Vacuum alignment and lattice artifacts: Wilson fermions

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Confinement in asymptotically free gauge theories is accompanied by the spontaneous breaking of the global flavor symmetry. If a subgroup of the flavor symmetry group is coupled weakly to additional gauge fields, the vacuum state tends to align such that the gauged subgroup is unbroken. Independently, a lattice discretization of the continuum theory typically reduces the manifest flavor symmetry, and, in addition, can give rise to new lattice-artifact phases with spontaneously broken symmetries. Here, we study the interplay of these two phenomena for Wilson fermions, using chiral Lagrangian techniques. We consider two examples: electromagnetic corrections to QCD, and a prototype composite Higgs model.

I. INTRODUCTION

Over the last few years, the computation of certain hadronic quantities using lattice QCD has become so accurate that electromagnetic effects, while typically small, need to be included in order to further improve on present errors [1]. A further reduction of the lattice spacing is also needed in order to suppress competing discretization effects.

There may indeed be a real competition between electromagnetic effects and lattice artifacts: both can have a nontrivial influence on the phase diagram of the lattice theory. First, given a strongly interacting gauge theory, let us weakly couple a subgroup of the flavor symmetry to dynamical gauge fields ("weak gauge fields," for short). It was observed long ago that the weak gauge fields can influence the symmetry-breaking pattern. Their coupling to unbroken flavor generators tends to stabilize the vacuum, whereas the coupling to broken generators tend to destabilize it, a phenomenon usually referred to as "vacuum alignment" [2]. Furthermore, depending on the resulting alignment, some of the Nambu-Goldstone bosons (NGBs) associated with the symmetry breaking may acquire a mass, thereby becoming pseudo-NGBs. An example is the QED-induced mass splitting between the charged and neutral pions in QCD.

Even without weak gauge fields, a nontrivial phase structure can also emerge at nonzero lattice spacing. An example is the possible appearance of a so-called Aoki phase in two-flavor QCD with Wilson fermions. Depending on the details of the regularization, a phase can appear in which isospin is spontaneously broken to a U(1) subgroup, along with parity [3–5].

It is interesting to study what happens when both effects are at work. For instance, in lattice QCD with two degenerate Wilson fermions, what would happen to the Aoki phase if QED is turned on, or if all the isospin generators are coupled to weak gauge fields?

Similar questions arise beyond the realm of QCD. The existence of a light Higgs particle has revived interest in composite Higgs models, in which a strongly coupled gauge theory breaks its flavor symmetry dynamically at the TeV scale, producing a massless meson with the quantum numbers of the Higgs among the NGBs associated with the breaking of the symmetry. The flavor currents of this strongly interacting theory can be coupled to a number of weak gauge fields, with the Standard Model's electroweak gauge fields among them. Electroweak symmetry breaking is then arranged to take place through the effective potential generated for the NGBs of the strongly coupled theory by the weak dynamics. A prototype example of such a theory is the "littlest Higgs" model of Ref. [6]. In this theory the flavor symmetry group is SU(5), spontaneously broken by the strong dynamics to SO(5). Weak gauge fields are coupled to an $[SU(2) \times U(1)]^2$ subgroup of SU(5), with the Standard Model's electroweak gauge fields coupling to the diagonal subgroup of $[SU(2) \times U(1)]^2$, which is also a subgroup of the unbroken SO(5).

A basic tool used in the phenomenological literature is the nonlinear sigma model describing the (pseudo-) NGBs (for recent reviews, see Refs. [7,8]). Such a low-energy effective theory requires an "ultraviolet completion." In many cases, the underlying theory can be taken to be a confining gauge theory, which, in turn, can be studied on the lattice. One can then use numerical methods in order to determine the low-energy constants (LECs) relevant for electroweak physics. Since not only the precise values of the LECs, but even their signs are usually outside the scope of the nonlinear sigma model, their determination is crucial if we are to confirm that the correct symmetry-breaking pattern indeed takes place.[1] Again, the question arises

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[1]For realistic studies of the phenomenology of such models, the top-quark sector should also be taken into account.

whether lattice artifacts might have an effect on the phase structure, possibly distorting the alignment properties of the continuum theory.

In this article, we consider these questions in the context of strongly coupled lattice gauge theories with Wilson fermions. The use of Wilson fermions means that axial symmetries are explicitly broken by the discretization, and they are recovered only in the continuum limit. In the two-flavor theory, this leads to the practical limitation that weak dynamical gauge fields can be coupled to isospin generators only, and not to the axial generators.

In order to realize the $SU(5)/SO(5)$ nonlinear sigma model we envisage a confining theory with five Weyl fermions in a real representation of the strong gauge group [2]. In the continuum, this strongly interacting theory can equivalently be formulated in terms of Majorana fermions. Transcribing the latter theory to the lattice is straightforward. But, once again, if we use Wilson fermions, only the $SO(5)$ flavor symmetry is preserved, because it is vectorial in the Majorana formulation. The remaining symmetries [which generate the coset $SU(5)/SO(5)$] are axial. They are explicitly broken by the Wilson mass term, again to be recovered only in the continuum limit. On the lattice we thus consider only dynamical weak gauge fields for subgroups of $SO(5)$. As we will see, this is sufficient to gain access to LECs of the low-energy effective theory that are of interest to phenomenology.

In Sec. II we will consider two-flavor QCD with Wilson fermions, and investigate what happens if we gauge all isospin generators, or if we gauge only the $U(1)$ subgroup for the I_3 component of the photon. We will consider the lowest-order pion effective potential, containing terms linear in the quark mass, quadratic in the lattice spacing, and linear in the fine-structure constant, assuming that these are all of a comparable magnitude. In Sec. III we will then consider the $SU(5)/SO(5)$ nonlinear sigma model, with the weak gauge fields being those of the Standard Model group $SU(2)_L \times U(1)_Y$, in a similar framework. Because of the more complicated form of the effective potential, we will not be able to fully explore the phase diagram that may arise from discretization effects. However, a quadratic fluctuation analysis around the vacuum of the continuum theory will still lead to nontrivial observations. The final section contains our conclusions, and a proof of vacuum alignment for the continuum $SU(5)/SO(5)$ theory is relegated to an Appendix.

II. TWO-FLAVOR QCD WITH WILSON FERMIONS

Following Ref. [5], we start from the effective potential for the pions of two-flavor lattice QCD with Wilson fermions,[2]

$$V_{\text{eff}} = -\frac{c_1}{4} \text{tr}(\Sigma + \Sigma^\dagger) + \frac{c_2}{16} (\text{tr}(\Sigma + \Sigma^\dagger))^2$$
$$= -c_1 \sigma + c_2 \sigma^2, \tag{2.1}$$

in which

$$\Sigma = \sigma + i \sum_a \tau_a \pi_a, \qquad \sigma^2 + \sum_a \pi_a^2 = 1 \tag{2.2}$$

is the nonlinear $SU(2)$ matrix built out of the isospin triplet of pion fields π_a, with τ_a the three Pauli matrices. The parameter c_1 is linear in the partially conserved axial current quark mass m, while c_2 is proportional to the square of the lattice spacing a.[3] Higher-order terms in the chiral expansion of V_{eff} will be neglected, since they do not qualitatively affect the phase diagram (unless at least one of the leading-order terms vanishes).

In the continuum limit, $c_2 = 0$, and there is a first-order phase transition when c_1, i.e., the quark mass m, changes sign: the condensate $\Sigma_0 = \langle \Sigma \rangle$ realigns from $\Sigma_0 = +1$ for $c_1 > 0$ to $\Sigma_0 = -1$ for $c_1 < 0$. At nonzero lattice spacing, this conclusion does not change if $c_2 < 0$, because the c_2 term in V_{eff} is minimized for $\Sigma_0 = \pm 1$, irrespective of the sign of Σ_0.[4] Compared to the continuum theory, the difference is that for $c_2 < 0$ the pion masses do not vanish at the transition; instead, they are all degenerate, and are of order $\sqrt{-c_2} \propto a$.

For $c_2 > 0$, the minimum of V_{eff} is reached at

$$\langle \sigma \rangle = \begin{cases} 1, & c_1 \geq 2c_2, \\ \frac{c_1}{2c_2}, & -2c_2 < c_1 < 2c_2, \\ -1, & c_1 \leq -2c_2. \end{cases} \tag{2.3}$$

For $|c_1| < 2c_2$ we find that $|\langle \sigma \rangle| < 1$, which implies that $\langle \pi_a \rangle \neq 0$. $SU(2)$ isospin is spontaneously broken to a $U(1)$ subgroup,[5] and two of the three pions become massless as the NGBs associated with this symmetry breaking. This region in the phase diagram is the Aoki phase. Clearly, in order to probe the Aoki phase transition, the couplings $c_1 \sim c_2$ have to be of the same magnitude. We may take the direction of symmetry breaking to point in the τ_3 direction, so that $\pi^\pm$ are the NGBs, while π^0 is massive inside the Aoki phase. At the phase boundaries $|c_1| = 2c_2$ all three pions are degenerate and massless, even though $c_1 \propto m$ does not vanish. In the continuum limit, $c_2 \propto a^2 \to 0$, and

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[2]For reviews of chiral perturbation theory for QCD with Wilson fermions, see Refs. [9–11].

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[3]Terms linear in the lattice spacing break the symmetry in exactly the same way as the term linear in the quark mass, and are thus absorbed into the term proportional to c_1. Since both c_1 and c_2 (and c_3 in Sec. II B below) have mass dimension equal to four, appropriate powers of Λ_{QCD} will always be understood.

[4]In the large-N_c limit, $c_2 < 0$ is excluded [12], but at finite N_c both signs are possible.

[5]For the reason why the Vafa-Witten theorem [13] does not apply inside the Aoki phase, see Ref. [5].

the Aoki phase shrinks to zero; the continuum limit at $c_2 = c_1 = 0$ yields QCD with two massless quarks.

Inside the Aoki phase of the lattice theory, parity is spontaneously broken as well. In the continuum, if we take the vacuum $\langle \Sigma \rangle = \pm 1$, parity acts as $\Sigma \to \Sigma^\dagger$. Since the symmetry is $SU(2)_L \times SU(2)_R$, any expectation value $\langle \Sigma \rangle \in SU(2)$ can be rotated to $\langle \Sigma \rangle = \pm 1$ using, e.g., an $SU(2)_L$ transformation. Thus, if we would want to expand around an equivalent vacuum $\langle \Sigma \rangle \neq \pm 1$, parity would merely take a more complicated form. By contrast, on the lattice the axial symmetries are explicitly broken. Vacua with different values of $\langle \sigma \rangle$ are inequivalent, and, for any $\langle \pi_a \rangle \neq 0$, parity is broken spontaneously.

A. Gauging isospin

We now consider what happens if we gauge isospin, with a gauge coupling g weak enough that we can analyze the effect on the phase diagram by considering the order-g^2 correction to $V_{\rm eff}$. We expect that nontrivial modifications of the scenarios reviewed above may occur when $g^2 \sim c_1 \sim c_2$, or, equivalently, $g^2 \sim m/\Lambda_{\rm QCD} \sim (a\Lambda_{\rm QCD})^2$.

In order to find the order-g^2 part of $V_{\rm eff}$ we proceed as follows. The lowest-order chiral effective action contains a term

$$\mathcal{L} = \frac{f^2}{8} {\rm tr}((D_\mu \Sigma)^\dagger D_\mu \Sigma), \qquad (2.4)$$

where f is the pion decay constant in the chiral limit, and

$$D_\mu \Sigma = \partial_\mu \Sigma + ig[V_\mu, \Sigma], \qquad (2.5)$$

with $V_\mu = \sum_a V_{\mu,a} \tau_a/2$ the isospin gauge field.[6] Upon working out the nonderivative part of $\mathcal{L}$,

$$\frac{g^2 f^2}{4} {\rm tr}(V_\mu^2 - V_\mu \Sigma V_\mu \Sigma^\dagger), \qquad (2.6)$$

we see, first of all, that the weak gauge fields V_μ remain massless on the isospin-symmetric vacua $\Sigma_0 = \pm 1$. Furthermore, integrating over the weak gauge fields, we find the leading-order contribution to the effective potential (2.1),[7]

$$\Delta V_{\rm eff} = -\frac{g^2 c_3}{8} \sum_a {\rm tr}(\tau_a \Sigma \tau_a \Sigma^\dagger), \qquad (2.7)$$

in which c_3 is independent of g^2 to leading order. From Ref. [14] we know that $c_3 > 0$. Using Eq. (2.2), we find for the effective potential

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[6]The gauging of the vector symmetries leads to explicit breaking of the axial symmetries.

[7]The effective potential due to the weak gauge fields always has a similar form, even if some of the weakly gauged symmetries are spontaneously broken. The reason is that the gauge bosons' mass will be proportional to gf, and thus gauge-field mass effects only show up in the effective potential at order g^4.

$$V_{\rm eff} + \Delta V_{\rm eff} = -c_1 \sigma + (c_2 - g^2 c_3)\sigma^2 + {\rm constant}. \quad (2.8)$$

The effect of the weak gauge fields V_μ on the phase diagram is very simple: the parameter c_2 gets shifted to $c_2 - g^2 c_3$. If $c_2 < 0$, the transition when c_1 goes through zero remains first order. Even in the continuum limit, when $c_2 = 0$, all pions acquire a mass proportional to $\sqrt{g^2 c_3} \propto g$. If $c_2 > 0$, the Aoki transition changes into a first-order transition when the lattice spacing becomes small enough such that $c_2 < g^2 c_3$. In other words, the Aoki phase gets pushed away from the continuum limit.

B. Coupling the photon

The situation changes when we restrict the gauge field to $V_\mu = A_\mu Q$, with $Q = {\rm diag}(2/3, -1/3) = 1/6 + \tau_3/2$, and $g = e$, the electric charge, so that A_μ is the photon field. In that case, the shift in the effective potential becomes

$$\Delta V_{\rm eff}^{\rm em} = -\frac{e^2 c_3}{8} {\rm tr}(\tau_3 \Sigma \tau_3 \Sigma^\dagger), \qquad (2.9)$$

with the same coefficient c_3 as in Eq. (2.7). Using Eq. (2.2) again,

$$V_{\rm eff} + \Delta V_{\rm eff}^{\rm em} = -c_1 \sigma + c_2 \sigma^2 - \frac{e^2 c_3}{2}(\sigma^2 + \pi_3^2). \quad (2.10)$$

Again, the analysis of this effective potential is very simple. Since $c_3 > 0$, minimizing the effective potential requires that $\langle \sigma \rangle^2 + \langle \pi_3 \rangle^2 = 1$, i.e., $\langle \pi_1 \rangle = \langle \pi_2 \rangle = 0$, irrespective of the values of c_1 and c_2. If $c_2 < 0$, $\langle \sigma \rangle = \pm 1$ depending on the sign of c_1, the phase transition is first order, and takes place at $c_1 = 0$. The term proportional to c_3 raises the charged pion mass relative to the neutral pion mass [15].

If $c_2 > 0$, and $|c_1| < 2c_2$ so that we are in the Aoki phase, again $\sigma = c_1/(2c_2)$ as in Eq. (2.3), and therefore

$$\langle \pi_3 \rangle = \sqrt{1 - \frac{c_1^2}{4c_2^2}}. \qquad (2.11)$$

Isospin is explicitly broken by the coupling to QED, but parity is spontaneously broken in the Aoki phase, and there still is a second-order phase transition. Inside the Aoki phase, the pion masses are

$$m_\pm^2 = e^2 c_3 f^{-2},$$
$$m_0^2 = 2c_2\left(1 - \frac{c_1^2}{4c_2^2}\right)f^{-2}. \qquad (2.12)$$

We see that, depending on the relative size of the parameters c_1, c_2 and $e^2 c_3$, the neutral pion might even be heavier than the charged pion, even though in the continuum limit Witten's inequality implies that this can never be the case [14]. The reason is that now we have a competition:

electromagnetic effects increase the charged pion mass relative to the neutral pion mass, whereas the lattice artifacts that give rise to the breaking of isospin in the Aoki phase create an opposite effect, since the charged pions are the NGBs of this symmetry breaking.

III. LITTLEST HIGGS

In this section we present an analysis of the littlest Higgs model of Ref. [6][8] that parallels what we did for QCD in Sec. II. First, we very briefly review the necessary ingredients of this theory in the continuum, in Sec. III A, including the coupling to the Standard Model gauge fields. We next consider the Aoki phase for this theory, without the weak gauge fields, in Sec. III B. In Sec. III C we then consider the competition between the effective potential generated by the weak gauge fields and that generated by lattice artifacts in the determination of the phase diagram.

A. Littlest Higgs—continuum

We consider a strongly coupled gauge theory with five Weyl fermions in a real representation of the (unspecified) strong gauge group. This theory has an SU(5) flavor symmetry which we assume to be broken to SO(5) by a bilinear fermion condensate, resulting in 14 NGBs parametrizing the coset SU(5)/SO(5). In order to construct the effective theory for these NGBs, we introduce the nonlinear field

$$\Sigma = \exp(i\Pi/f)\Sigma_0 \exp(i\Pi^T/f) = \exp(2i\Pi/f)\Sigma_0, \quad (3.1)$$

with $\Sigma_0 = \langle\Sigma\rangle$ given by[9]

$$\Sigma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.2)$$

Since the bilinear fermion condensate is symmetric in its SU(5) indices, so is Σ. Therefore, Σ transforms into $U\Sigma U^T$ with $U \in$ SU(5), and this leads to the form (3.1) for Σ in terms of the "pion" field Π, which satisfies $\Sigma_0\Pi^T = \Pi\Sigma_0$. The generators T of the unbroken SO(5) obey the relation $\Sigma_0 T^T = -T\Sigma_0$.

The Standard Model SU(2)$_L$ gauge fields $W_{\mu a}$ are coupled to an SU(2) subgroup of SO(5) generated by [6]

$$Q_a = \begin{pmatrix} \frac{1}{2}\tau_a & 0 & 0 \\ 0 & -\frac{1}{2}\tau_a^T & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a = 1, 2, 3, \quad (3.3)$$

where again τ_a are the Pauli matrices. The SU(2) generated by the Q_a is an invariant subgroup of the SO(4) group defined by embedding its elements in the upper left 4×4 block of the SO(5) matrices.

The leading-order effective potential for the Σ field, obtained by integrating over the W fields, is given by

$$V_{\text{weak}} = g^2 C_{\text{w}}\text{tr}(\Sigma Q_a \Sigma^* Q_a^*), \quad (3.4)$$

where a sum over a is implied. The low-energy constant C_{w} is analogous to the constant c_3 in Eq. (2.7), and it is positive, as we show in the Appendix, using the relevant result of Ref. [14]. In Ref. [6] more weak gauge fields were coupled to a subgroup of SU(5) in order to obtain the "collective" symmetry breaking typical of little Higgs models. However, the primary goal of a lattice investigation of this theory would presumably be the determination of the LEC C_{w}, which can be probed using any subgroup of SU(5), such as, for instance, the SU(2) group we introduced in Eq. (3.3). As we explain below, this allows us to maintain all gauged symmetries (strong and weak) on the lattice if we choose to work with Wilson fermions.

In Eq. (3.4), the minimum value for the trace, -3, is attained for $\Sigma = \Sigma_0$. Therefore the vacuum is aligned, i.e., the W fields do not move the vacuum away from Eq. (3.2). The potential V_{weak} is invariant under the SO(4) subgroup defined above: if we transform $\Sigma \to U\Sigma U^T$ with $U \in$ SO(4), we see that this is equivalent to keeping Σ fixed, while transforming $Q_a \to U^T Q_a U^* = R_{ab}Q_b$ inside the trace, with R in the fundamental representation of SO(3). Here we used the fact that the Q_a generate an invariant subgroup of SO(4). Using that $R_{ab}R_{ac} = (R^T R)_{bc} = \delta_{bc}$ the invariance follows.

We may also introduce the hypercharge weak gauge field, which gauges the U(1) symmetry generated by [6]

$$Y = \frac{1}{2} \text{diag}(1, 1, -1, -1, 0). \quad (3.5)$$

This breaks the SO(4) symmetry explicitly to SU(2)$_L \times$ U(1)$_Y$, with SU(2)$_L$ the group to which the W fields couple. The new contribution to the effective potential is

$$V_Y = g'^2 C_{\text{w}}\text{tr}(\Sigma Y \Sigma^* Y), \quad (3.6)$$

where the constant C_{w} is the same as in Eq. (3.4), and g' is the hypercharge gauge coupling.

In order to move to the lattice, the strongly interacting theory is first reformulated in terms of Majorana fermions instead of Weyl fermions. Now, because the fermions transform in a real representation of the strong gauge

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[8]See also Ref. [8] for a review.
[9]Relative to Ref. [6] we interchanged the third and fifth rows and columns in the form for Σ_0.

group, a gauge-invariant fermion mass term can be added to the theory, breaking $SU(5) \to SO(5)$ softly. Going to the lattice using Wilson fermions, it is then straightforward to augment this local mass term with a Wilson mass term as well, in order to avoid species doublers. The exact flavor symmetry of the lattice theory is therefore just $SO(5)$, regardless of the fermion mass. We expect the full $SU(5)$ symmetry to be restored in the continuum limit, provided that the single-site Majorana mass is tuned appropriately. These features are, of course, completely analogous to the usual case of Wilson-Dirac fermions.

On the lattice, before any weak gauge fields are coupled to the flavor currents and for a large enough positive quark mass, the fermion condensate will be proportional to the unit matrix (see Sec. III B below). Anticipating this, it is convenient to reformulate the (massless) continuum effective theory such that this is also the case there. Starting from Eq. (3.2) it is straightforward to find an element U of $SU(5)$ such that

$$\Sigma_0' = U\Sigma_0 U^T = \mathbf{1}. \qquad (3.7)$$

We also have to transform the generators Q_a and Y to the new basis, defining

$$W_a \equiv UQ_aU^\dagger, \qquad X \equiv UYU^\dagger. \qquad (3.8)$$

Since Σ_0' is proportional to the unit matrix, the W_a and X are antisymmetric and Hermitian, and thus purely imaginary. The potential $V_{\text{weak}} + V_Y$ can be written as

$$V_{\text{weak}} + V_Y = -g^2 C_{\text{w}}\text{tr}(\Sigma W_a\Sigma^* W_a) - g'^2 C_{\text{w}}\text{tr}(\Sigma X\Sigma^* X). \qquad (3.9)$$

After adding $V_{\text{weak}} + V_Y$ to the effective theory, the complete vacuum manifold is the $U(1)$ circle generated by $T = \text{diag}(1, 1, 1, 1, -4)$.[10]

For Majorana (equivalently, Weyl) fermions there are no separate C and P symmetries, only a CP symmetry. The role of CP parallels that of parity in the two-flavor theory of Sec. II. If we expand the nonlinear field around the unit matrix, CP acts on the pion field as $\Pi \to -\Pi$. Since the vacuum manifold contains the unit matrix, it follows that CP symmetry is unbroken in the continuum theory.

B. Littlest Higgs—lattice artifacts

In this subsection, we turn off the weak gauge fields, and consider only the strongly coupled theory on the lattice, using Wilson-Majorana fermions.

The construction of the effective potential representing the effects of a quark mass and lattice artifacts to order a^2

for the $SU(5)/SO(5)$ effective theory is very similar to the construction for the $(SU(2)_L \times SU(2)_R)/SU(2)$ case reviewed in Sec. II. The only difference is that more invariants proportional to a^2 exist, so that now the effective potential becomes [16]

$$V_{\text{Aoki}} = -\frac{c_1}{2}\,\text{tr}(\Sigma + \Sigma^\dagger) + \frac{c_2}{4}(\text{tr}(\Sigma + \Sigma^\dagger))^2$$
$$- \frac{c_3}{4}(\text{tr}(\Sigma - \Sigma^\dagger))^2 + \frac{c_4}{2}\,\text{tr}(\Sigma^2 + \Sigma^{\dagger 2}), \qquad (3.10)$$

in which c_1 is proportional to the (subtracted) quark mass, and $c_{2,3,4}$ are all proportional to a^2.[11] There is no symmetry relating the theory with $c_1 > 0$ to that with $c_1 < 0$, because no nonanomalous transformation exists that relates the two. We will therefore mostly limit ourselves to the choice $c_1 \geq 0$ in this article.

On our new basis the pion field Π in Eq. (3.1) is real and symmetric, and can thus be diagonalized by an $SO(5)$ transformation. It follows that in order to find the minimum of V_{Aoki} we may choose Σ in Eq. (3.10) to be diagonal,

$$\Sigma = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}, e^{i\phi_4}, e^{i\phi_5}), \qquad (3.11)$$

subject to the constraint

$$\sum_{i=1}^{5} \phi_i = 0 \quad (\text{mod } 2\pi). \qquad (3.12)$$

Substituting this into Eq. (3.10) yields

$$V_{\text{Aoki}} = -\sum_i(c_1\cos\phi_i - 2c_4\cos^2\phi_i) + c_2\left(\sum_i \cos\phi_i\right)^2$$
$$+ c_3\left(\sum_i \sin\phi_i\right)^2. \qquad (3.13)$$

This is not easily minimized, so we will begin with simplifying V_{Aoki} by omitting the double-trace terms, i.e., by setting $c_2 = c_3 = 0$. Even with only c_1 and c_4, the minimization of V_{Aoki} will not be a simple task, because of the constraint (3.12).

For $c_4 < 0$, the minimum is at $\phi_i = 0$, as in the case of Sec. II, and the pseudo-NGBs remain massive in the limit $c_1 \to 0$, as long as $c_4 \neq 0$; their mass is proportional to $\sqrt{c_1 - 4c_4}$.

For $c_4 > 0$, we will proceed in several steps. First we prove that for $c_1 > 4c_4$ the solution is again $\phi_i = 0$, so that no symmetry is spontaneously broken. We will then analyze the case that $c_1 = 4c_4 - 2\varepsilon$ with $\varepsilon > 0$ small, as well as the case that $c_1 = \varepsilon$ is small. Since we may take c_4

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[10]Since we gauge only the generators $Q_a = Q_1^a + Q_2^a$ and $Y = Y_1 + Y_2$ of Ref. [6], the Higgs field components of Π pick up a mass; see Sec. III C below.

[11]For $SU(2)$, the last three terms collapse to the one term in Eq. (2.1).

to set the overall scale of V_{Aoki}, we will set $c_4 = 1$ in most of the rest of this subsection.

The potential V_{Aoki} is extremized if

$$\sin\phi_i(c_1 - 4\cos\phi_i) = \lambda, \qquad (3.14)$$

where λ is a Lagrange multiplier enforcing the constraint. First, let us ignore the constraint, which is equivalent to setting $\lambda = 0$. Then, for $c_1 > 4$, Eq. (3.14) implies that $\phi_i = 0$, if we also demand the solution to be the minimum of V_{Aoki}. Since this solution satisfies the constraint (3.12), we have found the solution we are looking for. Also, since there is only one minimum for $c_1 > 4$, it follows by continuity that the same is true at $c_1 = 4$. Therefore, if a phase transition occurs at $c_1 = 4$, this phase transition is second order.

Next, we consider $c_1 = 4 - 2\epsilon$, with $\epsilon > 0$ small. Since only a continuous phase transition may take place, ϕ_i will be small as well, and we thus expand the left-hand side of Eq. (3.14) to order ϕ_i^3,

$$\phi_i(-\epsilon + \phi_i^2) = \lambda/2. \qquad (3.15)$$

From this, it follows that for any triple i, j, k, if ϕ_i is equal to neither ϕ_j nor ϕ_k, then

$$\phi_i^2 + \phi_i\phi_j + \phi_j^2 = \phi_i^2 + \phi_i\phi_k + \phi_k^2 = \epsilon. \qquad (3.16)$$

It follows that either $\phi_k = \phi_j$, or $\phi_k = -\phi_i - \phi_j$. This provides us with a finite list of options to check, and we find that V_{Aoki} is minimized for

$$\Sigma = \Sigma_0(4 - 2\epsilon) = \exp\left[i\,\text{diag}(\varphi, \varphi, \varphi, -3\varphi/2, -3\varphi/2)\right], \qquad (3.17a)$$

$$\phi^2 = \frac{9}{7}\epsilon. \qquad (3.17b)$$

Indeed, a second-order phase transition takes place at $c_1 = 4$, with, below that value, a symmetry-breaking pattern $SO(5) \to SO(3) \times SO(2)$. In addition, CP symmetry, $\Sigma \to \Sigma^*$, is spontaneously broken as well. We note that the solution (3.17) cannot be rotated to $\Sigma_0 = \mathbf{1}$, because on the lattice the $SU(5)$ transformation that would do this is not a symmetry.

We now turn to the case that $c_1 = 0$. If ϕ_0 is a solution of Eq. (3.14), i.e., $\sin 2\phi_0 = -\lambda/2$, then all possible solutions are

$$\phi_i = \phi_0, \qquad \phi_i = \pi/2 - \phi_0,$$
$$\phi_i = \pi + \phi_0, \qquad \phi_i = 3\pi/2 - \phi_0. \qquad (3.18)$$

Going through all possibilities for choosing the ϕ_i, $i = 1, \ldots, 5$ from this list, and demanding that any such choice satisfies the constraint (3.12), yields three degenerate minima for $c_1 = 0$:

$$\Sigma = \Sigma_0^{(1)}(0) = \exp\left[(2\pi i/5)\text{diag}(1,1,1,1,1)\right], \qquad (3.19a)$$

$$\Sigma = \Sigma_0^{(2)}(0) = \exp\left[(2\pi i/5)\text{diag}(1,1,1,-3/2,-3/2)\right], \qquad (3.19b)$$

$$\Sigma = \Sigma_0^{(3)}(0) = \exp\left[(2\pi i/5) \times \text{diag}(1,-3/2,-3/2,-3/2,-3/2)\right]. \qquad (3.19c)$$

Next, let us consider small $c_1 = \epsilon$. Once again, since the three global minima at $c_1 = 0$ are discrete, this can at most lead to a small shift $\delta\phi_i$ away from $2\pi/5$ or $-3\pi/5$ for each i. Expanding V_{Aoki}, we find

$$V_{\text{Aoki}}^{(1)} = \frac{5}{4}\left(3 - \sqrt{5} - \epsilon\left(\sqrt{5} - 1\right)\right) + \frac{1}{2}\left(1 + \sqrt{5}\right)\sum_i \delta\phi_i^2, \qquad (3.20a)$$

$$V_{\text{Aoki}}^{(2)} = \frac{5}{4}\left(3 - \sqrt{5} - \frac{\epsilon}{5}\left(\sqrt{5} - 1\right)\right) + \frac{1}{2}\left(1 + \sqrt{5}\right)\sum_i \delta\phi_i^2, \qquad (3.20b)$$

$$V_{\text{Aoki}}^{(3)} = \frac{5}{4}\left(3 - \sqrt{5} + \frac{3\epsilon}{5}\left(\sqrt{5} - 1\right)\right) + \frac{1}{2}\left(1 + \sqrt{5}\right)\sum_i \delta\phi_i^2, \qquad (3.20c)$$

where the superscript on V_{Aoki} refers to which solution in Eq. (3.19) we are expanding around. We have expanded to quadratic order in $\delta\phi_i$, dropping terms of order $\epsilon\delta\phi_i^2$, and we have used that $\sum_i \delta\phi_i = 0$ because of Eq. (3.12). For small $c_1 = \epsilon > 0$ the first minimum, $\Sigma_0^{(1)}(0)$, is the absolute minimum, and, since the coefficients of the $\delta\phi_i^2$ terms in Eq. (3.20) are always positive, the minimum stays at $\Sigma_0(c_1 > 0) = \Sigma_0^{(1)}(0)$. The vacuum $\Sigma_0^{(1)}(0)$ only breaks CP. For $c_1 < 0$, the solution $\Sigma_0^{(3)}(0)$ becomes the absolute minimum, and a first-order transition takes place at $c_1 = 0$. This solution breaks $SO(5)$ to $SO(4)$, giving rise to four NGBs.

The picture that arises is that there is an Aoki-like phase when $c_4 > 0$. For c_1 just below $4c_4$, $SO(5)$ breaks to $SO(3) \times SO(2)$, and there are six exact NGBs. CP is broken as well. We do not know what happens when c_1 is further decreased, but when we reach $c_1 = 0$ (while keeping $c_1 > 0$) only CP remains spontaneously broken.

We end this section with a few observations on what happens if c_2 and c_3 are turned back on. First, with $c_3 = 0$, Eq. (3.14) becomes

$$\sin\phi_i\left(c_1 - 4\cos\phi_i - 2c_2\sum_i\cos\phi_i\right) = \lambda. \qquad (3.21)$$

Our previous symmetric solution, $\phi_i = 0$, is still the only solution when $c_1 > 4c_4 + 10c_2$. Moreover, this remains true when $c_3 > 0$ as well.

Next, we consider in more detail what happens for smaller values of c_1 when c_2 and c_3 do not vanish, but are small. Near $c_1 = 0$, the potential remains equal to a constant plus a positive-definite quadratic form in $\delta\phi_i$. Denoting the new constant piece as $\delta V^{(i)}_{\text{Aoki}}$ we find

$$\delta V^{(i)}_{\text{Aoki}} = b^{(i)}\left(c_2 \frac{3-\sqrt{5}}{8} + c_3 \frac{5+\sqrt{5}}{8}\right), \qquad (3.22)$$

where $b^{(1)} = 25$, $b^{(2)} = 1$, and $b^{(3)} = 9$. Since $\delta V^{(2)}_{\text{Aoki}}$ is smaller than the other two, there will be regions where each of the solutions, now including $\Sigma^{(2)}_0(0)$, is the global minimum.

For c_1 near $4c_4$, we consider the case that $c_2 \sim c_3 \sim \varepsilon$. Expanding Eq. (3.21), we find the $c_2 \neq 0$ version of Eq. (3.15),

$$\phi_i(-(\epsilon + 5c_2) + \phi_i^2) = \lambda/2. \qquad (3.23)$$

The c_3 term does not contribute to this order, because of the constraint (3.12). The phase transition now takes place when $\epsilon + 5c_2$ becomes positive, or, equivalently, when c_1 becomes smaller than $4 + 10c_2$. Restoring c_4, the phase boundary gets shifted from $c_1 = 4c_4$ to $c_1 = 4c_4 + 10c_2$, consistent with what we already found above.

C. Combined phase diagram

We now combine the potentials V_{Aoki} of Eq. (3.10) and V_{weak} of Eq. (3.4).[12] The combined potential is invariant under SO(4), embedded in the upper left 4×4 block of the SO(5) matrices. The pion field Π, which transforms as the traceless, two-index symmetric representation of SO(5), decomposes into fields transforming as the traceless, two-index symmetric representation of SO(4) which we will denote by **9**, the fundamental representation, denoted by **4**, and a singlet, denoted by **1**.

A first observation is that, when $c_3 \geq 0$, the phase transition boundary is still at $c_1 = 4c_4 + 10c_2$. This is because for $c_1 \geq 4c_4 + 10c_2$, the minimum of both V_{Aoki} and V_{weak} is at $\phi_i = 0$. Then, substituting Eq. (3.1) into the sum of Eqs. (3.10) and (3.4), we find for the masses of each of these representations[13]

$$M_1^2 = (4/f^2)(c_1 - 4c_4 - 10c_2),$$
$$M_4^2 = (4/f^2)(c_1 - 4c_4 - 10c_2 + 3g^2C_w/4),$$
$$M_9^2 = (4/f^2)(c_1 - 4c_4 - 10c_2 + 2g^2C_w). \qquad (3.24)$$

We see that indeed V_{weak} leads to mass splittings consistent with the fact that it breaks SO(5) to SO(4). For $g = 0$, we recover the observation that one enters an Aoki phase when $c_1 - 4c_4 - 10c_2$ becomes negative, even though the masses alone are not sufficient to deduce the symmetry-breaking pattern found in Sec. III B. Furthermore, in the continuum limit ($c_2 = c_3 = c_4 = 0$) and for vanishing quark mass ($c_1 = 0$), Eq. (3.24) confirms that V_{weak} stabilizes the vacuum manifold that we have inferred from Eq. (3.9).

An important practical issue facing the lattice simulation of a UV completion of this model is how to tune the bare mass towards its critical value where the (Majorana) fermions become massless. Starting from large positive c_1, in the absence of weak gauge fields the fermion mass, as well as the masses of all pions, will vanish simultaneously when c_1 reaches $4c_4 + 10c_2$. Equation (3.24) tells us that, when the weak gauge fields are dynamical, this is no longer true. Clearly, the massless limit of the continuum theory corresponds to vanishing M_1^2, and therefore it is the singlet that must be tuned to criticality in a lattice simulation. By contrast, if one were to tune M_4^2 or M_9^2 to zero in the lattice theory, the curvature in the singlet direction would have become negative at the origin, implying that the SO(4) singlet field, η, has acquired a nonvanishing expectation value. Since this vacuum is still invariant under the SO(4) symmetry of the full potential V, there are no NGBs. However, CP symmetry is spontaneously broken, because for $\langle\eta\rangle \neq 0$ one has $\langle\Sigma\rangle \neq \langle\Sigma^*\rangle$ on the lattice.

We have not been able to minimize the full potential inside the Aoki phase. It appears likely that, as one moves towards more negative values of $c_1 - 4c_4 - 10c_2$, the SO(4) symmetry will break spontaneously, giving rise to some NGBs.

Imagine starting at some small but fixed $c_1 - 4c_4 - 10c_2 < 0$, and gradually turning on g. For $g = 0$, we have found that the vacuum is given by Eq. (3.17), with symmetry breaking SO(5) $\rightarrow$ SO(3) $\times$ SO(2). For $g \neq 0$ the symmetry of the theory is reduced to SO(4). As long as g is small enough, we expect that the vacuum (3.17) will be modified by continuous $O(g^2)$ corrections. One may speculate on how the SO(3) $\times$ SO(2) and SO(4) subgroups of SO(5) align relative to each other. One possibility is that the SO(3) is a subgroup of the SO(4), with the spontaneous symmetry breaking pattern SO(4) $\rightarrow$ SO(3). Instead of six, there will only be three exact NGBs. However, one can verify that in that case all the SU(2) generators in Eq. (3.8) are broken, and these three NGBs are thus eaten by the $W_{\mu a}$ gauge fields. Another possibility is that the SO(2) is a subgroup of the SO(4), while the SO(3) is explicitly broken to another SO(2). The spontaneous symmetry-breaking

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[12]We set the hypercharge gauge coupling $g' = 0$, since not much changes in our analysis when it is turned on.

[13]In the model of Ref. [6], thanks to the presence of more weak gauge fields, $M_4 = 0$ in the continuum, allowing its identification with the Higgs field.

pattern is now $SO(4) \to SO(2) \times SO(2)$, yielding four NGBs. In this case, only two out of three $SU(2)$ generators in Eq. (3.8) are broken. One W field stays massless, with two exact NGBs remaining in the spectrum. According to Ref. [2], this scenario would be favored.

We see that, deeper inside the Aoki phase, it is quite likely that one would encounter long-range effects mediated by exact NGBs. The existence of such exact NGBs is purely a lattice artifact.

IV. CONCLUSIONS

In an asymptotically free gauge theory with massless fermions one can consider a number of small perturbations. In the continuum, one can give the (Dirac or Majorana) fermions a mass. One can also couple the fermions to another dynamical gauge field gauging some of the flavor symmetries, such that, at the scale where the original gauge theory confines, the new gauge coupling is weak. In general, such perturbations break the flavor symmetry of the strong gauge theory explicitly.

In addition, in order to study such a theory nonperturbatively, one needs to consider the lattice discretization. Again, the lattice formulation usually breaks explicitly some of the flavor symmetries. Finally, the strong dynamics typically gives rise to spontaneous symmetry breaking.

In this article we investigated flavor symmetry breaking using effective field theory techniques in two examples, using Wilson fermions for the lattice formulation of the theory. In both cases, we allowed the weak gauge fields to couple only to a subgroup of the lattice flavor symmetry group, since otherwise we would need to consider a chiral gauge theory on the lattice.[14] In many applications to physics beyond the Standard Model, weak gauge fields coupling to broken, or axial, generators are also needed. Nevertheless, the restriction to weak gauge fields coupled to conserved lattice currents is not a severe limitation, as it already gives access to LECs whose values are phenomenologically interesting. The reason is that, thanks to its symmetry structure, the (continuum) effective theory is typically characterized by a very small number of LECs, which are common to weak gauge fields coupled to both vector and axial generators.

The two examples we considered are QCD with two light flavors where also (part of) the isospin symmetry group is gauged, and the littlest Higgs model of Ref. [6]. In the latter case, only the Standard Model subgroup of the flavor symmetry group was gauged, because, among the weak gauge fields of Ref. [6], only the electroweak fields couple to vector currents of the strongly interacting theory. Since a lattice gauge theory with Wilson fermions gives rise to a nontrivial phase diagram at nonzero lattice spacing, this phase structure can "interfere" with the expected effects of the continuum perturbations from the fermion masses and weak gauge fields.

In the QCD case, in the continuum limit, gauging isospin leads to all pions acquiring a mass. However, if lattice spacing effects, represented in the effective theory through the LEC c_2 in Sec. II, are large enough, one finds that some of the pions may remain massless as a pure lattice artifact. This happens if the LEC $c_2 > 0$ so that an Aoki phase exists near the continuum limit. Moreover, in that case parity is also spontaneously broken. In the case that only a $U(1)$ subgroup of isospin is gauged, all pions can remain massive even for vanishing quark mass, with the neutral pion mass of order the lattice spacing, but parity can still be spontaneously broken. Perhaps surprisingly, the neutral pion can be heavier than the charged pion.

Very similar conclusions are obtained in the case of the littlest Higgs model, which we studied in Sec. III. In the continuum, the weak gauge fields make most of the Nambu-Goldstone bosons of the strong gauge theory massive, but inside the Aoki phase of the lattice version of the theory, some of these mesons may again become massless, as a consequence of lattice artifacts. Moreover, inside the Aoki phase, CP is spontaneously broken as well. Because of the complicated structure of the effective potential in this case an exhaustive study of the phase diagram is more difficult, but the message is essentially the same as in the case of QCD with two flavors.

Our results lead us to the following conjecture. If a general subgroup of the unbroken flavor symmetries is gauged, we expect the boundary of the Aoki phase to stay at the same location, but the symmetry-breaking pattern inside the Aoki phase can change. If, however, an invariant subgroup of the unbroken flavor symmetry group is gauged, the potential will retain the same flavor symmetry, and, as a result, the boundary of the Aoki phase itself will shift its location. This includes the case where the full unbroken flavor symmetry group is gauged, as in Sec. II A.[15]

Clearly, these results have practical consequences for the lattice study of electromagnetic effects in hadronic physics and for composite Higgs models. The interplay between all three sources of symmetry breaking (weak gauge fields, fermion masses, and lattice artifacts) will have to be considered very carefully in order to arrive at valid conclusions about the continuum limit. For example, in the context of the littlest Higgs model of Sec. III, our analysis clarifies how to tune to the massless limit on the lattice. It should be straightforward to extend the analysis framework we developed in this article to gauge theories with different flavor symmetry groups.

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[14]The definition of chiral gauge theories on the lattice is as yet not a fully solved problem; see for example Ref. [17].

[15]An example of a nontrivial invariant subgroup would be an $SU(4)/SO(4)$ nonlinear sigma model, where the lattice flavor symmetry is $SO(4)$, and in which the invariant subgroup of Sec. III A is gauged.

Finally, our conclusions are not limited to lattice gauge theories with Wilson fermions. In a companion paper [18] we find that similar considerations apply to the use of staggered fermions as well, since staggered fermions also break continuum flavor symmetries,[16] and a nontrivial phase structure is possible in that case as well [20]. In addition, the same continuum mass matrix can arise from inequivalent choices of the staggered mass terms on the lattice, and this can also give rise to a competition with the effects of the weak gauge fields.

APPENDIX: PROOF THAT C_w IS POSITIVE

In this appendix we prove that C_w in Eq. (3.4) is positive. The structure of the proof is similar to the proof that the electromagnetic contribution to $m_{\pi^\pm}^2 - m_{\pi^0}^2$ in QCD is positive [14].

We will consider the case of a strongly coupled gauge theory with N_f Weyl fermions in a real representation of the strong gauge group. This theory has an $SU(N_f)$ flavor symmetry, which is spontaneously broken to $SO(N_f)$. For the purpose of this appendix, it is convenient to assemble each two-component Weyl fermion and its antifermion field into a four-component Majorana fermion χ_i, $i = 1, \ldots, N_f$. The continuum action is then $\frac{1}{2}\bar\chi_i\gamma_\mu D_\mu\chi_i$, where D_μ is the covariant derivative in the real representation. Here $\bar\chi_i = \chi_i^T C\mathcal{R}$ by definition, where C is the usual charge-conjugation matrix, and $\mathcal{R}$ is a matrix such that $\mathcal{R}^T = \mathcal{R}^\dagger = \mathcal{R}^{-1} = \mathcal{R}$ and $\mathcal{R}\mathcal{T}_\alpha\mathcal{R} = -\mathcal{T}_\alpha^T$ for the generators $\mathcal{T}_\alpha$ of the strong gauge group in the real representation. The fermion condensate $\langle\bar\chi_i\chi_j\rangle$ is symmetric in the indices i, j, and we will assume that $\langle\bar\chi_i\chi_j\rangle \propto \delta_{ij}$. Thus the unbroken $SO(N_f)$ generators are antisymmetric, $T_a^T = -T_a$, and the broken generators for the coset $SU(N_f)/SO(N_f)$ are symmetric, $T_a^T = T_a$.

We introduce an $SU(N_f)$ global spurion $Q = Q_aT_a$,[17] where T_a are the Hermitian generators of $SU(N_f)$. The spurion transforms as $Q \to UQU^\dagger$ for $U \in SU(N_f)$. The microscopic partition function is

$$Z(Q) = \int d[A]d[W]d[\chi]\exp[-S_S(A_\mu,\chi_i) - S_W(W_\mu,\chi_i,Q)],$$
(A1)

$$S_W(W_\mu,\chi_i,Q) = \frac{1}{4}F_{\mu\nu}^2 + gW_\mu Q_a J_{\mu a},$$
(A2)

$$J_{\mu a} = \bar\chi_i\gamma_\mu P_R T_{aij}\chi_j = \bar\chi_i\gamma_\mu P_L(-T^T)_{aij}\chi_j.$$
(A3)

Here A_μ is the strong gauge field, and S_S is the action for the strong dynamics. The field W_μ is the weak gauge field, with $F_{\mu\nu}$ its field strength. Since we work to order g^2, a single gauge field W_μ will be sufficient. Correspondingly, we may neglect the nonlinear part of $F_{\mu\nu}$. In this framework, the global $SU(N_f)$ transformations are carried by the spurions Q_a, whereas the field W_μ is invariant.[18]

The leading-order effective potential, bilinear in Q, is now

$$V_{\rm eff} = g^2 C_0 {\rm tr}(Q^2) + g^2 C_w {\rm tr}(Q\Sigma Q^*\Sigma^*),$$
(A4)

in which C_0 is another constant. The chiral field Σ is a unitary and symmetric matrix, and transforms as $\Sigma \to U\Sigma U^T$. Using $Q^* = Q^T$, on the vacuum $\Sigma_0 = \mathbf{1}$ this expression collapses to

$$V_{\rm vac} = g^2 C_0 {\rm tr}(Q^2) + g^2 C_w {\rm tr}(QQ^T).$$
(A5)

Introducing general linear combinations Q^V and Q^A of the unbroken and broken $SU(N_f)$ generators,

$$Q^V = \sum_{T_a = -T_a^T} Q_a^V T_a, \qquad Q^A = \sum_{T_a = +T_a^T} Q_a^A T_a.$$
(A6)

and using the fact that $Q = Q^V + Q^A$ and $Q^T = -Q^V + Q^A$, we may write $V_{\rm vac}$ as

$$V_{\rm vac} = g^2 C_0 {\rm tr}(Q^V Q^V + Q^A Q^A) - g^2 C_w {\rm tr}(Q^V Q^V - Q^A Q^A).$$
(A7)

Differentiating twice yields the linear combinations

$$\frac{\partial}{\partial Q_a^V}\frac{\partial}{\partial Q_b^V}V_{\rm vac} = g^2\delta_{ab}(C_0 - C_w),$$
(A8a)

$$\frac{\partial}{\partial Q_a^A}\frac{\partial}{\partial Q_b^A}V_{\rm vac} = g^2\delta_{ab}(C_0 + C_w),$$
(A8b)

from which we may extract C_0 and C_w separately.

In the microscopic theory

$$\langle J_{\mu a}(x)J_{\nu b}(0)\rangle = -{\rm tr}\langle\gamma_\mu T_a P_R[\chi(x)\bar\chi(0)]\gamma_\nu T_b P_R[\chi(0)\bar\chi(x)]\rangle$$
$$+ {\rm tr}\langle\gamma_\mu T_a P_R[\chi(x)\bar\chi(0)]\gamma_\nu T_b^T P_L[\chi(0)\bar\chi(x)]\rangle$$
(A9a)

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[16]For reviews, see Refs. [9,19] and references therein.
[17]For the QCD plus QED case, see Refs. [7,18,21].

[18]It is also possible to promote the flavor symmetry to a local symmetry (at least classically), by introducing the gauge fields $W_{\mu a}T_a$, see Ref. [7].

$$= -\text{tr}(T_a T_b)\text{tr}\langle\gamma_\mu P_R[\chi(x)\bar\chi(0)]\gamma_\nu P_R[\chi(0)\bar\chi(x)]\rangle$$
$$+ \text{tr}(T_a T_b^T)\text{tr}\langle\gamma_\mu P_R[\chi(x)\bar\chi(0)]\gamma_\nu P_L[\chi(0)\bar\chi(x)]\rangle, \quad \text{(A9b)}$$

where $[\chi(x)\bar\chi(y)]$ is the Majorana fermion propagator. In Eq. (A9b), in each term the first trace is over flavor indices, and the second over Dirac and strong gauge group indices.

The reason for the two terms on the right-hand side is that, with χ_i being Majorana, two different contractions contribute. In the first term on the right-hand side of Eq. (A9), we express both currents using the first expression on the right-hand side of Eq. (A3), and then contract the fermion fields cyclically. The second term is obtained by first rewriting $J_{\nu b}(0)$ using the second expression on the right-hand side of Eq. (A3), before cyclically contracting the fermions.

Unlike in the case of QCD, the same two-current correlation function now has both symmetry-preserving and symmetry-breaking parts. But these two parts have a different flavor structure. Indeed, the flavor structure of the two terms in Eq. (A9b) reproduces that obtained at the effective potential level (A5). Defining form factors from the contractions ($P_{\mu\nu}^\perp$ is the transverse projector)

$$q^2 P_{\mu\nu}^\perp \Pi_0(q^2) = -\int d^4x e^{iqx}\text{tr}\langle\gamma_\mu P_R[\chi(x)\bar\chi(0)]$$
$$\times \gamma_\nu P_R[\chi(0)\bar\chi(x)]\rangle, \quad \text{(A10a)}$$

$$q^2 P_{\mu\nu}^\perp \Pi_w(q^2) = \int d^4x e^{iqx}\text{tr}\langle\gamma_\mu P_R[\chi(x)\bar\chi(0)]$$
$$\times \gamma_\nu P_L[\chi(0)\bar\chi(x)]\rangle, \quad \text{(A10b)}$$

one finds that

$$C_0 = \frac{1}{16\pi^2}\int_0^\infty dq^2 q^2 \Pi_0(q^2), \quad \text{(A11a)}$$

$$C_w = \frac{1}{16\pi^2}\int_0^\infty dq^2 q^2 \Pi_w(q^2). \quad \text{(A11b)}$$

Finally, we observe that the Dirac structure in Eq. (A10b) is identical to that of $\Pi_{LR}(q^2)$ in QCD, and therefore the proof in Ref. [14] that $\Pi_{LR}(q^2) \geq 0$ applies to $\Pi_w(q^2)$ as well, with the consequence that $C_w > 0$. We note that the first of these two integrals is UV divergent, but the second, being an order parameter, is finite.

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