Cosmology: The search for twenty-four (or more) functions

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We enumerate the $4(1 + F) + 2S$ independent arbitrary functions of space required to describe a general relativistic cosmology containing an arbitrary number of noninteracting fluid (F) and scalar fields (S). Results are also given for arbitrary space dimension and for higher-order gravity theories, where the number increases to $16 + 4F + 2S$. Both counts are subject to assumptions about whether the dark energy is a cosmological constant. A more detailed analysis is provided when global homogeneity is assumed and the functions become constants. This situation is also studied in the case where the flat and open universes have compact spatial topologies. This changes the relative generalities significantly and places new constraints on the types of expansion anisotropy that are permitted. The most general compact homogeneous universes containing Friedmann models are spatially flat and described by $8 + 4F + 2S$ constants. Comparisons are made with the simple six-parameter lambda-CDM model and physical interpretations provided for the parameters needed to describe the most general cosmological models.

I. INTRODUCTION

There have been several attempts to reduce the description of the astronomical universe to the determination of a small number of measurable parameters. Typically, these will be the free parameters of a well defined cosmological model that uses the smallest number of constants that can provide a best fit to the available observational evidence. Specific examples are the popular characterizations of cosmology as a search for "nine numbers" [1], "six numbers" [2], or the six-parameter minimal ΛCDM model used to fit the WMAP [3] and Planck data sets [4]. In all these, and other, cases of simple parameter counting there are usually many simplifying assumptions that amount to ignoring other parameters or setting them to zero, for example, by assuming a flat Friedmann background universe or a power-law variation of density inhomogeneity in order to reduce the parameter count and any associated degeneracies. The assumption of a power-law spectrum for inhomogeneities will reduce a spatial function to two constants, while the assumption that the universe is described by a Friedmann metric plus small inhomogeneous perturbations both reduces the number of metric unknowns and converts functions into constant parameters. In this paper we are going to provide some context for the common minimal parameter counts cited above by determining the total number of spatial functions that are needed to prescribe the structure of the universe if it is assumed to contain a finite number of simple matter fields. We are not counting fundamental constants of physics, like the Newtonian gravitation constant, the coupling constants defining quadratic Lagrangian extensions of general relativistic gravity, or the 19 free parameters that define the behaviors of the 61 elementary particles in the standard three-generation $U(1) \times SU(2) \times SU(3)$ model of particle physics. However, there is some ambiguity in the status in some quantities. For example, is the dark energy equivalent to a true cosmological constant (a fundamental constant), or to some effective fluid or scalar field, or some other emergent effect [5]? Some fundamental physics parameters, like neutrino masses, particle lifetimes, or axion phases, can also play a part in determining cosmological densities but that is a secondary use of the cosmological observable. Here we will take an elementary approach that counts the number of arbitrary functions needed to specify the general solution of the Einstein equations (and its generalizations). This will give a minimalist characterization that can be augmented by adding any number of additional fields in a straightforward way. We will also consider the count in higher-order gravity theories as well as for general relativistic cosmologies. We enumerate the situation in spatially homogeneous universes in detail so as to highlight the significant impact of their spatial topology on evaluations of their relative generality.

II. SIMPLE FUNCTION COUNTING

The cosmological problem can be formulated in general relativity using a metric in a general synchronous reference system [6]. Assume that there are F matter fields which are noninteracting and each behaves as a perfect fluid with some equation of state $p_i(\rho_i)$, $i = 1, \ldots F$. They will each have a normalized 4-velocity field, $(u_a)_i$, $a = 0, 1, 2, 3$. These will in general be different and noncomoving. Thus each matter field is defined on a spacelike surface of constant time by four arbitrary functions of three spatial variables, x^α since the u_0 components are determined by the normalizations $(u_a u^a)_i = 1$. This means that the initial data for the F noninteracting fluids are specified by $4F$ functions of three spatial variables. If we were in an

N-dimensional space, then each fluid would require $N + 1$ functions of N spatial variables and F fluids would require $(N + 1)F$ such functions to describe them in general.

The 3D metric requires the specification of six $g_{\alpha\beta}$ and six $\dot{g}_{\alpha\beta}$ for the symmetric spatial 3×3 metric in the synchronous system but these may be reduced by using the four coordinate covariances of the theory and a further four can be eliminated by using the four constraint equations of general relativity. This leaves four independently arbitrary functions of three spatial variables [6] which is just twice the number of degrees of freedom of the gravitational spin-2 field. The general transformation between synchronous coordinate systems maintains this number of functions [6]. This is the number required to specify the general vacuum solution of the Einstein equations in a three-dimensional space. In an N-dimensional space we would require $N(N + 1)$ functions of N spatial variables to specify the initial data for $g_{\alpha\beta}$ and $\dot{g}_{\alpha\beta}$. This could be reduced by $N + 1$ coordinate covariances and $N + 1$ constraints to leave $(N - 2)(N + 1)$ independent arbitrary functions of N variables [7]. This even number is equal to twice the number of degrees of freedom of the gravitational spin-2 field in $N + 1$ dimensional spacetime.

When we combine these counts we see that the general solution in the synchronous system for a general relativistic cosmological model containing F fluids requires the specification of $(N - 2)(N + 1) + F(N + 1) = (N + 1) \times (N + F - 2)$ independent functions of N spatial variables. If there are also S noninteracting scalar fields, $\phi_j, j = 1, ..., S$, present with self-interaction potentials $V(\phi_j)$, then two further spatial functions are required (ϕ_j and $\dot{\phi}_j$) to specify each scalar field and the total becomes $(N + 1) \times (N + F - 2) + 2S$. For the observationally relevant case of $N = 3$, this reduces to $4(F + 1) + 2S$ spatial functions.

For example, if we assume a simple realistic scenario in which the universe contains separate baryonic, cold dark matter, photon, neutrino and dark energy fluids, all with separate noncomoving velocity fields, but no scalar fields, then $F = 5$ and our cosmology needs 24 spatial functions in the general case. If the dark energy is not a fluid, but a cosmological constant with constant density and $u_i = \delta_i^0$, then the dark energy "fluid" description reduces to the specification of a single constant, $\rho_{\text{DE}} = \Lambda/8\pi G$, rather than four functions and reduces the total to 21 independent spatial functions. However, if the cosmological constant is an evolving scalar field, then we would have $F = 4$ and $S = 1$, and now 22 spatial functions are required. Examples of full function asymptotic solutions were found for perturbations around de Sitter spacetime by Starobinsky [8], the approach to "sudden" finite-time singularities [9] by Barrow, Cotsakis and Tsokaros [10], and near quasi-isotropic singularities with $p > \rho$ "fluids" by Heinzle and Sandin [11].

These function counts of 21–24 should be regarded as lower bounds. They do not include the possibility of a cosmological magnetic field or some other unknown matter fields. They also treat all light ($\ll 1$ MeV) neutrinos as if they are identical (heavy neutrinos can be regarded as CDM if they provide the largest contribution to the matter density but if they are not responsible for the dominant dark matter then they should be counted as a further contribution to F). If there are matter fields which are not simple fluids with $p(\rho)$, for example an imperfect fluid possessing a bulk viscosity or a gas of free particles with anisotropic pressures, then additional parameters are required to specify them—although there can still be overall constraints—a trace-free energy-momentum tensor, for example, in the cases of electric and magnetic fields or Yang-Mills fields—and we would just count the number of independent terms in the symmetric energy-momentum tensor.

In the case of the Planck or WMAP mission data analyses, six constants are chosen to define the standard (minimal) ΛCDM model. For WMAP [3], these are the present-day Hubble expansion rate H_0, the densities of baryons and cold dark matter, the optical depth τ, at a fixed redshift, and the amplitude and slope of an assumed power-law spectrum of curvature inhomogeneities on a specified reference length scale. This is equivalent to including three matter fields (radiation, baryons, cold dark matter) but the standard ΛCDM assumes zero spatial curvature k, *ab initio* so a relaxation of this would add a curvature term or a dark energy field, because when $k \neq 0$ the latter could no longer be deduced from the other densities and the critical density defined by H_0. The light neutrino densities are assumed to be calculable from the radiation density using the standard isotropically expanding thermal history, so there are effectively $F = 5$ matter fields (with k set to zero in the base model and a metric time derivative determined by H). All deviations from isotropy and homogeneity enter only at the level of perturbation theory and are characterized by the spectral amplitude and slope on large scales; the amplitude on small scales ("acoustic peaks" in the power spectrum) is determined from that on large scales by an $e^{-2\tau}$ damping factor determined by the optical depth parameter τ. The Planck parameter choice is equivalent [4].

Although a general solution of the Einstein equations requires the full complement of arbitrary functions, different parts of the general solution space can have behaviors of quite different complexity. For example, when $N \leq 9$ there are homogeneous vacuum universes which are dynamically chaotic but the chaotic behavior disappears when $N \geq 10$ even though the number of arbitrary constants remains maximal for each N [12].

III. MORE GENERAL GRAVITY THEORIES

There has been considerable interest in trying to explain the dark energy as a feature of a higher-order gravitational theory that extends the Lagrangian of general relativity in a nonlinear fashion [13–16]. This offers the possibility of

introducing a Lagrangian that is a function of $L = f(R, R_{ab}R^{ab})$ of the scalar curvature R and/or the Ricci scalar $R_{ab}R^{ab}$ in anisotropic models, with the property that it contributes a slowly varying dark energylike behavior at late times without the need to specify an explicit cosmological constant. However, these higher-order Lagrangian theories (excluding the Lovelock Lagrangians in which the variation of the higher-order terms contribute pure divergences [17]) all have fourth-order field equations in three-dimensional space when $f \neq A + BR$, with A, B constants. This means that the initial data set for such theories is considerably enlarged because we must specify $\ddot{g}_{\alpha\beta}$ and $\dddot{g}_{\alpha\beta}$ in addition to $g_{\alpha\beta}$ and $\dot{g}_{\alpha\beta}$. In N dimensions, this results in a further $N(N+1)$ functions of N variables and so a general cosmological model with F fluids and S scalar fields requires a specification of $2(N^2 - 1) + F(N+1) + 2S = (N+1)(F + 2N - 2) + 2S$ independent arbitrary spatial functions. For $N = 3$, this is $16 + 4F + 2S$. General relativity with four matter fields plus a cosmological constant requires 20 spatial functions plus one constant, in general, whereas a higher-order gravity theory with four matter fields and no scalar fields (and no cosmological constant because it should presumably emerge from the metric behavior) requires the specification of 32 spatial functions in general.

IV. REDUCING FUNCTIONS TO CONSTANTS

The most common simplification used to reduce the size of the cosmological characterisation problem is to turn the spatial functions into constants. This simplification will be an exact if the universe is assumed to be spatially homogeneous. The set of possible spatially homogeneous and isotropic universes with natural topology is based upon the classification of homogeneous 3-spaces created by Bianchi [18–21] (together with the exceptional case of Kantowski-Sachs-Kompayeets-Chernov with $S^1 \times S^2$ topology [22,23] which we will ignore here).

The most general Bianchi type universes are those of types VI_h, VII_h, $VIII$ and IX. Of these, only types VII_h and IX, respectively, contain open and closed isotropic Friedmann subcases. These most general Bianchi types are all defined by four arbitrary constants in vacuum plus a further four for each noninteracting perfect fluid source. Therefore, in three-dimensional spaces, the most general spatially homogeneous universes containing F fluids are defined by $4(1 + F)$ arbitrary constants. This suggests that they might be the leading-order term in a linearization of the general inhomogeneous solution in the homogeneous limit. However, things might not be so simple. The 4-function space of solutions to Einstein's models like type IX with compact spaces has a conical structure at points with Killing vectors and so linearization about the points must control an infinite number of spurious linearizations that are not the leading-order term in any series expansion that converges to a true solution [24,25].

The Bianchi classification of spatially homogeneous universes derives from the classification of the group of isometries with three-dimensional subgroups that act simply transitively on the manifold. Intuitively, these give cosmological histories that look the same to observers in different places on the same hypersurface of constant time.

The Bianchi types are subdivided into two classes [26]: Class A contains types $I(1 + F)$, $II(2 + 3F)$, $VI_0(3 + 4F)$, $VII_0(3 + 4F)$, and $IX(4 + 4F)$, while class B contains types $V(1 + 4F)$, $IV(3 + 4F)$, $III(3 + 4F)$, $VI_{-1/9}(4 + 3F)$, $VI_h(4 + 4F)$ and $VII_h(4 + 4F)$. The brackets following each Roman numeral labeling the Bianchi type geometry contain the number of constants defining the general solution when F noninteracting perfect fluids, each with $p > -\rho$, are present, so $F = 0$ defines the vacuum case. For example, Bianchi type I denoted by $I(1 + F)$ is defined by one constant in vacuum (when it is the Kasner metric) and one additional constant for the value of the density when each matter field is added. We have ignored scalar fields here for simplicity but to restore that consideration simply add $2S$ inside each pair of brackets. The Euclidean geometry in the type I case requires $R_{0\alpha} = 0$ and so the three noncomoving velocities (and hence any possible vorticity) must be identically zero. This contains the zero-curvature Friedmann model as the isotropic (zero parameter) special case. In the next simplest case, of type V, the general vacuum solution found by Saunders [27] contains one parameter, but each additional perfect fluid adds four parameters because it requires specification of a density and three nonzero u_α components. The spatial geometry is a Lobachevsky space of constant negative isotropic curvature. The isotropic subcases of type V are the zero-parameter Milne universe in vacuum and the F-parameter open Friedmann universe containing F fluids.

In practice, one cannot find exact homogeneous general solutions containing the maximal number of arbitrary constants, although the qualitative behaviors are fairly well understood, and many explorations of the observational effects use the simplest Bianchi I or V models (usually without including noncomoving velocities) because they possess isotropic three-curvature and add only a simple fast-decaying anisotropy term (one constant parameter) to the Friedmann equation. The most general anisotropic metrics which contain isotropic special cases, of types VII and IX, possess both expansion anisotropy (shear) and anisotropic three-curvature. Their shear falls off more slowly and the observational bounds on it are much weaker [28–33].

V. LINKS TO OBSERVABLES

The free spatial functions (or constants) specifying inhomogeneous (homogeneous) metrics have simple physical interpretations. In the most general cases the four vacuum parameters can be thought of as giving two shear modes (i.e. time derivatives of metric anisotropies) and two

parts of the anisotropic spatial curvature (composed of ratios and products of metric functions). In the simplest vacuum models of type I and V the three-curvature is isotropic so there is only one shear parameter. It describes the allowed metric shear and in the V model a second parameter is the isotropic three-curvature (which is zero in type I). When matter is added then there is always a single ρ (or p) for each perfect fluid and then up to three noncomoving fluid velocity components. If the fluid is comoving, as in type I, only the density parameter is required for each fluid; in type V there can also be three noncomoving velocities. The additional parameters control the expansion shear anisotropy, anisotropic three-curvature. They may all contribute to temperature anisotropy in the cosmic microwave background (CMB) radiation but the observed anisotropy is determined by an integral down the past null cone over the shear (effectively the shear to Hubble rate ratio at last scattering of the CMB), rather than the Weyl curvature modes driven by the curvature anisotropy (which can be oscillatory [34] and so can be periodically be very small), while the velocities contribute dipole variations. Thus, it is difficult to extract complete information about all the anisotropies from observations of the lower multipoles of the CMB alone in the most general cases [31,35–37].

At present, the observational focus is upon testing the simplest possible ΛCDM model, defined by the smallest number of parameters. As observational sensitivity increases it will become possible to place specific bounds or make determinations of the full spectrum of defining functions (or constants). In an inflationary model they can be identified from the spatial functions defining the asymptotic expansion around the de Sitter metric [8].

There have also been interesting studies of the observational information needed to determine the structure of our past null cone rather than constant-time hypersurfaces in the Universe [38], extending earlier investigations of the links between observables and general metric expansions by McCrea [39] and by Kristian and Sachs [40]. The high level of isotropy in the visible Universe, possibly present as a consequence of a period of inflation in the early Universe [41], or special initial conditions [11,32,42–44], is what allows several of the defining functions of a generic cosmological model to be ignored on the grounds that they are too small to be detected with current technology. An inflationary theory of the chaotic or eternal variety, in which inflation only ends locally, will lead to some complicated set of defining functions that exhibit large smooth isotropic regions within a complicated global structure which is beyond our visual horizon and unobservable (although not necessarily falsifiable within a particular cosmological model).

In practice, there is a divide between the complexity of inhomogeneity in the Universe on small and large scales. On large scales there has been effectively no processing of the primordial spectrum of inhomogeneity by damping or nonlinear evolution. Its description is well approximated by replacing a smooth function by a power law defined by two constants, as for the microwave background temperature fluctuation spectrum or the two-point correlation function of galaxy clustering. Here, the defining functions may be replaced by statistical distributions for specific features, like peak or voids in the density distribution. On small scales, inhomogeneities that entered the horizon during the radiation era can be damped out by photon viscosity or diffusion and may leave distortions in the background radiation spectrum as witness to their earlier existence. The baryon distribution may provide baryon acoustic oscillations which yield potentially sensitive information about the baryon density [3,4]. On smaller scales that enter the horizon later, where damping and nonlinear self-interaction has occurred, the resulting distributions of luminous and dark matter are more complicated. However, they are correspondingly more difficult to predict in detail and numerical simulations of ensembles of models are used to make predictions down to the limit of reliable resolution. Predicting their forms also requires a significant extension of the simple, purely cosmological enumeration of free functions that we have discussed so far. Detailed physical interactions, 3D hydrodynamics, turbulence, shocks, protogalaxy shapes, magnetic fields, and collision orientations all introduce additional factors that may increase the parameters on which observable outcomes depend. The so-called bias parameter, equal to the ratio of luminous matter density to the total density, is in reality a spatial function that is being used to follow the ratio of two densities because one (the dark matter) is expected to be far more smoothly distributed than the other. All these small scale factors combine to determine the output distribution of the baryonic and nonbaryonic density distributions and their associated velocities.

VI. TOPOLOGY

So far, we have assumed that the cosmological models in question take the natural topology, that is, R^3 for the three-dimensional flat and negatively curved spaces and S^3 for the closed spaces. However, compact topologies can also be imposed upon flat and open universes and there has been considerable interest in this possibility and its observational consequences for optical images of galaxies and the CMB [45–47]. The classification of compact negatively curved spaces is a challenging mathematical problem and when the permitted compact spatial topologies are imposed on flat and open homogeneous cosmologies it produces a major change in their relative generalities and the numbers of constants needed to specify them in general and in the differences between the counts for vacuum and perfect fluid models.

The most notable consequences of imposing a compact topology on three-dimensional homogeneous spaces is that

the Bianchi universes of types IV and VI_h no longer exist and open universes of Bianchi types V and VII_h must be isotropic with spaces that are quotients of a space of constant negative curvature, as required by Mostow's rigidity theorem [48–52]. The only universes with nontrivial structure that differs from that of their universal covering spaces are those of Bianchi types I, II, III, VI_0, VII_0 and $VIII$. The numbers of parameters needed to determine their general cosmological solutions when F noninteracting fluids are present and the spatial geometry is compact are given by $I(10 + F)$, $II(6 + 3F)$, $III(2 + N'_m + F)$, $VI_0(4 + 4F)$, $VII_0(8 + 4F)$ and $VIII(4 + N_m + 4F)$, again with $F = 0$ giving the vacuum case, as before, and an addition of $2S$ to each prescription if S scalar fields are included. Here, N_m is the number of moduli degrees of freedom which measures of the complexity of the allowed topology, with $N_m \equiv 6g + 2k - 6 \equiv N'_m - 2g$, where g is the genus and k is the number of conical singularities of the underlying orbifold [50,51]. It can be arbitrarily large. The rigidity restriction that compact types V and VII_h must be isotropic means that compactness creates general parameter dependencies of $V(F)$ and $VII_h(F)$ which are the same as those for the open isotropic Friedmann universe, or the Milne universe in vacuum when $F = 0$. It is possible that a similar restriction occurs for open compact inhomogeneous universes and they may only exist when they are isotropic and homogeneous. However, the situation there is much more complicated and the theorem of Fischer, Marsden and Moncrief indicates that linearization instabilities will also beset attempts to perturb away from homogeneous spaces because of the coexistence of Killing symmetries and spatial compactness [24,25].

The resulting classification is shown in Table 1. We see that the introduction of compact topology for the simplest type I spaces produces a dramatic increase in relative generality. Indeed, they become the most general vacuum models by the parameter-counting criterion. An additional nine parameters are required to describe the compact type I universe compared to the case with noncompact Euclidean R^3 topology. The reason for this increase is that at any time the compact 3-torus topology requires three identification scales in orthogonal directions to define the torus and three angles to specify the directions of the vectors generating this lattice plus all their time derivatives. This gives 12 parameters, of which two can be removed using a time translation and a constraint equation, leaving ten in vacuum compared to the one required in the noncompact Kasner vacuum case.

The following general points are worth noting: (i) The imposition of a compact topology changes the relative generalities of homogeneous cosmologies; (ii) the compact flat universes are more general than the open or closed ones; (iii) type $VIII$ universes, which do not contain Friedmann special cases but can in principal become arbitrarily close to isotropy, are the most general compact universes. The most general case that contains isotropic

TABLE I. The number of independent arbitrary constants required to prescribe the general three-dimensional spatially homogeneous Bianchi type universes containing F perfect fluid matter sources in cases with noncompact and compact spatial topologies. The vacuum cases arise when $F = 0$. If S scalar fields are also present, then each parameter count increases by $2S$. The type IX universe does not admit a noncompact geometry and compact universes of Bianchi types IV and VI_h do not exist. Types III and $VIII$ have potentially unlimited topological complexity and arbitrarily large numbers of defining constant parameters through the unbounded topological parameters $N_m \equiv 6g + 2k - 6$ and $N'_m = N_m + 2g$, where g is the genus and k is the number of conical singularities of the underlying orbifold.

| Cosmological Bianchi type | No. of defining parameters with F noninteracting fluids | |
	Noncompact topology	Compact topology
I	$1 + F$	$10 + F$
II	$2 + 3F$	$6 + 3F$
VI_0	$3 + 4F$	$4 + 4F$
VII_0	$3 + 4F$	$8 + 4F$
$VIII$	$4 + 4F$	$4 + N_m + 4F$
IX	$\cdots$	$4 + 4F$
III	$3 + 4F$	$2 + N'_m + F$
IV	$3 + 4F$	$\cdots$
V	$1 + 4F$	F
VI_h	$4 + 4F$	$\cdots$
VII_h	$4 + 4F$	F

special cases is that of type VII_0—recall that the VII_h metrics are forced to be isotropic so open Friedmann universes now become asymptotically stable [50] and approach the Milne metric whereas in the noncompact case they are merely stable and approach a family of anisotropic vacuum plane waves [53].

VII. INHOMOGENEITY

The addition of inhomogeneity turns the constants defining the cosmological problem into functions of three space variables. For example, we are familiar with the linearized solutions for small density perturbations of a Friedmann universe with natural topology which produces two functions of space that control temporally growing and decaying modes. The function of space in front of the growing mode is typically written as a power law in length scale (or wave number) and so has arbitrary amplitude and power index (both usually assumed to be scale-independent constants to first or second order) which can fitted to observations. Clearly there is no limit to the number of parameters that could be introduced to characterize the density inhomogeneity function by means of a series expansion around the homogeneous model (and the same could be done for any vortical or gravitational-wave perturbation modes). Further analysis of the function

characterizing the radiation density is seen in the attempts to measure and calculate the deviation of its statistics from Gaussianity [54] and to reconstruct the past light-cone structure of the Universe [36]. Any different choice of specific spatial functions to characterize inhomogeneity in densities or gravitational waves requires some theoretical motivation. What happens in the inhomogeneous case if open or flat universes are given compact spatial topologies is not known. As we have just seen, the effects of topology on the spatially homogeneous anisotropic models were considerable whereas the effects on the overall evolution of isotropic models (as opposed to multiple image optics) are insignificant. It is generally just assumed that realistically inhomogeneous universes with nonpositive curvature (or curvature of varying sign) can be endowed with a compact topology and, if so, this places no constraints on their dynamics. However, both assumptions would be untrue for homogeneous universes and would necessarily fail for inhomogeneous ones in the homogeneous limit. It remains to be determined what topological constraints arise in the inhomogeneous cases. They could be weaker because inhomogeneous anisotropies can be local (far smaller in scale than the topological identifications) or they could be globally constrained like homogeneous anisotropies. Newtonian intuitions can be dangerous because compactification of a Newtonian Euclidean cosmological space seems simple but if we integrate Poisson's equation over the compact spatial volume, we see that the total mass of matter must be zero [55].

VIII. CONCLUSIONS

We have provided a simple analysis of the number of independent arbitrary functions of space required to specify a general cosmological model, with natural topology, containing a specified number of noninteracting fluids (F) and scalar fields (S) on a spacelike surface of constant time. This number is equal to $4(1 + F) + 2S$ in general relativistic cosmologies and increases to $16 + 4F + 2S$ in higher-order gravity theories with three space dimensions. Generalizations to universes with N space dimensions were also found. When the assumption of spatial homogeneity is introduced these maximal counts remain true for the most general cosmologies but the spatial functions are replaced by constants. We enumerate these constants for each of the homogeneous Bianchi type universes containing noninteracting fluids and scalar fields. When the spaces of the flat and open homogeneous universes are compactified the classification changes dramatically. Some homogeneous geometries are no longer permitted and other important cases, including those of the anisotropic universes containing open Friedmann universes, are constrained to be exactly isotropic. The hierarchy of generality changes and the number of constants required to specify the most general compact topologies increases significantly. For universes containing isotropic particular cases it is largest for the flat universes of type VII_0, where $8 + 4F + 2S$ constants are required, and type I, where $10 + F + 2S$ are required, but can be arbitrarily large in the case of type $VIII$ because of the unlimited topological complexity. How these topological constraints change when inhomogeneities are present remains an open question. These results provide a wider context for the parameter counts in ΛCDM where, for CDM, baryons, radiation and neutrinos F is at least 4 or 5 depending on assumptions about the nature of the dark energy but would necessarily be larger with nonstandard topology permitted.

[1] M. Rowan Robinson, *The Nine Numbers of the Cosmos* (Oxford University, New York, 1999).

[2] M. J. Rees, *Just Six Numbers: The Deep Forces That Shape the Universe* (Phoenix, London, 2001).

[3] D. N. Spergel *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **148**, 175 (2003).

[4] P. A. R. Ade *et al.* (Planck Collaboration), arXiv:1303.5076.

[5] J. D. Barrow and D. Shaw, Phys. Rev. Lett. **106**, 101302 (2011).

[6] L. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), 4th rev. ed..

[7] J. D. Barrow, in *The Physical Universe: The Interface between Cosmology, Astrophysics and Particle Physics*, edited by J. D. Barrow, A. Henriques, M. Lago, and M. Longair (Springer-Verlag, Berlin, 1991), pp. 1–20.

[8] A. A. Starobinsky, Sov. Phys. JETP Lett. **37**, 66 (1983).

[9] J. D. Barrow, Classical Quantum Gravity **21**, L79 (2004).

[10] J. D. Barrow, S. Cotsakis, and A. Tsokaros, Classical Quantum Gravity **27**, 165017 (2010).

[11] M. Heinzle and P. Sandin, Commun. Math. Phys. **313**, 385 (2012).

[12] J. Demaret, M. Henneaux, and P. Spindel, Phys. Lett. **164B**, 27 (1985).

[13] J. D. Barrow and A. C. Ottewill, J. Phys. A **16**, 2757 (1983).

[14] V. Faraoni and T. Sotiriou, Rev. Mod. Phys. **82**, 451 (2010).

[15] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Phys. Rep. 513, 1 (2012).

[16] T. Clifton and J. D. Barrow, Phys. Rev. D 72, 123003 (2005).

[17] D. Lovelock, J. Math. Phys. (N.Y.) 12, 498 (1971).

[18] L. Bianchi, Mem. Mat. Fis. Soc. Ital. Sci., Ser. Terza 11, 267 (1898); reprinted in Gen. Relativ. Gravit. 33, 2157 (2001).

[19] A. H. Taub, Ann. Math. 53, 472 (1951).

[20] M. A. H. MacCallum, in General Relativity: An Einstein Centenary Survey, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979), pp. 533–576.

[21] G. F. R. Ellis, S. T. C. Siklos, and J. Wainwright, in Dynamical Systems in Cosmology, edited by J. Wainwright and G. F. R. Ellis (Cambridge University Press, Cambridge, England, 1997), pp. 11–42.

[22] R. Kantowski and R. K. Sachs, J. Math. Phys. (N.Y.) 7, 443 (1966).

[23] A. S. Kompaneets and A. S. Chernov, Sov. Phys. JETP 20, 1303 (1964).

[24] A. E. Fischer, J. E. Marsden, and V. Moncrief, Ann. Inst. Henri Poincaré 33, 147 (1980).

[25] J. D. Barrow and F. J. Tipler, Phys. Rep. 56, 371 (1979).

[26] G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. 12, 108 (1969).

[27] P. T. Saunders, Mon. Not. R. Astron. Soc. 142, 213 (1969).

[28] C. B. Collins and S. W. Hawking, Mon. Not. R. Astron. Soc. 162, 307 (1972).

[29] A. Doroshkevich, V. Lukash, and I. D. Novikov, Sov. Phys. JETP 37, 739 (1973).

[30] J. D. Barrow, Mon. Not. R. Astron. Soc. 175, 359 (1976).

[31] J. D. Barrow, R. Juszkiewicz, and D. N. Sonoda, Mon. Not. R. Astron. Soc. 213, 917 (1985).

[32] J. D. Barrow, Phys. Rev. D 51, 3113 (1995).

[33] J. D. Barrow, Phys. Rev. D 55, 7451 (1997).

[34] W. C. Lim, U. S. Nilsson, and J. Wainwright, Classical Quantum Gravity 18, 5583 (2001).

[35] W. Stoeger, M. Araujo, and T. Gebbie, Astrophys. J. 476, 435 (1997).

[36] R. Maartens, Phil. Trans. R. Soc. A 369, 5115 (2011).

[37] U. S. Nilsson, C. Uggla, J. Wainwright, and W. C. Lim, Astrophys. J. 522, L1 (1999).

[38] G. F. R. Ellis, S. D. Nell, R. Maartens, W. R. Stoeger, and A. P. Whitman, Phys. Rep. 124, 315 (1985).

[39] W. H. McCrea, Z. Astrophys. 9, 290 (1934); 18, 98 (1939); reprinted in Gen. Relativ. Gravit. 30, 315 (1998).

[40] J. Kristian and R. K. Sachs, Astrophys. J. 143, 379 (1966).

[41] A. Guth, Phys. Rev. D 23, 347 (1981).

[42] J. D. Barrow, Nature (London) 272, 211 (1978).

[43] R. Penrose, Cycles of Time (Bodley Head, London, 2010); P. Tod, arXiv:1309.7248.

[44] J. Hartle and S. W. Hawking, Phys. Rev. D 28, 2960 (1983).

[45] G. F. R. Ellis, Gen. Relativ. Gravit. 2, 7 (1971).

[46] R. Aurich, S. Lustig, F. Steiner, and H. Then, Classical Quantum Gravity 21, 4901 (2004).

[47] P. A. R. Ade et al. (Planck Collaboration), arXiv:1303.5086.

[48] A. Ashtekar and J. Samuel, Classical Quantum Gravity 8, 2191 (1991).

[49] H. V. Fagundes, Gen. Relativ. Gravit. 24, 199 (1992).

[50] J. D. Barrow and H. Kodama, Int. J. Mod. Phys. D 10, 785 (2001).

[51] J. D. Barrow and H. Kodama, Classical Quantum Gravity 18, 1753 (2001).

[52] H. Kodama, Prog. Theor. Phys. 107, 305 (2002).

[53] J. D. Barrow and D. H. Sonoda, Phys. Rep. 139, 1 (1986).

[54] P. A. R. Ade et al. (Planck Collaboration), arXiv:1303.5084.

[55] This follows from Poisson's equation since $0 = \int_V \nabla^2 \Phi dV = 4\pi G \int_V \rho dV = 4\pi G M$, where V is the compact spatial volume and Φ is the Newtonian gravitational potential.