate at higher PN order some other terms possibly of more complicated form. The details of this iteration depend on the adopted coordinate system. We shall see that at 5.5PN order it plays a crucial role in harmonic coordinates, but can be avoided by choosing appropriately another coordinate system.

III. SOURCE TERMS FOR THE TAIL-OF-TAIL INTERACTION

Tails arise from a quadratic interaction between the mass monopole moment or ADM mass M of the source, and STF nonstatic (propagating) multipole moments I_P, J_P for which $p \geq 2$: dominantly the mass quadrupole I_{ij}, subdominantly the mass octupole I_{ijk} and current quadrupole J_{ij}, and so on. If we consider only source terms that are relevant for the dominant tail interaction, the gothic metric deviation in harmonic coordinates (i.e. satisfying $\partial_\beta h^{\alpha\beta} = 0$), in the vacuum region outside the matter source, obeys

$$\Box h^{\alpha\beta}_{M\times I_{ij}} = \Lambda^{\alpha\beta}_{M\times I_{ij}}, \tag{3.1}$$

where $\Box \equiv \Box_\eta$ is the flat d'Alembertian operator, and $\Lambda_{M\times I_{ij}}$ is the gravitational source term composed of quadratic products involving derivatives of linear terms, h_M and $h_{I_{ij}}$, solutions of the linearized vacuum field equations. The metric solution of Eq. (3.1) diverges at the origin $r = 0$ located inside the matter source, and is supposed to be matched to the actual post-Newtonian expansion of the field inside the source.

At cubic order the gothic metric for the tail-of-tail interaction obeys

$$\Box h^{\alpha\beta}_{M\times M\times I_{ij}} = \Lambda^{\alpha\beta}_{M\times M\times I_{ij}}, \tag{3.2}$$

where $\Lambda_{M\times M\times I_{ij}}$ is made of quadratic products between $h_{M\times M}$ and $h_{I_{ij}}$ and between h_M and $h_{M\times I_{ij}}$, as well as cubic products between h_M, h_M and $h_{I_{ij}}$. This source term has been computed in Eqs. (2.14)–(2.16) of Ref. [16], where it is split into a local (instantaneous) part $\mathcal{I}_{M\times M\times I_{ij}}$ and a nonlocal (hereditary) part $\mathcal{H}_{M\times M\times I_{ij}}$:

$$\Lambda^{\alpha\beta}_{M\times M\times I_{ij}} = \mathcal{I}^{\alpha\beta}_{M\times M\times I_{ij}} + \mathcal{H}^{\alpha\beta}_{M\times M\times I_{ij}}. \tag{3.3}$$

Clearly the hereditary part comes from the tails that are already present in $h_{M\times I_{ij}}$ and interact with h_M to contribute to the cubic source term $\Lambda_{M\times M\times I_{ij}}$.

The solutions of Eqs. (3.1) and (3.2) are obtained iteratively by applying the flat retarded integral operator, denoted $\Box^{-1}_{\mathrm{ret}}$, on the source term, but after multiplying it by a regularization factor r^B to cope with the divergence of the multipole expansion when $r \to 0$. Analytic continuation in $B \in \mathbb{C}$ is invoked and the finite part when $B \to 0$ provides a certain particular solution. To ensure that the harmonic coordinate condition is satisfied at each step, one must add to the latter solution a specific homogeneous retarded solution [15], which does not generate tail integrals when expanded in the near zone, and can be safely ignored.

The instantaneous part of the cubic source term (3.2) explicitly reads [16][2]

$$\mathcal{I}^{00}_{M\times M\times I_{ij}} = M^2 n_{ab} r^{-7}\{-516 I_{ab} - 516 r I^{(1)}_{ab} - 304 r^2 I^{(2)}_{ab} - 76 r^3 I^{(3)}_{ab} + 108 r^4 I^{(4)}_{ab} + 40 r^5 I^{(5)}_{ab}\}, \tag{3.4a}$$

$$\begin{aligned}\mathcal{I}^{0i}_{M\times M\times I_{ij}} &= M^2 \hat{n}_{iab} r^{-6}\left\{4 I^{(1)}_{ab} + 4 r I^{(2)}_{ab} - 16 r^2 I^{(3)}_{ab} + \frac{4}{3} r^3 I^{(4)}_{ab} - \frac{4}{3} r^4 I^{(5)}_{ab}\right\}\\ &\quad + M^2 n_a r^{-6}\left\{-\frac{372}{5} I^{(1)}_{ai} - \frac{372}{5} r I^{(2)}_{ai} - \frac{232}{5} r^2 I^{(3)}_{ai} - \frac{84}{5} r^3 I^{(4)}_{ai} + \frac{124}{5} r^4 I^{(5)}_{ai}\right\},\end{aligned} \tag{3.4b}$$

[2] From now on we generally pose $G = c = 1$.

$$\begin{aligned}\mathcal{I}^{ij}_{M\times M\times I_{ij}} &= M^2\hat{n}_{ijab}r^{-5}\{-190I^{(2)}_{ab} - 118rI^{(3)}_{ab} - \frac{92}{3}r^2I^{(4)}_{ab} - 2r^3I^{(5)}_{ab}\}\\ &+ M^2\delta_{ij}n_{ab}r^{-5}\left\{\frac{160}{7}I^{(2)}_{ab} + \frac{176}{7}rI^{(3)}_{ab} - \frac{596}{21}r^2I^{(4)}_{ab} - \frac{160}{21}r^3I^{(5)}_{ab}\right\}\\ &+ M^2\hat{n}_{a(i}r^{-5}\left\{-\frac{312}{7}I^{(2)}_{j)a} - \frac{248}{7}rI^{(3)}_{j)a} + \frac{400}{7}r^2I^{(4)}_{j)a} + \frac{104}{7}r^3I^{(5)}_{j)a}\right\}\\ &+ M^2r^{-5}\left\{-12I^{(2)}_{ij} - \frac{196}{15}rI^{(3)}_{ij} - \frac{56}{5}r^2I^{(4)}_{ij} - \frac{48}{5}r^3I^{(5)}_{ij}\right\}. \qquad (3.4c)\end{aligned}$$

Here, all the time derivatives of the quadrupole moment $I^{(p)}_{ab}$ are evaluated at the current retarded time $t-r$, hence the instantaneous character of this term. The hereditary part of the source term (3.2) is

$$\mathcal{H}^{00}_{M\times M\times I_{ij}} = M^2n_{ab}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{96Q_0I^{(4)}_{ab} + \left[\frac{272}{5}Q_1 + \frac{168}{5}Q_3\right]rI^{(5)}_{ab} + 32Q_2r^2I^{(6)}_{ab}\right\}, \qquad (3.5a)$$

$$\begin{aligned}\mathcal{H}^{0i}_{M\times M\times I_{ij}} &= M^2\hat{n}_{iab}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-32Q_1I^{(4)}_{ab} + \left[-\frac{32}{3}Q_0 + \frac{8}{3}Q_2\right]rI^{(5)}_{ab}\right\}\\ &+ M^2n_ar^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{96}{5}Q_1I^{(4)}_{ai} + \left[\frac{192}{5}Q_0 + \frac{112}{5}Q_2\right]rI^{(5)}_{ai} + 32Q_1r^2I^{(6)}_{ai}\right\}, \qquad (3.5b)\end{aligned}$$

$$\begin{aligned}\mathcal{H}^{ij}_{M\times M\times I_{ij}} &= M^2\hat{n}_{ijab}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-32Q_2I^{(4)}_{ab} + \left[-\frac{32}{5}Q_1 - \frac{48}{5}Q_3\right]rI^{(5)}_{ab}\right\}\\ &+ M^2\delta_{ij}n_{ab}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-\frac{32}{7}Q_2I^{(4)}_{ab} + \left[-\frac{208}{7}Q_1 + \frac{24}{7}Q_3\right]rI^{(5)}_{ab}\right\}\\ &+ M^2\hat{n}_{a(i}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{96}{7}Q_2I^{(4)}_{j)a} + \left[\frac{2112}{35}Q_1 - \frac{192}{35}Q_3\right]rI^{(5)}_{j)a}\right\}\\ &+ M^2r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{32}{5}Q_2I^{(4)}_{ij} + \left[\frac{1536}{75}Q_1 - \frac{96}{75}Q_3\right]rI^{(5)}_{ij} + 32Q_0r^2I^{(6)}_{ij}\right\}. \qquad (3.5c)\end{aligned}$$

The kernels of the above tail integrals are made of Legendre functions of the second kind, Q_m, which are here computed at x, while the quadrupole moments $I^{(p)}_{ab}$, all appearing inside the integrals, are evaluated at time $t-rx$. Since x ranges from 1 to $+\infty$ the hereditary character of all terms in Eq. (3.5) is evident. The Legendre function $Q_m(x)$ has a branch cut from $-\infty$ to 1 and is conveniently expressed in terms of the usual Legendre polynomial $P_m(x)$ by means of the explicit formula

$$Q_m(x) = \frac{1}{2}P_m(x)\ln\left(\frac{x+1}{x-1}\right) - \sum_{j=1}^{m}\frac{1}{j}P_{m-j}(x)P_{j-1}(x). \qquad (3.6)$$

IV. GENERAL FORMULA FOR INTEGRATING THE SOURCE TERMS

For any source term of the type $\hat{n}_LS(r,t-r)$, i.e. which has some definite multipolarity ℓ, and is sufficiently regular when $r\to 0$, we can write the usual retarded integral $\Box^{-1}_{\mathrm{ret}}$ of this source as [15]

$$\begin{aligned}u_L(\mathbf{x},t) &\equiv \Box^{-1}_{\mathrm{ret}}\left[\hat{n}_LS(r,t-r)\right]\\ &= \int_{-\infty}^{t-r}\mathrm{d}s\hat{\partial}_L\left\{\frac{R(\frac{t-r-s}{2},s) - R(\frac{t+r-s}{2},s)}{r}\right\}, \qquad (4.1a)\end{aligned}$$

$$\text{where } R(\rho,s) = \rho^\ell\int_0^\rho\mathrm{d}\lambda\frac{(\rho-\lambda)^\ell}{\ell!}\left(\frac{2}{\lambda}\right)^{\ell-1}S(\lambda,s). \qquad (4.1b)$$

In the present case we have to apply this formula to two types of source terms, either instantaneous or hereditary, which can generically be written as

$$S(r,t-r) = r^{B-k}F(t-r), \qquad (4.2a)$$

$$\text{or} \quad S(r,t-r) = r^{B-k}\int_1^{+\infty}\mathrm{d}xQ_m(x)F(t-rx). \qquad (4.2b)$$

Here F represents some time derivative $I^{(p)}_{ab}$ of the quadrupole moment. Notice the important factor r^B which is systematically included and, when $\Re(B)$ is large enough, ensures the regularity of the source term as $r \to 0$ as well as the applicability of the integration formula (4.1). Complex analytic continuation in $B \in \mathbb{C}$ is assumed throughout.

Since, ultimately, we shall be interested in the metric at the location of one of the particles, our goal is to compute the near-zone expansion of the solution (4.1) when $r \to 0$. For that purpose it is not necessary to control the full solution $u_L(\mathbf{x}, t)$. Indeed we can obtain this expansion directly from the near-zone expansion of the corresponding source thanks to the following formula [19]:

$$u_L(\mathbf{x}, t) = \hat{\partial}_L \left\{ \frac{G(t-r) - G(t+r)}{r} \right\} + \Box^{-1}_{\text{inst}}[\hat{n}_L \overline{S(r, t-r)}], \tag{4.3a}$$

$$\text{with} \quad G(u) = \int_{-\infty}^{u} \mathrm{d}s R\left(\frac{u-s}{2}, s\right). \tag{4.3b}$$

The first term in Eq. (4.3a) will be of primary interest. It is a homogeneous solution of the wave equation which is of retarded-minus-advanced type and is thus regular when $r \to 0$. Clearly such a solution will be directly valid inside the matter source by virtue of a matching argument. For later reference we note that the near-zone expansion $r \to 0$ of this term is

$$\hat{\partial}_L \left\{ \frac{G(t-r) - G(t+r)}{r} \right\} = -2\hat{x}_L \sum_{k=0}^{+\infty} \frac{r^{2k}}{(2k)!!(2k+2\ell+1)!!} G^{(2k+2\ell+1)}(t). \tag{4.4}$$

The second term in (4.3a) is a particular solution of the inhomogeneous equation which is defined by means of the operator of "instantaneous" potentials as

$$\Box^{-1}_{\text{inst}}[\hat{n}_L \overline{S(r, t-r)}] = \sum_{i=0}^{+\infty} \left(\frac{\partial}{\partial t}\right)^{2i} \Delta^{-1-i}[\hat{n}_L \overline{S(r, t-r)}]. \tag{4.5}$$

Such operator acts directly on the formal near-zone expansion of the source, indicated by the overbar, namely

$$\overline{S(r, t-r)} = \sum_{j=0}^{+\infty} \frac{(-r)^j}{j!} S^{(j)}(t). \tag{4.6}$$

Note that the instantaneous operator (4.5) is always well defined when acting to source terms of the type (4.2) that are multiplied by the regularization factor r^B. As usual we apply repeatedly the Poisson operators Δ^{-1} on source terms $\sim \hat{n}_L r^{B+j}$ using analytic continuation, and consider at the end the finite part when $B \to 0$. An important point is that the term (4.5) diverges when $r \to 0$ and cannot be extended inside the matter source. It should be matched to a full-fledged solution of the field equations inside the source. As we shall prove in the Appendix, this term will actually contribute only at integral PN orders. Therefore the only effect at the half-integral 5.5PN order comes from the first term in Eq. (4.3a) containing the function G, which we now compute.

In order to apply the formulas (4.3) explicitly we need to find the expression of the function $G(u)$ for source terms of the type (4.2). This is easily done for the instantaneous source terms (4.2a) but is more tricky for the tail terms (4.2b). Here we shall give the result only for the tail terms (4.2b). The case for the instantaneous terms can be deduced from it by replacing the Legendre function $Q_m(x)$ by a truncated delta function $\delta_+(x-1)$ such that $\int_1^{+\infty} \mathrm{d}x \delta_+(x-1)\phi(x) = \phi(1)$, i.e. given formally by $\delta_+(x-1) = Y(x-1)\delta(x-1)$ where Y is Heaviside's function.

To get $G(u)$ we have to manipulate three integrations: one in the definition of the function G, Eq. (4.3b); one in the definition of the function R, Eq. (4.1b); and one present in the source term itself, Eq. (4.2b). These three integrations can be rearranged after appropriate commutations of integrals, changes of variables and integrations by parts, as

$$G(u) = C_{k,\ell,m}(B) \int_0^{+\infty} \mathrm{d}\tau \tau^B F^{(k-\ell-2)}(u-\tau), \tag{4.7}$$

where the B-dependent coefficient is given by

$$C_{k,\ell,m}(B) = \frac{2^\ell}{\ell!} \frac{\Gamma(B-k+\ell+3)}{\Gamma(B+1)} \int_0^{+\infty} \mathrm{d}y Q_m(1+y) \times \int_0^1 \mathrm{d}z \frac{z^{B-k-\ell+1}(1-z)^\ell}{(2+yz)^{B-k+\ell+3}}, \tag{4.8}$$

Γ being the usual Eulerian function. Notice that, depending on the values of k and ℓ, the function $F(u-\tau)$ in Eq. (4.7) will appear either with multitime derivatives or multitime antiderivatives. The formula for the coefficient (4.8), thanks to the use of Γ functions, is able to treat both cases at the same time, and is valid in either case. Again, we have finally to take the finite part of the Laurent expansion of the result when $B \to 0$. An alternative form of Eq. (4.8), in which one integration is explicitly performed, reads

$$C_{k,\ell,m}(B) = \frac{\Gamma(B-k-\ell+2)}{2\Gamma(B+1)} \times \sum_{i=0}^{\ell} \frac{(\ell+i)!}{i!(\ell-i)!} \frac{\Gamma(B-k+\ell+3)}{\Gamma(B-k+i+3)} \times \int_0^{+\infty} \mathrm{d}y \left(\frac{y}{2}\right)^i \frac{Q_m(1+y)}{(2+y)^{B-k+2}}. \tag{4.9}$$

Suppose that the coefficient (4.8) or equivalently (4.9) admits the singular Laurent expansion when $B \to 0$

$$C_{k,\ell,m}(B) = \sum_{i=-q}^{+\infty} \alpha_{(i)} B^i, \tag{4.10}$$

with finite part coefficient $\alpha_{(0)}$, residue coefficient $\alpha_{(-1)}$, and so on. Applying the finite part at $B = 0$ we see that the function $G(u)$ reads

$$G(u) = -\alpha_{(0)} F^{(k-\ell-3)}(u) + \sum_{j=1}^{q} \frac{\alpha_{(-j)}}{j!} \int_0^{+\infty} \mathrm{d}\tau (\ln \tau)^j F^{(k-\ell-2)}(u-\tau). \tag{4.11}$$

We have performed directly the integration over τ in the first term. It can be checked that there are always enough time derivatives on the quadrupole moment in $F = I_{ab}^{(p)}$ so that this term is made of some time derivative (and not antiderivative) of this moment. Note also that we have discarded the contribution at $\tau = +\infty$ assuming that the quadrupole moment becomes constant in the remote past. Finally the first term in (4.11) is purely instantaneous and cannot contribute at any half-integral PN order for circular orbits as has been shown from Eq. (2.1) by dimensionality arguments.

The terms in Eq. (4.11) with $j \geq 1$ correspond to tails. As we have seen, only tails (and tails-of-tails) can contribute for circular orbits at the 5.5PN order. Thus, what we have to do is control the *pole* part when $B \to 0$ of the B-dependent coefficients (4.8), and we must do that for all the source terms in Eqs. (3.4) and (3.5). We find that only simple poles appear for all these terms at 5.5PN order, i.e. only the term $j = 1$ in (4.11) contributes. Hence we require

$$G^{\text{tail}}(u) = \alpha_{(-1)} \int_0^{+\infty} \mathrm{d}\tau \ln \tau F^{(k-\ell-2)}(u-\tau). \tag{4.12}$$

V. CONTROL OF THE 5.5PN TERM IN THE REDSHIFT FACTOR

In the Appendix we show that we do not have to consider the second term in Eq. (4.3a), defined by (4.5), since it contributes only at integral PN orders (4PN, 5PN, 6PN, etc.). This situation is fortunate: we have obtained this term only in the form of a multipole expansion valid outside the matter source and diverging when $r \to 0$, and to control it we would need to invoke matching to the actual post-Newtonian field inside the physical source.

Gathering all the results for the functions $G^{\text{tail}}(u)$ defined by (4.12) for all the terms in Eqs. (3.4) and (3.5), we obtain the tail-of-tail contributions in the gothic metric as

$$(h^{00})_{M\times M\times I_{ij}} = \frac{116}{21} \frac{G^3 M^2}{c^8} \int_0^{+\infty} \mathrm{d}\tau \ln \tau \partial_{ab} \left[\frac{I_{ab}^{(3)}(t-r-\tau) - I_{ab}^{(3)}(t+r-\tau)}{r} \right], \tag{5.1a}$$

$$\begin{aligned}(h^{0i})_{M\times M\times I_{ij}} = {}& \frac{4}{105} \frac{G^3 M^2}{c^7} \int_0^{+\infty} \mathrm{d}\tau \ln \tau \hat{\partial}_{iab} \left[\frac{I_{ab}^{(2)}(t-r-\tau) - I_{ab}^{(2)}(t+r-\tau)}{r} \right] \\ & - \frac{416}{75} \frac{G^3 M^2}{c^9} \int_0^{+\infty} \mathrm{d}\tau \ln \tau \partial_a \left[\frac{I_{ia}^{(4)}(t-r-\tau) - I_{ia}^{(4)}(t+r-\tau)}{r} \right],\end{aligned} \tag{5.1b}$$

$$\begin{aligned}(h^{ij})_{M\times M\times I_{ij}} = {}& -\frac{32}{21} \frac{G^3 M^2}{c^8} \int_0^{+\infty} \mathrm{d}\tau \ln \tau \delta_{ij} \partial_{ab} \left[\frac{I_{ab}^{(3)}(t-r-\tau) - I_{ab}^{(3)}(t+r-\tau)}{r} \right] \\ & + \frac{104}{35} \frac{G^3 M^2}{c^8} \int_0^{+\infty} \mathrm{d}\tau \ln \tau \hat{\partial}_{a(i} \left[\frac{I_{j)a}^{(3)}(t-r-\tau) - I_{j)a}^{(3)}(t+r-\tau)}{r} \right] \\ & + \frac{76}{15} \frac{G^3 M^2}{c^{10}} \int_0^{+\infty} \mathrm{d}\tau \ln \tau \frac{I_{ij}^{(5)}(t-r-\tau) - I_{ij}^{(5)}(t+r-\tau)}{r}.\end{aligned} \tag{5.1c}$$

At this stage we have the important verification that the latter piece of the metric should be separately divergence free, i.e. $(\partial_\beta h^{\alpha\beta})_{M\times M\times I_{ij}} = 0$. This verification is important because it tests the rather involved formulas (4.8)–(4.9).

Once we have the metric (5.1) we compute its near-zone expansion $r \to 0$ thanks to the formula (4.4). We need in fact only the leading term in that formula, corresponding to $k = 0$ in (4.4). In anticipation of our change from the gothic metric $h^{\alpha\beta}$ to the usual covariant metric $g_{\alpha\beta}$, we shall include the contribution of the spatial trace $h^{ii} \equiv \delta_{ij} h^{ij}$ together with the h^{00} component. Then we get at leading order when $r \to 0$ the expressions

$$(h^{00}+h^{ii})_{M\times M\times I_{ij}} = -\frac{824}{1575}\frac{G^3M^2}{c^{13}}x^{ab}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I^{(8)}_{ab}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{15}}\right), \tag{5.2a}$$

$$(h^{0i})_{M\times M\times I_{ij}} = \frac{832}{225}\frac{G^3M^2}{c^{12}}x^{a}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I^{(7)}_{ia}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{14}}\right), \tag{5.2b}$$

$$(h^{ij})_{M\times M\times I_{ij}} = -\frac{152}{15}\frac{G^3M^2}{c^{11}}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I^{(6)}_{ij}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{13}}\right). \tag{5.2c}$$

The powers of $1/c$ show that this indeed corresponds to a 5.5PN term. However, we notice that in harmonic coordinates the ij component of the metric, which is of order $1/c^{11}$, can be coupled to a Newtonian term $h^{00}=-4U_{\mathrm{ext}}/c^2+\mathcal{O}(1/c^4)$, where U_{ext} is the Newtonian potential as seen from the exterior of the source, to produce from the next iteration a term of order $1/c^{13}$ comparable to that in the 00 and ii components of the metric. The exterior Newtonian potential U_{ext}, together with the associated "superpotential" χ_{ext} such that $\Delta\chi_{\mathrm{ext}}=2U_{\mathrm{ext}}$, are defined by their multipole expansions,

$$U_{\mathrm{ext}}(\mathbf{x},t) = G\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}I_L(t)\partial_L\left(\frac{1}{r}\right), \tag{5.3a}$$

$$\chi_{\mathrm{ext}}(\mathbf{x},t) = 2\Delta^{-1}U_{\mathrm{ext}} = G\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}I_L(t)\partial_L(r). \tag{5.3b}$$

Thus we see that, in harmonic coordinates, we shall also have a contribution from *quartic* interactions of the type $M\times M\times I_{ij}\times I_L$. This includes, in the particular case $\ell=0$, the interaction $M\times M\times M\times I_{ij}$ which can be viewed as a kind of "tail-of-tail-of-tail." The equation determining this quartic interaction is readily found to be

$$\Delta[(h^{00}+h^{ii})_{M\times M\times I_{ij}\times I_L}] = -\frac{608}{15}\frac{G^3M^2}{c^{13}}\partial_{ij}U_{\mathrm{ext}}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I^{(6)}_{ij}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{15}}\right), \tag{5.4}$$

and is immediately integrated as

$$(h^{00}+h^{ii})_{M\times M\times I_{ij}\times I_L} = -\frac{304}{15}\frac{G^3M^2}{c^{13}}\partial_{ij}\chi_{\mathrm{ext}}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I^{(6)}_{ij}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{15}}\right). \tag{5.5}$$

Next we want to extend these results inside the matter source. This is straightforward for the piece (5.2) which is regular inside the source and is valid there as it stands. However this requires a matching argument for the extra piece (5.5) since it diverges when $r\to 0$. Fortunately the problem of the matching is easily solved by noticing that the exterior Newtonian potential U_{ext} and superpotential χ_{ext} represent the multipole expansions of the usual Poisson potential U and superpotential χ given by

$$U(\mathbf{x},t) = G\int\frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}',t), \tag{5.6a}$$

$$\chi(\mathbf{x},t) = 2\Delta^{-1}U = G\int\mathrm{d}^3\mathbf{x}'|\mathbf{x}-\mathbf{x}'|\rho(\mathbf{x}',t), \tag{5.6b}$$

where ρ is the Newtonian mass density of the source. Here we neglect post-Newtonian corrections, and have simply used the fact that the mass moments I_L take on their usual Newtonian expressions in the Newtonian limit. Once the metric is matched, i.e. U_{ext} and χ_{ext} are replaced by U and χ, we can consider the case where the source is a point-particles binary for which we have, at Newtonian order,

$$U(\mathbf{x},t) = \frac{Gm_1}{r_1}+\frac{Gm_2}{r_2}, \tag{5.7a}$$

$$\chi(\mathbf{x},t) = Gm_1r_1+Gm_2r_2. \tag{5.7b}$$

The metric is then complete. Coming back to the usual covariant metric $g_{\alpha\beta}$ we find the following contributions at 5.5PN order:

$$g_{00}^{5.5\mathrm{PN}} = \frac{412}{1575}\frac{G^3M^2}{c^{13}}x^{ab}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I_{ab}^{(8)}(t-\tau) + \frac{152}{15}\frac{G^3M^2}{c^{13}}\partial_{ab}\chi\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I_{ab}^{(6)}(t-\tau) + \mathcal{O}\left(\frac{1}{c^{15}}\right), \tag{5.8a}$$

$$g_{0i}^{5.5\mathrm{PN}} = \frac{832}{225}\frac{G^3M^2}{c^{12}}x^{a}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I_{ia}^{(7)}(t-\tau) + \mathcal{O}\left(\frac{1}{c^{14}}\right), \tag{5.8b}$$

$$g_{ij}^{5.5\mathrm{PN}} = \frac{152}{15}\frac{G^3M^2}{c^{11}}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I_{ij}^{(6)}(t-\tau) + \mathcal{O}\left(\frac{1}{c^{13}}\right). \tag{5.8c}$$

A priori this metric will contain both conservative and dissipative (radiation-reaction) effects. Here we want to keep only the conservative effects, that are compatible with the helical symmetry and exactly circular orbits. We shall assume that the split between conservative and dissipative effects is equivalent to a split between "time-symmetric" and "time-antisymmetric" contributions in the following sense. Namely, we decompose the tail integrals in (5.8) into conservative and dissipative pieces defined by

$$\left(\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I_{ab}^{(p)}(t-\tau)\right)_{\mathrm{cons}} = \frac{1}{2}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau[I_{ab}^{(p)}(t-\tau) + I_{ab}^{(p)}(t+\tau)], \tag{5.9a}$$

$$\left(\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau I_{ab}^{(p)}(t-\tau)\right)_{\mathrm{diss}} = \frac{1}{2}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau[I_{ab}^{(p)}(t-\tau) - I_{ab}^{(p)}(t+\tau)]. \tag{5.9b}$$

This will be justified later when we check that the equations of motion associated with the conservative/symmetric piece of the metric are indeed conservative, i.e. that the acceleration is purely radial. Notice that there should be a logarithm $\ln r$ associated with the conservative part of the tail integral, exactly as at 4PN and 5PN orders [11,21]. However this logarithm is an instantaneous 5.5PN term and therefore is zero for circular orbits by the argument (2.1). Finally the conservative part of the metric at the 5.5PN order is

$$(g_{00}^{5.5\mathrm{PN}})_{\mathrm{cons}} = \frac{206}{1575}\frac{G^3M^2}{c^{13}}x^{ab}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau[I_{ab}^{(8)}(t-\tau) + I_{ab}^{(8)}(t+\tau)] + \frac{76}{15}\frac{G^3M^2}{c^{13}}\partial_{ab}\chi\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau[I_{ab}^{(6)}(t-\tau) + I_{ab}^{(6)}(t+\tau)] + \mathcal{O}\left(\frac{1}{c^{15}}\right), \tag{5.10a}$$

$$(g_{0i}^{5.5\mathrm{PN}})_{\mathrm{cons}} = \frac{416}{225}\frac{G^3M^2}{c^{12}}x^{a}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau[I_{ia}^{(7)}(t-\tau) + I_{ia}^{(7)}(t+\tau)] + \mathcal{O}\left(\frac{1}{c^{14}}\right), \tag{5.10b}$$

$$(g_{ij}^{5.5\mathrm{PN}})_{\mathrm{cons}} = \frac{76}{15}\frac{G^3M^2}{c^{11}}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau[I_{ij}^{(6)}(t-\tau) + I_{ij}^{(6)}(t+\tau)] + \mathcal{O}\left(\frac{1}{c^{13}}\right), \tag{5.10c}$$

where we recall that the superpotential χ is given by Eq. (5.7b).

The metric (5.10) corresponds to harmonic coordinates. In harmonic coordinates we have obtained a "quartic" nonlinear contribution at 5.5PN order given by the second term in (5.10a). However let us introduce new coordinates, which have the desirable property of canceling the latter quartic nonlinear contribution, and removing the $0i$ and ij components of the metric. The coordinate transformation vector from the harmonic coordinates to the new ones is given by

$$\eta_0 = \frac{77}{225}\frac{G^3M^2}{c^{12}}x^{ab}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau[I_{ab}^{(7)}(t-\tau) + I_{ab}^{(7)}(t+\tau)] + \mathcal{O}\left(\frac{1}{c^{14}}\right), \tag{5.11a}$$

$$\eta_i = -\frac{38}{15}\frac{G^3M^2}{c^{11}}x^{a}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau[I_{ia}^{(6)}(t-\tau) + I_{ia}^{(6)}(t+\tau)] + \mathcal{O}\left(\frac{1}{c^{13}}\right). \tag{5.11b}$$

The coordinate transformation at the requested order, including the nonlinear correction with respect to a linear gauge transformation [see e.g. Eqs. (6.9)–(6.10) in Ref. [22]], reads

$$(g'^{5.5\mathrm{PN}}_{00})_{\mathrm{cons}} = (g^{5.5\mathrm{PN}}_{00})_{\mathrm{cons}} + \frac{2}{c}\partial_t\eta_0 + \frac{2}{c^2}\partial_i\eta_j\partial_{ij}\chi + \mathcal{O}\left(\frac{1}{c^{15}}\right), \tag{5.12a}$$

$$(g'^{5.5\mathrm{PN}}_{0i})_{\mathrm{cons}} = (g^{5.5\mathrm{PN}}_{0i})_{\mathrm{cons}} + \frac{1}{c}\partial_t\eta_i + \partial_i\eta_0 + \mathcal{O}\left(\frac{1}{c^{14}}\right), \tag{5.12b}$$

$$(g'^{5.5\mathrm{PN}}_{ij})_{\mathrm{cons}} = (g^{5.5\mathrm{PN}}_{ij})_{\mathrm{cons}} + \partial_i\eta_j + \partial_j\eta_i + \mathcal{O}\left(\frac{1}{c^{13}}\right). \tag{5.12c}$$

The nonlinear term in Eq. (5.12a) cancels the second term in Eq. (5.10a) and we find the simple new metric

$$(g'^{5.5\mathrm{PN}}_{00})_{\mathrm{cons}} = \frac{428}{525}\frac{G^3M^2}{c^{13}}x^{ab}\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau[I^{(8)}_{ab}(t-\tau) + I^{(8)}_{ab}(t+\tau)] + \mathcal{O}\left(\frac{1}{c^{15}}\right), \tag{5.13a}$$

$$(g'^{5.5\mathrm{PN}}_{0i})_{\mathrm{cons}} = \mathcal{O}\left(\frac{1}{c^{14}}\right), \tag{5.13b}$$

$$(g'^{5.5\mathrm{PN}}_{ij})_{\mathrm{cons}} = \mathcal{O}\left(\frac{1}{c^{13}}\right). \tag{5.13c}$$

The computations to follow have been performed with the two metrics (5.10) and (5.13) giving identical results.

Following Eq. (1.2) we compute the components of the metric [either (5.10) or (5.13)] at the location of the particle 1. For this, we simply replace x^i by y_1^i and thus (in a center-of-mass frame) by $X_2x_{12}^i$ where $x_{12}^i = y_1^i - y_2^i$ and $X_2 = m_2/m$. At linear order in the mass ratio ν we can assume that $X_2 = 1 + \mathcal{O}(\nu)$. The term $\partial_{ab}\chi$ in the harmonic-coordinate metric necessitates a regularization and reads $(\partial_{ab}\chi)_1 = m_2(\delta^{ab} - n_{12}^{ab})/r_{12}$ on particle 1.

On the other hand the quadrupole moment is given by the usual Newtonian expression $I_{ij} = m\nu\hat{x}_{12}^{ij}$ and its time derivatives are computed for circular orbits using the Newtonian equations of motion. Similarly the ADM mass is given with this approximation by $M = m$. Then the quadrupole moment is to be evaluated in the past and in the future, at advanced and retarded times $t\pm\tau$. To do that we relate the separation vector and relative velocity at earlier and future times to the current values for circular orbits by using

$$x_{12}^i(t\pm\tau) = \cos(\Omega\tau)x_{12}^i(t) \pm \sin(\Omega\tau)v_{12}^i(t)/\Omega, \tag{5.14a}$$

$$v_{12}^i(t\pm\tau) = \mp\Omega\sin(\Omega\tau)x_{12}^i(t) + \cos(\Omega\tau)v_{12}^i(t), \tag{5.14b}$$

where Ω is the orbital frequency of the circular motion. We are then left with the integrals

$$\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau\cos(2\Omega\tau) = -\frac{\pi}{4\Omega}, \tag{5.15a}$$

$$\int_0^{+\infty}\mathrm{d}\tau\,\ln\tau\sin(2\Omega\tau) = -\frac{1}{2\Omega}[\ln(2\Omega) + \gamma_\mathrm{E}]. \tag{5.15b}$$

We shall find that, for the conservative part of the dynamics, only the first integral (with the factor π) contributes. The other integral (with Euler's constant γ_E) will not be needed.

It is important also to consider the modification of the equations of motion which is induced by the 5.5PN metric (5.10). We find that with the conservative symmetrized (half-retarded plus half-advanced) expression (5.10) the modification is purely conservative; i.e. it only affects the relation between the orbital frequency Ω and the coordinate separation r_{12}. This is a confirmation of our prescriptions (5.9). Writing only the Newtonian and 5.5PN terms we get

$$\Omega^2 = \frac{Gm}{r_{12}^3}\left[1 + \frac{27392}{525}\nu\pi\gamma^{11/2}\right], \tag{5.16}$$

where $\gamma = Gm/(r_{12}c^2)$. The inverse relation in terms of $x = (Gm\Omega/c^3)^{2/3}$ is

$$\gamma = x\left[1 - \frac{27392}{1575}\nu\pi x^{11/2}\right]. \tag{5.17}$$

We have checked that the modification of the motion does not affect the position of the center of mass so we can use the usual formulas when going to the center-of-mass frame.

At last we have the metric on particle 1 and we insert it into Eq. (1.2). We then go to the frame of the center of mass and reduce the expression to circular orbits, mindful of the modification (5.17) to the relation between orbital separation and frequency—which we find does not actually contribute to the final result. Posing then $q = m_1/m_2$ and $y = (Gm_2\Omega/c^3)^{2/3} = x(1+q)^{-2/3}$, we define the SF part to the redshift factor as $u_1^T = u_{\mathrm{Schw}}^T(y) + qu_{\mathrm{SF}}^T(y) + \mathcal{O}(q^2)$ and find that the 5.5PN contribution therein is

$$u_{\mathrm{SF}}^T = y\left[1 - \frac{13696}{525}\pi y^{11/2}\right]. \tag{5.18}$$

We have written only the Newtonian and 5.5PN terms. This result is in perfect agreement with the high-precision numerical and analytic computation of the gravitational self-force reported in Eq. (20) of the companion paper [12].

Analytical self-force calculations, essentially extending those in Refs. [12,23] and based on the Regge-Wheeler equation, have recently obtained exact results up to order 6PN [24], in precise agreement with the high-precision results of [12]. While such an approach is applicable to all PN orders at first order in perturbation theory, our methods in principle apply to arbitrarily high order in the mass ratio, while also extending to higher PN order.

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APPENDIX: PROOF THAT CERTAIN SPECIFIC TERMS DO NOT CONTRIBUTE AT 5.5PN ORDER

The second term in Eq. (4.3a) is a particular solution of the wave equation defined by means of the operator of the instantaneous potentials $\Box^{-1}_{\text{inst}}$ given in Eq. (4.5). It is crucial that such an operator acts directly on the near-zone expansion of the source term (4.6), where the source term itself is given for this application by Eq. (4.2b) namely

$$S(r,t-r) = r^{B-k}\int_1^{+\infty}\mathrm{d}x Q_m(x)F(t-rx). \tag{A1}$$

We are looking for the hereditary tail part of the metric. Since the operator $\Box^{-1}_{\text{inst}}$ is instantaneous, i.e. it does not involve any integral extending over time, the only possible tail integrals will come from the tails that are already present in the near-zone expansion [namely $\overline{S(r,t-r)}$ when $r\to 0$] of the source term (A1).

Let us first note that one cannot compute the near-zone expansion of (A1) by directly expanding $F(t-rx)$ under the integral sign because the coefficients in the expansion will involve the integral of the Legendre function $Q_m(x)$ multiplied by arbitrary powers of x, which will become divergent at some stage. Hence we split the integral (A1) into a "recent" part from $x=1$ to K, where K is a constant such that $K\gg 1$, and a "remote" part from K up to $+\infty$. Now we are allowed to perform the Taylor expansion of $F(t-rx)$ when $r\to 0$ into the recent part. That expansion will be made of time derivatives $F^{(n)}(t)$ with coefficients given by finite integrals from 1 to K of some $x^n Q_m(x)$. Hence the expansion of the recent part is purely instantaneous and does not contain tails. Looking for hereditary tails we can thus concentrate our attention to the remote part of the integral, namely

$$\overline{S(r,t-r)}|_{\text{tail}} = r^{B-k}\int_K^{+\infty}\mathrm{d}x Q_m(x)F(t-rx). \tag{A2}$$

In the right-hand side an overbar is implicitly understood, meaning that the expression should be considered in the form of a near-zone expansion. Since we assumed $K\gg 1$ we are allowed to replace the Legendre function $Q_m(x)$ by its formal expansion when $x\to\infty$, which is of the type $Q_m(x)\sim\sum_{p=0}^{+\infty}x^{-m-2p-1}$, with some constant coefficients that we shall not need to consider here. Thus,

$$\overline{S(r,t-r)}|_{\text{tail}} \sim \sum_{p=0}^{+\infty} r^{B-k}\int_K^{+\infty}\frac{\mathrm{d}x}{x^{m+2p+1}}F(t-rx). \tag{A3}$$

Next we repeatedly integrate the latter integrals by parts. The all-integrated terms will be some functions $F^{(k)}(t-Kr)$ which can be Taylor-expanded when $r\to 0$ without problem. They do not contain tails so we ignore them. After $m+2p+1$ integrations by parts we get

$$\overline{S(r,t-r)}|_{\text{tail}} \sim \sum_{p=0}^{+\infty} r^{B-k+m+2p+1} \times\int_K^{+\infty}\mathrm{d}x\,\ln x F^{(m+2p+1)}(t-rx). \tag{A4}$$

As before we do not need to write the detailed (B-dependent) coefficients in front of each term. Posing next $\tau = rx$ we obtain

$$\overline{S(r,t-r)}|_{\text{tail}} \sim \sum_{p=0}^{+\infty} r^{B-k+m+2p} \times\int_{rK}^{+\infty}\mathrm{d}\tau\,\ln \tau F^{(m+2p+1)}(t-\tau), \tag{A5}$$

where again, a nontail term (proportional to $\ln r$) has been ignored. Finally we note that the recent part of the latter integral, from 0 to rK, can also be expanded without tails. It is then convenient to add it back in order to complete our result (A5). Thus we have identified the tail part of the source as

$$\overline{S(r,t-r)}|_{\text{tail}} \sim \sum_{p=0}^{+\infty} r^{B-k+m+2p} \times\int_{0}^{+\infty}\mathrm{d}\tau\,\ln \tau F^{(m+2p+1)}(t-\tau). \tag{A6}$$

Following the prescription (4.5), it remains to apply the operator $\Box^{-1}_{\text{inst}}$. This gives

$$\Box^{-1}_{\text{inst}}[\hat{n}_L\overline{S(r,t-r)}]|_{\text{tail}} \sim \sum_{i=0}^{+\infty}\sum_{p=0}^{+\infty}\Delta^{-i-1}(\hat{n}_L r^{B-k+m+2p}) \times\int_{0}^{+\infty}\mathrm{d}\tau\,\ln \tau F^{(m+2p+2i+1)}(t-\tau). \tag{A7}$$

The iterated Poisson operators Δ^{-i-1} are straightforwardly computed and, as usual, we consider the finite part when $B\to 0$. This yields some powers of r and possibly some $\ln r$ due to poles $\sim 1/B$. Thus we get (with $a=0$ or 1)

$$\Box^{-1}_{\mathrm{inst}}[\hat{n}_L \overline{S(r,t-r)}]|_{\mathrm{tail}} \sim \sum_{i=0}^{+\infty}\sum_{p=0}^{+\infty} \hat{n}_L r^{-k+m+2p+2i+2} (\ln r)^a \times \int_0^{+\infty} \mathrm{d}\tau \ln \tau F^{(m+2p+2i+1)}(t-\tau). \tag{A8}$$

If we restore all the powers of c's and G's together with the fact that F is composed of a mass squared M^2 times a time derivative of a quadrupole moment I_{ab}, we end up with

$$\Box^{-1}_{\mathrm{inst}}[\hat{n}_L \overline{S(r,t-r)}]|_{\mathrm{tail}} \sim G^3 M^2 \sum_{i,p} \frac{\hat{n}_L r^{-k+m+2p+2i+2}(\ln r)^a}{c^{13-k+m+2p+2i}} \times \int_0^{+\infty} \mathrm{d}\tau \ln \tau I_{ab}^{(8-k+m+2p+2i)}(t-\tau). \tag{A9}$$

Let us look at the actual source for that particular interaction $M^2 \times I_{ab}$ as given by Eqs. (3.5)—as explained already, we can ignore the nontail part (3.4) of the source. We observe that, for all the terms in Eqs. (3.5), the combination $k+m+\ell$ is always an *odd* integer. Furthermore, using the fact that the space indices among $\alpha\beta = 00$, $0i$, ij must be distributed between the indices of $\hat{n}_L$ and $I_{ab}^{(p)}$, we see that ℓ must be even in the 00 and ij components of the metric and odd in the $0i$ components; see also the discussion above Eq. (2.4). We thus conclude from Eq. (A9) that the powers of $1/c$ are even in the 00 and ij components and odd in the $0i$ components, which means precisely that all the terms in Eq. (A9) have necessarily integral PN orders. Closer inspection of (A9) with the explicit values of k, m and ℓ in the source (3.5) shows that these terms are necessarily of order 4PN, 5PN, 6PN and so on, but can never arise at 5.5PN order.

In conclusion, we have proved that only the first term in Eq. (4.3a) contributes at the 5.5PN order, and this is what we have computed in the text. A related issue is that the second term in Eq. (4.3a), that we have investigated in this appendix, is in fact divergent when $r \to 0$; indeed see e.g. (A9) which involves negative powers of r, when $k = +3$ say, which is a typical term in the source (3.5). Thus the second term in (4.3a) cannot be continued inside the source by itself. It has to be matched to the actual PN expansion of the field inside the source. Only the first term in Eq. (4.3a), which is a regular homogeneous solution of the wave equation, is valid inside the source and can be continued there. This is why we could compute it at the location of one of the particles in a binary system. The other term, by contrast, necessitates a matching procedure which we do not control in the present work. However, past experience with tails (e.g. in Ref. [11]) indicates that one does not need a complete matching in order to compute the tails inside the source, essentially because they contribute to the radiation reaction and can be determined as "boundary conditions" set outside the source. Therefore we do not expect that the second term in Eq. (4.3a) should contribute to the present tail-of-tail effect. Regardless, in this appendix we have directly proven that the PN order of such a term is necessarily integral and cannot be 5.5PN.

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