PHYSICAL REVIEW D **89,** 061301(R) (2014)

Scaling laws and sum rules for the *B*-mode polarization

Massimo Giovannini*

Department of Physics, Theory Division, CERN, 1211 Geneva 23, Switzerland and INFN, Section of Milan-Bicocca, 20126 Milan, Italy

(Received 3 February 2014; published 4 March 2014)

The formation of the microwave background polarization anisotropies is investigated when the stochastic Faraday rate is stationary, random and Markovian. The scaling properties of the polarization power spectra and of their nonlinear combinations are scrutinized as a function of the comoving frequency. It is argued that each frequency channel of a given experiment measuring simultaneously the E-mode and the B-mode spectra can be analyzed in this framework with the aim of testing the physical origin of the polarization in a model-independent perspective.

DOI: 10.1103/PhysRevD.89.061301 PACS numbers: 98.70.Vc, 78.20.Ek, 95.75.Hi, 98.80.Cq

Synchrotron sources are known to emit polarized radiation [1] that is stochastically rotated by the Faraday effect [1,2]. To obtain a suitable physical description of the frequency scaling, the corresponding polarization observables are customarily averaged over the rotation rate [2]. Unlike the case of synchrotron emission, the degree of linear polarization of the cosmic microwave background (CMB in what follows) stems directly from the adiabatic initial conditions of the Einstein-Boltzmann hierarchy, implying that the position of the first acoustic peak in the TT correlations[1] must be roughly $4/3$ times larger than the position of the first anticorrelation peak in the TE angular power spectrum. This prediction has been observationally established by the various data releases of the WMAP Collaboration [3] and later confirmed by independent polarization experiments [4]. The aim of this investigation is to discuss the formation of the CMB polarization by characterizing the Faraday rate as a stationary and random process with approximate Markovian behaviour. Within this novel approach, exploiting the analogies with the stochastic rotation of the synchrotron polarization, various scaling properties of the corresponding angular power spectra can be derived and eventually tested if and when multifrequency measurements of the B mode become available.

It is relevant to remark that in the minimal concordance scenario the B-mode polarization is strictly vanishing, since the tensor modes are absent from the fit. The tensor-to-scalar ratio fixes the relative amplitude of the tensor and of the scalar power spectra, and it is introduced as a further parameter; in this framework the B-mode polarization does not vanish. We shall discuss primarily the minimal concordance paradigm with no tensors and briefly comment on the modifications required if a tensor background is included.

We shall consistently work in a conformally flat spacetime whose metric tensor can be written as $g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric; $a(\tau)$ shall denote the scale factor and τ is the conformal time coordinate. With these necessary specifications, in the concordance paradigm the Faraday rotation rate can be expressed as

$$X_F(\vec{x},\tau) = \frac{\bar{\omega}_{Be}}{2}\left(\frac{\bar{\omega}_{pe}}{\bar{\omega}}\right)^2 = \frac{e^3}{2\pi}\left(\frac{n_e}{m_e^2 a^2}\right)\left(\frac{\vec{B}\cdot\hat{n}}{\bar{\nu}^2}\right), \quad (1)$$

where $\bar{\omega} = 2\pi\bar{\nu}$ is the (comoving) angular frequency, while $\bar{\omega}_{Be}$ and $\bar{\omega}_{pe}$ denote the comoving Larmor and plasma frequencies. The WMAP experiment observes the microwave sky in five frequency channels ranging from 23 to 94 GHz. The Planck satellite explores instead the microwave sky in nine frequency channels: three of them are at low frequency (between 30 and 70 GHz), while the remaining six are located between 100 and 857 GHz. Let us introduce the differential optical depth, $\epsilon' = \tilde{n}_e x_e \sigma_{e\gamma} a(\tau)$, where $\sigma_{e\gamma}$ denotes the electron-photon scattering cross section and x_e the ionization fraction. The comoving and the physical electron concentrations appearing in Eq. (1) and in the definition of the differential optical depth ϵ', respectively, are related as $n_e = a^3(\tau)x_e\tilde{n}_e$. To gauge the magnitude of the rate, it is useful to express X_F in units of ϵ', i.e.

$$\frac{X_F(\vec{x},\tau)}{\epsilon'} = 35.53\left(\frac{\vec{B}\cdot\hat{n}}{\text{nG}}\right)\left(\frac{\text{GHz}}{\bar{\nu}}\right)^2, \quad (2)$$

showing that the actual value of X_F/ϵ' is not necessarily much smaller than 1, and it is $\mathcal{O}(1)$ for comoving field strengths of a few nG (i.e. 1 nG $= 10^{-9}$ G) and frequencies $\mathcal{O}(10)$ GHz.

The scalar modes of the geometry induced by an inhomogeneous magnetic field are the leading source of

*massimo.giovannini@cern.ch

[1] Following the established terminology, the B-mode autocorrelations are denoted by BB. With similar logic, we shall mention throughout the text the TT, TE and EE angular power spectra denoting, respectively, the autocorrelations of the temperature, the autocorrelations of the E mode, and their mutual cross correlations.

distortion of the TT, EE and TE angular power spectra that have been derived for the magnetized adiabatic mode and compared with the available experimental data with the aim of pinning down the properties of the magnetic field. In Ref. [5] this analysis has been performed, for the first time, using the WMAP five-year data, and later confirmed by other analyses and different data sets (see, e.g., Ref. [6] and references therein). The purpose here is not to predict the Faraday effect given the current bounds on the magnetic field: this exercise has been already explored in the past (see, e.g., Ref. [7] and references therein), and it has its own limitations. The idea is instead to reverse the problem and use the heuristic power of the stochastic description to infer the origin of the B-mode polarization even without a detailed knowledge of the properties of the magnetic field.

The Faraday rate introduced in Eq. (1) enters directly the evolution of the magnetized brightness perturbations (see, e.g., Refs. [7,8] and references therein), whose explicit form in the conformally flat case is given by

$$\Delta'_{\pm} + (\epsilon' + n^i\partial_i)\Delta_{\pm} = \mathcal{M}(\vec{x},\tau) \mp 2iX_F(\vec{x},\tau)\Delta_{\pm}, \quad (3)$$

where $\Delta_{\pm}(\vec{x},\tau) = \Delta_Q(\vec{x},\tau) \pm i\Delta_U(\vec{x},\tau)$; $\Delta_Q(\vec{x},\tau)$ and $\Delta_U(\vec{x},\tau)$ define the brightness perturbations of the corresponding Stokes parameters. In Eq. (3), the prime denotes a derivation with respect to the conformal time coordinate τ; the source term $\mathcal{M}(\vec{x},\tau)$ is determined by the electron-photon scattering cross section and by the properties of the magnetic field. The conventional discussion assumes that first the polarization is formed and then it is Faraday-rotated with $X_F \ll 1$, as it happens if the ambient magnetic field is not too high and the observational frequency is not too small. The goal of the latter approach is to derive a set of phenomenological bounds on the comoving magnetic field $\hat{n}\cdot\vec{B}$ that must be, *a priori*, smaller than the nG scale to comply with the assumed smallness of the Faraday rate.

Rather than deriving a further bound of the magnetic field intensity, the purpose here is to explore a different approach, where the Faraday rate is described as a random, stationary and approximately Markovian process. The randomness implies that $X_F(\tau)$ is not a deterministic variable but rather a stochastic process which is stationary insofar as the autocorrelation function $\Gamma(\tau_1,\tau_2) = \langle X_F(\tau_1)X_F(\tau_2)\rangle$ only depends on time differences, i.e. $\Gamma(\tau_1,\tau_2) = \Gamma(|\tau_1-\tau_2|)$; furthermore, we shall also assume that the process has zero mean, even if this is not strictly necessary for the consistency of the whole approach. If τ_b defines the time scale of variation of the brightness perturbations of the polarization observables, the physical situation investigated here corresponds to $\tau_b \gg \tau_c$, where τ_c is the correlation time scale of X_F. In the simplest case of a Gaussian-correlated process, the autocorrelation function $\Gamma(\tau_1-\tau_2) = F(\tau_1)\tau_c\delta(\tau_1-\tau_2)$. If the time scale of spatial variation of the rate is comparable with the time scale of spatial variation of the gravitational fluctuations, X_F can be considered only time dependent (i.e. a stochastic process). In the opposite situation, the Faraday rate must be considered fully inhomogeneous (i.e. a stochastic field). These two possibilities will be separately considered hereunder. On a purely logical ground, X_F can just be a random variable characterized by a given probability distribution, and this is somehow the most naive case that has been already analyzed in the framework of the synchrotron emission (see, e.g., the second paper of Ref. [2]) and that will not be treated here.

If $X_F(\tau)$ is interpreted as a stochastic process, Eq. (3) becomes, in Fourier space,

$$\delta'_{\pm} + (ik\mu + \epsilon')\delta_{\pm} = \frac{3}{4}(1-\mu^2)\epsilon' S_P(\vec{k},\tau) \mp 2iX_F(\tau)\delta_{\pm}, \quad (4)$$

where $S_P(\vec{k},\tau) = (\delta_{I2} + \delta_{P0} + \delta_{P2})$, and $\delta_{\pm}(\vec{k},\tau)$ denotes the Fourier transform of $\Delta_{\pm}(\hat{n},\tau)$; δ_{P0} and δ_{P2} are the monopole and the quadrupole of δ_P, and δ_{I2} is the quadrupole of the brightness perturbation related to the intensity of the radiation field. Equation (4) must be complemented by the evolution of the brightness perturbations of the intensity (i.e. δ_I) that can be used to solve approximately the system in the tight-coupling limit [9]. The source term $S_P(\vec{k},\tau)$ depends on the frequency of the channel, since the magnetic field modifies the trajectories of the electrons scattering the CMB the photons; for the sake of simplicity, this effect (that is also frequency dependent) shall be neglected in what follows, but it is described in detail in the last paper of Ref. [7], and it can be easily included.

For equal times (but for different Fourier modes), the fluctuations of the brightness perturbations are random with the power spectrum determined by the (nearly scale-invariant) spectrum of Gaussian curvature perturbations [3]. Thus, in the absence of Faraday mixing, $\delta_{\pm}$ obeys then a deterministic evolution in time, while the spatial fluctuations of the polarization are randomly distributed and fixed by the correlation properties of the adiabatic curvature perturbations. Conversely, since $X_F(\tau)$ is now treated as a stochastic process, Eq. (4) becomes a stochastic differential equation [10] in time, and its formal solution is obtainable by iteration:

$$\delta_{\pm}(\vec{k},\tau) = \sum_{n=0}^{\infty}\delta_{\pm}^{(n)}(\vec{k},\tau), \qquad \delta_{\pm}^{(0)}(\vec{k},\tau) = \delta_P(\vec{k},\tau). \quad (5)$$

Equations (4) and (5) imply the following recurrence relations:

$$\delta_P(\vec{k},\tau) = \frac{3}{4}(1-\mu^2)\int_0^{\tau} e^{-ik\mu(\tau-\tau_1)}\mathcal{K}(\tau_1)S_P(\vec{k},\tau_1), \quad (6)$$

$$\delta_{\pm}^{(n+1)}(\vec{k},\tau) = \pm 2i \int_0^{\tau} e^{-ik\mu(\tau-\tau_1)} \mathcal{K}(\tau_1) X_F(\tau_1) \delta_{\pm}^{(n)}(\vec{k},\tau_1). \tag{7}$$

The differential optical depth directly enters the visibility function, giving the probability that a photon is emitted between τ and $\tau + d\tau$:

$$\mathcal{K}(\tau_1) = \epsilon'(\tau_1) e^{-\epsilon(\tau_1,\tau)}, \qquad \epsilon(\tau_1,\tau) = \int_{\tau_1}^{\tau} x_e \tilde{n}_e \sigma_{e\gamma} \frac{a(\tau')}{a_0}. \tag{8}$$

The full solution of Eq. (4) is formally expressible as

$$\delta_{\pm}(\vec{k},\tau) = \frac{3}{4}(1-\mu^2) \times \int_0^{\tau} e^{-ik\mu(\tau-\tau_1)} \mathcal{K}(\tau_1) S_P(\vec{k},\tau_1) \mathcal{A}_{\pm}(\tau,\tau_1) d\tau_1,$$
$$\mathcal{A}_{\pm}(\tau,\tau_1) = e^{\mp 2i \int_{\tau_1}^{\tau} X_F(\tau') d\tau'}. \tag{9}$$

The visibility function adopted for the analytic estimates has the approximate shape of a double Gaussian whose first peak arises around last scattering (i.e. for $\tau \simeq \tau_r$), while the second (smaller) peak occurs for the reionization epoch at a typical redshift of about 11 [3,9]. The finite thickness of the last scattering surface does not affect the ratios between the different combinations of polarization power spectra discussed here, so that the limit of sudden recombination can be safely be adopted; in this limit the first and more pronounced Gaussian profile tends to a Dirac delta function.

The statistical properties of $\mathcal{A}_{\pm}$ follow directly from the correlation properties of $X_F(\tau)$. If, for instance, $X_F(\tau)$ obeys a stationary and Gaussian process, for any set of n Faraday rates (characterized by different conformal times) the correlator $\langle X_F(\tau_1)X_F(\tau_2)\ldots X_F(\tau_n)\rangle$ vanishes if n is odd; if n is even, the same correlator equals

$$\sum_{\text{pairings}} \langle X_F(\tau_1)X_F(\tau_2)\rangle\langle X_F(\tau_3)X_F(\tau_4)\rangle\ldots\langle X_F(\tau_{n-1})X_F(\tau_n)\rangle, \tag{10}$$

where the sum is performed over all the $(n-1)!$ pairings. In the Gaussian case, the evaluation of the averages can be performed by first doing the standard moment expansion and by the resumming the obtained result. As an example, from the explicit expression of $\mathcal{A}_{\pm}$, it follows that

$$\langle \mathcal{A}_{\pm}(\tau,\tau_r)\mathcal{A}_{\pm}(\tau,\tau_r)\rangle = \langle e^{\pm 4i \int_{\tau_r}^{\tau} X_F(\tau') d\tau'}\rangle = \sum_{n=0}^{\infty} \frac{(-2\omega_F)^n}{n!}, \tag{11}$$

where ω_F is given by

$$\omega_F = 4 \int_{\tau_r}^{\tau} d\tau_1 \int_{\tau_r}^{\tau} d\tau_2 \langle X_F(\tau_1) X_F(\tau_2)\rangle. \tag{12}$$

It follows from Eq. (12) that even if $X_F \leq 1$, ω_F is not bound to be smaller than 1.

If the stationary process is only approximately Markovian, the result of Eq. (11) still holds, but in an approximate sense. While the standard moment expansion can be formally adopted in specific cases (like the Gaussian one), it cannot be used to provide successive approximations. The reason is that any finite number of terms constitutes a bad representation of the function defined by the whole series. This difficulty is overcome with the use of the cumulants that are certain combinations of the moments. Dropping the functions and keeping only their corresponding arguments, we have that the relations between the ordinary moments and the cumulants (denoted by $\langle\langle\ldots\rangle\rangle$) is $\langle 1\rangle = \langle\langle 1\rangle\rangle$, $\langle 12\rangle = \langle\langle 1\rangle\rangle\langle\langle 2\rangle\rangle + \langle\langle 12\rangle\rangle$, $\langle 123\rangle = \langle\langle 1\rangle\rangle\langle\langle 2\rangle\rangle\langle\langle 3\rangle\rangle + \langle\langle 12\rangle\rangle\langle\langle 3\rangle\rangle + \langle\langle 31\rangle\rangle\langle\langle 2\rangle\rangle + \langle\langle 23\rangle\rangle\langle\langle 1\rangle\rangle + \langle\langle 123\rangle\rangle$, and so on and so forth for the other moments of the cluster expansion. Substituting the naive moment expansion with the cumulant expansion, we have that the average of Eq. (11) is given by

$$\langle \mathcal{A}_{\pm}(\tau,\tau_r)\mathcal{A}_{\pm}(\tau,\tau_r)\rangle = \exp\left[\sum_{m=1}^{\infty} \frac{(\pm 4i)^m}{m!} \int_{\tau_r}^{\tau} d\tau_m \left\langle\left\langle X_F(\tau_1)X_F(\tau_2)\ldots X_F(\tau_m)\right\rangle\right\rangle\right]. \tag{13}$$

As first suggested by Van Kampen (see Ref. [10], third and fourth paper) in the approximately Markovian case, the averages of certain stochastic processes will be given by an exponential whose exponent is a series of successive cumulants of X_F. All the cumulants beyond the second are zero in the case of an exactly Gaussian process, and the result reported in Eq. (11) is recovered. Since each integrand in Eq. (13) virtually vanishes unless $\tau_1, \tau_2, \ldots, \tau_m$ are close together, the only contribution to the integral comes from a tube with a diameter of order τ_c along the diagonal in the m-dimensional integration space. More generally, the mth cumulant vanishes as soon as the sequence of times $\tau_1, \tau_2, \ldots, \tau_m$ contains a gap that is large compared to τ_c. This is the reason why, in a nutshell, the concept of cumulant is rather practical also in our case.

As an example of stationary process that is not delta correlated, consider the case where $\Gamma(\tau_1 - \tau_2) = \langle X_F(\tau_1)X_F(\tau_2)\rangle$ can take only two values, $\bar{x}_F^2$ and $-\bar{x}_F^2$, and let us suppose that $X_F(\tau)$ has switched an even number of times in the interval between τ_1 and τ_2, so that $\Gamma(\tau_1 - \tau_2) = \bar{x}_F^2$, whereas the correlation function gives $-\bar{x}_F^2$ if there have been an odd number of switches. If $p(n, \Delta\tau)$ is the probability of n switches in the interval $\Delta\tau = \tau_1 - \tau_2$, it follows that

$$\Gamma(\Delta\tau) = \bar{x}_F^2 \sum_{n=0,2,4...}^{\infty} p(n,\Delta\tau) - \bar{x}_F^2 \sum_{n=1,3,5...}^{\infty} p(n,\Delta\tau) = \bar{x}_F^2 \sum_{n=0}^{\infty} (-1)^n p(n,\Delta\tau). \quad (14)$$

As the switches are random with average rate r, the function $p(n,\Delta\tau)$ is nothing but a Poisson distribution with a mean number of switches $\bar{n} = r\Delta\tau$, i.e. $p_n = \bar{n}^n e^{-\bar{n}}/n!$. This means that $\Gamma(\Delta\tau) = \bar{x}_F^2 \exp[-2r\Delta\tau]$. This is an example of a dichotomic Markov process [10] applied to the case of a stochastic Faraday rate.

If X_F is a stochastic field rather than a stochastic process, the discussion is mathematically slightly different but physically equivalent as far as the frequency scaling is concerned. More specifically, the evolution equations for $\delta_\pm$ will now contain a convolution and can be written as

$$\delta_\pm' + (ik\mu + \epsilon')\delta_\pm = \frac{3}{4}(1-\mu^2)\epsilon' S_P(\vec{k},\tau) \mp i b_F(\bar{\nu},\tau) \times \int d^3p\, \delta_\pm(\vec{k}+\vec{p},\tau) n^i B_i(\vec{p},\tau), \quad (15)$$

where we define, for convenience, $b_F(\bar{\nu},\tau) = 2e^3 n_e/[(2\pi)^{5/2} m_e^2 a^2(\tau)\bar{\nu}^2]$. The iterative solution of Eqs. (5) and (6)–(7) becomes, in this case,

$$\partial_\tau \delta_\pm^{(0)} + (ik\mu + \epsilon')\delta_\pm^{(0)} = \frac{3}{4}(1-\mu^2)\epsilon' S_P(\vec{k},\tau), \quad (16)$$

$$\partial_\tau \delta_\pm^{(1)} + (ik\mu + \epsilon')\delta_\pm^{(1)} = \mp i b_F(\bar{\nu},\tau) \int d^3p\, \delta_P(\vec{k}+\vec{p},\tau) n^i B_i(\vec{p},\tau), \quad (17)$$

$$\partial_\tau \delta_\pm^{(2)} + (ik\mu + \epsilon')\delta_\pm^{(2)} = \mp i b_F(\bar{\nu},\tau) \int d^3p'\, \delta_P(\vec{k}+\vec{p}',\tau) n^i B_i(\vec{p},\tau) n^j B_j(\vec{p}',\tau). \quad (18)$$

To compute the averages, we must therefore specify the correlation properties of the Faraday rate. Even if the spatial dependence may reside in all the terms contributing to the Faraday rate, it is reasonable to presume that the leading effect may come from the magnetic field whose correlation function will then be parametrized as

$$\langle B_i(\vec{q},\tau_1) B_j(\vec{p},\tau_2)\rangle = \frac{2\pi^2}{p^3} P_{ij}(\hat{p}) \bar{P}_B(p) \Gamma(|\tau_1 - \tau_2|) \delta^{(3)}(\vec{q}+\vec{p}), \quad (19)$$

where $\Gamma(|\tau_1-\tau_2|) = \tau_c \delta(\tau_1 - \tau_2)$ in the delta-correlated case. In the same approximation exploited before and using Eq. (19), ω_F becomes now

$$\omega_F = \frac{8\bar{b}_F^2}{3\bar{\nu}^4} \int \frac{dp}{p} \bar{P}_B(p) \int_{\tau_r}^{\tau} d\tau_1 \int_{\tau_r}^{\tau} d\tau_2 \frac{\Gamma(|\tau_1-\tau_2|)}{a^2(\tau_1) a^2(\tau_2)}, \quad (20)$$

where the constant $\bar{b}_F = b_F(\bar{\nu},\tau) a^2(\tau) \bar{\nu}^2$ has been introduced in order to draw special attention to the frequency scaling that is the most relevant aspect of Eq. (20), at least in the present approach.

The dependence of the polarization observables upon ω_F can now be determined. Since $\Delta_\pm$ transform as fluctuations of spin weight ± 2, they can be expanded in terms of spin-± 2 spherical harmonics ${}_{\pm 2}Y_{\ell m}(\hat{n})$, with coefficients $a_{\pm 2,\ell m}$. The E- and B-modes are, up to a sign, the real and the imaginary parts of $a_{\pm 2,\ell m}$, i.e. $a_{\ell m}^{(E)} = -(a_{2,\ell m} + a_{-2,\ell m})/2$ and $a_{\ell m}^{(B)} = i(a_{2,\ell m} - a_{-2,\ell m})/2$. The real-space fluctuations constructed from $a_{\ell m}^{(E)}$ and $a_{\ell m}^{(B)}$ have the property of being invariant under rotations on a plane orthogonal to $\hat{n}$. They are therefore scalars and must be expanded in terms of (ordinary) spherical harmonics:

$$\Delta_E(\hat{n},\tau) = \sum_{\ell m} N_\ell^{-1} a_{\ell m}^{(E)} Y_{\ell m}(\hat{n}),$$
$$\Delta_B(\hat{n},\tau) = \sum_{\ell m} N_\ell^{-1} a_{\ell m}^{(B)} Y_{\ell m}(\hat{n}), \quad (21)$$

where $N_\ell = \sqrt{(\ell-2)!/(\ell+2)!}$. Within these notations, the EE and BB angular power spectra are defined as

$$C_\ell^{(EE)} = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \langle a_{\ell m}^{(E)*} a_{\ell m}^{(E)}\rangle,$$
$$C_\ell^{(BB)} = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \langle a_{\ell m}^{(B)*} a_{\ell m}^{(B)}\rangle, \quad (22)$$

while the cross-correlation power spectrum is constructed from $\langle a_{\ell m}^{(E)*} a_{\ell m}^{(B)} + a_{\ell m}^{(E)} a_{\ell m}^{(B)*}\rangle$. The repeated application of generalized ladder operators whose action either raises or lowers the spin weight of a given fluctuation [11] (see also Ref. [7]) leads to a direct connection between Δ_E, Δ_B and $\Delta_\pm$:

$$\Delta_E(\hat{n},\tau) = -\frac{1}{2}\partial_\mu^2\left[(1-\mu^2)(\Delta_+ + \Delta_-)\right],$$
$$\Delta_B(\hat{n},\tau) = \frac{i}{2}\partial_\mu^2\left[(1-\mu^2)(\Delta_+ - \Delta_-)\right], \quad (23)$$

where ∂_μ^2 denotes the second derivative with respect to $\mu = \cos\vartheta$. From Eqs. (21) and (23), we can finally determine $a_{\ell m}^{(E)}$ and $a_{\ell m}^{(B)}$ within the set of conventions followed here:

$$a_{\ell m}^{(E)} = -\frac{N_\ell}{2(2\pi)^{3/2}}\int d\hat{n} Y_{\ell m}^{*}(\hat{n}) \times \int d^3k \partial_\mu^2 \left\{(1-\mu^2)\left[\delta_+(\vec{k},\tau) + \delta_-(\vec{k},\tau)\right]\right\},$$
$$a_{\ell m}^{(B)} = \frac{iN_\ell}{2(2\pi)^{3/2}}\int d\hat{n} Y_{\ell m}^{*}(\hat{n}) \times \int d^3k \partial_\mu^2 \left\{(1-\mu^2)\left[\delta_+(\vec{k},\tau) - \delta_-(\vec{k},\tau)\right]\right\}. \quad (24)$$

Inserting Eq. (24) into Eq. (22), the angular power spectra of the E-mode and of the B-mode polarizations can be derived, and they are

$$C_\ell^{(EE)}(\omega_F) = e^{-\omega_F}\cosh\omega_F \bar{C}_\ell^{(EE)},$$
$$C_\ell^{(BB)}(\omega_F) = e^{-\omega_F}\sinh\omega_F \bar{C}_\ell^{(EE)}; \quad (25)$$

the cross-correlation power spectrum is instead vanishing $C_\ell^{(EB)} = 0$. In Eq. (25), $\bar{C}_\ell^{(EE)}$ is the E-mode autocorrelation produced by the standard adiabatic mode and in the absence of Faraday mixing. Equation (25) shows that the B-mode and the E-mode polarizations are both frequency dependent. Despite the fact that these formulas hold also when $\omega_F \geq 1$, in the limit $\omega_F \ll 1$ the standard results are recovered, and only the B mode depends on the frequency [7]. From Eq. (25), the following sum rules for the angular power spectra can be easily established:

$$C_\ell^{(EE)}(\omega_F) + C_\ell^{(BB)}(\omega_F) = \bar{C}_\ell^{(EE)}, \quad (26)$$

$$C_\ell^{(EE)}(\omega_F) - C_\ell^{(BB)}(\omega_F) = e^{-2\omega_F}\bar{C}_\ell^{(EE)}. \quad (27)$$

Introducing now the properly normalized angular power spectra $\mathcal{G}_{E\ell}(\omega_F)$ and $\mathcal{G}_{B\ell}(\omega_F)$,

$$\mathcal{G}_{E\ell}(\omega_F) = \frac{\ell(\ell+1)}{2\pi} C_\ell^{(EE)}(\omega_F),$$
$$\mathcal{G}_{B\ell}(\omega_F) = \frac{\ell(\ell+1)}{2\pi} C_\ell^{(BB)}(\omega_F), \quad (28)$$

the following ratio of nonlinear combinations has well-defined scaling properties with ω_F:

$$\mathcal{L}_0(\omega_F) = \frac{\mathcal{G}_{E\ell}^2(\omega_F) - \mathcal{G}_{B\ell}^2(\omega_F)}{[\mathcal{G}_{E\ell}(\omega_F) + \mathcal{G}_{B\ell}(\omega_F)]^2} \to e^{-2\omega_F}. \quad (29)$$

Equation (29) does not assume that the Faraday rate is much smaller than 1, and it does not even assume a specific form of the Markov process. For an exactly Gaussian process or for a dichotomic Markov process [see, e.g., Eq. (14)], the explicit expressions of ω_F can be rather different, but the frequency dependence will be always the same: since ω_F is quadratic in the rates, it will always scale as $1/\bar{\nu}^4 \simeq \lambda^4$, where λ denotes the wavelength of the channel. Since the scale factor is normalized in such a way that $a_0 = 1$, physical and comoving frequencies coincide today but not in the past. The combination reported in Eq. (29) is not unique, and different expressions can be envisaged depending on the actual features of the measurement. Two further combinations explicitly depending on ω_F are

$$\mathcal{L}_1(\omega_F) = \frac{\mathcal{G}_{E\ell}(\omega_F) - \mathcal{G}_{B\ell}(\omega_F)}{\mathcal{G}_{E\ell}(\omega_F) + \mathcal{G}_{B\ell}(\omega_F)} \to e^{-2\omega_F},$$
$$\mathcal{L}_2(\omega_F) = \frac{\mathcal{G}_{E\ell}^2(\omega_F) + \mathcal{G}_{B\ell}^2(\omega_F)}{\mathcal{G}_{E\ell}^2(\omega_F) - \mathcal{G}_{B\ell}^2(\omega_F)} \to \cosh 2\omega_F. \quad (30)$$

Since $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ contain ratios of the angular power spectra, the finite thickness of the last scattering surface is not expected to affect these conclusions in any significant manner.

The discussion has been conducted, so far, in the framework of the standard concordance paradigm. If a tensor component is added, the relations reported in Eq. (25) are modified by the presence of $\bar{C}_\ell^{(BB)}$, denoting the B-mode contribution stemming from the tensor background. This means that, overall, the B-mode power spectrum will be given as $C_\ell^{(BB)}(\omega_F) \to e^{-\omega_F}\sinh\omega_F\bar{C}_\ell^{(EE)} + \bar{C}_\ell^{(BB)}$. The sum rules and the scaling relations obtained above can be generalized to this case by following the same logic. The important point to bear in mind is that while the Faraday contribution does depend on the comoving frequency, $\bar{C}_\ell^{(BB)}$ is frequency independent.

In the concordance paradigm with no tensors, the stochastic Faraday rotation can be tested through multifrequency observations once the measurements of the B-mode polarization become available. If the E-mode and the B-mode autocorrelations are independently measured in each frequency channel of a given experiment, both scale-invariant and scale-dependent combinations of the angular power spectra can be constructed frequency by frequency. So, for instance, the combination $\mathcal{L}_0 + \mathcal{L}_2 \to 2$ is scale invariant in the limit $\omega_F \ll 1$. Similarly, $\mathcal{L}_2/(\mathcal{L}_0 + \mathcal{L}_0^{-1})$ is scale invariant in spite of the value of ω_F. Equations (29) and (30) illustrate, then, a possible redundant set of physical observables that can be used to discriminate between the frequency dependence induced by the stochastic Faraday effect and that induced by other concurrent forms of frequency scaling caused either by the known or by the yet unknown foregrounds.

If the B mode is frequency independent, as expected if and when it comes from the tensor modes of the geometry, the internal linear combination (ILC) technique can be applied indifferently for all the channels of the experiment that eventually detects the B mode. In practice, the ILC map is a weighted linear combination over the smoothed maps obtained from each of the different frequency channels. If the signal is inherently frequency dependent, as our considerations predict, the ILC cannot be blindly applied. It is our opinion that, in these matters, the scaling relations of Eqs. (29)–(30) (and their descendants) are crucial if we intend to disentangle the real physical effects from potential foregrounds.

The present investigation described the Faraday effect of the CMB as a random, stationary and quasi-Markovian process. The stochastic treatment of this physical phenomenon has been explored in analogy with the case of synchrotron polarization. The obtained results encompass and complement previous analyses where the formation of Faraday effect has been customarily presented as a purely deterministic process in time. Apart from the discussion of the frequency scaling of the polarization observables, further applications of the approach developed here seem both physically plausible and technically feasible.

[1] B. J. Burn, Mon. Not. R. Astron. Soc. **133**, 67 (1966); A. G. Pacholczyk and T. L. Swihart, Astrophys. J. **150**, 647 (1967); **161**, 415 (1970); V. N. Sazonov, Zh. Eksp. Teor. Fiz. **56**, 1065 (1969) [, Sov. Phys. JETP **29**, 578 (1969)]; T. Jones and A. O'Dell, Astrophys. J. **214**, 522 (1977); **215**, 236 (1977).

[2] J. Simonetti, J. Cordes, and S. Sprangler, Astrophys. J. **284**, 126 (1984); D. Melrose and J. Macquart, Astrophys. J. **505**, 921 (1998); D. Lai and W. C. G. Ho, Astrophys. J. **588**, 962 (2003).

[3] D. N. Spergel *et al.*, Astrophys. J. Suppl. Ser. **148**, 175 (2003); **170**,377 (2007); C. L. Bennett *et al.*, Astrophys. J. Suppl. Ser. **192**, 17 (2011); B. Gold *et al.*, Astrophys. J. Suppl. Ser. **192**, 15 (2011); E. Komatsu *et al.*, Astrophys. J. Suppl. Ser. **192**, 18 (2011).

[4] M. L. Brown *et al.*, Astrophys. J. **705**, 978 (2009); D. Araujo *et al.*, Astrophys. J. **760**, 145 (2012).

[5] M. Giovannini, Phys. Rev. D **74**, 063002 (2006); **79**,121302 (2009); **79**,103007 (2009); Classical Quantum Gravity **27**, 105011 (2010).

[6] D. G. Yamazaki, K. Ichiki, T. Kajino, and G. J. Mathews, Phys. Rev. D **81**, 023008 (2010); D. Paoletti and F. Finelli, Phys. Rev. D **83**, 123533 (2011); R. R. Caldwell, L. Motta, and M. Kamionkowski, Phys. Rev. D **84**, 123525 (2011); M. Giovannini, Classical Quantum Gravity **30**, 205017 (2013).

[7] M. Giovannini, Phys. Rev. D **56**, 3198 (1997); **71**,021301 (2005); M. Giovannini and K. E. Kunze, Phys. Rev. D **78**, 023010 (2008); M. Giovannini, Phys. Rev. D **79**, 103007 (2009); Classical Quantum Gravity **27**, 225016 (2010).

[8] K. Enqvist, Int. J. Mod. Phys. D **07**, 331 (1998); M. GiovanniniInt. J. Mod. Phys. D **13**, 391 (2004); Classical Quantum Gravity **23**, R1 (2006); J. D. Barrow, R. Maartens, and C. G. Tsagas, Phys. Rep. **449**, 131 (2007).

[9] P. J. E. Peebles and J. T. Yu, Astrophys. J. **162**, 815 (1970); R. A. Sunyaev and Y. B. Zeldovich, Astrophys. Space Sci. **7**, 3 (1970); U. Seljak, Astrophys. J. **435**, L87 (1994); P. Naselsky and I. Novikov, Astrophys. J. **413**, 14 (1993); B. Jones and R. Wyse, Astron. Astrophys. **149**, 144 (1995); H. Jorgensen, E. Kotok, P. Naselsky, and I. Novikov, Astron. Astrophys. **294**, 639 (1995).

[10] R. Kraichnan, J. Math. Phys. (N.Y.) **2**, 124 (1961); R. Zwanzig, J. Chem. Phys. **33**, 1338 (1960); N. G. Van Kampen, Physica (Utrecht) **74**, 215 (1974); R. H. Terweil, Physica (Utrecht) **74**, 248 (1974).

[11] J. Goldberg, A. Macfarlane, E. Newman, F. Rohrlich, and E. Sudarshan, J. Math. Phys. (N.Y.) **8**, 2155 (1967); M. Zaldarriaga and U. Seljak, Phys. Rev. D **55**, 1830 (1997); D. N. Spergel and D. M. Goldberg, Phys. Rev. D **59**, 103001 (1999); **59**,103002 (1999).