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’t Hooft operators on an interface and bubbling D5-branes

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We consider a brane configuration consisting of a D5-brane, D1-branes and D3-branes. According to the anti-de Sitter/conformal field theory (AdS/CFT) correspondence this system realizes a ’t Hooft operator embedded in the interface in the gauge theory side. In the gravity side the near-horizon geometry is $\mathrm{AdS}_5 \times S^5$. The D5-brane is treated as a probe in the $\mathrm{AdS}_5 \times S^5$ and the D1-branes become the gauge flux on the D5-brane. We examine the condition for preserving an appropriate amount of supersymmetry and derive a set of differential equations which is the sufficient and necessary condition. This supersymmetric configuration shows bubbling behavior. We try to derive the relation between the probe D5-brane and the Young diagram which labels the corresponding ’t Hooft operator. We propose the dictionary of the correspondence between the Young diagram and the probe D5-brane configuration.

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I. INTRODUCTION

Nonlocal operators play an important role in studying the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1]. These operators are classified by the dimensionalities of the operators. There must be a string theory counterpart of each operator according to the AdS/CFT correspondence. For example, a Wilson loop operator [2] is a 1-dimensional nonlocal operator and corresponds to a fundamental string [3,4] or a probe D-brane [3,5–8]. A surface operator is a 2-dimensional nonlocal operator and corresponds to a D3-brane [9–12]. An example of a 3-dimensional nonlocal operator is “an interface” [13–26].

The Wilson loop operator has a bubbling geometry description in the gravity side [27–30] which is an analogue of the bubbling AdS geometry for local operators [31]. In this case the total geometry is described as a fiber bundle over 2-dimensional base space. This base space with the boundary carries the information of the representation of the Wilson loop, or the Young diagram.

On the other hand, the interface is a 3-dimensional nonlocal operator in the gauge theory. It is known that in the AdS/CFT scenario this operator is introduced by adding a probe D5-brane to the original multiple D3-brane system [16]. As a result, some of the D3-branes can end on the D5-brane. The gauge theory realized from this system consists of two gauge theories with different gauge groups divided by a wall—the interface.

In this paper we consider another type of 1-dimensional nonlocal operators—’t Hooft operators [32–35] on the interface. They have magnetic charges while the Wilson operators have electric charges. The ’t Hooft operators correspond to D1-branes in the string theory. So we can construct the system consisting of D3-branes, a D5-brane and D1-branes so that the supersymmetry is preserved as shown later. The expected correspondence is as follows. In the previous bubbling geometry scenario the boundary of the bubbling structure is related to the Young diagram which classifies the Wilson operators. In the same way we expect the boundary of the bubbling D5-brane is related to the Young diagram which classifies the ’t Hooft operators. We note that the world volume of the probe D5-brane has the bubbling structure while in the bubbling geometries the spacetime geometries have the bubbling structure. Our goal is to relate the D1-brane system corresponding to the ’t Hooft operator to the Young diagram using the probe D5-brane.

We examine the supersymmetry condition of the D1-D5 bound state in the $\mathrm{AdS}_5 \times S^5$ and obtain a set of differential equations which determines the configuration of the D5-brane world volume embedding and the gauge flux on it. To solve these equations we require the boundary condition. This condition determines the shape of the Young diagram.

The outline of this paper is as follows. In Sec. II we introduce the brane configuration used in our investigation. In Sec. III, we study that the structure of the D-branes is restricted by the condition for preserving supersymmetry and derive a set of differential equations. In Sec. IV we carefully look at the equations determining the brane structure and find the independent smaller set of equations. In Sec. V we investigate the boundary of the bound state of the multiple brane system in order to see how to relate the brane system to the Young diagram. In Sec. VI we summarize the result of this paper and propose future works.

II. BRANE CONFIGURATION

The AdS/CFT correspondence with a probe D5-brane has been studied in [16]. Let us first briefly review this correspondence. This system consists of N D3-branes and a

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TABLE I. The brane system. In this table "$\circ$" denotes the directions along which branes extend.

	0	1	2	3	4	5	6	7	8	9
D3	$\circ$	$\circ$	$\circ$	$\circ$						
D5	$\circ$	$\circ$	$\circ$		$\circ$	$\circ$	$\circ$			
D1	$\circ$				$\circ$					

D5-brane. The D3-branes extend along the directions 0123 in 10-dimensional spacetime and the D5-brane extends 012456 (see Table I). The D5-brane does not extend in the direction 3, so D3-branes can end on the D5-brane in this direction. Let k D3-branes out of N end on this D5-branes, and suppose $k \ll N$. This system can be seen from two different points of view: the gravity side and the gauge theory side. These two theories are conjectured to be equivalent.

In the gravity side, these multiple D3-branes warp the spacetime and give rise to $\mathrm{AdS}_5 \times S^5$ spacetime in the near horizon limit. Meanwhile, the backreaction of the D5-brane is negligible, and therefore the D5-brane is treated as a probe brane. Consequently, this system describes the superstring theory with the probe D5-brane in the $\mathrm{AdS}_5 \times S^5$.

In the gauge theory side, the D5-brane is regarded as a wall between gauge theories with different gauge groups $SU(N)$ and $SU(N-k)$ where N is the total number of the D3-branes and k is the number of D3-branes which end on the D5-brane. This wall gives the boundary condition of each gauge theory and is called "an interface."

In this paper we would like to insert a 't Hooft operator on the interface in the gauge theory. This corresponds to adding D1-branes ending on the D3-branes in string theory. The total system is then made of N D3-branes, a D5-brane and D1-branes as shown in Table I.

Similar to the previous case, the D3-branes forming the spacetime give $\mathrm{AdS}_5 \times S^5$ geometry, while the D5-brane and the D1-branes are treated as probes. The D1-branes are embedded as a world volume flux in the D5-brane and there is a symmetry $U(1) \times U(1) \times SO(3)$ related to the rotations in the directions 12, 56 and 789, respectively. This configuration preserves $1/4$ of original supersymmetry in the near-horizon.

III. CONDITION FOR SUSY

In this section we study the supersymmetric embedding of the D5-brane. First, we investigate the supersymmetry (SUSY) of the bulk spacetime $\mathrm{AdS}_5 \times S^5$. Second, a part of the bulk supersymmetry is broken when a D-brane is added. The remaining supersymmetry is analyzed by the kappa symmetry projection. Finally, this condition gives the restriction to the embedding of the D5-brane in the bulk spacetime.

A. Supersymmetry in the bulk spacetime

We first consider the supersymmetry of the bulk spacetime $\mathrm{AdS}_5 \times S^5$. We concentrate on the S^2 part of the S^5. The metric is

$$ds^2 = \frac{1}{y^2}(-dt^2 + dy^2 + dr^2 + r^2 d\psi^2 + dx_3^2) + d\theta^2 + \sin^2\theta d\phi^2. \tag{3.1}$$

The Ramond-Ramond five-form flux $F^{(5)}$ takes nonzero value:

$$F^{(5)} = 4(\mathrm{vol}(\mathrm{AdS}_5) + \mathrm{vol}(S^5)), \tag{3.2}$$

where $\mathrm{vol}(\mathrm{AdS}_5)$ and $\mathrm{vol}(S^5)$ are volume forms of AdS_5 and S^5, respectively. Here we use the unit where the radius of the AdS spacetime equals unity. The condition for preserving supersymmetry is that the supersymmetry transformations of the fermions are zero. The dilatino condition is trivially satisfied in the above background, while the gravitino condition for the SUSY parameter ϵ can be written as

$$\nabla_M \epsilon + \frac{i}{2^4}\Gamma^{M_1 M_2 \ldots M_5} F^{(5)}_{M_1 M_2 \ldots M_5} \Gamma_M \epsilon = 0. \tag{3.3}$$

Here the covariant derivative is defined as

$$\nabla_M = \partial_M + \frac{1}{4}\Omega_M{}^{AB}\Gamma_{AB}, \tag{3.4}$$

where $\Omega_M{}^{AB}$ are the spin connections which are related to the vielbein E^M, $M = 0, 1, \ldots 9$ of the metric (3.1) as

$$\begin{aligned} &dE^A = -\Omega^A{}_B E^B, \qquad \Omega^{AB} = -\Omega^{BA}, \\ &\Omega^{AB} = \Omega_M{}^{AB} E^M. \end{aligned} \tag{3.5}$$

We choose the vielbein as

$$\begin{aligned} &E^0 = \frac{dt}{y}, \quad E^1 = \frac{dr}{y}, \quad E^2 = \frac{r d\psi}{y}, \quad E^3 = \frac{dx_3}{y}, \\ &E^4 = \frac{dy}{y}, \quad E^5 = d\theta, \quad E^6 = \sin\theta d\phi. \end{aligned} \tag{3.6}$$

For the detailed calculation of Eq. (3.3), see Appendix A. Here we only show the result. The bulk space preserves the SUSY generated by the parameter

$$\epsilon = e^{-\frac{\theta}{2}\gamma\Gamma_{45}} e^{\frac{\phi}{2}\Gamma_{56}} e^{-\frac{1}{2}\ln y \cdot \gamma} e^{r\frac{1+\gamma}{2}\Gamma_{14}} e^{x_3\frac{1+\gamma}{2}\Gamma_{34}} e^{t\frac{1+\gamma}{2}\Gamma_{04}} e^{\frac{\psi}{2}\Gamma_{12}} \epsilon_0, \tag{3.7}$$

where $\gamma = -i\Gamma_{0123}$ and ϵ_0 is a constant spinor as we calculated in Appendix A.

B. Ansatz for D5-brane

We consider a bound state of a D5-brane and D1-branes in the $\mathrm{AdS}_5 \times S^5$ spacetime. The D1-branes are realized as the world volume gauge flux on the D5-brane. Thus we consider a probe D5-brane with the world volume gauge flux. We define the world volume coordinates of the D5-brane as $(t, y, \psi, \phi, u_1, u_2)$ where the coordinates (t, y, ψ, ϕ) are identified with the coordinates of the bulk spacetime. According to the symmetry $U(1)^2 \times SO(3)$ we put the ansatz on the embedding as:

$$r = ys(u), \qquad x_3 = yz(u), \qquad \theta = \theta(u), \tag{3.8}$$

where $s(u)$, $z(u)$ and $\theta(u)$ are unknown functions of coordinates u^i, $i = 1, 2$. Since (u^1, u^2) are not fixed yet, there remains the general coordinate transformation symmetry of (u^1, u^2). Some of the D3-branes end on the D5-brane. Thus the ansatz for the world volume gauge flux is written as

$$\mathcal{F} = dP \wedge d\psi + dQ \wedge d\phi, \tag{3.9}$$

where potentials P and Q are functions of u. Then we have unknown functions of u

$$s(u), \qquad z(u), \qquad \theta(u), \qquad P(u), \qquad Q(u). \tag{3.10}$$

Our goal is to determine these functions.

C. Condition for SUSY

In this subsection we try to obtain the condition for preserving supersymmetry. When a Dp-brane exists, a part of the original supersymmetry is broken. The remaining supersymmetry parameters are spinors of the form (3.7) which satisfy the relation [36–42]

$$\Gamma\epsilon = \epsilon. \tag{3.11}$$

This is called "the kappa symmetry projection" where the operator Γ is determined for a Dp-brane as

$$d^{p+1}\xi \cdot \Gamma := (-e^{-\Phi}(-\det(G_{\mathrm{ind}} + \mathcal{F}))^{-1/2} e^{\mathcal{F}}\chi)|_{(p+1)-\mathrm{form}}, \tag{3.12}$$

$$\chi := \sum_n \frac{1}{(2n)!} \hat{E}^{a_s} \ldots \hat{E}^{a_1} \Gamma_{a_1 \ldots a_s} K^n(-i), \tag{3.13}$$

where ξ^i, $i = 0, \ldots, p$, are world volume coordinates, Φ is the dilaton, G_{ind} is the induced metric of the Dp-brane and $\hat{E}^A$ is the pullback of E^A defined as $\hat{E}^A := E^A_M \frac{\partial X^M}{\partial \xi^i} d\xi^i$. We calculated examples for a D5-brane and for a D1-brane in Appendix B and we use the relations obtained in these examples in the following calculation.

We calculate the kappa symmetry projection operator Γ defined above under our ansatz given in Sec. III B. Here we only show the result

$$\Gamma = \frac{1}{W}\{s \sin\theta \mathcal{A}\Gamma_{62} K(-i)\Gamma_{04} + \sin\theta \mathcal{B}(-i)\Gamma_{60} - s\mathcal{C}(-i)\Gamma_{20} + \mathcal{D}K(-i)\Gamma_{04}\}. \tag{3.14}$$

For the detailed calculation, see Appendix C. Here we defined a y independent function W as

$$W := y^2\sqrt{-\det(G_{\mathrm{ind}} + \mathcal{F})}. \tag{3.15}$$

The induced metric for the D5-brane is

$$\begin{aligned} ds^2 &= -\frac{1}{y^2} dt^2 + s^2 d\psi^2 + \sin^2\theta d\phi^2 + \frac{\beta}{y^2} dy^2 \\ &\quad + h_{ij} du^i du^j + \frac{\partial_a \beta}{y} du^a dy, \\ \beta &:= 1 + s^2 + z^2, \qquad h_{ij} := \sum_{\lambda = s,z,\theta} \partial_i \lambda \partial_j \lambda. \end{aligned} \tag{3.16}$$

In the expression (3.14), $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are the following matrices.

$$\mathcal{A} := -\{s,z\}\Gamma_{13} - \{s,\theta\}\Gamma_{15} - \{z,\theta\}\Gamma_{35} + s^2\left\{\frac{z}{s}, \theta\right\}\Gamma_{1345}, \tag{3.17a}$$

$$\mathcal{B} := -\left\{P, \frac{z}{s}\right\}\Gamma_{13} + \{P,s\}\Gamma_{14} + \{P,z\}\Gamma_{34} - s\{P,\theta\}\Gamma_{15} - z\{P,\theta\}\Gamma_{35} - \{P,\theta\}\Gamma_{45}, \tag{3.17b}$$

$$\mathcal{C} := -\left\{Q, \frac{z}{s}\right\}\Gamma_{13} + \{Q,s\}\Gamma_{14} + \{Q,z\}\Gamma_{34} - s\{Q,\theta\}\Gamma_{15} - z\{Q,\theta\}\Gamma_{35} - \{Q,\theta\}\Gamma_{45}, \tag{3.17c}$$

$$\mathcal{D} := -\{P,Q\}(1 + s\Gamma_{14} + z\Gamma_{34}), \tag{3.17d}$$

where $\mathcal{C}$ is obtained from $\mathcal{B}$ by replacing all P's by Q's. We use the notation of "Poisson bracket"

$$\{A,B\} := \epsilon^{ab}\frac{\partial A}{\partial u^a}\frac{\partial B}{\partial u^b} = \frac{\partial A}{\partial u^1}\frac{\partial B}{\partial u^2} - \frac{\partial A}{\partial u^2}\frac{\partial B}{\partial u^1}. \tag{3.18}$$

Under our ansatz the parameter ϵ in Eq. (3.7) is decomposed by the dependence of y and t:

$$\begin{aligned}\epsilon &= e^{-\frac{\theta}{2}\gamma\Gamma_{45}}e^{\frac{\phi}{2}\Gamma_{56}}e^{-\frac{1}{2}\ln y\cdot\gamma}e^{r\frac{1+\gamma}{2}\Gamma_{14}}e^{x_3\frac{1+\gamma}{2}\Gamma_{34}}e^{t\frac{1+\gamma}{2}\Gamma_{04}}e^{\frac{\psi}{2}\Gamma_{12}}\epsilon_0\\
&= e^{-\frac{\theta}{2}\gamma\Gamma_{45}}e^{-\frac{1}{2}\ln y\cdot\gamma}\left(1+ys\frac{1+\gamma}{2}\Gamma_{14}\right)\left(1+yz\frac{1+\gamma}{2}\Gamma_{34}\right)\left(1+t\frac{1+\gamma}{2}\Gamma_{04}\right)\xi\\
&= e^{-\frac{\theta}{2}\gamma\Gamma_{45}}e^{-\frac{1}{2}\ln y\cdot\gamma}(\xi+ys\Gamma_{14}\xi_-+yz\Gamma_{34}\xi_-+t\Gamma_{04}\xi_-)\\
&= e^{-\frac{\theta}{2}\gamma\Gamma_{45}}\left(\frac{1}{\sqrt{y}}\xi_++\sqrt{y}\xi_-+\frac{1}{\sqrt{y}}(ys\Gamma_{14}\xi_-+yz\Gamma_{34}\xi_-+t\Gamma_{04}\xi_-)\right)\\
&=: \sqrt{y}\epsilon_1+\frac{1}{\sqrt{y}}\epsilon_2+\frac{t}{\sqrt{y}}\epsilon_3,\end{aligned}\tag{3.19}$$

where we define $\xi := e^{\frac{\phi}{2}\Gamma_{56}}e^{\frac{\psi}{2}\Gamma_{12}}\epsilon_0$ in the second line and $\xi_\pm := \frac{1\pm\gamma}{2}\xi$. The explicit forms of $\epsilon_1,\epsilon_2,\epsilon_3$ are written as

$$\epsilon_1 = e^{-\frac{\theta}{2}\gamma\Gamma_{45}}(1+s\Gamma_{14}+z\Gamma_{34})\xi_-,\tag{3.20a}$$

$$\epsilon_2 = e^{-\frac{\theta}{2}\gamma\Gamma_{45}}\xi_+,\tag{3.20b}$$

$$\epsilon_3 = e^{-\frac{\theta}{2}\gamma\Gamma_{45}}\Gamma_{04}\xi_-.\tag{3.20c}$$

Since the kappa symmetry operator of Eq. (3.14) is independent of y and t, we can impose the projection condition (3.11) for each ϵ_i :

$$\Gamma\epsilon_i = \epsilon_i,\qquad i=1,2,3.\tag{3.21}$$

The kappa symmetry projections for the D5-brane and the D1-brane give the conditions (B13) and (B18), respectively, which are obtained in Appendix B.

$$\text{D5 condition}\Leftrightarrow(K\Gamma_{3456}+\gamma)\xi=0,\tag{3.22}$$

$$\text{D1 condition}\Leftrightarrow(iK\Gamma_{04}-1)\xi=0.\tag{3.23}$$

We want to obtain the condition for the functions (3.10) such that all spinors restricted by the Eqs. (3.22) and (3.23) satisfy the projection condition (3.21). The condition (3.21) is equivalent to

$$\begin{aligned}e^{\frac{\theta}{2}\gamma\Gamma_{45}}\{&s\sin\theta\mathcal{A}\Gamma_{62}K(-i)\Gamma_{04}+\sin\theta\mathcal{B}(-i)\Gamma_{60}-s\mathcal{C}(-i)\Gamma_{20}\\
&+\mathcal{D}K(-i)\Gamma_{04}-W\}\epsilon_i=0,\qquad i=1,2,3.\end{aligned}\tag{3.24}$$

For ϵ_2 (3.20b),

$$\begin{aligned}(3.24)\Leftrightarrow\{&s\sin\theta\mathcal{A}\cdot\Gamma_{51}+\sin\theta\mathcal{B}\Gamma_{53}e^{\theta\Gamma_{45}}-s\mathcal{C}\cdot\Gamma_{31}\\
&+\mathcal{D}e^{\theta\Gamma_{45}}-W\}\xi_+=0,\end{aligned}\tag{3.25}$$

where we used relations obtained from (3.22), (3.23) and $\gamma\xi_\pm=-i\Gamma_{0123}\xi_\pm=\pm\xi_\pm$,

$$\Gamma_{62}\xi_\pm=\Gamma_{51}\xi_\pm,\tag{3.26a}$$

$$\Gamma_{60}\xi_\pm=\pm i\Gamma_{53}\xi_\pm,\tag{3.26b}$$

$$\Gamma_{20}\xi_\pm=\pm i\Gamma_{31}\xi_\pm.\tag{3.26c}$$

The left-hand side of (3.25) can be written only by using $\Gamma_1,\Gamma_3,\Gamma_4,\Gamma_5$ and 1 (identity matrix) and their products. Each coefficient of independent matrices gives the conditions:

$$\begin{aligned}&s\{s,\cos\theta\}-\sin\theta\{P,z\sin\theta\}+s^3\left\{Q,\frac{z}{s}\right\}\\
&-\cos\theta\{P,Q\}-W=0,\end{aligned}\tag{3.27a}$$

$$s\{z,\theta\}-\{P,s\sin\theta\}=0,\tag{3.27b}$$

$$s\sin^2\theta\left\{P,\frac{z}{s}\right\}+\{Q,z\}+\cos\theta\{P,Q\}=0,\tag{3.27c}$$

$$s^2\sin\theta\cos\theta\left\{P,\frac{z}{s}\right\}+sz\{Q,\theta\}-s\sin\theta\{P,Q\}=0,\tag{3.27d}$$

$$\begin{aligned}&s^3\left\{\frac{z}{s},\cos\theta\right\}+\frac{1}{2}\{P,\cos^2\theta\}-s\{Q,s\}+z\cos\theta\{P,Q\}\\
&=0,\end{aligned}\tag{3.27e}$$

$$\{P,z\cos\theta\}-\{P,Q\}=0,\tag{3.27f}$$

$$\sin\theta\{P,s\cos\theta\}-s\{Q,\theta\}=0.\tag{3.27g}$$

In this equation (3.27d) is not independent and can be lead from (3.27f) and (3.27g). For ϵ_3, a similar calculation gives the same conditions. For ϵ_1, the calculation is a bit complicated, but we can do it in the same way.

$$-\frac{s^4}{2}\left\{\frac{\beta}{s^2},\cos\theta\right\}+\frac{z^3\sin^2\theta}{2}\left\{P,\frac{\beta}{z^2}\right\}-\beta\cos\theta\{P,Q\}-W=0, \tag{3.28a}$$

$$\frac{sz^3}{2}\left\{\frac{\beta}{z^2},\cos\theta\right\}+\frac{s^3\sin^2\theta}{2}\left\{P,\frac{\beta}{s^2}\right\}+\frac{s}{2}\{Q,\beta\}=0, \tag{3.28b}$$

$$\frac{1}{2}\{\beta,\cos\theta\}-\frac{z^3}{2}\left\{Q,\frac{\beta}{z^2}\right\}-W=0, \tag{3.28c}$$

$$\frac{1}{2}\{P,\beta\sin^2\theta\}-\frac{s^4}{2}\left\{Q,\frac{\beta}{s^2}\right\}+zW=0, \tag{3.28d}$$

$$\cos\theta\{s^2,z\}+\{P,\beta\cos^2\theta\}=0, \tag{3.28e}$$

$$\frac{1}{4}\{s^2,z^2\}+\frac{z^3\cos\theta}{2}\left\{P,\frac{\beta}{z^2}\right\}+\beta\{P,Q\}=0, \tag{3.28f}$$

$$s\sin\theta\{s,z\}+\frac{s^2\sin\theta\cos\theta}{2}\left\{P,\frac{\beta}{s^2}\right\}+\beta\{Q,\theta\}=0. \tag{3.28g}$$

Consequently, we obtain the 14 equations (3.27a)–(3.27g) and (3.28a)–(3.28g). We find an independent set of these equations in the next section.

IV. SOLVING SUSY CONDITION

One can check the last seven equations (3.28a)–(3.28g), are derived from Eqs. (3.27a)–(3.27g). So we only have to consider Eqs. (3.27a)–(3.27g) which are rewritten as

$$\{s,z\}=-\frac{1}{s\cos\theta}\{P,\beta\cos^2\theta\}, \tag{4.1a}$$

$$\{s,\theta\}=-\frac{1}{s}\{P,z\sin\theta\}+\frac{s^2}{\sin\theta}\left\{Q,\frac{z}{s}\right\}-\frac{1}{s}\cot\theta\{P,Q\}-\frac{1}{s\sin\theta}W, \tag{4.1b}$$

$$\{z,\theta\}=\frac{1}{s}\{P,s\sin\theta\}, \tag{4.1c}$$

$$\{Q,s\}=s^2\left\{\frac{z}{s},\cos\theta\right\}+\frac{1}{2s}\{P,\cos^2\theta\}+\frac{1}{2s}\{P,z^2\cos^2\theta\}, \tag{4.1d}$$

$$\{Q,z\}=-s\sin^2\theta\left\{P,\frac{z}{s}\right\}-\cos\theta\{P,z\cos\theta\}, \tag{4.1e}$$

$$\{Q,\theta\}=\frac{\sin\theta}{s}\{P,s\cos\theta\}, \tag{4.1f}$$

$$\{P,Q\}=\{P,z\cos\theta\}. \tag{4.1g}$$

By the definition of the Poisson bracket (3.18), the bracket can be rewritten in terms of differential forms as

$$\{A,B\}du^1\wedge du^2=\partial_iA\partial_jB\epsilon^{ij}du^1du^2=dA\wedge dB=d(A\wedge dB) \tag{4.2}$$

Then Eqs. (4.1a)–(4.1g) are expressed in terms of differential forms as follows.

$$d(\sqrt{\beta}(dz-\cos\theta dP))=0, \tag{4.3a}$$

$$sds\wedge d(\cos\theta)-\sin\theta dP\wedge d(z\cos\theta)+s^3dQ\wedge d\left(\frac{z}{s}\right)-\cos\theta dP\wedge dQ-Wdu^1\wedge du^2=0, \tag{4.3b}$$

$$sdz\wedge d(\cos\theta)+\sin\theta dP\wedge d(s\sin\theta)=0, \tag{4.3c}$$

$$d(P+Q)\wedge\frac{dz}{z}-\frac{\sin^2\theta}{s}dP\wedge ds-\sin\theta dP\wedge d(\sin\theta)=0, \tag{4.3d}$$

$$sdQ\wedge ds-\frac{1}{2}dP\wedge d((z^2+1)\cos^2\theta)-s^3d\left(\frac{z}{s}\right)\wedge d(\cos\theta)=0, \tag{4.3e}$$

$$sdQ\wedge d(\cos\theta)+\sin^2\theta dP\wedge d(s\cos\theta)=0, \tag{4.3f}$$

$$d(dP(Q-z\cos\theta))=0. \tag{4.3g}$$

Since Eq. (4.3f) can be written as a total derivative, it is expressed as the derivative of a appropriate function ω according to Poincaré's lemma:

$$d\left(-(Q+\sin^2\theta P)\frac{d\theta}{\sin\theta\cos\theta}+P\frac{ds}{s}\right)=0$$
$$\Leftrightarrow-(Q+\sin^2\theta P)\frac{d\theta}{\sin\theta\cos\theta}+P\frac{ds}{s}=d\omega. \tag{4.4}$$

Eqs. (4.3c), (4.3g) lead to the relation

$$z\cos\theta=P+Q. \tag{4.5}$$

Furthermore, Eqs. (4.3d) and (4.3e) are equivalent to Eqs. (4.3g) and (4.3a), respectively. We also substitute the explicit form of W into Eq. (4.1b). Then our equations are simplified as follows.

$$d(\sqrt{\beta}(dz-\cos\theta dP))=0, \tag{4.6a}$$

$$d\left(-(Q+\sin^2\theta P)\frac{d\theta}{\sin\theta\cos\theta}+P\frac{ds}{s}\right)=0, \tag{4.6b}$$

$$z\cos\theta=P+Q, \tag{4.6c}$$

$$\left(\frac{s^2}{\cos^2\theta}+1\right)\{P,\cos\theta\}^2+\frac{s^2}{\cos^2\theta}\{Q,\theta\}^2+\frac{s}{\cos\theta}\{s,\cos\theta\}(\{P,Q\}-z\{P,\cos\theta\})+z\{P,\cos\theta\}\{P,Q\}+2\frac{s^2}{\cos^2\theta}\{P,\cos\theta\}\{Q\cos\theta\}=0. \tag{4.6d}$$

This is one of the main results of this paper.

A. Special case

Let us check the consistency of these equations in the well-known case [16] where

$$P=0,\qquad Q=\kappa\cos\theta,\qquad z=\kappa. \tag{4.7}$$

We can easily check that this configuration satisfies Eqs. (4.6a)–(4.6d).

This configuration contains no D1-brane and corresponds to the 't Hooft operator with the trivial Young diagram.

V. BOUNDARY BEHAVIOR

We have to give boundary conditions to solve Eqs. (4.6a)–(4.6d). The boundary of the u-plane [the base 2-dimensional space coordinated by (u^1,u^2)] is given by $s=0$ or $\sin\theta=0$. The boundary condition is not arbitrary and it contains the detailed information of the associated operators in the gauge theory as in [27–31]. We explain the relation between the boundary behavior of our system and Young diagrams which label the 't Hooft operators.

The structure of the D5-brane world volume is a fiber bundle over the u-plane with the fiber $S^1\times S^1$ coordinated by ϕ and ψ. Each point of the boundary is distinguished by whether $s=0$ or $\sin\theta=0$ and the boundary is divided into segments as shown in Fig. 1. Let I_i, $i=1,\ldots,\ell$ denote the ith $s=0$ segment and J_j, $j=1,\ldots,\ell-1$ denote the jth $\sin\theta=0$ segment. The pullback $dP|_{I_i}$ vanishes and P is a constant P_i on I_i for smoothness since $d\psi$ is singular at I_i and $dP\wedge d\psi$ must vanish. The pullback $dQ|_{J_j}$ also vanishes and Q is a constant Q_j on J_j in the same way. Thus the gauge flux reduces to $\mathcal{F}=dQ\wedge d\phi$ at I_i and $\mathcal{F}=dP\wedge d\psi$ at J_j. At each internal point on I_i the fiber reduces to S^1 coordinated by ϕ and at both endpoints of I_i the radius of this S^1 fiber vanishes. Therefore these S^1 fibers make a noncontractible S^2 cycle denoted by S_i^2. There is also a noncontractible S^2 cycle (denoted by $\tilde{S}_j^2$) on J_j in the same way.

The charge is defined as the integration of the flux on each noncontractible S^2 and we define these quantities as

$$n_i:=\frac{\sqrt{\lambda}}{(2\pi)^2}\int_{S_i^2}dQ\wedge d\varphi=\frac{\sqrt{\lambda}}{2\pi}\int_{I_i}dQ=\frac{\sqrt{\lambda}}{2\pi}(Q_i-Q_{i-1}), \tag{5.1}$$

$$m_j:=\frac{\sqrt{\lambda}}{(2\pi)^2}\int_{\tilde{S}_j^2}dP\wedge d\psi=\frac{\sqrt{\lambda}}{2\pi}\int_{J_j}dP=\frac{\sqrt{\lambda}}{2\pi}(P_{j+1}-P_j). \tag{5.2}$$

Here Q_0 is defined as the value of Q on the first $\theta=0$ half line J_0. The normalization is determined so that n_i and m_j are integers as follows. In a general D5-brane with world volume flux the number of the D3-branes and the number of the D1-branes are calculated by the integration of the gauge flux as seen from the Wess-Zumino term of the D5-brane action.

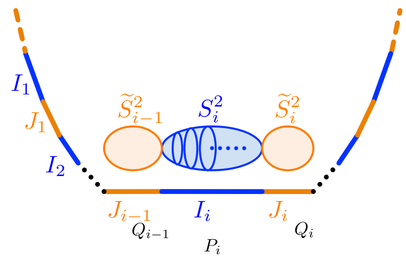

Fig. 1 (color online). The boundary line and 2-spheres composed of ψ and ϕ-cycles.

$$(\text{number of D3-branes}) = \frac{T_5}{T_3}\int_{\mathcal{M}_2}\mathcal{F} = \frac{1}{(2\pi)^2\alpha'}\int_{\mathcal{M}_2}\mathcal{F}, \tag{5.3}$$

$$\begin{aligned}(\text{number of D1-branes}) &= \frac{T_5}{T_1}\int_{\mathcal{M}_4}\frac{1}{2}\mathcal{F}\wedge\mathcal{F}\\ &= \frac{1}{32\pi^4\alpha'^2}\int_{\mathcal{M}_4}\mathcal{F}\wedge\mathcal{F},\end{aligned} \tag{5.4}$$

where the integral over $\mathcal{M}_2$ or $\mathcal{M}_4$ denotes the integral over the perpendicular directions to D3-branes or D1-branes on the D5-brane world volume. We also use the Dp-brane tension T_p

$$T_p = \frac{1}{(2\pi)^p\alpha'^{(p+1)/2}g_s}, \tag{5.5}$$

and $\alpha' = 1/\sqrt{\lambda}$ in our unit. Here g_s is the string coupling constant.

Since the quantities n_i and m_j are integers, these can be related to the number of boxes in the Young diagram as follows. First we deform the boundary as stepwise by bending it at the edges of each segment. After that deformation this boundary line can be interpreted as the right down edge of the Young diagram as shown in Fig. 2. The integers n_i and m_j correspond to each length of the edge of the Young diagram.

Let us consider the relation between the number of branes and the Young diagram for a consistency check. The number of the D3-branes ending on the D5-brane, denoted by k, is related to the vertical length of the Young diagram as follows.

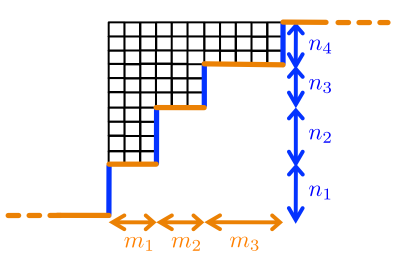

Fig. 2 (color online). The relation between a deformed boundary line and the Young diagram.

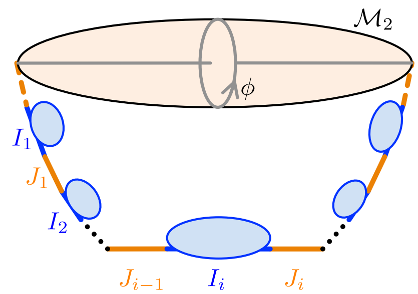

Fig. 3 (color online). $\mathcal{M}_2$ is a 2-dimensional manifold located sufficiently far from the center. It can be deformed into 2-spheres located in the boundary without changing the value of the integral.

$$\begin{aligned}k &= \frac{\lambda}{4\pi^2}\int_{\mathcal{M}_2}\mathcal{F}\\ &= \frac{\lambda}{4\pi^2}\sum_i\int_{S_i^2}dQ\wedge d\phi\\ &= \frac{\lambda}{2\pi}\sum_i\int_{I_i}dQ\\ &= \sum_i n_i,\end{aligned} \tag{5.6}$$

where $\mathcal{M}_2$ is a 2-cycle shown in Fig. 3.

On the other hand, the number of the D1-branes k' can be interpreted as the total number of boxes in the Young diagram which characterize the boundary condition as expected. This relation is derived as follows.

$$\begin{aligned}k' &= \frac{\lambda}{32\pi^4}\int_{\mathcal{M}_4}2dP\wedge d\psi\wedge dQ\wedge d\varphi\\ &= -\frac{\lambda}{4\pi^2}\int_{u-\text{plane}}dP\wedge dQ\\ &= -\frac{\lambda}{4\pi^2}\int_{u-\text{plane}}d(P\wedge dQ)\\ &= -\frac{\lambda}{4\pi^2}\int_{\partial(u-\text{plane})}P\wedge dQ\\ &= -\frac{\lambda}{4\pi^2}\sum_i P_i\int_{I_i}dQ\\ &= -\frac{\lambda}{4\pi^2}\sum_{i\geq 2}\left(\sum_{j\leq i-1}\frac{2\pi}{\sqrt{\lambda}}m_j\right)\left(\frac{2\pi}{\sqrt{\lambda}}n_i\right)\\ &= -\sum_{i\geq 2}\left(\sum_{j\leq i-1}m_j\right)n_i\\ &= -\left(\sum_{j\leq 1}m_j\right)n_2 - \left(\sum_{j\leq 2}m_j\right)n_3\ldots.\end{aligned} \tag{5.7}$$

Here $\mathcal{M}_4$ is a 4-cycle coordinated by u^1, u^2, ψ, ϕ. In the 5th line we used the fact that P is a constant at each I_i. Then in the next line the integral can be rewritten by (5.1) and the potential functions P_i can be translated by adding a constant to all P_i. Using this ambiguity we set $P_1 = 0$. The first term of the final expression (5.7) ($i = 2$) is equal to the number of the boxes in the lowest set of columns of the corresponding Young diagram. The second term is equal to the number of the boxes in the second lowest set of columns, and so forth (Fig. 2).

From the above calculations (5.6), (5.7), we see a correspondence between the brane configuration and the number of the boxes in the Young diagram. Namely, k, the number of the D3-branes ending on the D5-brane, corresponds to the vertical length of the Young diagram, and k', the number of the D1-branes embedded on the D5-brane, is the total number of the boxes in the Young diagram. These are consistent with our conjectured relation.

VI. CONCLUSION

In this paper we propose the relation between the Young diagram and the brane configuration, and find the method to determine the brane configuration from the Young diagram. This relates the shape of the Young diagram, described by integers (5.1) and (5.2), and the brane configuration as follows. These numbers are the number of boxes in the Young diagram. The correspondence is that k, the number of the D3-branes ending on the D5-brane, corresponds to the vertical length of the Young diagram (5.6) and k', the number of the D1-branes embedded on the D5-brane corresponds to the total number of the boxes in the Young diagram. Once we are given a certain 't Hooft operator, we obtain a Young diagram describing that operator. This information about the Young diagram directly requires the boundary condition via n_i and m_j (5.1) and (5.2). Then we relate this operator to a brane configuration according to Eqs. (4.6a)–(4.6d).

We can propose some interesting future works. First, we can try to confirm this correspondence by a concrete calculation as in Sec. IV A for the simplest case.

Second, it is also an interesting future work to calculate physical quantities, such as expectation values of these 't Hooft operators and correlation functions with other operators in the string theory side and the gauge theory side. In the gauge theory side, we can make use of localization technique [43,44]. In the string theory side we can compute these quantities from the classical action of the D5-brane. It will be very interesting to compare these two results and check the AdS/CFT correspondence.

Finally, another interesting application is to consider a deformed 't Hooft operator. In this paper we only consider the simplest path—straight line for the 't Hooft operator. When this path is deformed into a knotted configuration, the brane configuration becomes much more complicated. This topic is related to the knot homology as recently studied in [45,46].

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APPENDIX A: SUSY IN BULK SPACE

We investigate supersymmetry in $\mathrm{AdS}_5 \times S^5$ spacetime with metric

$$ds^2 = \frac{1}{y^2}(-dt^2 + dy^2 + dr^2 + r^2 d\psi^2 + dx_3^2) + d\theta^2 + \sin^2\theta d\phi^2. \tag{A1}$$

In order to preserve supersymmetry, the gravitino transformation must give zero,

$$\nabla_M \epsilon + \frac{i}{2^4}\Gamma^{M_1 M_2 \ldots M_5} F^{(5)}_{M_1 M_2 \ldots M_5} \Gamma_M \epsilon = 0, \tag{A2}$$

$$\nabla_M = \partial_M + \frac{1}{4}\Omega_M{}^{AB}\Gamma_{AB}, \tag{A3}$$

where gamma matrices with indices $M = t, r, \psi, x_3, y, \theta, \phi$ are $\Gamma_t \coloneqq E_t^A \Gamma_A = \frac{1}{y}\Gamma_0$ and so on. Γ_A, $A = 0, \ldots, 9$, are constant gamma matrices in 10-dimensional spacetime. They satisfy $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$ where $\eta_{AB} = \mathrm{diag}(-1, +1, \ldots, +1)$. We use the notation for antisymmetrized products of gamma matrices as

$$\Gamma_{A_1 A_2 \ldots A_n} \coloneqq \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma)\Gamma_{A_{\sigma(1)}}\Gamma_{A_{\sigma(2)}}\ldots\Gamma_{A_{\sigma(n)}}. \tag{A4}$$

The SUSY parameter ϵ is a complex Weyl spinor which satisfies $\Gamma^{01\ldots 9}\epsilon = \epsilon$. In this paper we choose vielbein as

$$E^0 = \frac{dt}{y}, \quad E^1 = \frac{dr}{y}, \quad E^2 = \frac{r d\psi}{y}, \quad E^3 = \frac{dx_3}{y}, \quad E^4 = \frac{dy}{y}, \quad E^5 = d\theta, \quad E^6 = \sin\theta d\phi. \tag{A5}$$

The spin connections $\Omega^{AB} = \Omega_M{}^{AB} E^M$ are related to vielbein as $dE^A = -\Omega^A_B E^B$, and calculated using this relation as follows.

$$\Omega^{04} = -\frac{dt}{y}, \qquad \Omega^{12} = -d\psi, \qquad \Omega^{14} = -\frac{dr}{y},$$
$$\Omega^{24} = -\frac{rd\psi}{y}, \qquad \Omega^{34} = -\frac{dx_3}{y}, \qquad \Omega^{56} = -\cos\theta d\phi, \tag{A6}$$

and the other components are zero. The equations (A2) for 7 components, $M = t,\ r, \psi, x_3, y, \theta, \phi$, are

$$\partial_t \epsilon - \frac{1+\gamma}{2y}\Gamma_{04}\epsilon = 0, \tag{A7a}$$

$$\partial_r \epsilon - \frac{1+\gamma}{2y}\Gamma_{14}\epsilon = 0, \tag{A7b}$$

$$\partial_\psi \epsilon - \frac{1}{2}\Gamma_{12}\epsilon - \frac{1+\gamma}{2y}\Gamma_{24}\epsilon = 0, \tag{A7c}$$

$$\partial_{x_3} \epsilon - \frac{1+\gamma}{2y}\Gamma_{34}\epsilon = 0, \tag{A7d}$$

$$\partial_y \epsilon + \frac{1}{2y}\gamma\epsilon = 0, \tag{A7e}$$

$$\partial_\theta \epsilon + \frac{1}{2}\gamma\Gamma_{45}\epsilon = 0, \tag{A7f}$$

$$\partial_\phi \epsilon - \frac{1}{2}e^{-\gamma\Gamma_{45}}\Gamma_{56}\epsilon = 0, \tag{A7g}$$

where we used the matrix $\gamma \coloneqq -i\Gamma_{0123}$. Solving Eqs. (A7a)–(A7g) we obtain the supersymmetry parameter in the bulk spacetime.

$$\epsilon = e^{-\frac{\theta}{2}\gamma\Gamma_{45}} e^{\frac{\phi}{2}\Gamma_{56}} e^{-\frac{1}{2}\ln y\cdot\gamma} e^{r\frac{1+\gamma}{2}\Gamma_{14}} e^{x_3\frac{1+\gamma}{2}\Gamma_{34}} e^{t\frac{1+\gamma}{2}\Gamma_{04}} e^{\frac{\psi}{2}\Gamma_{12}}\epsilon_0, \tag{A8}$$

where ϵ_0 is an arbitrary constant complex Weyl spinor. For convenience, we define $\xi \coloneqq e^{\frac{\phi}{2}\Gamma_{56}} e^{\frac{\psi}{2}\Gamma_{12}}\epsilon_0$. Then ϵ is rewritten as

$$\epsilon = e^{-\frac{\theta}{2}\gamma\Gamma_{45}} e^{-\frac{1}{2}\ln y\cdot\gamma} e^{r\frac{1+\gamma}{2}\Gamma_{14}} e^{x_3\frac{1+\gamma}{2}\Gamma_{34}} e^{t\frac{1+\gamma}{2}\Gamma_{04}}\xi. \tag{A9}$$

APPENDIX B: KAPPA SYMMETRY PROJECTION FOR D5-BRANES AND D1-BRANES

The kappa symmetry projection [36–42] plays a crucial role in our research. The supersymmetry with the parameters (A9) which satisfy

$$\Gamma\epsilon = \epsilon \tag{B1}$$

survives in the presence of a D-brane. Here the projection operator Γ is defined for a Dp-brane in type IIB string theory as

$$d^{p+1}\xi\cdot\Gamma \coloneqq (-e^{-\Phi}(-\det(G_{\mathrm{ind}} + \mathcal{F}))^{-1/2}e^{\mathcal{F}}\chi)|_{(p+1)-\mathrm{form}}, \tag{B2}$$

$$\chi \coloneqq \sum_n \frac{1}{(2n)!}\hat{E}^{a_{2n}}\ldots\hat{E}^{a_1}\Gamma_{a_1\ldots a_s}K^n(-i), \tag{B3}$$

where ξ^i, $i = 0, \ldots, p$, are world volume coordinates, Φ is the dilaton which is zero now, G_{ind} is the induced metric of the Dp-brane and $\hat{E}^A$ is the pull back of E^A defined as $\hat{E}^A \coloneqq E^A_M \frac{\partial X^M}{\partial \xi^i} d\xi^i$.

In this section we calculate the two cases of them—a D5-brane and a D1-brane. We now consider the situation in $\mathrm{AdS}_5 \times S^5$ spacetime formed by multiple D3-branes with metric

$$ds^2 = \frac{1}{y^2}(-dt^2 + dy^2 + dr^2 + r^2 d\psi^2 + dx_3^2) + d\theta^2 + \sin^2\theta d\phi^2, \tag{B4}$$

where we concentrate on the S^2 part of the S^5 and the AdS radius is set to unity.

1. D5-brane

First, let us consider the D5-brane with ansatz [16,42]

$$x_3 = \kappa y, \qquad \mathcal{F} = f\sin\theta d\theta\wedge d\phi, \qquad (\kappa, f\colon \mathrm{constant}). \tag{B5}$$

The induced metric of the D5-brane with coordinates $(t, r, \psi, y = \frac{1}{\kappa}x_3, \theta, \phi)$ is

$$ds^2_{D5} = \frac{1}{y^2}(-dt^2 + dr^2 + r^2 d\psi^2 + (\kappa^2 + 1)dy^2) + d\theta^2 + \sin^2\theta d\phi^2. \tag{B6}$$

We need to calculate the determinant of

$$G_{\mathrm{ind}} + \mathcal{F} = \begin{bmatrix} -1/y^2 & & & & & \\ & 1/y^2 & & & & \\ & & r^2/y^2 & & & \\ & & & \left(1+\frac{1}{\kappa^2}\right)/y^2 & & \\ & & & & 1 & f\sin\theta \\ & & & & -f\sin\theta & \sin^2\theta \end{bmatrix}, \tag{B7}$$

where all empty components denote zeros. The result is

$$\sqrt{-\det(G_{\mathrm{ind}}+\mathcal{F})} = \frac{r\sin\theta}{y^4}\sqrt{1+1/\kappa^2}\sqrt{1+f^2}. \tag{B8}$$

Since $e^{\mathcal{F}} = 1 + f\sin^2\theta d\theta\wedge d\phi$,

$$\begin{aligned} d^6\xi\cdot\Gamma_{D5} &= \left(-\frac{1}{\sqrt{1+1/\kappa^2}\sqrt{1+f^2}}\frac{y^4}{r\sin\theta}(1+f\sin\theta d\theta\wedge d\phi)\chi\right)\Big|_{6-\mathrm{form}} \\ &= -\frac{1}{\sqrt{1+1/\kappa^2}\sqrt{1+f^2}}\frac{y^4}{r\sin\theta}(\chi|_{6-\mathrm{form}} + f\sin\theta d\theta\wedge d\phi\cdot\chi|_{4-\mathrm{form}}). \end{aligned} \tag{B9}$$

$\chi|_{6-\mathrm{form}}$ and $\chi|_{4-\mathrm{form}}$ are

$$\chi|_{6-\mathrm{form}} = dtdrd\psi dyd\theta d\phi Ki\frac{r\sin\theta}{y^4}\left(\Gamma_{012356} + \frac{1}{\kappa}\Gamma_{012456}\right), \tag{B10}$$

$$\chi|_{4-\mathrm{form}} = dtdrd\psi dy(-i)\frac{r}{y^4}\left(\Gamma_{0123} + \frac{1}{\kappa}\Gamma_{0124}\right). \tag{B11}$$

We obtain the following result by putting them together.

$$\Gamma_{D5} = \frac{-1}{\sqrt{(\kappa^2+1)(f^2+1)}}\gamma(K\Gamma_{56}+f)(\Gamma_{34}+\kappa). \tag{B12}$$

The necessary and sufficient condition for ϵ to satisfy $\Gamma_{D5}\epsilon = \epsilon$ is $\kappa = -f$ and

$$(K\Gamma_{3456}+\gamma)\xi = 0. \tag{B13}$$

2. D1-brane

Next, let us calculate the D1-brane case. The induced metric for the D1-brane with world volume coordinates (t, y) is

$$ds^2_{\mathrm{D1}} = \frac{1}{y^2}(-dt^2+dr^2). \tag{B14}$$

Since the dilaton Φ is zero and there is no flux, $\mathcal{F} = 0$,

$$d^2\xi\cdot\Gamma = -y^2\chi|_{2-\mathrm{form}}. \tag{B15}$$

Substituting

$$\chi|_{2-\mathrm{form}} = -dtdrK(-i)\frac{1}{y^2}\Gamma_{04}, \tag{B16}$$

we obtain

$$\Gamma_{\mathrm{D1}} = \Gamma_{04}K(-i). \tag{B17}$$

The necessary and sufficient condition for satisfying $\Gamma_{D1}\epsilon = \epsilon$ is

$$(iK\Gamma_{04}-1)\xi = 0. \tag{B18}$$

Both the conditions (B13) and (B18) are satisfied in our bound state of a D5-brane and D1-branes.

Appendix C: DERIVATION OF Γ

We calculate Γ defined in (B2) and (B3) for a D5-brane with world volume coordinates $(t, \psi, \phi, y, u^1, u^2)$. There is a flux on the D5-brane,

$$\mathcal{F} = dP(u)\wedge d\psi + dQ(u)\wedge d\phi. \tag{C1}$$

In our situation, the dilation is zero, and

$$\begin{aligned} e^{\mathcal{F}} &= 1 + \partial_a P du^a\wedge d\psi + \partial_a Q du^a\wedge d\phi \\ &\quad - \partial_a P\partial_b Q\epsilon^{ab}d\psi\wedge d\phi\wedge du^1\wedge du^2, \end{aligned} \tag{C2}$$

$$
\begin{aligned}
(e^{\mathcal{F}}\chi)|_{6-\text{from}} &= \chi|_{6-\text{form}} + \chi|_{4-\text{form}}\cdot\partial_a P du^a\wedge d\psi + \chi|_{4-\text{form}}\cdot\partial_a Q du^a\wedge d\phi \\
&\quad + \chi|_{2-\text{form}}\cdot(-\partial_a P\partial_b Q\epsilon^{ab})d\psi\wedge d\phi\wedge du^1\wedge du^2.
\end{aligned}
\tag{C3}
$$

Here the first $\chi|_{4-\text{form}}$ in the expression (C3) is proportional to $dt\wedge d\phi\wedge dy\wedge du^b$, $(b\neq a)$, while the second is proportional to $dt\wedge d\psi\wedge dy\wedge du^b$, $(b\neq a)$ and we use the notation

$$
\{A,B\} \coloneqq \epsilon^{ab}\partial_a A\partial_b B = \frac{\partial A}{\partial u^1}\frac{\partial B}{\partial u^2} - \frac{\partial A}{\partial u^2}\frac{\partial B}{\partial u^1},
\tag{C4}
$$

in the following. Each term of Eq. (C3) is calculated as follows.

$$
\chi|_{6-\text{form}} = d^6\xi\cdot\frac{s\sin\theta}{y^2}\left(\{z,\theta\}\Gamma_{35} + \{s,\theta\}\Gamma_{15} - s^2\left\{\frac{z}{s},\theta\right\}\Gamma_{1345} + \{s,z\}\Gamma_{13}\right)\Gamma_{04}\Gamma_{62}K(-i),
\tag{C5a}
$$

$$
\begin{aligned}
\chi|_{4-\text{form}}\cdot\partial_a P du^a d\psi &= \frac{\sin\theta}{y^2}\left(s^2\left\{P,\frac{z}{s}\right\}\Gamma_{13} - \{P,s\}\Gamma_{14} - \{P,z\}\Gamma_{34} + s\{P,\theta\}\Gamma_{15}\right. \\
&\quad \left. + z\{P,\theta\}\Gamma_{35} + \{P,\theta\}\Gamma_{45}\right)\Gamma_{60}(-i)d^6\xi,
\end{aligned}
\tag{C5b}
$$

$$
\begin{aligned}
\chi|_{4-\text{form}}\cdot\partial_a Q du^a d\psi &= \frac{s}{y^2}\left(-s^2\left\{Q,\frac{z}{s}\right\}\Gamma_{13} + \{Q,s\}\Gamma_{14} + \{Q,z\}\Gamma_{34}\right. \\
&\quad \left. - s\{Q,\theta\}\Gamma_{15} - z\{Q,\theta\}\Gamma_{35} - \{Q,\theta\}\Gamma_{45}\right)\Gamma_{20}(-i)d^6\xi,
\end{aligned}
\tag{C5c}
$$

$$
\chi|_{2-\text{form}}\cdot(-\partial_a P\partial_b Q\epsilon^{ab})d\psi d\phi dt dy = \frac{1}{y^2}(\epsilon^{ab}\partial_a P\partial_b Q)(s\Gamma_{14} + z\Gamma_{34} + 1)\Gamma_{04}K(-i)\cdot d^6\xi,
\tag{C5d}
$$

where $d^6\xi = dt\wedge d\psi\wedge d\phi\wedge dy\wedge du^1\wedge du^2$.

In the definition (B2), $\mathcal{L}_{\text{DBI}}$ is

$$
\mathcal{L}_{\text{DBI}} = \sqrt{-\det(G_{\text{ind}} + \mathcal{F})} = :\frac{W}{y^2}.
\tag{C6}
$$

Under our ansatz, see Eq. (3.8) in Sec. III B, the induced metric G_{ind} is

$$
ds^2_{\text{ind}} = -\frac{1}{y^2}dt^2 + s^2 d\psi^2 + \sin^2\theta d\phi^2 + \frac{\beta}{y^2}dy^2 + h_{ij}du^i du^j + \frac{\partial_a\beta}{y}du^a dy,
\tag{C7}
$$

$$
h_{ij} \coloneqq \sum_{\lambda=s,z,\theta}\partial_i\lambda\partial_j\lambda.
\tag{C8}
$$

We define a convenient variable $\beta\coloneqq 1+s^2+z^2$. W is calculated as the following determinant.

$$
\begin{aligned}
W^2 = -y^4 \det &\begin{bmatrix} -1/y^2 & & & & & \\ & s^2 & & & -J_1 & -J_2 \\ & & \sin^2\theta & & -L_1 & -L_2 \\ & & & \frac{\beta}{y^2} & \frac{1}{2y}\partial_1\beta & \frac{1}{2y}\partial_2\beta \\ & J_1 & L_1 & \frac{1}{2y}\partial_1\beta & h_{11} & h_{12} \\ & J_2 & L_2 & \frac{1}{2y}\partial_2\beta & h_{21} & h_{22} \end{bmatrix} \\
= \det &\begin{bmatrix} s^2 & & & J_1 & -J_2 \\ & \sin^2\theta & & -L_1 & -L_2 \\ & & \beta & \frac{1}{2}\partial_1\beta & \frac{1}{2}\partial_2\beta \\ J_1 & L_1 & \frac{1}{2y}\partial_1\beta & h_{11} & h_{12} \\ J_2 & L_2 & \frac{1}{2y}\partial_2\beta & h_{21} & h_{22} \end{bmatrix},
\end{aligned}
\tag{C9}
$$

where $J_a \coloneqq \partial P/\partial u^a$ and $L_a \coloneqq \partial Q/\partial u^a$. To calculate this determinant the following formula is convenient.

$$
\det\begin{bmatrix} A & D \\ C & B \end{bmatrix} = \det A \cdot \det(B - CA^{-1}D). \tag{C10}
$$

We use this formula for We use this formula for

$$
A = \begin{bmatrix} s^2 & & \\ & \sin^2\theta & \\ & & \beta \end{bmatrix}, \qquad D = \begin{bmatrix} -J_1 & -J_2 \\ -L_1 & -L_2 \\ \frac{1}{2}\partial_1\beta & \frac{1}{2}\partial_2\beta \end{bmatrix}, \qquad C = \begin{bmatrix} J_1 & L_1 & \frac{1}{2}\partial_1\beta \\ J_2 & L_2 & \frac{1}{2}\partial_2\beta \end{bmatrix}, \qquad B = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}.
$$

Then W is written explicitly as

$$
\begin{aligned}
W^2 = {} & s^2\sin^2\theta\{s,z\}^2 \\
& + s^2\sin^2\theta((z^2+1)\{s,\theta\}^2 + (s^2+1)\{z,\theta\}^2 - 2sz\{s,\theta\}\{z,\theta\}) \\
& + \sin^2\theta((z^2+1)\{s,P\}^2 + (s^2+1)\{z,P\}^2 - 2sz\{s,P\}\{z,P\}) \\
& + s^2((z^2+1)\{s,Q\}^2 + (s^2+1)\{z,Q\}^2 - 2sz\{s,Q\}\{z,Q\}) \\
& + \beta\{P,Q\}^2.
\end{aligned}
\tag{C11}
$$

Summarizing the above, the operator Γ is

$$
\Gamma = \frac{1}{W}\{s\sin\theta\mathcal{A}\Gamma_{62}K(-i)\Gamma_{04} + \sin\theta\mathcal{B}(-i)\Gamma_{60} - s\mathcal{C}(-i)\Gamma_{20} + \mathcal{D}K(-i)\Gamma_{04}\}, \tag{C12}
$$

where

$$
\mathcal{A} \coloneqq -\{s,z\}\Gamma_{13} - \{s,\theta\}\Gamma_{15} - \{z,\theta\}\Gamma_{35} + s^2\left\{\frac{z}{s},\theta\right\}\Gamma_{1345}, \tag{C13a}
$$

$$
\mathcal{B} \coloneqq -\left\{P,\frac{z}{s}\right\}\Gamma_{13} + \{P,s\}\Gamma_{14} + \{P,z\}\Gamma_{34} - s\{P,\theta\}\Gamma_{15} - z\{P,\theta\}\Gamma_{35} - \{P,\theta\}\Gamma_{45}, \tag{C13b}
$$

$$\mathcal{C} := -\left\{Q, \frac{z}{s}\right\}\Gamma_{13} + \{Q, s\}\Gamma_{14} + \{Q, z\}\Gamma_{34} - s\{Q, \theta\}\Gamma_{15} - z\{Q, \theta\}\Gamma_{35} - \{Q, \theta\}\Gamma_{45}, \tag{C13c}$$

$$\mathcal{D} := -\{P, Q\}(1 + s\Gamma_{14} + z\Gamma_{34}), \tag{C13d}$$

$\mathcal{C}$ is obtained by replacing all P's in $\mathcal{B}$ by Q's, and W is given by Eq. (C11).

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