

Physics at the surface of a star in Eddington-inspired Born-Infeld Gravity

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We study phenomena happening at the surface of a star in Eddington-inspired Born-Infeld (EiBI) gravity. The star is made of particles, which are effectively described by a polytropic fluid. The EiBI theory was known to have a pathology that singularities happen at a star surface. We suggest that the gravitational backreaction on the particles cures the problem. Strong tidal forces near the (surface) singularity modify the effective equation of state of the particles or make the surface be unstable depending on its matter contents. The geodesic deviation equations take after Hooke's law, where its frequency squared is proportional to the scalar curvature at the surface. For a positive curvature, a particle collides with a probing wall more often and increases the pressure. With the increased pressure, the surface is no longer singular. For a negative curvature, the matters around the surface experience repulsions with infinite accelerations. Therefore, the EiBI gravity is saved from the pathology of a surface singularity.

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I. INTRODUCTION

General relativity (GR) predicts that space-time singularities can be formed from regular initial data, e.g., in the gravitational collapse of massive stars and in the early universe. Recently, a modified theory of gravity, the so-called Eddington-inspired Born-Infeld (EiBI) theory was proposed in order to resolve these singularities [1]. The theory was further studied diversely [2–17]. EiBI gravity is equivalent to GR in vacuum and does not propagate any degrees of freedom other than massless gravitons. On the other hand, the theory introduces nonlinear couplings to the matter fields [2–4], which resolve some of the singularities appearing in GR. In the presence of a perfect fluid, the big-bang singularity in the early universe is replaced by a freezing, exponentially inflating, or a bouncing behavior of the cosmological scale factor, depending on the equation of state (EOS) of the fluid [1,5]. The gravitational collapse of noninteracting particles does not lead to singular states in the nonrelativistic limit [2,6]. The stability of EiBI compact stars was considered in Ref. [7], where it was shown that the standard results of stellar stability theory still hold. The structure and physical properties of specific classes of neutrons, quark and “exotic” stars in EiBI gravity were studied [8]. They showed that the EiBI gravity stars are more massive than their GR counterparts. Constraints on the theory have been considered using a solar model [9] and cosmological observations [10,12]. A tensor instability of the homogeneous and isotropic universe was found in Ref. [13]. In the presence of a scalar field, a regular initial state in EiBI gravity is studied to present the inflation paradigm [18,19], where the tensor instability was shown not to be a problem. Metric perturbations on the

background of a homogeneous and isotropic universe based on EiBI gravity were studied in Ref. [20,21].

With all these merits, flaws of the theory were also discovered. In Ref. [11], it was shown that the EiBI theory is reminiscent of Palatini $f(\mathcal{R})$ gravity [22,23] and that it shares the pathology such as the curvature singularity at the surface of polytropic stars. In Ref. [17], similar singularities were shown to exist when a phase transition happens inside a star. These observations cast serious doubt on the viability of EiBI gravity. In this article, we reexamine the problem by checking the motion of particles at the surface. In the presence of the curvature singularity, we show that one of the following two things happens depending on the signature of the curvature. For positive curvatures, the (effective) EOS of particles near the surface is modified so that the singularity is removed. This happens because the pressure by the particles increases due to rapid oscillations in geodesic deviation equations. For negative curvatures, on the other hand, the surface becomes unstable.

EiBI gravity is described by the action [1],

$$S_{\text{EiBI}} = \frac{1}{\kappa} \int d^4x \left[\sqrt{-|g_{\mu\nu} + \kappa \mathcal{R}_{\mu\nu}(\Gamma)|} - \lambda \sqrt{-|g_{\mu\nu}|} \right] + S_M(g, \Phi), \quad (1)$$

where $S_M(g, \Phi)$ is the matter action, Φ generically denotes any matter field, $\mathcal{R}_{\mu\nu}(\Gamma)$ is the Ricci tensor built from the connection Γ , $|\mathcal{G}_{\mu\nu}| = \det \mathcal{G}_{\mu\nu}$, λ is a dimensionless parameter related with the cosmological constant by $\Lambda = (\lambda - 1)/\kappa$, κ is the extra EiBI parameter which has dimension of length squared, and we set $8\pi G = 1 = c$. The equations of motion are obtained by varying the action (1) with respect to the fields $g_{\mu\nu}$ and $\Gamma_{\mu\nu}^\rho$, respectively,

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$$\sqrt{\frac{|q|}{|g|}}q^{\mu\nu} = \lambda g^{\mu\nu} - \kappa T^{\mu\nu}, \quad (2)$$

$$g_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}, \quad (3)$$

where $q_{\mu\nu}$ is the auxiliary metric by which the connection $\Gamma_{\mu\nu}^\rho$ is defined, and $q^{\mu\nu}$ is the matrix inverse of $q_{\mu\nu}$.

In Sec. II, we review the problem of surface singularity for a star which consists of an ideal polytropic fluid. In Sec. III, we calculate the gravitational backreaction on the motions of particles by investigating the geodesic deviation equation. We show that the surface singularity problem can be cured. We summarize the results in Sec. IV and discuss their physical implications.

II. THE SINGULARITY AT THE SURFACE OF A POLYTROPIC STAR

Because we are interested in stars in a flat background, we set $\lambda = 1$. We also restrict our interests to the case with positive κ , because the gravitational collapse does not lead to singularities [6]. We consider a spherically symmetric star made of particles which are effectively described by a perfect fluid with polytropic EOS

$$p = p_{\text{poly}} \equiv K\rho^\Gamma, \quad (4)$$

where K is a constant compensating dimensional difference and Γ is a positive dimensionless constant. The stress tensor of the fluid is given by $T^{\mu\nu} = (\rho + p)u^a u^b + p g^{ab}$. The metric and auxiliary metrics for a spherically symmetric star are

$$g_{\mu\nu} dx^\mu dx^\nu = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + R^2 e^{\chi(r)} d\Omega^2, \quad (5)$$

$$q_{\mu\nu} dx^\mu dx^\nu = -e^{\beta(r)} dt^2 + e^{\alpha(r)} dr^2 + r^2 d\Omega^2, \quad (6)$$

where $\nu(r)$, $\lambda(r)$, $\chi(r)$, $\alpha(r)$, and $\beta(r)$ are arbitrary metric functions of the radial coordinate r , $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, and R is a constant representing the radius of the star. The problem was analyzed in Refs. [7,8,17] recently. We follow their convention except for the metric function χ , where $f(r)$ was used in place of $R^2 e^\chi$ in Ref. [8]. The scalar curvature for the metric $g_{\mu\nu}$ is given by

$$R_g = -e^{-\lambda}(\nu'' + 2\chi'') - \frac{e^{-\lambda}}{2}[-\nu'\lambda' + \nu'^2 + 2(\nu' - \lambda')\chi' + 3\chi'^2] + 2R^{-2}e^{-\chi}, \quad (7)$$

where the prime denotes the derivative with respect to r . Outside the star $r > R$, the metric should be the vacuum Schwarzschild solution with

$$e^\nu = e^\beta = 1 - \frac{2M}{r} = e^{-\lambda} = e^{-\alpha}, \quad R^2 e^\chi = r^2, \quad (8)$$

where M is the mass of the star. The continuity equation gives

$$p' + \frac{\nu'}{2}(\rho + p) = 0. \quad (9)$$

With the EOS (4), the continuity equation in Eq. (9) is integrated to give

$$\rho = \left[\frac{1}{K} \exp\left(\frac{\Gamma-1}{2\Gamma}(\nu_0 - \nu)\right) - \frac{1}{K} \right]^{1/(\Gamma-1)}. \quad (10)$$

For $\Gamma \leq 1$, p vanishes at the places where $\nu \rightarrow \infty$. The energy density and pressure exponentially decrease with ν . For $\Gamma > 1$, the surface of a star is defined by the place $r = R$ where the pressure vanishes at $\nu(R) = \nu_0$.

Let us describe how the curvature singularity happens at the surface of a star composed of an ideal fluid described by Eq. (4) in EiBI gravity. In Appendix A, the equations of motion (2) and (3) are reduced into portable forms. Based on the results, an EiBI generalization of the Tolman-Oppenheimer-Volkoff equation in GR was developed in Refs. [7,8,17]. Differentiating Eq. (A1), we get

$$\begin{aligned} \lambda' &= \alpha' - \frac{a'}{a} - \frac{b'}{b}, & \nu' &= \beta' + \frac{a'}{a} - \frac{3b'}{b}, \\ \chi' &= \frac{2}{r} - \frac{a'}{a} - \frac{b'}{b}, \end{aligned} \quad (11)$$

where

$$a = \sqrt{1 + \kappa\rho}, \quad b = \sqrt{1 - \kappa p}. \quad (12)$$

One can also show that the function α is differentiable and β is of class C^2 for positive r . The continuity of their first derivatives can be ensured from Eqs. (A7) and (A8). Differentiating Eq. (A8), we express β'' as a sum of continuous functions

$$2\beta'' = \frac{4e^{\alpha-\beta}}{\kappa}(e^\nu - e^\beta) - \beta'(\beta' - \alpha') - \frac{4\beta'}{r}. \quad (13)$$

This guarantees the continuity of β'' at the star surface.

Because $\alpha \in C^1$ and $\beta \in C^2$, possible divergent contributions to the curvature scalar in Eq. (7) come from the discontinuity of the derivatives of ρ and p . Noting $\frac{\nu'}{\rho} = \frac{dp}{d\rho} = K\Gamma\rho^{\Gamma-1} \rightarrow 0$ for $\Gamma > 1$, we find that most singular contributions to Eq. (7) come from ρ'' , which are included in ν'' and χ'' . After calculating the explicit form for ρ' and ρ'' in Appendix B, one can ensure that ρ'^2 , ρ''^2 , p'' , and $\rho'p'$ do not contain a divergent contribution. The curvature scalar at the stellar surface, as $r \rightarrow R$ and $\rho \rightarrow 0$, is

$$\begin{aligned}
R_g &\approx -e^{-\lambda}(\nu'' + 2\chi'') \\
&= -e^{-\lambda} \left[\beta'' - \frac{4}{r^2} - \left(\frac{a'}{a} \right)' - 5 \left(\frac{b'}{b} \right)' \right] \\
&\approx \frac{\kappa e^{-\lambda}}{2} \rho'', \tag{14}
\end{aligned}$$

$$R_g \approx \frac{\kappa}{2} \left(1 - \frac{2M}{R} \right) \begin{cases} 0, & 0 < \Gamma \leq 1 \\ \frac{(2-\Gamma)\beta'^2}{(2\Gamma K)^2} \left(\rho^{3-2\Gamma} - \frac{\kappa}{4\Gamma K} \rho^{5-3\Gamma} \right), & 1 < \Gamma < 2 \\ \frac{-\beta''}{4K+\kappa/2} + \frac{\beta'^2(4K-\kappa)(2K-\kappa)}{2(4K+\kappa/2)^3}, & \Gamma = 2 \\ -\frac{2\beta''}{\kappa} + \frac{4\beta'^2}{\kappa} \left(3 + 5\Gamma - 3\Gamma^2 - \frac{4\Gamma(\Gamma-2)K}{\kappa^2} \rho^{\Gamma-3} \right), & \Gamma > 2. \end{cases} \tag{15}$$

We find that the curvature scalar diverges for $3/2 < \Gamma < 2$ and $2 < \Gamma < 3$. This implies that the surface singularity is happening for a star composed of an ideal polytropic fluid. The result for $0 < \Gamma < 2$ was known in Ref. [11]. On the other hand, the precise divergent structure for $2 < \Gamma < 3$ and the absence of divergences for $\Gamma = 2$ and $\Gamma > 3$ are new results in this work.

III. GRAVITATIONAL BACKREACTION THROUGH THE GEODESIC DEVIATION EQUATION

An important example of polytropic fluid with diverging surface curvature is the nonrelativistic degenerate Fermi gas with $\Gamma = 5/3$, which happens for dense Fermi particles when the Fermi energy exceeds the temperature by far. Examples are the electron gas in metals and in white dwarf stars and the neutron star, whose density is so high that the neutron gas is degenerate. We derive the EOS for the Fermi gas roughly and then discuss its validity. At low temperatures and/or high densities we approach the uncertainty principle. In such a case, the Pauli exclusion principle will cause the pressure to be higher than that inferred by the temperature. The complete degenerate case happens when two electrons are occupied for each phase space volume. The average magnitude of the momentum of a Fermi particle is roughly given by the Fermi momentum, $P_F = m v_F = \hbar(3\pi^2 n)^{1/3}$, where n and m are the number density and mass of a Fermi particle, respectively. The pressure, the total momentum transfer on a unit surface area per unit time, is

$$p_F \approx P_F n v_F = \frac{\hbar^2}{m} (3\pi^2)^{2/3} n^{5/3}. \tag{16}$$

where we keep only the potentially divergent parts in the first and third equalities and ignore $\kappa\rho$ and κp compared to one. Assuming the continuity of $\lambda(r)$, we may replace it to its surface value by using $e^{-\lambda(R)} = 1 - 2M/R$.

The second derivative of ρ is obtained in Appendix B. From the result in Eq. (B4) and Eq. (14), we get

From this equation, we take the EOS for the degenerate Fermi gas as $p_F = (\rho/K)^{5/3}$ because $n \propto \rho$.

A. The geodesic deviation equations

Let us examine how robust the degenerated Fermi gas approximation at the surface of the star is. The approximation holds only when the temperature is smaller than the Fermi energy. Therefore, the degenerate Fermi gas approximation does not hold for low number densities. This observation presents one reason to suspect the validity of the polytropic fluid approximation because the surface singularity happens in the low density regime. However, if a star is very cold, the approximation will be valid even for an extremely low energy density. Let us check how the geometry reacts on the motions of the particles constituting the fluid. The EOS (16) is derived from the motions of Fermi particles in a flat space-time. On the basis that the energy density and the pressure are local quantities, the extension to curved space-times is justified by the principle of general covariance. However, we would like to argue that the pressure may not be local in the presence of a high curvature. Note that two distant objects, a colliding particle and a probing wall, are necessary to define the pressure microscopically. Therefore, if some high curvature effects probe/modify the microscopic structure, the covariant justification for the pressure may not hold, which is the situation happening near the surface singularity. To examine the effect, let us study the geodesic deviation equations. The relative acceleration $a^a (\equiv d^2 X^a / d\lambda^2)$ of the separation X^a with the motions $u^a = (\partial/\partial\lambda)^a$ is given by

$$\begin{aligned}
a^0 &= \frac{2\nu'' + \nu'^2 - \lambda'\nu'}{4} u^1 (X^0 u^1 - X^1 u^0) - \frac{R^2 e^{\chi-\lambda}}{4} \nu' \chi' u^3 (X^0 u^3 - X^3 u^0), \\
a^1 &= \frac{e^{\nu-\lambda}}{4} (2\nu'' + \nu'^2 - \lambda'\nu') u^0 (X^0 u^1 - X^1 u^0) - \frac{R^2 e^{\chi-\lambda}}{4} [\lambda'\chi' - 2\chi'' - \chi'^2] u^3 (X^1 u^3 - X^3 u^1), \\
a^3 &= \frac{e^{\nu-\lambda} \nu' \chi'}{4} u^0 (X^0 u^3 - X^3 u^0) + \frac{1}{4} [\chi'\lambda' - 2\chi'' - \chi'^2] u^1 (X^1 u^3 - X^3 u^1), \tag{17}
\end{aligned}$$

where we assume $X^2 = 0 = u^2$ without loss of generality because of the spherical symmetry. Keeping only the dominant parts, we get

$$\begin{aligned} a^0 &\simeq \frac{\kappa\rho''}{4}u^1(X^0u^1 - X^1u^0), & a^3 &\simeq \frac{\kappa\rho''}{4}u^1(X^1u^3 - X^3u^1), \\ a^1 &\simeq \frac{\kappa\rho''e^{\nu-\lambda}}{4}u^0(X^0u^1 - X^1u^0) \\ &\quad - \frac{\kappa\rho''R^2e^{\lambda-\lambda}}{4}u^3(u^3X^1 - u^1X^3), \end{aligned} \quad (18)$$

where we use $\nu'' \approx \kappa\rho''/2 \approx -\chi'' > 0$ for $\Gamma = 5/3$. Consider two geodesics staying away radially so that $X^\mu = (0, X^1, 0, 0)$. Consider time evolution only with $u^\mu = (1, 0, 0, 0)$. Then, the geodesic deviation equation takes the form

$$a^1 \approx -\frac{\kappa\rho''e^{\nu(R)-\lambda(R)}}{4}X^1, \quad a^0 = 0 = a^3, \quad (19)$$

where we set the values of ν and λ to be those at the surface of the star. A similar expression can be obtained for a^3 if we use $u^\mu = (1, 1, 0, 0)$ and $X^\mu = (0, 0, 0, X^3)$. These are reminiscent of Hooke's law with frequency

$$f = \frac{1}{2\pi} \sqrt{\frac{R_g e^{\nu(R)}}{2}}. \quad (20)$$

For the case with $3/2 < \Gamma < 2$, R_g is positive. By using Eqs. (8) and (15), we get

$$f \approx \frac{\epsilon_0}{4\pi R} \left(\frac{M}{R}\right) \left(\frac{\sqrt{\kappa}\rho^{3/2-\Gamma}}{K}\right), \quad (21)$$

where each term in the parenthesis is dimensionless. In the second equality in Eq. (21), we replace an order one Γ dependent term with ϵ_0 to denote that the calculation is approximate. We should mention that ϵ_0 is an order one dimensionless constant, which depends on the species of the particle constituting the fluid. Following the interpretation of Hooke's law, any two nearby geodesics (e.g., a wall and a particle) will cross each other f times per unit period of time irrespective of the separation distance. For the cases with $2 < \Gamma < 3$, the frequency becomes imaginary because $R_g < 0$. We discuss this case in Sec. III C.

B. Modification of the equation of state for the case with $3/2 < \Gamma < 2$

Now, let us calculate the pressure once more as we have done in Eq. (16). Because of the oscillations in the geodesic deviation equation (21), all Fermi particles inside an oscillation scale collide with the wall f times. The pressure due to the geodesic deviation motion can be estimated as

$$p_{\text{gd}} \approx f\ell n P_{\text{F}}, \quad (22)$$

where ℓ is the characteristic scale of the oscillations. Because Eq. (21) is free from scales, ℓ appears to be nothing but the star radius, R . Let us try to set $\ell = R$ and examine what happens. After setting $P_{\text{F}} = mv_{\text{F}} = A\rho^{1/3}$, where A is a constant, and $n = p_{\text{F}}/(v_{\text{F}}P_{\text{F}}) = mK/A^2 \times \rho$, we get $p_{\text{gd}} \propto \rho^{7/6}$. The eventual expression for the pressure will be given by the sum, $p_{\text{F}} + p_{\text{gd}}$. In low densities, p_{gd} dominates the pressure over p_{F} . Because p_{gd} acts as if it is that of a polytropic fluid with $\Gamma = 7/6 < 3/2$, the surface singularity disappears. However, when the curvature becomes too small so that ρ'' term is subdominant, there is no reason for the pressure p_{gd} to exist. Therefore, once again, the EOS will be given by that of the degenerate Fermi gas (4) and the surface singularity happens once more. To avoid this awkward situation, a delicate balance should be taken between the diverging curvature effect and the modification of the EOS so that the curvature is not too small and not too large. In the presence of such a balance, the curvature must be finite. In addition, the characteristic scale may not be the radius of the star but be smaller than that. It is not an easy task to exactly determine how the characteristic size will be decreased through the balance between the gravity and the Fermi motions. However, a good measure will be the ratio of the correlation scales between the degenerate Fermion and the gravity per unit time, given by v_{F}/c , where we restored c for comparison with v_{F} . One may expect the characteristic length will be decreased by the power of the form, $\ell = R(v_{\text{F}}/c)^k$ with $k > 0$. In this article, we propose $k = 1$. Explicitly, after setting $\ell = Rv_{\text{F}}/c$, we get

$$\begin{aligned} p_{\text{gd}} &= \frac{Rf}{c} n v_{\text{F}} P_{\text{F}} = \frac{Rf}{c} P_{\text{F}} = \epsilon (\rho c^2)^{3/2}; \\ \epsilon &= \frac{\epsilon_0}{4\pi} \frac{GM}{Rc^2} \sqrt{\frac{8\pi G\kappa}{c^4}}, \end{aligned} \quad (23)$$

where we restore G and c with the prescription $\kappa \rightarrow \frac{8\pi G\kappa}{c^4}$, $\frac{M}{R} \rightarrow \frac{GM}{Rc^2}$, $\rho \rightarrow \rho c^2$, and $1/R \rightarrow c/R$. Later in this work, we return to use the unit with $8\pi G = 1 = c$.

The above discussions may also hold for all other matters composing the polytropic fluid with $3/2 < \Gamma < 2$ because this is due to the geometric effect. Therefore, for the corresponding particles we propose a new EOS as

$$p = K\rho^\Gamma + \epsilon\rho^{3/2}. \quad (24)$$

Note that ϵ decreases with the radius of the star but increases with the mass of the star. In addition, the correction term vanishes in the GR limit, $\kappa \rightarrow 0$. With the EOS in Eq. (24), the second term dominates the pressure when $\rho < \rho_c = (\epsilon/K)^{2/(2\Gamma-3)}$. In that region, the EOS takes after that of the polytropic fluid with $\Gamma = 3/2$, in which case no singularity appears at the star surface. Higher curvature induces higher pressure. As the pressure becomes higher, the EOS is modified so that Γ

decreases. When Γ arrives at $3/2$, the singularity disappears and further modification will not happen. Therefore, the form in Eq. (24) appears to be appropriate intuitively once we accept the fact that the curvature modifies the EOS.

In Appendix C, we obtain ρ'' at the star surface with the new EOS (24). Then, the curvature scalar at the surface takes the form

$$R_g \approx \frac{16\pi^2}{9\epsilon_0^2(1-2M/R)} \frac{1}{R^2}. \quad (25)$$

For an $O(1)$ number ϵ_0 , this value is acceptable. Interestingly, the curvature scalar does not contain κ even though it originates from the EiBI gravity. In GR, the scale of the surface curvature is $R_{GR} \propto \rho \approx M/R^3$. R_g is larger than R_{GR} by the factor R/M . For the case of a neutron star, this ratio is not large.

C. Instability of the surface for the case with $2 < \Gamma < 3$

For the matters with $2 < \Gamma < 3$, the situation is different from that of the previous subsection. This is because the signature of the curvature in Eq. (15) is negative contrary to the previous case. Following Eq. (19), two nearby geodesics do not oscillate but repel each other with infinite acceleration. In the presence of such infinity repulsion, no particle may stay at the surface even in the presence of supporting pressure. A particle at the surface will be repelled to the center of the star with infinite acceleration. In the absence of a particle, the radius of the surface decreases, which happens for any value of radius. We interpret this as a signature of the instability of the surface. A stable star surface may not exist for that matter. One may find similar instability happens in the geodesic deviations around a spherically symmetric Newtonian star. Let us consider the situation that two radially displaced massive particles fall into the star, which were at rest initially. Outside the star, the two particles recede relatively because the gravity is stronger inside. Even in the presence of a constant pressure, their relative distance may not decrease. On the other hand, inside a static star, they draw nearer because the gravity becomes weaker inside. The given pressure may help keep the particles stay closer. Therefore, we may conclude that a star fails to have a stable surface composed of matters with $2 < \Gamma < 3$, even though such matters can appear inside.

IV. SUMMARY AND DISCUSSIONS

In summary, we studied the phenomena happening at the surface of a star in the EiBI theory, where the star is made of particles which are effectively described by a polytropic fluid, $p \propto \rho^\Gamma$. For a star composed of a specific kind of ideal-polytropic fluid with $3/2 < \Gamma < 2$ and $2 < \Gamma < 3$, a surface singularity is shown to exist. Noting that the fluid is a macroscopic description of the collective motions of

particles, we demonstrated the possibility that a gravitational backreaction on the matter dynamics modifies the effective descriptions. The effects are captured by means of the geodesic deviation equation, which resembles Hooke's law, where its frequency squared is proportional to the curvature. For $3/2 < \Gamma < 2$, the curvature is positive and Hooke's law makes a particle collide with a probing wall more often, which modifies the effective EOS. With the modified EOS, the surface is no longer singular and the size of the curvature at the surface is shown to be acceptable. On the other hand, for the matters with $2 < \Gamma < 3$, the curvature is negative and the seeming Hooke's law provides infinite repulsions of the particles. As a result, stable star surfaces do not form with the matter. Note, however, that this result does not imply that the star composed of the corresponding matter does not exist. That matter may constitute a core of the star leaving other matter to form the surface. These two results lead to the conclusion that the surface singularity problem is absent in the EiBI gravity.

Accepting that the EiBI gravity is sound theoretically, one of the most urgent tasks is to find out observable effects which discriminate the EiBI gravity from GR. An answer is given in Ref. [8]. There, it was shown that the EiBI stars are more massive than their GR counterparts. They appeal that some stellar-mass black hole candidates could be in fact EiBI neutron or quark stars. However, it is not easy to distinguish this modified-gravity effect from other effects in GR because it is related to the internal structures of the star. Another prediction is given in Sec. III C, where it was argued that no star exists with its surface composed of polytropic fluid with $2 < \Gamma < 3$. A third observable signature is the relatively large surface curvature (larger than that in GR by the factor R/M) for $3/2 < \Gamma < 2$ stars, where the neutron stars are included. The curvature is independent of κ and is almost independent of the mass of the (large) star. The merit for these two new observations is that the surface curvature is independent of the complex internal structures of the star but is determined by the physics at the surface only. One may design a scattering experiment of light which grazes the surface of a neutron star. The light will bear information of the surface curvature, which can be observed from a distant place independently from other information. Therefore, this fairly large surface curvature will provide an opportunity to distinguish the EiBI theory from GR by means of observation.

In Ref. [7], a similar pathology as the surface singularity was shown to happen during the phase transition inside a star. Because the origin of the singularity is the same as that of the present one, the problem may also be cured with the same prescription. Similar surface singularities were known to exist in the Palatini $f(\mathcal{R})$ gravity [22–24] and gravity with auxiliary field [25]. Therefore, we expect that those singularities will be removed with the same way. An interesting observation was presented in Ref. [26]

on the resolution of the singularity for the Palatini $f(\mathcal{R})$ gravity.

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APPENDIX A: THE EQUATION OF MOTION

The equation of motion (2) presents relations between the auxiliary metric, the physical metric, and sources as

$$e^\alpha = abe^\lambda, \quad e^\beta = \frac{b^3}{a} e^\nu, \quad Re^\chi = \frac{r^2}{ab}. \quad (\text{A1})$$

The curvatures obtained from the auxiliary metric $q_{\mu\nu}$ are

$$R_{00} = \frac{e^{\beta-\alpha}}{4} \left[\beta'(-\alpha' + \beta') + \frac{4}{r}\beta' + 2\beta'' \right], \quad (\text{A2})$$

$$R_{11} = -\frac{\beta'^2}{4} - \frac{1}{2}\beta'' + \frac{\alpha'\beta'}{4} + \frac{\alpha'}{r}, \quad (\text{A3})$$

$$R_{22} = 1 + \frac{r}{e^\alpha} \left[-\frac{1}{r} + \frac{1}{2}(\alpha' - \beta') \right]. \quad (\text{A4})$$

Therefore, Eq. (3) becomes

$$\begin{aligned} -e^\beta &= -e^\nu + \frac{\kappa e^{\beta-\alpha}}{4} \left[\beta'(\beta' - \alpha') + \frac{4\beta'}{r} + 2\beta'' \right], \\ e^\alpha &= e^\lambda + \kappa \left[\frac{\alpha'}{r} + \frac{\alpha'\beta'}{4} - \frac{\beta''}{2} - \frac{\beta'^2}{4} \right], \\ r^2 &= f + \kappa + \frac{\kappa r}{e^\alpha} \left[-\frac{1}{r} + \frac{1}{2}(\alpha' - \beta') \right]. \end{aligned} \quad (\text{A5})$$

Combining the first two lines of Eq. (A5), we get

$$\alpha' + \beta' = \frac{re^\alpha}{\kappa} [e^{\nu-\beta} - e^{\lambda-\alpha}]. \quad (\text{A6})$$

Combining this with the last equation of Eq. (A5) and using Eq. (A1), we get

$$(re^{-\alpha})' = 1 - \frac{r^2}{2\kappa} \left(2 - \frac{3}{ab} + \frac{a}{b^3} \right), \quad (\text{A7})$$

$$(1 + r\beta')e^{-\alpha} = 1 - \frac{r^2}{2\kappa} \left(2 - \frac{1}{ab} - \frac{a}{b^3} \right). \quad (\text{A8})$$

Equation (A7) can be rewritten as

$$e^{-\alpha} = 1 - \frac{2m(r)}{r}, \quad (\text{A9})$$

where the function $m(r)$ is obtained as

$$\frac{dm}{dr} = \frac{r^2}{4\kappa} \left(2 - \frac{3}{ab} + \frac{a}{b^3} \right). \quad (\text{A10})$$

The outside geometry of the star is nothing but that of the Schwarzschild geometry. Therefore, $m(R) = M$.

APPENDIX B: CALCULATION OF THE SURFACE CURVATURE

Imposing Eqs. (4) and (11) into Eq. (9), we obtain an analytic expression for ρ' as

$$\begin{aligned} \rho'(r) &= -\beta'(r)g(\rho); \\ g(\rho) &\equiv \frac{2\rho(K + \rho^{1-\Gamma})}{4\Gamma K + \kappa a^{-2}\rho^{2-\Gamma} + \kappa KH(\rho)}, \end{aligned} \quad (\text{B1})$$

where $H(\rho)$ denotes the homogeneous function

$$H(\rho) = (a^{-2} + 3\Gamma b^{-2})\rho + 3K\Gamma b^{-2}\rho^\Gamma. \quad (\text{B2})$$

For $\rho \sim 0$, we have

$$g \simeq \begin{cases} \frac{\rho}{2\Gamma}, & 0 < \Gamma \leq 1 \\ \frac{K+1}{2K}\rho, & \Gamma = 1 \\ \frac{1}{2\Gamma K}\rho^{2-\Gamma}, & 1 < \Gamma < 2 \\ \frac{1}{4K+\kappa/2}, & \Gamma = 2 \\ \frac{2}{\kappa}, & \Gamma > 2. \end{cases} \quad (\text{B3})$$

The second derivative of the energy density becomes

$$\begin{aligned} \rho'' &= -\beta''g(\rho) + \beta'^2g(\rho)^2 \left(\frac{1}{\rho} + \frac{1-\Gamma}{\rho + K\rho^\Gamma} \right. \\ &\quad \left. - \kappa \frac{(2-\Gamma)a^{-2}\rho^{1-\Gamma} - \kappa a^{-4}\rho^{2-\Gamma} + KH'(\rho)}{4\Gamma K + \kappa a^{-2}\rho^{2-\Gamma} + \kappa KH(\rho)} \right), \end{aligned} \quad (\text{B4})$$

where $H'(\rho)$ denotes the derivative of H with respect to ρ ,

$$\begin{aligned} H'(\rho) &= a^{-2} + 3\Gamma b^{-2} - \kappa a^{-4}\rho \\ &\quad + 3K\Gamma^2 b^{-2}\rho^{\Gamma-1} \left(1 - \frac{\kappa\rho}{b^2} + \frac{\kappa K\rho^\Gamma}{b^2} \right). \end{aligned}$$

Near the surface of the star, we may use $\beta' = 2M/R^2 \times (1 - 2M/R)^{-1}$ and $\beta'' = -4M(R - M)/R^2(R - 2M)^2$, which are the surface values of the corresponding outer solution. Taking the $\rho \rightarrow 0$ limit, we have

$$\rho'' \simeq \begin{cases} 0, & 0 < \Gamma \leq 1 \\ \frac{(2-\Gamma)\beta'^2}{(2\Gamma K)^2} (\rho^{3-2\Gamma} - \frac{\kappa}{4\Gamma K} \rho^{5-3\Gamma}), & 1 < \Gamma < 2 \\ \frac{-\beta''}{4K+\kappa/2} + \frac{\beta'^2(4K-\kappa)(2K-\kappa)}{2(4K+\kappa/2)^3}, & \Gamma = 2 \\ -\frac{2\beta''}{\kappa} + \frac{4\beta'^2}{\kappa} \left(1 + (2-\Gamma)H'(0) - \frac{4\Gamma(\Gamma-2)K}{\kappa^2} \rho^{\Gamma-3} \right), & \Gamma > 2, \end{cases} \quad (\text{B5})$$

where $H'(0) = 3\Gamma + 1$.

APPENDIX C: THE SURFACE CURVATURE WITH MODIFIED EQUATION OF STATE

Let us calculate the curvature scalar at the surface once more with the new EOS (24). The first derivative ρ' becomes

$$\rho' = -\beta' g_m(\rho); \quad g_m(\rho) \equiv \frac{2\rho(K + \rho^{1-\Gamma} + \epsilon\rho^{3/2-\Gamma})}{4\Gamma K + \kappa a^{-2}\rho^{2-\Gamma} + \kappa KH(\rho) + \epsilon\rho^{3/2-\Gamma}G(\rho)}, \quad (\text{C1})$$

where

$$G(\rho) = 6 + \kappa \left[\left(\frac{1}{a^2} + \frac{9}{2b^2} \right) \rho + \frac{3K(2\Gamma + 3)}{2b^2} \rho^\Gamma + \frac{9\epsilon}{2b^2} \rho^{3/2} \right]. \quad (\text{C2})$$

Note that, when $\epsilon = 0$, Eq. (C1) reproduces Eq. (B1). The second derivative is given by

$$\rho'' = -\beta'' g_m(\rho) + \beta'^2 g_m(\rho)^2 \left(\frac{1K + (2-\Gamma)\rho^{1-\Gamma} + (5/2-\Gamma)\epsilon\rho^{3/2-\Gamma}}{K + \rho^{1-\Gamma} + \epsilon\rho^{3/2-\Gamma}} - \frac{\epsilon(3/2-\Gamma)\rho^{1/2-\Gamma}G(\rho) + \epsilon\rho^{3/2-\Gamma}G'(\rho) + \kappa[(2-\Gamma)a^{-2}\rho^{1-\Gamma} - \kappa a^{-4}\rho^{2-\Gamma} + KH'(\rho)]}{4\Gamma K + \kappa a^{-2}\rho^{2-\Gamma} + \kappa KH(\rho) + \epsilon\rho^{3/2-\Gamma}G(\rho)} \right). \quad (\text{C3})$$

For $3/2 < \Gamma < 2$, $\rho'' \simeq \beta'^2/(18\epsilon^2)$.

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