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Shear viscosity to relaxation time ratio in SU(3) lattice gauge theory

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We evaluate the ratio of the shear viscosity to the relaxation time of the shear current above but near the critical temperature T_c in SU(3) gauge theory on the lattice. The ratio is related to Kubo's canonical correlation of the energy-momentum tensor in Euclidean space with the relaxation-time approximation and an appropriate regularization. Using this relation, we evaluate the ratio on the lattice in Euclidean space. We obtain the ratio with reasonable statistics for the range of temperature $1.5 \lesssim T/T_c \lesssim 4$. We find that the characteristic speed of the transverse plane wave propagating in a gluon medium v_η is almost constant, $v_\eta^2 \simeq 0.5$, for $T/T_c \gtrsim 1.5$, which ensures the causality in this mode in second-order dissipative hydrodynamics.

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I. INTRODUCTION

One of the highlights at the Relativistic Heavy Ion Collider (RHIC) is the success of relativistic hydrodynamic models for the description of the space-time evolution of the quark-gluon matter and final-state observables in high-energy heavy ion collisions [1–3]. This revealed that the quark-gluon matter is a strongly correlated system. The LHC has also provided new experimental results of ultra-relativistic heavy ion collisions. As a whole, the transverse-momentum dependence of the elliptic flow observed at the LHC is similar to that at the RHIC [4]. This result implies that the hydrodynamic description of the quark-gluon matter at the LHC is as good as that at the RHIC.

Recently the importance of dissipative effects in the hydrodynamic models has been recognized [5–7]. The simplest relativistic dissipative hydrodynamics is the first-order theory [8,9], which in the nonrelativistic limit reduces to Navier-Stokes theory. The first-order theory, however, is plagued with the problem of acausality and instability [10]. One of the strategies to evade these problems is to extend the theory to the second order with respect to the dissipative currents. In the second-order theory, however, there appear many phenomenological parameters as second-order transport coefficients in addition to the first-order ones. These transport coefficients cannot be determined within the framework of hydrodynamics. *Ab initio* calculations based on microscopic theory, i.e., QCD in our case, are required to constrain the parameters in dissipative hydrodynamic models.

Since the temperature range realized at the RHIC and LHC is not within the reach of perturbative QCD, we need to employ nonperturbative approaches to investigate the transport properties of hot QCD matter created at the RHIC and LHC. At present, lattice QCD simulation is the only systematic way to calculate physical quantities in such a nonperturbative region. There are several pioneering studies that analyzed transport coefficients on the lattice [11–15]. These studies used Green-Kubo formulas, which relate the transport coefficients to the low-energy behavior of the corresponding spectral functions. In this method, one needs to extract the spectral functions from Euclidean correlators obtained in lattice QCD simulations. This step is, however, nontrivial, because it is an ill-posed problem [16].

In this paper, we focus on the ratio of the shear viscosity to the relaxation time of the shear current. In Refs. [17,18], it was argued that the ratio is related to a static fluctuation of the energy-momentum tensor by rewriting the Green-Kubo formula for classical systems with the relaxation-time approximation. We extend their arguments to treat quantum systems and present a relation that relates the ratio to Kubo's canonical correlation of the energy-momentum tensors. The relation enables us to relate transport properties of the medium to Euclidean observables on the lattice directly without analyzing the spectral functions. Whereas the ratio itself is not a transport coefficient, its determination reduces the number of free parameters in hydrodynamic equations.

The canonical correlation of the energy-momentum tensor, which is related to the ratio in this relation, however, is ultraviolet divergent. We thus need to regularize the canonical correlation to obtain a physical quantity. We remove the divergence by a vacuum subtraction. In addition to this prescription, it is argued that one must take care of the contact terms which exist in the canonical correlation in Euclidean space. The canonical correlation is affected by these terms. We will see later that this contribution should be removed in the analysis to obtain the transport property separately from the vacuum subtraction.

The structure of this paper is as follows. In Sec. II we review Israel-Stewart theory [19], which is one of the second-order theories of dissipative hydrodynamics. We show how to introduce the relaxation times of the dissipative currents in the theory. In Sec. III we relate the ratio to Kubo's canonical correlation of the energy-momentum tensors with the relaxation-time approximation for quantum field theory. In Sec. IV we discuss how to remove the unphysical contributions existing in the canonical correlation on the lattice. We then present numerical results of SU(3) lattice gauge simulations in Sec. V. The last section is devoted to conclusions and discussion.

II. ISRAEL-STEWART THEORY AND RELAXATION TIME

In this section, we give a brief review of Israel-Stewart (IS) theory and the role of relaxation times in this theory. Hydrodynamics consists of equations of motion for conserved currents. The only conserved current in pure gauge theory is the energy-momentum tensor $T^{\mu\nu}$, for which the conservation law reads

$$\partial_\mu T^{\mu\nu} = 0. \tag{1}$$

In the dissipative hydrodynamics, the energy-momentum tensor is decomposed as

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)(g^{\mu\nu} - u^\mu u^\nu) + \pi^{\mu\nu}, \tag{2}$$

where ϵ and P are the energy density and the pressure in equilibrium, respectively, and Π and $\pi^{\mu\nu}$ are the dissipative currents of bulk and shear channels, respectively. The metric $g^{\mu\nu}$ is defined by $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$. Since pure gauge theory does not have conserved currents except for $T^{\mu\nu}$, the natural choice of the flow vector u^μ is that in the energy frame [9], where u^μ is proportional to the energy flow with the normalization $u_\mu u^\mu = 1$.

In second-order theory, Π and $\pi^{\mu\nu}$ are regarded as dynamical variables, and additional equations of motion for these variables are introduced besides Eq. (1). One of the arguments to obtain these equations is a phenomenological one proposed in Ref. [19]. In this derivation, the evolution equations are obtained as a consequence of the second law of thermodynamics [19–21],

$$\partial_\mu s^\mu \geq 0, \tag{3}$$

where $s^\mu = s u^\mu$, with s being the entropy density in the local rest frame. By choosing the entropy density to be the equilibrium one, $s = s_{\mathrm{eq}}$, and assuming linearity in the relation between the dissipative currents and the thermodynamic forces, we obtain the first-order theory with Eq. (3). To extend the theory to second order, it is assumed that the entropy density near equilibrium is given by a function of Π and $\pi^{\mu\nu}$, and that it is analytic near equilibrium, $\Pi = \pi^{\mu\nu} = 0$ [19]. The entropy density near equilibrium is then given by the Taylor expansion with respect to Π and $\pi^{\mu\nu}$. Up to the second order, it reads

$$s \equiv s_{\mathrm{eq}} - \frac{1}{2T}(\beta_0 \Pi^2 + \beta_2 \pi_{\mu\nu}\pi^{\mu\nu}), \tag{4}$$

where β_0 and β_2 are phenomenological parameters. The second law of thermodynamics, Eq. (3), with Eq. (4) leads to a set of evolution equations, which are referred to as IS equations. In the shear channel, for example, the evolution equation reads

$$\dot{\pi}_{\mu\nu} = -\frac{1}{\tau_\pi}[\pi_{\mu\nu} + \alpha], \tag{5}$$

where $\dot{\pi}_{\mu\nu} = u^\rho \partial_\rho \pi_{\mu\nu}$ is the derivative along the temporal direction in the local rest frame and α denotes contributions that vanish in a uniform medium.

The evolution equation (5) implies that the temporal correlator of the spatial average of the shear current,

$$\bar{\pi}_{\mu\nu}(t) = \frac{1}{V}\int_V d^3x \pi_{\mu\nu}(t, \vec{x}), \tag{6}$$

behaves as

$$\langle \bar{\pi}_{\mu\nu}(t)\bar{\pi}_{\mu\nu}(0)\rangle \simeq e^{-t/\tau_\pi}\langle \bar{\pi}_{\mu\nu}(0)\bar{\pi}_{\mu\nu}(0)\rangle, \tag{7}$$

where V is the volume of the system. Since $\pi_{\mu\nu}$ damps with a time scale τ_π in a uniform medium, τ_π is called the relaxation time of the shear current. τ_π is related to β_2 in Eq. (4) as

$$\tau_\pi = 2\eta\beta_2, \tag{8}$$

where η is the shear viscosity [19]. The corresponding first-order equation is obtained by taking the limit $\tau_\pi \to 0$ in Eq. (5). Similarly the evolution equation in the bulk channel includes the relaxation time of the bulk current τ_Π, which is related to β_0 in Eq. (4) as $\tau_\Pi = \zeta\beta_0$ with the bulk viscosity ζ. We note that $\bar{\pi}_{\mu\nu}(t)$ in Eq. (7) should be regarded as a classical variable in hydrodynamical equations. We will extend this equation to quantum correlators in Sec. III.

The second-order theory can become causal because the relaxation times make the hydrodynamic equations hyperbolic. The wave-front speed (characteristic speed) of the transverse plane wave v_η in the IS theory is given by [21]

$$v_\eta^2 = \frac{\eta}{\tau_\pi(\epsilon + P)} = \frac{\eta}{\tau_\pi T s}. \tag{9}$$

For causality to be satisfied, the causality condition $v_\eta^2 \leq 1$ is required.

III. RATIO OF SHEAR VISCOSITY TO RELAXATION TIME

A. Relaxation-time approximation

We now relate the ratio η/τ_π to Kubo's canonical correlation of $T_{\mu\nu}$ in Euclidean space. The original idea of the following discussion was proposed within the framework of classical theory in Refs. [17,18]. We modify their arguments to treat quantum systems.

We start from the Green-Kubo formula for the shear viscosity,

$$\eta = \frac{V}{\hbar}\int_0^\infty dt \int_0^{\hbar/T} d\lambda \langle \bar{\pi}_{12}(-i\lambda)\bar{\pi}_{12}(t)\rangle, \tag{10}$$

where $\langle O\rangle = \mathrm{Tr}[e^{-H/T}O]/\mathrm{Tr}[e^{-H/T}]$ is the statistical average in equilibrium, with H being the Hamiltonian. Here we set Boltzmann's constant $k_\mathrm{B} = 1$. In order to modify this formula as a feasible one for lattice simulations, we assume that, in analogy with Eq. (7), the correlator of $\bar{\pi}_{12}$ damps exponentially with real time,

$$\int_0^\infty dt \int_0^{\hbar/T} d\lambda \langle \bar{\pi}_{12}(-i\lambda)\bar{\pi}_{12}(t)\rangle \simeq \int_0^\infty dt \int_0^{\hbar/T} d\lambda e^{-t/\tau_\pi} \langle \bar{\pi}_{12}(-i\lambda)\bar{\pi}_{12}(0)\rangle_{\mathrm{reg}}, \tag{11}$$

for $t > 0$. On the right-hand side of Eq. (11), we labeled the statistical average with a subscript "reg," since the λ integral of the Euclidean correlator $\langle \bar{\pi}_{12}(-i\lambda)\bar{\pi}_{12}(0)\rangle$ is ultraviolet divergent and ill-defined as we will see later. In order to make Eq. (11) meaningful, therefore, an appropriate regularization of the right-hand side has to be introduced. Moreover, $\langle \bar{\pi}_{12}(-i\lambda)\bar{\pi}_{12}(0)\rangle$ has a delta function at $\lambda = 0$, i.e., the contact term. In the next section we show that the contribution of this term must be removed from the Euclidean correlator. We postpone the discussion on the details of these issues to the next section, and for the moment assume that the right-hand side of Eq. (11) can be defined appropriately.

In this study, we refer to Eq. (11) as the relaxation-time approximation, owing to its analogy to the standard approximation in nonequilibrium statistical mechanics, $\langle O(t)O(0)\rangle \simeq e^{-t/\tau}\langle O(0)^2\rangle$. Note, however, that in Eq. (11) the correlator between operators separated by complex time is involved, while the standard relaxation-time approximation is for real-time separation. Since the correlator in Eq. (11) should reduce to Eq. (7) for the range of t where the second-order hydrodynamics is applicable, the time scale appearing in the exponential function in Eq. (11) is identified as the relaxation time τ_π in Eq. (5).

With Eqs. (10) and (11), we obtain

$$\begin{aligned}\eta &\simeq \frac{V}{\hbar}\int_0^\infty dt e^{-t/\tau_\pi} \int_0^{\hbar/T} d\lambda \langle \bar{\pi}_{12}(-i\lambda)\bar{\pi}_{12}(0)\rangle_{\mathrm{reg}} \\ &= \frac{V}{\hbar}\tau_\pi \int_0^{\hbar/T} d\lambda \langle \bar{\pi}_{12}(-i\lambda)\bar{\pi}_{12}(0)\rangle_{\mathrm{reg}}.\end{aligned} \tag{12}$$

In the local rest frame of the medium, one can replace $\bar{\pi}_{12}$ by the spatial component of the energy-momentum tensor $\bar{T}_{12}$. Equation (12) then means that the ratio η/τ_π is given by

$$\begin{aligned}\frac{\eta}{\tau_\pi} &= \frac{V}{T}\langle \tilde{T}_{12}^2\rangle_{\mathrm{reg}} \\ &\equiv \frac{V}{\hbar}\int_0^{\hbar/T} d\lambda \langle \bar{T}_{12}(-i\lambda)\bar{T}_{12}(0)\rangle_{\mathrm{reg}}.\end{aligned} \tag{13}$$

Here $\langle \tilde{A}^2\rangle$ denotes the imaginary time average of Kubo's canonical correlation for the spatial averages of an operator A,

$$\langle \tilde{A}^2\rangle \equiv \frac{T}{\hbar}\int_0^{\hbar/T} d\lambda \langle \bar{A}(\lambda)\bar{A}(0)\rangle. \tag{14}$$

Equation (13) shows that the shear viscosity to relaxation time ratio is related to Kubo's canonical correlation of $\bar{T}_{12}$ with an appropriate regularization. The same argument is applicable to the bulk channel, which leads to a formula to represent the ratio ζ/τ_Π with diagonal components of $T_{\mu\nu}$. We note that Eq. (13) is also derived by the projection-operator method [22].

B. Classical limit

In the classical limit $\hbar/T \to 0$, one can replace the integral $\int_0^{\hbar/T} d\lambda$ with $\hbar/T$ in Eq. (12). Equation (13) then reduces to

$$\frac{\eta}{\tau_\pi} = \frac{V}{T}\langle \bar{T}_{12}^2\rangle. \tag{15}$$

η/τ_π is, therefore, given by the static fluctuation of $\bar{T}_{12}$ in the classical limit. Equation (15) is the formula derived in Refs. [17,18]. Here we understand that appropriate subtractions, which will be described below, are carried out in Eq. (15).

In classical systems, Eq. (15) is also derived by using Einstein's principle and the entropy density in the IS theory (4) [17,18]. According to Einstein's principle, the probability distribution $P(x)$ of a state variable x in equilibrium is given by $P(x) \sim e^{S(x)}$, where $S(x)$ is the nonequilibrium entropy in a volume V. If one identifies the IS entropy (4) to be that in Einstein's principle, the probability distribution for a state variable $\bar{\pi}_{12}$ in equilibrium in V is given by

$$P_\pi(\bar{\pi}_{12}) \sim \exp\left[-\frac{V}{T}\beta_2\bar{\pi}_{12}^2\right]. \tag{16}$$

With this distribution function, the fluctuation of $\bar{\pi}_{12}$ in the rest frame of the medium is calculated to be

$$\langle \bar{\pi}_{12}^2\rangle = \frac{\int d\bar{\pi}_{12}\bar{\pi}_{12}^2 P_\pi(\bar{\pi}_{12})}{\int d\bar{\pi}_{12} P_\pi(\bar{\pi}_{12})} = \frac{T}{2\beta_2 V} = \frac{\eta T}{\tau_\pi V}, \tag{17}$$

which is identical to Eq. (15) with $\bar{\pi}_{12} = \bar{T}_{12}$.

It should, however, be noted that Einstein's principle is not applicable to systems where quantum fluctuations of the state variable x are not negligible [23]. In quantum mechanics, when a state variable x does not commute with the Hamiltonian, x and the energy cannot be determined simultaneously. This means that the entropy cannot be defined as a function of nonconserving state variables in quantum systems. Einstein's principle therefore can be utilized only when the uncertainty to determine x together with the total energy is sufficiently small.

IV. REGULARIZATION

A. Operator product expansion

In this section, we discuss the regularization introduced in Eq. (11). From now on we set $\hbar = 1$. As already mentioned in the last section, the canonical correlation $\langle \tilde{T}_{12}^2 \rangle_{\rm reg}$ in Eq. (13) is not identical to the naive definition

$$\langle \tilde{T}_{12}^2 \rangle = \frac{T}{V} \int d^4x \langle T_{12}(x) T_{12}(0) \rangle, \tag{18}$$

with $x = (-i\lambda, \vec{x})$, for two reasons. First, the integral in Eq. (18) is ultraviolet divergent because of the short-distance behavior of the Euclidean correlator of T_{12}. Second, while the Euclidean correlator includes contact terms, which are proportional to the δ function, this term should be subtracted in Eq. (11), as discussed below. Since both of these effects are related to the short-distance behavior of the Euclidean correlator $\langle T_{12}(x) T_{12}(0) \rangle$, we take advantage of the operator product expansion (OPE) to investigate them.

The product of T_{12} at short distance behaves as

$$T_{12}(x) T_{12}(0) \simeq C_T \frac{1}{x^8} + C_{\mu\nu} T_{\mu\nu}(0) \delta^{(4)}(x) + \cdots, \tag{19}$$

where C_T and $C_{\mu\nu}$ are c-number Wilson coefficients, which are determined perturbatively. For dimensional reasons, the first term on the right-hand side of Eq. (19) is proportional to x^{-8} and leads to the ultraviolet divergence of Eq. (18) unless $C_T = 0$. This term, however, does not have medium effects and it does not affect the long-distance behavior responsible for hydrodynamics. Therefore the divergence is not related to the hydrodynamical behavior of the matter and should be removed from Eq. (18) when discussing its transport properties. This removal is simply accomplished by the vacuum subtraction, defined by the difference between the correlators at finite temperature and zero temperature,

$$\int d^4x \langle T_{12}(x) T_{12}(0) \rangle_0 \equiv \int d^4x \langle T_{12}(x) T_{12}(0) \rangle_T - \int d^4x \langle T_{12}(x) T_{12}(0) \rangle_{T=0}. \tag{20}$$

The next terms of the OPE in Eq. (19) are proportional to dimension-four operators, which are components of $T_{\mu\nu}$ in pure gauge theory. These terms are the contact terms, which are proportional to $\delta^{(4)}(x)$ and shown in Eq. (19). As discussed in the next subsection, the integral in the Green-Kubo formula (10) does not contain the contribution from the contact terms. The contribution thus should also be subtracted in Eq. (13), which is derived from the Green-Kubo formula. Note that the contact terms cannot be removed by the vacuum subtraction, because the statistical average of the contact terms is proportional to $\langle T_{\mu\nu} \rangle$, which depends on the temperature. By taking these effects into account, one finally finds that the canonical correlation in Eq. (13) is given by

$$\int d^4x \langle T_{12}(x) T_{12}(0) \rangle_{\rm reg} \equiv \int d^4x \langle T_{12}(x) T_{12}(0) \rangle_0 - (\text{contact terms}). \tag{21}$$

For the dimensional reason, the terms with a higher dimensional operator than four in Eq. (19) (shown by the dots) do not yield an ultraviolet divergence and are not singular.

The coefficients of contact terms are most conveniently evaluated by taking the high-frequency limit of the correlator with zero spatial momentum and by making use of the Riemann-Lebesgue lemma. In the leading-order OPE result [24–26], they are given by

$$\begin{aligned} C_{\mu\nu} \langle T_{\mu\nu} \rangle_0 &= G_0(q_4 \to \infty, \vec{q} = \vec{0}) \\ &= \frac{1}{2} \langle T_{44} - T_{33} \rangle_0 + \frac{1}{6} \langle F^2 \rangle_0, \end{aligned} \tag{22}$$

where $G_0(q_4, \vec{q}) = \int d^4x e^{i(q_4\tau - \vec{q}\cdot\vec{x})} \langle T_{12}(x) T_{12}(0) \rangle_0$ and $\langle F^2 \rangle = \langle F^a_{\mu\nu} F^a_{\mu\nu} \rangle$ is the gluon condensate.

From Eqs. (21) and (22), we obtain the ratio

$$\begin{aligned} \frac{\eta}{\tau_\pi} &= \int d^4x \langle T_{12}(x) T_{12}(0) \rangle_0 - C_{\mu\nu} \langle T_{\mu\nu} \rangle_0 \\ &= \frac{V}{T} \langle \tilde{T}_{12}^2 \rangle_0 + \frac{1}{2}(\epsilon + P) - \frac{1}{6} \langle F^2 \rangle_0. \end{aligned} \tag{23}$$

The right-hand side of Eq. (23) consists of the canonical correlation of $T_{\mu\nu}$ and the thermodynamic quantities, both of which can be calculated on the lattice.

B. Contact terms in the Green-Kubo formula

We investigate here the contribution of the contact terms to the Green-Kubo formula (10), and show that these terms do not contribute to the integral in Eq. (10). From Eq. (19), the sum of the contact terms in Euclidean space is given by

$$g(-i\lambda, \vec{x}) \equiv C_{\mu\nu} \langle T_{\mu\nu} \rangle_0 \delta^{(4)}(x). \tag{24}$$

To evaluate the contribution of the contact terms to Eq. (10), we perform the analytic continuation $\lambda \to \lambda + it$

in Eq. (24). In order to obtain the analytic continuation of the δ function to complex arguments, we utilize the Poisson equation in four-dimensional space,

$$\delta^{(4)}(x) = -\frac{1}{2\pi^2}\partial^2 \frac{1}{x^2}, \quad (25)$$

where $\partial^2 = \partial_\lambda^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$ and $x^2 = \lambda^2 + x_1^2 + x_2^2 + x_3^2$. The contact term is then rewritten as

$$g(-i\lambda, \vec{x}) = -\frac{1}{2\pi^2} C_{\mu\nu}\langle T_{\mu\nu}\rangle_0 \partial^2 \frac{1}{x^2+\epsilon}, \quad (26)$$

where we added an infinitesimal quantity $\epsilon > 0$ to regularize the singularity at $x^2 = 0$. Before the analytic continuation, we carry out the differentiation and the spatial integration of $1/(x^2+\epsilon)$,

$$\int d^3x \partial^2 \frac{1}{x^2+\epsilon} = -\frac{2\pi^2\epsilon}{(\lambda^2+\epsilon)^{3/2}}, \quad (27)$$

where we have used the formula

$$\int dr \frac{r^2}{(r^2+c)^3} = -\frac{r}{4(r^2+c)^2} + \frac{r}{8c(r^2+c)} + \frac{1}{8c}\int \frac{dr}{r^2+c}. \quad (28)$$

Performing the analytic continuation $\lambda \to \lambda + it$, the contact term becomes

$$g(t - i\lambda; \epsilon) = C_{\mu\nu}\langle T_{\mu\nu}\rangle_0 \frac{\epsilon}{\{(\lambda+it)^2+\epsilon\}^{3/2}}. \quad (29)$$

To evaluate the contribution of this term to the Green-Kubo formula, we perform the temporal integrals of $g(t-i\lambda;\epsilon)$,

$$\left|\int_0^\infty dt \int_0^{\frac{1}{T}} d\lambda g(t-i\lambda;\epsilon)\right| = \left|C_{\mu\nu}\langle T_{\mu\nu}\rangle_0 \int_0^\infty dt \int_{0+it}^{\frac{1}{T}+it} dz \frac{\epsilon}{(z^2+\epsilon)^{3/2}}\right| \to 0 \quad (\epsilon \to 0), \quad (30)$$

with $z = \lambda + it$.

In this way, one concludes that the contact terms (24) do not contribute to the Green-Kubo formula, and as a result they do not affect the ratio η/τ_π. On the other hand, the canonical correlation (18) measured on the lattice involves their contribution. Therefore the contribution should be removed as in Eq. (23) in numerical analyses.

V. LATTICE MEASUREMENTS

A. Formulation on the lattice

In this section, we analyze the ratio η/τ_π on the lattice by evaluating the three observables on the right-hand side of Eq. (23).

We have performed the lattice simulations for SU(3) pure gauge theory with the standard Wilson gauge action. In Table I we list the lattice parameters used in the present study. To investigate the lattice spacing and spatial volume dependencies, simulations have been carried out on four isotropic lattices of spatial volume N_σ^3. The lattice spacing, a, in Table I for each β is determined by the string tension $\sqrt{\sigma} = 460$ MeV and the parametrization of $a\sqrt{\sigma}$ in Ref. [27]. All observables discussed in the next subsection, however, are dimensionless and do not depend on this physical scale. We use $T_c/\sqrt{\sigma} = 0.63$ [28,29] and normalize T by T_c. The temporal length $aN_\tau = 1/T$ of each lattice corresponds to the range of temperature $0.5 \lesssim T/T_c \lesssim 4$, which covers those realized in heavy ion collisions at the RHIC and LHC. Gauge configurations were updated by the heatbath and overrelaxation algorithms. We have generated 300 000–2000 000 configurations for each parameter. Statistical errors were estimated by the jackknife method with bin sizes up to 1000. To perform the vacuum subtraction, we regarded the lattice with the largest N_τ for each β to be the vacuum one and subtracted the expectation value on this lattice from the one at each N_τ. In fact, the lattice with the largest N_τ for each β corresponds to a temperature well below T_c, where medium effects on the expectation value are well suppressed.

To calculate the first and second terms in Eq. (23), we employed the traceless definition of the energy-momentum tensor,

$$T_{\mu\nu} = 2\,\mathrm{Tr}\left[F_{\mu\rho}F_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}F_{\rho\sigma}F_{\rho\sigma}\right], \quad (31)$$

with the field strength $F_{\mu\nu}$ on the lattice defined by the clover type plaquette. This definition conforms to the parity of the spatial coordinate. In non-Abelian gauge theory, the diagonal components of $T_{\mu\nu}$ are modified due to the trace anomaly. Since the first term in Eq. (23) consists of only an off-diagonal component of $T_{\mu\nu}$, the definition Eq. (31) does

TABLE I. Simulation parameters. N_σ and N_τ are the numbers of the lattice sites in the spatial and temporal directions, respectively. a and L_σ are the lattice spacing and spatial lattice size, respectively.

$\beta = 6/g^2$	N_σ	N_τ	a [fm]	L_σ [fm]
6.499	32	4, 6, 8, 32	0.046	1.5
6.205	32	4, 6, 8, 32	0.068	2.2
6.000	32	4, 6, 8, 16	0.094	3.0
6.000	16	4, 6, 8, 16	0.094	1.5

not affect the validity of the result for this term. The trace anomaly also cancels out in the calculation of the second term between ε and P.

On the lattice, due to the discretized translation symmetry the energy-momentum tensor receives a renormalization factor Z compared to the operator in the continuum limit, $T^{(\mathrm{cont})}_{\mu\nu} = ZT^{(\mathrm{lat})}_{\mu\nu}$. Assuming that the finite lattice spacing effects in $\langle T^{(\mathrm{lat})}_{\mu\nu}\rangle_0$ are expressed solely in this form with a β-dependent renormalization factor $Z(\beta)$, it is determined so as to satisfy the thermodynamic relation [30]. For the traceless part of the energy-momentum tensor (31), this procedure is nicely accomplished with the relation $\epsilon + P = Z(\beta)(\langle T^{(\mathrm{lat})}_{33}\rangle_0 - \langle T^{(\mathrm{lat})}_{44}\rangle_0)$, where the left-hand side is the thermodynamic quantity after the continuum extrapolation [29,31]. To determine Z for each β, we used $N_\tau = 6$ data as a representative case. The operator on the lattice can also receive additive renormalization, $T^{(\mathrm{cont})}_{\mu\nu} = Z(\beta)T^{(\mathrm{lat})}_{\mu\nu} + C(\beta)\delta_{\mu\nu}$, where the tensor structure of the second term is constrained by Euclidean symmetry. The existence of C, however, does not affect $\langle \tilde{T}^2_{12}\rangle_0$ and $\epsilon + P$ in Eq. (23) because the former does not depend on C and in the latter C is canceled out.

The gluon condensate $\langle F^2\rangle_0$ in Eq. (23) has been measured from the gauge action using the relation

$$\frac{\langle F^2\rangle_0}{T^4} = N_\tau^4 T\frac{d\beta}{dT}S_0^{\mathrm{G}} = -6N_\tau^4 a\frac{dg^{-2}}{da}S_0^{\mathrm{G}}, \qquad (32)$$

where S_0^{G} is the lattice gauge action constructed from the standard plaquette. For the values of adg^{-2}/da, we have used those determined in Ref. [29].

B. Numerical results

In Fig. 1, we show the temperature dependence of each term on the right-hand side of Eq. (23), $\langle \tilde{T}^2_{12}\rangle_0$, $(\epsilon + P)/2$, and $\langle F^2\rangle_0/6$. In these plots, we show the results without the correction of the renormalization factor $Z(\beta)$ discussed in the previous subsection. Figure 1(a) shows that the canonical correlation $\langle \tilde{T}^2_{12}\rangle_0$ takes negative values for $\beta = 6.000$ at all T values analyzed. On the finer lattices with $\beta = 6.499$ and 6.205, the negative values are also observed for $N_\tau = 4$, which corresponds to the highest temperature for each β. For $N_\tau \geq 6$, the statistical error is large and the expectation values are consistent with zero within statistics.

The thermodynamic quantities presented in the lower two panels of Fig. 1 are determined with good statistics. We have checked that the numerical results in Fig. 1(c) are consistent with the previous analyses [29,31]. The finite spatial volume effect on these results is estimated by comparing numerical results on the two lattices with different spatial dimensions L_σ with the same $\beta = 6.000$. Figure 1 shows that the spatial volume dependence is almost negligible except in the vicinity of T_c. In the middle

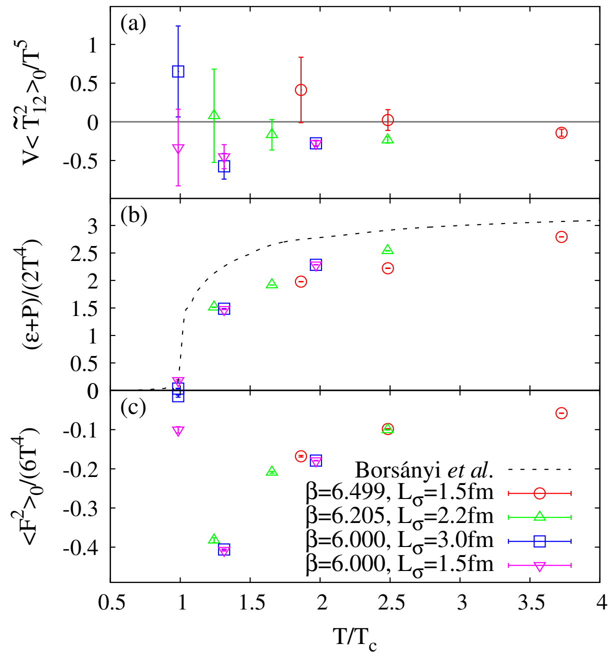

FIG. 1 (color online). Temperature dependencies of three terms in Eq. (23). (a) Kubo's canonical correlation with the vacuum subtraction $\langle \tilde{T}^2_{12}\rangle_0$. (b) The second term, $(\epsilon + P)/2 = (\langle T_{33}\rangle_0 - \langle T_{44}\rangle_0)/2$. The results qualitatively reproduce the result of the QCD equation of state in the literature. (c) The third term, the gluon condensate $\langle F^2\rangle_0/6$ determined by the conventional approach with the standard plaquette.

panel of Fig. 1, the value of $(\epsilon + P)/2T^4$ evaluated with the thermodynamic relation after the continuum extrapolation [31] is also shown. The figure shows that our result slightly underestimates the thermodynamic quantity. This difference can be absorbed by introducing the renormalization factor $Z(\beta)$ discussed above, which is taken into account in the following numerical analysis. Our numerical result shows that the value of $Z(\beta)$ is in the range $1.3 \lesssim Z(\beta) \lesssim 1.5$ for the values of β we used.

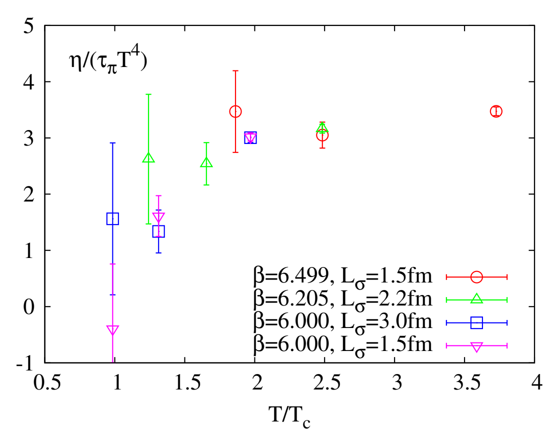

FIG. 2 (color online). Temperature dependence of the shear viscosity to relaxation time ratio η/τ_π. The contributions from vacuum and contact terms are subtracted at each temperature.

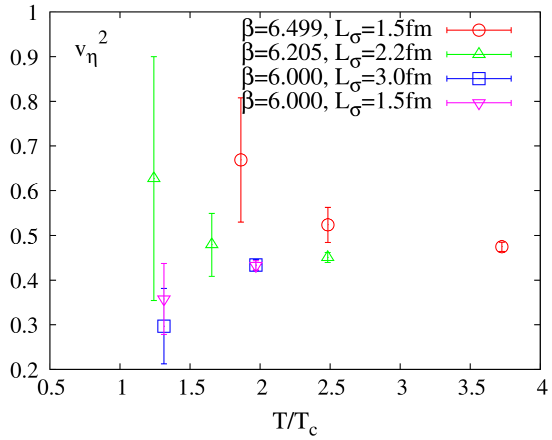

FIG. 3 (color online). Temperature dependence of the square of the characteristic speed $v_\eta^2 = \eta/\{\tau_\pi(\epsilon + P)\}$. The results show that $v_\eta^2 \simeq 0.5$ for $1.5 \lesssim T/T_c \lesssim 4$.

In Fig. 2, we show the temperature dependence of the ratio η/τ_π given by Eq. (23). From the figure, one sees that the ratio η/τ_π shows a rapid increase in the vicinity of T_c. The results in Fig. 1 show that this behavior of η/τ_π is dominated by the second term in Eq. (23). Figure 2 also shows that the results of η/τ_π with all sets of configurations coincide within the statistical error, which indicates that the lattice spacing dependence is well suppressed in the present result. Comparing the two results with $\beta = 6.000$, we observe that the spatial volume dependence is also small. In Fig. 3, we show the temperature dependence of v_η^2 determined by Eq. (9). While the effect of $Z(\beta)$ is taken into account in this plot, the effect is smaller than the statistical errors. The figure shows that v_η^2 takes almost constant values around $v_\eta^2 \simeq 0.5$, for $1.5 \lesssim T/T_c \lesssim 4$. This result indicates that the causality condition is satisfied in the second-order hydrodynamics for this mode in this range of the temperature. The statistical error, however, increases as the temperature approaches T_c from above.

It is of interest to compare the present result on v_η^2 with those obtained in various models. Grad's 14-moment approximation predicts $v_\eta^2 \simeq 1/6$ in the high-energy limit [19], and the AdS/CFT correspondence $v_\eta^2 = 1/2(2 - \ln 2) \simeq 0.36$ [32,33]. The projection-operator method for a pion gas provides $v_\eta^2 = P/(\epsilon + P) \simeq 0.25$ near T_c [22]. The measured values of v_η^2 in our lattice gauge simulation are higher than any of the other theoretical predictions.

Finally, let us consider the implication of our results on the magnitude of the relaxation time τ_π. It has been conjectured that the shear viscosity to entropy density ratio η/s has the universal lower bound $1/4\pi$, which is the value obtained by the AdS/CFT correspondence [34]. On the other hand, it has been suggested by the comparison between the results of viscous hydrodynamic models and the experiments at the RHIC and LHC that η/s is less than $\sim 5\times$ the lower bound above T_c [5,6]. If one substitutes these values in Eq (9) and uses $v_\eta^2 \simeq 0.5$, the relaxation time τ_π is roughly estimated to be $\tau_\pi \simeq \mathcal{O}(10^{-2}) - \mathcal{O}(10^{-1})$ fm in the temperature range. It is notable that this value of τ_π is significantly shorter than the typical timescale in relativistic heavy ion collisions [1–3]. Introducing τ_π into hydrodynamic simulations would lead to little effect on the result. We also note that the value of τ_π estimated above is almost ten times smaller than the result for τ_π in the pion gas below T_c [21]. This is another indication that the deconfined phase above T_c is a strongly interacting system.

VI. CONCLUSION

In this paper, we evaluated the ratio of the shear viscosity to the relaxation time of the shear current, η/τ_π, for the range of temperature $0.5 \lesssim T/T_c \lesssim 4$ with a SU(3) lattice gauge simulation. The ratio is related to the canonical correlation of T_{12}. The canonical correlation, however, is ultraviolet divergent. By studying the short-distance behavior with the operator product expansion, we found that the canonical correlation receives contributions from the vacuum and temperature-dependent contact terms in Euclidean space and that they need to be subtracted. Using the relation, we obtained the temperature dependence of η/τ_π in SU(3) lattice gauge theory. Thus, the number of the free parameters in relativistic dissipative hydrodynamic models has been reduced by the first-principle calculation for the first time in this analysis.

We also investigated the temperature dependence of the characteristic speed of the transverse mode v_η^2 in the same temperature range. We found that v_η^2 in the hot gluonic medium takes $v_\eta^2 \simeq 0.5$ at $T/T_c \gtrsim 1.5$. This result indicates that there is no causality problem in the propagation of the transverse mode in the Israel-Stewart theory of second-order hydrodynamics.

The same analysis performed in this work is applicable to the bulk channel. By repeating the same procedure as in Secs. III and IV, one obtains the ratio of the bulk viscosity to the corresponding relaxation time of the bulk current ζ/τ_Π from Kubo's canonical correlation of the diagonal components of $T_{\mu\nu}$. This study is currently in progress.

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