

Sailing through the big crunch-big bang transition

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In a recent series of papers, we have shown that theories with scalar fields coupled to gravity (e.g., the standard model) can be lifted to a Weyl-invariant equivalent theory in which it is possible to unambiguously trace the classical cosmological evolution through the transition from big crunch to big bang. The key was identifying a sufficient number of finite, Weyl-invariant conserved quantities to uniquely match the fundamental cosmological degrees of freedom across the transition. In doing so we had to account for the well-known fact that many Weyl-invariant quantities diverge at the crunch and bang. Recently, some authors rediscovered a few of these divergences and concluded based on their existence alone that the theories cannot be geodesically complete. In this paper, we show that this conclusion is invalid. Using conserved quantities we explicitly construct the complete set of geodesics and show that they pass continuously through the big crunch-big bang transition.

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In the standard big bang inflationary model [1], the cosmic singularity problem is left unresolved and the cosmology is geodesically incomplete. Consequently, the origin of space and time and the peculiar, exponentially fine-tuned initial conditions required to begin inflation [2,3] are not explained. In a recent series of papers [4–11], we have shown how to construct the complete set of homogeneous classical cosmological solutions of the standard model coupled to gravity, in which the cosmic singularity is replaced by a bounce: the smooth transition from contraction and big crunch to big bang and expansion. These are *generic* geodesically complete solutions that can, for example, naturally incorporate the cyclic theory of the Universe [12,13] in which it is proposed that large-scale smoothness, flatness and nearly scale-invariant perturbations are generated during the periods of slow contraction preceding each big bang.

The key to our construction of classical geodesically complete solutions was to “lift” the action (e.g., the standard model coupled to Einstein gravity) to a Weyl-invariant equivalent theory. We then identified a number of Weyl-invariant finite quantities [7] that are conserved near cosmological singularities for symmetry reasons. Our proposal was to match these quantities across the singularities which separate the patches of spacetime describing the big crunch-big bang transition. We showed there were sufficiently many such conserved quantities to ensure a unique match for all cosmological fields. In doing so, we necessarily had to pay attention to the well-known fact [14–17] that many Weyl-invariant quantities diverge at the bounce, such as the Weyl curvature, $C_{\nu\lambda\delta}^{\mu}$. Recently, Carrasco *et al.* [18] and Kallosh and Linde [19] rediscovered some of these divergences and, without paying attention to our discussion of conserved Weyl-invariant finite quantities, claimed the divergences necessarily spoil

the geodesic completeness of our proposed big crunch-big bang transition. In this paper, we demonstrate that this naive claim is incorrect.

To be sure, what is presented here is a straightforward elaboration of what was already proven in our earlier papers [4–11]. Once we identified a sufficient number of conserved finite Weyl-invariant quantities to determine a unique continuation of all the fundamental cosmological fields (e.g., scalar fields and metric) across all patches of field space, it should be obvious that the spacetime is geodesically complete because the geodesics of particles in the theory are all expressed in terms of these cosmological fields, as detailed below. It remains true that there exist infinitely many Weyl-invariant quantities that diverge at the crunch or bang, but these are irrelevant to the geodesic completeness. In fact, even for these quantities, the field continuation proposed in our papers uniquely determines their evolution before and after they go singular.

To illustrate the point, we focus on the vicinity of the big crunch or big bang where it suffices to consider a simplified standard model with a Higgs-like scalar field conformally coupled to gravity plus radiation. Following the procedure in [4–11], the “lifted” action of the standard model is achieved by adding an extra scalar field ϕ and imposing Weyl symmetry so the number of gauge invariant physical degrees of freedom remain the same. What is achieved in this way is the inclusion of all patches of the fields that leads to a geodesically complete cosmology as explained in more detail in [10,11]. The leading contributions near the big crunch-big bang transition are [7]:

$$\int d^4x \sqrt{-g} \left[\frac{1}{2} ((\partial\phi)^2 - (\partial h)^2) + \frac{1}{12} (\phi^2 - h^2) R \right], \quad (1)$$

where $h^2 \equiv H^\dagger H$ for Higgs doublet H . The complete description also includes a term describing the radiation (see below). All other contributions to the action, including matter fields, as well as density perturbations become negligible in this limit, and the cosmic evolution becomes *smoothly ultralocal*, meaning that spatial gradients become dynamically negligible [20]. The fact that the space-time may be treated as spatially homogeneous near the singularity also allows us to find all of its geodesics.

The action is invariant under the local gauge transformations $g_{\mu\nu} \rightarrow \Omega^{-2}(x^\mu)g_{\mu\nu}$, $\phi \rightarrow \Omega(x^\mu)\phi$ and $h \rightarrow \Omega(x^\mu)h$. Although the lift introduces a second scalar field ϕ with wrong-sign kinetic energy, it is obviously a gauge-artifact since fixing a Weyl gauge in which $\phi_E \equiv (\sqrt{6}/\kappa) \cosh \kappa\sigma/\sqrt{6}$ and $h_E \equiv (\sqrt{6}/\kappa) \sinh \kappa\sigma/\sqrt{6}$, where ($\kappa^2 \equiv 8\pi G$, with G Newton's constant) converts the action to Einstein gravity plus canonical scalar field σ with no ghost degrees of freedom in the incoming and outgoing cosmological states. Using the Bianchi I, VIII or IX metrics including anisotropy, the line element near the big crunch and bang is [7]

$$ds^2 = a^2(\tau)[-d\tau^2 + e^{-\sqrt{8/3}\kappa\alpha_1}d\sigma_3^2 + e^{\sqrt{2/3}\kappa\alpha_1}(e^{\sqrt{2}\kappa\alpha_2}d\sigma_1^2 + e^{-\sqrt{2}\kappa\alpha_2}d\sigma_2^2)], \quad (2)$$

where τ is the conformal time and $\alpha_{1,2}(\tau)$ parametrize the anisotropy. The $d\sigma_{1,2,3}$ generically include information about the spatial curvature; however, since the spatial curvature is negligible near a big crunch or big bang, $d\sigma_{1,2,3}$ reduce locally to $dx_{1,2,3}$, respectively, resulting in the Kasner-type metric.

Another useful gauge choice, dubbed γ -gauge fixes $g_\gamma = -1$ or equivalently the scale factor $a_\gamma = 1$, in which case the action is

$$\int d\tau \left(\frac{1}{2e} \left[-\dot{\phi}_\gamma^2 + \dot{h}_\gamma^2 + \frac{\kappa^2}{6}(\phi_\gamma^2 - h_\gamma^2)(\dot{\alpha}_1^2 + \dot{\alpha}_2^2) \right] - e\rho_r \right), \quad (3)$$

where $e(\tau)$ is the lapse function and the radiation density is $\rho_r/a^4(\tau)$ where ρ_r is constant.

In this gauge, it is straightforward to find the complete set of solutions that continuously track the evolution of ϕ_γ , h_γ and $\alpha_{1,2}$ through a big crunch, a brief interlude of antigravity, and then a big bang. Expressing the solution in terms of Einstein gauge fields (indicated by the subscript E) we obtain:

$$a_E^2(\tau) = 2|\tau||p + \rho_r\tau|, \quad (4)$$

$$\alpha_{1,2}(\tau) = \frac{p_{1,2}}{2p} \ln \left| \frac{\tau}{T_{1,2}(p + \rho_r\tau)} \right|, \quad (5)$$

$$a_E^2 h_E^2 = \frac{1}{2} |\tau| |p + \rho_r\tau| \times \left(\left| T_3 \left(\rho_r + \frac{p}{\tau} \right) \right|^{p_3/2p} - \left| T_3 \left(\rho_r + \frac{p}{\tau} \right) \right|^{-p_3/2p} \right)^2, \quad (6)$$

where τ is conformal time in units where $\kappa = \sqrt{6}$ and $T_{1,2,3}$ are integration constants. Note that the solution for $a^2 h^2 = a_E^2 h_E^2$ given above is Weyl invariant. The crunch occurs at $\tau_c = -p/\rho_r$ and the bang at $\tau = 0$ with the period of antigravity in between. The constants $p_{1,2,3}$ are the finite *conserved* values of the canonical momenta of the *fields* at the crunch or bang: $\pi_3 = a_E^2 \dot{h}_E \rightarrow p_3$, $\pi_{1,2} = a_E^2 \dot{\alpha}_{1,2} \rightarrow p_{1,2}$, and $p \equiv \sqrt{p_1^2 + p_2^2 + p_3^2}$. A $|p_3|$ which is larger than $\sqrt{15}(p_1^2 + p_2^2)^{1/2}$ ensures the avoidance of the mixmaster behavior [21,22] even when spatial curvature is present [6]. In Ref. [7], we describe a total of 15 conserved Noether charges that are finite at the crunch and the bang and whose conservation across the singularities is sufficient to uniquely determine the solutions given above.

We now consider the geodesics of massive particles in the standard model in this geometry. The *Weyl-invariant* action for a particle moving in a gravitational background can be expressed as

$$\mathcal{S}_{\text{particle}} = - \int d\lambda m(x) \sqrt{-\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x)}, \quad (7)$$

where $x^\mu(\lambda)$ is a function of the affine parameter λ . Note that $m(x)$ is generally x -dependent in theories like the standard model when the Higgs field contributes to the mass of particles: $m(x) = g_p h(x)$, where g_p is a dimensionless coupling of the particle field to the Higgs field. From this action, an explicit expression for *all* geodesics in an anisotropic Kasner universe can be derived, exploiting the spatial homogeneity of the metric [5,10]:

$$x^i(\tau) = q^i + \int^\tau d\tau' \frac{g_3^{ij}(\tau') k_j}{\sqrt{g_3^{kl}(\tau') k_k k_l + m^2(\tau') a^2(\tau')}}, \quad (8)$$

where k_i are the spatial components of the *conserved* particle momentum, q_i is the initial position, and $g_3^{ij}(\alpha_1(\tau), \alpha_2(\tau))$ is the inverse of the Kasner space-space metric appearing inside the square brackets in Eq. (2).

With this expression, Eq. (8), and noting that the integral converges for all physical parameters of the fields that determine the spacetime ($p_{1,2,3}, T_{1,2,3}, g_p, \rho_r$) and all parameters of the geodesics ($k_{1,2,3}, q^{1,2,3}$), we are effectively done with the proof of geodesic completeness. We stress that our complete solutions for $\alpha_{1,2}$ and the gauge invariant combination $a(\tau)m(\tau) = g_p a_E(\tau) h_E(\tau) = g_p h_\gamma(\tau)$ are continuous and sufficiently well behaved at the

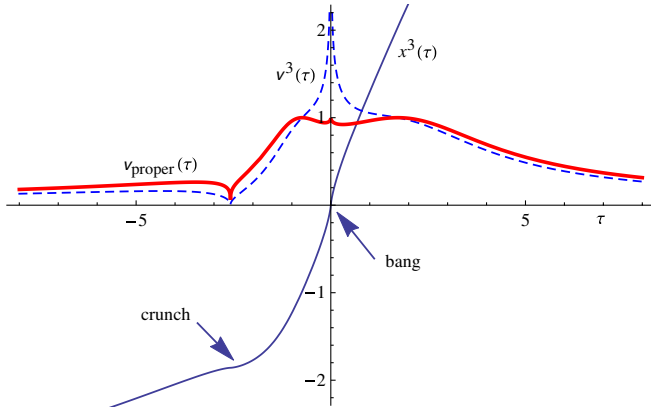


FIG. 1 (color online). A typical massive particle geodesic, showing the continuous passage through the big crunch and big bang singularities. In this example, the coordinate velocity $v^3 = \dot{x}^3$ goes to zero at the crunch and infinity at the bang. The proper speed of the particle with respect to particles comoving with the x^i coordinates, $v_{\text{proper}} = \sqrt{g_{3ij}\dot{x}^i\dot{x}^j} / \sqrt{g_{3ij}\dot{x}^i\dot{x}^j + m^2 a^2}$, never exceeds unity and touches zero at the crunch and unity at the bang. Likewise, the coordinate x^3 is finite and continuous throughout. The numerical values of the parameters used to generate this plot are $p_1 = -1/4$, $p_2 = 0$, $p_3 = 1$, $T_1 = 1$, $T_2 = 1$, $T_3 = 1$, $k_1 = 0$, $k_2 = 0$, $k_3 = 1$, $g_p = 1$, $\rho_r = 0.4$.

crunch and bang; this insures that the geodesics constructed from them are also continuous. The fact that the Weyl curvature and the Weyl-invariant quantities discussed in Refs. [18,19] diverge does not affect this conclusion one iota. This is clearly exemplified in Fig. 1 where we have chosen a typical geodesic and computed the behavior of the geodesic component $x^3(\tau)$ and the proper speed. Both are continuous throughout the big crunch-big bang transition and the proper speed is bounded by the speed of light, all despite the fact that the Weyl curvature and other quantities diverge and the Weyl-invariant metric is singular. Even though we do not expect to find a coordinate system to remove all curvature singularities, these are not sufficiently severe in our case to prevent the geodesics from completing their cosmic journeys.

We emphasize that this paper pertains to *all* the solutions of the field equations in the vicinity of the singularities and *all* the geodesics in those geometries, as obtained using our proposal [23]. This includes both *massive* and *massless* particles [23]. (In our general geodesic expression Eq. (8) that is expressed in terms of momenta, all that is needed to cover the case of lightlike geodesics is to set the mass or Higgs coupling g_p to zero. As is well known, this can also be obtained from the particle action in Eq. (7) by first defining the canonical momenta and keeping the momenta fixed while taking the zero mass limit; or, equivalently,

rewriting the action Eq. (7) in the first order formalism and then taking the limit.)

Our central point is that the continuation of the geometry beyond the singularity is established because we have shown in our case that all geodesics go through the relevant singularities. Classically, this geodesic completeness is not affected by the divergent curvatures that we [24] and others have identified [23]. In fact, the completion of the geometry is not supposed to eliminate the curvature singularities. Rather, it is supposed to show that, despite the curvature singularities, physical information can and does journey from cycle to cycle through the cosmological singularities [23]. Hence, we can claim the geodesic completeness for all *homogeneous* cosmological field configurations of the standard model coupled to gravity. Further details and a more thorough discussion of geodesics, geodesic deviation and geodesic completeness in Weyl-invariant theories are given in Ref. [25].

Of course, our purely classical analysis does not include strong quantum gravity effects near the singularities because the technology does not yet exist to do those computations [23]. Nevertheless, finding classical geodesic completeness and the complete set of classical solutions is very useful. Having the guidance and physical insight of classical analysis is often a reasonable starting point in understanding quantum theoretic descriptions of physical phenomena, especially when there are indications of new physics, as presented here. For example, suppose one wished to pursue physics near the singularity in the framework of string theory. Our solutions provide the starting point because they provide geodesically complete cyclic background geometries (metric and dilaton) consistent with perturbative world sheet conformal symmetry as required by the quantization of the string moving in backgrounds. Our classical calculations suggest exciting new phenomena to be explored. Based on historical precedents, it is reasonable to suppose that some or all qualitative features will survive quantum (string) corrections.

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- [23] In a Note Added in Proof in Ref. [18] written after our preprint, the authors introduce new arguments (and repeat some past ones). However, all new points (as well as old ones) were already addressed in our original preprint and earlier papers. For clarity, though, we have edited our concluding remarks and marked with this footnote to indicate where the direct response to each new point can be found.
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