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Scattering amplitudes of massive $\mathcal{N} = 2$ gauge theories in three dimensions

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We study the scattering amplitudes of mass-deformed Chern-Simons theories and Yang-Mills-Chern-Simons theories with $\mathcal{N} = 2$ supersymmetry in three dimensions. In particular, we derive the on-shell supersymmetry algebras which underlie the scattering matrices of these theories. We then compute various 3 and 4-point on-shell tree-level amplitudes in these theories. For the mass-deformed Chern-Simons theory, odd-point amplitudes vanish and we find that all of the 4-point amplitudes can be encoded elegantly in superamplitudes. For the Yang-Mills-Chern-Simons theory, we obtain all of the 4-point tree-level amplitudes using a combination of perturbative techniques and algebraic constraints and we comment on difficulties related to computing amplitudes with external gauge fields using Feynman diagrams. Finally, we propose a Britto-Cachazo-Feng-Witten recursion relation for mass-deformed theories in three dimensions and discuss the applicability of this proposal to mass-deformed $\mathcal{N} = 2$ theories.

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I. INTRODUCTION

Over the past few years, there has been a great deal of progress in understanding the scattering amplitudes of three-dimensional gauge theories. The study of scattering amplitudes of Chern-Simons-Matter theories with $\mathcal{N} \geq 4$ supersymmetry and supersymmetric Yang-Mills theories (SYM) with $\mathcal{N} \geq 2$ supersymmetry (initiated in [1] and [2–4], respectively) shows that the S-matrices of three-dimensional supersymmetric gauge theories contain fascinating simplifying aspects that are not manifest in their traditional Lagrangian descriptions. For instance, it was shown in [1] that the four-particle amplitudes of a large family of Chern-Simons-Matter theories have the same formal structure as the scattering matrix of the spin chain that is the large-N dilatation operator of $\mathcal{N} = 4$ SYM in $d = 4$. Furthermore, amplitudes of the $\mathcal{N} = 8$ superconformal Chern-Simons theory known as the Bagger-Lambert-Gustavsson theory [5,6] were studied in [7].

More recently, a BCFW recursion relation [8,9] for three-dimensional gauge theories with massless fields was developed in [10], and used to show that a $\mathcal{N} = 6$ superconformal Chern-Simons theory known as the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [11] has dual superconformal symmetry both at tree [12,13] and loop-level. Dual superconformal symmetry [14–16] is inequivalent to ordinary superconformal symmetry and generates Yangian symmetry when combined with ordinary superconformal symmetry [17]. In 4d $\mathcal{N} = 4$ SYM, dual superconformal symmetry corresponds to the ordinary superconformal symmetry of null-polygonal Wilson loops that are dual to the amplitudes [18–22]. The Yangian symmetry of $\mathcal{N} = 4$ SYM can be made manifest using a Grassmannian integral formula developed in [23]. An analogous formula for the ABJM theory was proposed in [24]. This formula involves an integral over the orthogonal Grassmannian. Some evidence for an amplitude/Wilson loop duality in the ABJM theory was found in [25–27]. Recently, 1-loop amplitudes were computed in the ABJM theory and shown to exhibit new structures which do not appear in 4d $\mathcal{N} = 4$ SYM theory, notably sign functions of the kinematic variables [28–30].

The recursion relation proposed in [10] was also used to show that three-dimensional maximal SYM amplitudes have dual conformal covariance [4]. Note that three-dimensional SYM theories do not have ordinary superconformal symmetry because the Yang-Mills coupling constant is dimensionful in three dimensions. Three-dimensional SYM theories exhibit a number of other surprising properties. In particular, Refs. [3,4] showed that they have helicity structure and Ref. [3] showed that their 4-point amplitudes have enhanced R-symmetry which originates from the duality between scalars and vectors in three dimensions. Furthermore, the loop amplitudes of three-dimensional maximal SYM theory have a similar structure to those of the ABJM theory. In particular, 1-loop corrections are finite or vanish in both theories [4]. Furthermore, the 2-loop 4-point amplitudes of both theories can be matched in the Regge-limit [31]. It was recently shown that three-dimensional supergravity amplitudes can

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be obtained as double copies of both three-dimensional supersymmetric Chern-Simons theories and three-dimensional SYM theories [32,33]. All these remarkable developments provide ample motivation for further investigation into the scattering amplitudes of gauge theories in three spacetime dimensions.

It may be fair to say that most of the investigations mentioned above are largely confined to the studies of massless theories with high degrees of supersymmetry. In this paper, we explore a complementary part of the landscape of $d = 3$ gauge theories from the point of view of scattering amplitudes. Specifically, we investigate mass-deformed $\mathcal{N} = 2$ gauge theories with adjoint matter fields. The two theories that span this category are mass-deformed Chern-Simons theory (hereafter referred to as CSM theory) and Yang-Mills-Chern-Simons theory with $\mathcal{N} = 2$ supersymmetry (YMCS), and we investigate their tree-level color-ordered scattering amplitudes in this paper. Whereas the gauge field has no propagating degrees of freedom in the Chern-Simons theory, in the Yang-Mills-Chern-Simons theory the gauge field has one propagating degree of freedom, which is massive. In particular, the Chern-Simons term provides a topological mass for the gauge field without breaking gauge invariance or locality [34,35].

From the point of view of scattering amplitudes, these theories are interesting for a number of reasons. We find that there are two different on-shell $\mathcal{N} = 2$ superalgebras that can potentially arise as symmetries of these theories. The first of these is the standard $\mathcal{N} = 2$ superalgebra with the schematic structure $\{Q^I, Q^J\} \sim \delta^{IJ} P$; $(I, J) = (1, 2)$. In the case of a flavor $SO(2)$ R symmetry, one can also have a "mass-deformed" algebra where the supercharges close on the momentum as well as the R symmetry generator. Or, schematically, $\{Q^I, Q^J\} \sim \delta^{IJ} P + m\epsilon^{IJ} R$. Such mass-deformed algebras—though rare in the list of all possible superalgebras—have been shown to arise as symmetries of three-dimensional gauge theories in a number of previous investigations [1,36,37]. In the present work we find that the mass-deformed $\mathcal{N} = 2$ algebra is the underlying symmetry algebra for the CSM theory. We find a convenient single particle representation of this algebra and find that all of the 4-point tree-level amplitudes can be encoded in superamplitudes (note that the odd-point amplitudes in the Chern-Simons theory have external legs which are gauge fields and therefore vanish on-shell). While component amplitudes of $\mathcal{N} \geq 4$ massive CSM theories have been studied in great detail (for example in [1]) the $\mathcal{N} = 2$ CSM theory studied in this paper is distinguished from the class of models investigated in [1] by virtue of allowing the matter fields to be in the adjoint representation (which is typically not possible for higher supersymmetry). The component amplitudes obtained in this paper are not known to be obtained by a trivial truncation of the $\mathcal{N} \geq 4$ amplitudes either. Furthermore, to the best of our knowledge the superamplitude presented in this paper for the massive $\mathcal{N} = 2$ theory is the first concrete formulation of a superamplitude for a massive gauge theory in $d = 3$.

In the case of the YMCS theory, we find that the underlying supersymmetry algebra is the *undeformed* one where the supercharges close on momenta alone. This is to be expected as there is no flavor symmetry in the bosonic sector of the theory, since the Lagrangian has only one scalar field. We derive an on-shell representation of this algebra and use it to obtain constraints on 4-point amplitudes (the on-shell algebra does not constrain the 3-point amplitudes). We find that that the relations among the 4-point amplitudes of the YMCS theory are considerably more complicated than those in the CSM theory. The root of the complication has to do with the absence of the extra $SO(2)$ symmetry in the bosonic sector. Nevertheless we are able to compute a number of these amplitudes and verify that the computed amplitudes are consistent with the on-shell algebra. Although we do not compute amplitudes with external gauge fields using Feynman diagrams, we are nevertheless able to deduce the 4-point amplitudes with external gauge fields using the on-shell algebra.

We also propose a BCFW recursion relation for mass-deformed three-dimensional theories which reduces to the proposal in [10] when the mass goes to zero. This recursion relation involves deforming two external legs of an on-shell amplitude by a complex parameter z. In order for the recursion relation to be applicable, the amplitudes must vanish as $z \to \infty$. We show that 4-point superamplitudes of the CSM theory have good large-z behavior, so our proposed recursion relation may be applicable to this theory. However, the proposed relations do not seem to apply to the YMCS theory and we comment on the relevant issues in the corresponding section of the paper.

Three-dimensional $\mathcal{N} = 2$ gauge theories are also interesting from various other points of view. In particular, they exhibit Seiberg duality [38–40], F-maximization [41], and a F-theorem [42]. The gravity duals of these theories are also known and have been studied in [42–44]. Finally, three-dimensional $\mathcal{N} = 2$ superconformal gauge theories arise from compactifications of the 6d (2, 0) CFT compactified on 3-manifolds [45–47]. It would be very interesting to make contact with these results from the point of view of scattering amplitudes.

The structure of this paper is as follows. In Sec. II we describe some general aspects of the CSM and YMCS theories whose amplitudes we compute in this paper. We pay special attention to the derivation of the on-shell supersymmetry algebras in this section. In particular, we derive the on-shell representation of the algebra for the YMCS system in some detail following canonical quantization. This derivation relies on a careful analysis of the implementation of the Gauss-law constraints on the physical Hilbert space, which is described in some detail. In Sec. III, we compute the 4-point amplitudes of the CSM theory and show that they can be encoded in superamplitudes. We also describe the symmetries of the 4-point superamplitudes which we expect to hold for higher-point superamplitudes. In Sec. IV, we compute various 3- and 4-point amplitudes of the YMCS theory at tree level and use the on-shell superalgebra to deduce the remaining 4-point amplitudes. We also comment on the complications that arise when trying to compute amplitudes with external gauge fields in the YMCS theory using perturbative techniques. In Sec. V, we propose a BCFW recursion relation for mass-deformed 3d theories and discuss its applicability to the theories studied in this paper. In Sec. VI, we present our conclusions and describe some future directions. In Appendix A, we describe our conventions, Feynman rules, and various other useful formulas. In Appendix B we provide more details about the calculation of various 4-point amplitudes.

II. MASS-DEFORMED $\mathcal{N} = 2$ GAUGE THEORIES

In this section we review some general aspects of the mass-deformed three-dimensional supersymmetric theories whose scattering amplitudes we study in this paper. The gauge field which appears in these theories has a Chern-Simons term

$$S_{\mathrm{CS}} = \kappa \int \epsilon^{\mu\nu\rho} \mathrm{tr}\left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho\right). \tag{1}$$

As is well known, a Chern-Simons gauge field is not parity invariant and does not have any propagating degrees of freedom. On the other hand, a Yang-Mills gauge field respects parity and has one massless degree of freedom in three dimensions. When taken in conjunction with the Yang-Mills action, the Chern-Simons term breaks parity and gives rise to a mass for the three-dimensional gluon [34,35]. There are alternate Lorentz invariant mass-terms for gluons that one can consider in three dimensions (see [48,49] for examples). But they typically lead to nonlocal terms in the action. A quadratic mass term for the gauge field could also arise via the Higgs mechanism, but this would break gauge invariance. In the present paper we consider the only known mass-term for a gauge field which is Lorentz invariant, gauge invariant, and local in three dimensions, namely S_{CS}. Note that S_{CS} admits two different supersymmetric completions leading to supersymmetric Chern-Simons and Yang-Mills-Chern-Simons theories. In the first case, the gauge field does not have propagating degrees of freedom and the physical on-shell degrees of freedom consist of matter hypermultiplets. In the latter case there are Yang-Mills kinetic terms and the Chern-Simons term provides a topological mass for the gauge field (which contributes to the on-shell degrees of freedom). We will study scattering processes in both the theories while restricting ourselves to the case of $\mathcal{N} = 2$ supersymmetry.

Before discussing mass-deformed $\mathcal{N} = 2$ gauge theories in greater detail, we briefly review the 3d spinor formalism. The three-dimensional spinor formalism can be obtained by dimensional reduction of the four-dimensional spinor formalism [4]. We begin by writing a 4d null momentum in bispinor form

$$p^{\alpha\dot{\beta}} = \lambda^\alpha \bar{\lambda}^{\dot{\beta}}, \tag{2}$$

where $\alpha = 1, 2$ and $\dot{\beta} = 1, 2$ are $SU(2)$ indices which arise from the fact that the Lorentz group is $SO(4) \sim SU(2)_L \times SU(2)_R$. When reducing to three dimensions, the distinction between dotted and undotted indices disappears because the Lorentz group is $SU(2) = [SU(2)_L \times SU(2)_R]_{\mathrm{diagonal}}$. Alternatively, we can reduce to three dimensions by modding out by translations along a vector field $T^{\alpha\dot{\beta}}$, as described in [4]. Using the vector field to change dotted indices to undotted indices in (2) and symmetrizing the indices then gives

$$p^{\alpha\beta} = \lambda^{(\alpha} \bar{\lambda}^{\beta)}. \tag{3}$$

We symmetrize the indices in order to remove the component of the momentum along the direction $T^{\alpha\dot{\beta}}$. The resulting momentum is a 2×2 symmetric object, which has three components.

We denote inner products of the spinors using bracket notation

$$\langle \lambda_i \lambda_j \rangle = \epsilon_{\beta\alpha} \lambda_i^\alpha \lambda_j^\beta.$$

If we square (3), we find that

$$\langle \lambda \bar{\lambda} \rangle^2 = -4m^2. \tag{4}$$

Hence, if the particle is massless, then $\lambda \propto \bar{\lambda}$ and the momentum can be written in bispinor form as $p^{\alpha\beta} = \lambda^\alpha \lambda^\beta$. More generally, for a massive particle in three dimensions, the momentum is given by (3). Equations (3) and (4) can be summarized as follows

$$\lambda^\alpha \bar{\lambda}^\beta = p^{\alpha\beta} - im\epsilon^{\alpha\beta}.$$

In particular, $\langle \lambda \bar{\lambda} \rangle = -2im$.

For later convenience, we will denote $\lambda = u$ and $\bar{\lambda} = -v$. The two spinors u and v are solutions of the free massive Dirac equation and are given in (A4))[1]. They satisfy

$$v^\alpha u^\beta = -p^{\alpha\beta} - im\epsilon^{\alpha\beta}, \tag{5}$$

where $P^{\alpha\beta} = -(p_\mu \gamma^\mu C^{-1})^{\beta\alpha}$ is given explicitly by

$$P^{\alpha\beta} = P^{\beta\alpha} = \begin{pmatrix} -p_0 - p_1 & p_2 \\ p_2 & -p_0 + p_1 \end{pmatrix}. \tag{6}$$

[1]An exhaustive list of the properties of these spinors can be found in [1].

A. $\mathcal{N} = 2$ massive Chern-Simons-Matter (CSM) theory

The CSM theory is described by the action

$$\begin{aligned} S_{\rm CSM} = \kappa \int \epsilon^{\mu\nu\rho} {\rm tr}\left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho\right) \\ - 2\int {\rm tr}|D_\mu \Phi|^2 + 2i \int {\rm tr}\bar{\Psi}(D_\mu \gamma^\mu \Psi + m\Psi) \\ - \frac{2}{\kappa^2}\int {\rm tr}(|[\Phi,[\Phi^\dagger,\Phi]] + e^2\Phi|^2) \\ + \frac{2i}{\kappa}\int {\rm tr}([\Phi^\dagger,\Phi][\bar{\Psi},\Psi] + 2[\bar{\Psi},\Phi][\Phi^\dagger,\Psi]), \end{aligned} \tag{7}$$

where,

$$\kappa = \frac{k}{4\pi}, \qquad m = e^2/\kappa. \tag{8}$$

Note that the Chern-Simons term is odd under parity, so the theory is not parity invariant. The parameter k is the Chern-Simons level. The matter couples to the gauge field with coupling constant $1/\sqrt{k}$. The parameter e sets the mass-scale in the superpotential. Even though it is a dimensionful number, it does not run in the superrenormalizable theory and can be regarded as a parameter of the theory. Taking the mass to zero or infinity while holding the coupling $1/\sqrt{k}$ constant corresponds to taking e to zero or infinity. In the massless limit this theory reduces to a conformal $\mathcal{N} = 2$ Chern-Simons-matter theory. In the infinitely massive limit, the theory reduces to a pure Chern-Simons theory with no propagating degrees of freedom. The conventions underlying the above action assume that all the fields are in the adjoint representation of the gauge group. Furthermore, we assume the generators of the gauge group t^a [which we can take to be $SU(N)$] to be Hermitian. We then have

$$\begin{aligned} &A = A^a t^a, \quad \Phi = \Phi^a t^a, \quad \Psi = \Psi^a t^a, \quad {\rm tr}(t^a t^b) = \frac{1}{2}\delta^{ab}, \\ &[t^a, t^b] = i f^{abc} t^c, \quad D_\mu = \partial_\mu - i[A_\mu,]. \end{aligned} \tag{9}$$

In terms of real variables,

$$\Phi = \frac{1}{\sqrt{2}}(\Phi^1 + i\Phi^2), \qquad \Psi = \frac{1}{\sqrt{2}}(\Psi^1 + i\Psi^2), \tag{10}$$

where Φ^i and Ψ^i are real and Majorana, respectively.

We can immediately see that the free (Abelian) part of the action is invariant under

$$\begin{aligned} &\delta_{\bar{\epsilon}}\Phi = \bar{\epsilon}\Psi, \quad \delta_\epsilon \Phi^\dagger = \bar{\Psi}\epsilon, \quad \delta_\epsilon \Psi = +i(\partial^\mu \gamma_\mu - m)\Phi\epsilon, \\ &\delta_{\bar{\epsilon}}\bar{\Psi} = -i\bar{\epsilon}(\partial^\mu\gamma_\mu + m)\Phi^\dagger, \end{aligned} \tag{11}$$

where $\delta_\epsilon = [\bar{Q}\epsilon,]$, $\delta_{\bar{\epsilon}} = [\bar{\epsilon}Q,]$. All other supersymmetry variations vanish. In the non-Abelian/interacting theory, the SUSY variation of the scalar fields remains as above, but the variations of the fermions and the gauge fields are given by

$$\begin{aligned} \delta_\epsilon \Psi &= \left(i(\partial^\mu\gamma_\mu - m)\Phi - \frac{i}{\kappa}[\Phi,[\Phi^\dagger,\Phi]]\right)\epsilon, \\ \delta_\epsilon A_\mu &= -\frac{i}{\kappa}[\Phi, \bar{\Psi}\gamma_\mu \epsilon]. \end{aligned} \tag{12}$$

The $\delta_{\bar{\epsilon}}$ variations in the non-Abelian case can be obtained from the ones given above by conjugation. The fundamental anticommutation relation between the supercharges is

$$\{Q^{\beta J}, Q^{\alpha I}\} = \frac{1}{2}(P^{\alpha\beta}\delta^{IJ} + m\epsilon^{\beta\alpha}\epsilon^{JI}R), \tag{13}$$

where R is the $SO(2) = U(1)$ symmetry generator which rotates (Φ^1, Φ^2) and (Ψ^1, Ψ^2).

For the mass-deformed Chern-Simons theory, the on-shell asymptotic states are those of the complex scalar Φ and fermion Ψ. In our notation, the asymptotic momentum-space states of Φ and Ψ are denoted $|a_+\rangle$ and $|\chi_+\rangle$, respectively. Using the mode expansions for these fields, which are given by (A3), in the supersymmetry algebra (11), we see that the supersymmetry variations of the on-shell states are given by

$$\begin{aligned} Q_I|\Phi_1\rangle &= \frac{1}{2}v|\chi_I\rangle, \\ Q_I|\Phi_2\rangle &= \frac{1}{2}v\epsilon^{IJ}|\chi_J\rangle, \\ Q_I|\chi_J\rangle &= \frac{1}{2}\delta_{IJ}u|\Phi_1\rangle + \frac{1}{2}\epsilon^{IJ}u|\Phi_2\rangle, \end{aligned} \tag{14}$$

where u and v are spinors defined in (A4)[2].

We can express these transformations in a way that makes the $U(1)$ R-symmetry of the theory manifest by forming complex combinations of the fields and supercharges

$$\begin{aligned} &a_\pm = \frac{1}{\sqrt{2}}(\Phi_1 \pm i\Phi_2), \qquad \chi_\pm = \frac{1}{\sqrt{2}}(\Psi_1 \pm i\Psi_2), \\ &Q_\pm = \frac{1}{\sqrt{2}}(Q_1 \pm iQ_2). \end{aligned} \tag{15}$$

We then obtain

$$\begin{aligned} &Q_+|a_+\rangle = \frac{1}{\sqrt{2}}v|\chi_+\rangle, \qquad Q_+|\chi_-\rangle = \frac{1}{\sqrt{2}}u|a_-\rangle, \\ &Q_-|a_-\rangle = \frac{1}{\sqrt{2}}v|\chi_-\rangle, \qquad Q_-|\chi_+\rangle = \frac{1}{\sqrt{2}}u|a_+\rangle, \\ &Q_-|a_+\rangle = Q_+|\chi_+\rangle = Q_+|a_-\rangle = Q_-|\chi_-\rangle = 0. \end{aligned} \tag{16}$$

[2]For a detailed discussion of the on-shell representation of three-dimensional massive $\mathcal{N} \geq 4$ superalgebras, we refer to [1].

It is important to emphasize that the superalgebra (13) is a noncentral extension of the standard $\mathcal{N}=2$ superalgebra. In particular, the anticommutator of the charges does not close onto the momentum generator alone, as it also involves the R symmetry generator as part of the fundamental supersymmetry algebra. Such mass-deformed algebras frequently arise in the context of three-dimensional gauge theories with mass-gaps; in particular in $\mathcal{N}\geq 4$ Chern-Simons-Matter theories [1]. It is instructive to see how the algebra described above is embedded in the supersymmetry algebra of the massive $\mathcal{N}=6$ theory. In the notation of [1], the matter content of $\mathcal{N}=5$, 6 CSM theories is given by scalars ϕ_a, $\tilde{\phi}_{\dot{a}}$ and fermions $\psi_{\dot{a}}$, $\tilde{\psi}_a$ where a, $\dot{a}$ are two different $SU(2)$ indices. The fields denoted by tildes are part of the twisted hypermultiplets, while those without the tildes form the untwisted hypermultiplets. In the case of $\mathcal{N}=6$ supersymmetry, one has—as part of the full supersymmetry algebra—supercharges $Q^{\pm}_{\alpha}$ that transform scalars and fermions belonging to the twisted and untwisted hypermultiplets to each other, while acting trivially on the $SU(2)$ indices (see the discussion in Sec. 2.4 of [1]). These supercharges (for a fixed value of the $SU(2)$ index) generate the massive $\mathcal{N}=2$ algebra considered here[3]. It should be noted, however, that the theory considered here is distinguished from the class of models studied in [1] by virtue of all the matter fields being in the adjoint representation, which, typically is not possible for $\mathcal{N}=4$ and higher supersymmetry.

We also note that just as the supersymmetric Chern-Simons theories are not known to be obtained as the dimensional reduction of higher dimensional gauge theories, this massive superalgebra is *not* what one obtains by the dimensional reduction of the free $\mathcal{N}=1$ theory in four dimensions. In fact, if one takes the massive $\mathcal{N}=1$ $d=4$ free action given by

$$S_{d=4}=-\frac{1}{2}\int_{R^4}(\partial_\mu\Phi_I\partial^\mu\Phi_I+m^2\Phi_I\Phi_I+i\bar{\Psi}\Gamma_\mu\partial^\mu\Psi+im\bar{\Psi}\Psi), \tag{17}$$

which is invariant under

$$\delta_\alpha\Psi=\frac{1}{2}(\Gamma^\mu\partial_\mu-m)\Phi_1\alpha+\frac{i}{2}(\Gamma^5\Gamma^\mu\partial_\mu+m\Gamma^5)\Phi_2\alpha,$$
$$\delta_\alpha\Phi_1=\frac{i}{2}\bar{\alpha}\Psi,\qquad \delta_\alpha\Phi_2=\frac{1}{2}\bar{\alpha}\Gamma^5\Psi, \tag{18}$$

it is easy to see that the algebra closes on only the momentum generators without any extensions

$$[\delta_\beta,\delta_\alpha]\Phi_I=\frac{i}{2}(\bar{\alpha}\Gamma^\mu\beta)\partial_\mu\Phi_I. \tag{19}$$

The algebra retains this standard form even after dimensional reduction to $d=3$, however the fermion mass-term in $d=3$ derived from the $SO(1,3)$-invariant four-dimensional mass-term would be given in the three-dimensional notation by $\int(\bar{\Psi}_1\Psi_1-\bar{\Psi}_2\Psi_2)$. This is different from the term we have in (7) where the mass terms for both the fermions have the same sign.

In other words, in $d=3$ we can choose between two different fermion mass-terms

$$M_1=\int(\bar{\Psi}_1\Psi_1-\bar{\Psi}_2\Psi_2),\quad\text{or}\quad M_2=\int\bar{\Psi}_I\Psi_I. \tag{20}$$

The choice M_1—the parity conserving option—leads to the standard $\mathcal{N}=2$ algebra without extensions while M_2 leads to a mass-deformed algebra and violates parity. However the Chern-Simons term, which is present in the gauge theories we study, violates parity. Thus it is natural that the fermionic mass terms resulting from the supersymmetric completion of the Chern-Simons term violate parity as well. It is apparently this interplay between the parity invariance of the theory and the fermionic mass term that leads to the massive nature of the on-shell algebra in this case.

B. $\mathcal{N}=2$ Yang-Mills-Chern-Simons (YMCS) theory

The second theory of relevance to this paper is the well-known $\mathcal{N}=2$ YMCS theory described off shell by the action $S_{\mathrm{YMCS}}=S_{\mathrm{YM}}+S_{\mathrm{CS}}$ where [4]

$$S_{\mathrm{YM}}=\frac{1}{e^2}\int\bigg[-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}-\frac{1}{2}D_\mu\Phi^aD^\mu\Phi^a+\frac{1}{2}F^aF^a+\frac{i}{2}\bar{\Psi}^a_I\gamma^\mu D_\mu\Psi^a_I+\frac{i}{2}f^{abc}\epsilon_{AB}\bar{\Psi}^a_A\Phi^b\Psi^c_B\bigg],$$
$$S_{\mathrm{CS}}=\frac{m}{2e^2}\int\bigg[\epsilon^{\mu\nu\rho}A^a_\mu\partial_\nu A^a_\rho-\frac{1}{3}f^{abc}\epsilon^{\mu\nu\rho}A^a_\mu A^b_\nu A^c_\rho+i\bar{\Psi}^a_I\Psi^a_I+2F^a\Phi^a\bigg]. \tag{21}$$

Here $m=\frac{ke^2}{4\pi}$ where k is the Chern-Simons level, $A=1,2$ is an $SO(2)$ index, the scalar field Φ and auxiliary field F are real, and the fermions are Majorana, so that $\bar{\Psi}_A\equiv\Psi^\dagger_A\gamma^0=\Psi^T_AC$, where the charge conjugation matrix $C=\gamma^0$. For further details about our conventions, see Appendix A. Note that the matter fields Φ and Ψ have mass dimension 1 and $3/2$. If they are rescaled by a factor of e, i.e. if $(\Phi,\Psi)\to e(\Phi,\Psi)$, then they will have mass dimensions $1/2$ and 1. F is an auxiliary field whose elimination generates the standard quadratic mass term of the real scalar Φ, while the Chern-Simons term gives a topological mass

[3] There are presumably other distinct embeddings of the $\mathcal{N}=2$ superalgebra in the larger $\mathcal{N}=6$ superalgebra as well.

[4] We have taken the trace using (9) and also have chosen to rescale the fields by the coupling constant, in comparison to (7).

to the gauge field. Both the Chern-Simons term and the fermionic mass term are odd under parity, so the theory is not parity invariant when the mass is nonzero. Taking the mass to zero or infinity while holding the Yang-Mills coupling constant corresponds to taking k to zero or infinity. In the massless limit, the theory reduces to $\mathcal{N} = 2$ Yang-Mills theory and parity is restored. In the infinitely massive limit, the theory reduces to a pure Chern-Simons theory. What is not transparent from the action given above, but is nevertheless true, is that the asymptotic physical states also involve a second massive scalar with the same mass. This scalar is nothing but the physical gauge invariant degree of freedom encoded in the gauge field.

Note that for the free theory, the supersymmetry transformation laws are

$$\delta_\eta A_\mu = -\frac{i}{2}(\bar{\eta}_I\gamma_\mu\Psi_I), \qquad \delta_\eta\Phi = \frac{1}{2}\bar{\eta}_I\Psi_J\epsilon_{IJ},$$
$$\delta_\eta\Psi_I = \frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}\eta_I - \frac{i}{2}(\gamma^\mu\partial_\mu - m)\epsilon_{IJ}\eta_J\Phi. \tag{22}$$

The anticommutator of the supercharges in these off-shell transformation laws closes onto the momentum operator alone

$$\{Q_I^\alpha, Q_J^\beta\} = -\frac{1}{2}\delta_{IJ}P^{\alpha\beta}. \tag{23}$$

Note that there is no $U(1)$ extension as there was for the superalgebra in the mass-deformed Chern-Simons theory. This is a consequence of the fact that the YMCS theory, while enjoying a $SO(2)$ R-symmetry which rotates the two fermionic fields in the theory, does not have a corresponding symmetry acting on the two bosonic fields, i.e. the scalar and gauge field. Indeed, we will find that the on-shell amplitudes of the YMCS theory exhibit $SO(2)$ R-symmetry in the fermionic sector. We do note, however, that both the algebras collapse to the same massless algebra when k in the YMCS theory and e in the CSM model are set to zero, which is consistent with the 4-point amplitudes of undeformed three-dimensional SYM exhibiting an enhanced $SO(2)$ symmetry [3,4].

We now focus on the on-shell superalgebra for this theory. Assuming that the on-shell degree of freedom associated with the YMCS gauge field corresponds to a massive scalar field (which we will justify shortly) and that the superalgebra in (22) is realized on the single particle asymptotic states, the transformation laws for the scattering states can be taken to be

$$Q_I|a_1\rangle = \frac{1}{2}u|\chi_I\rangle, \qquad Q_I|a_2\rangle = \frac{1}{2}\epsilon_{IJ}v|\chi_J\rangle,$$
$$Q_J|\chi_I\rangle = \frac{1}{2}\delta_{IJ}v|a_1\rangle - \frac{1}{2}\epsilon_{IJ}u|a_2\rangle. \tag{24}$$

We denote the massive scalar corresponding to the gauge field by a_1.

One can give an argument in favor of the algebra above being the appropriate one for the YMCS theory as follows. If one starts with the part of the algebra involving the variation of a_2, namely $Q_I|a_2\rangle = \frac{1}{2}\epsilon_{IJ}\tau|\chi_J\rangle$, there is an ambiguity about what the spinor τ is. This ambiguity can be resolved by applying the oscillator expansion of the fields to the off-shell transformation $\delta\Phi = \frac{1}{2}\bar{\eta}_I\Psi_J\epsilon_{IJ}$. In our convention, this fixes $\tau = v$. Once this is fixed, the closure of the algebra on a_2 fixes the transformation properties $Q_1|\chi_2\rangle = +u/2|a_2\rangle$ and $Q_2|\chi_1\rangle = -u/2|a_2\rangle$. With this part of the on-shell supersymmetry algebra determined, one can make the following ansatz for the supersymmetry algebra

$$Q_I|a_1\rangle = \frac{1}{2}\omega|\chi_I\rangle, \qquad Q_I|a_2\rangle = \frac{1}{2}\epsilon_{IJ}v|\chi_J\rangle,$$
$$Q_J|\chi_I\rangle = \frac{1}{2}\delta_{IJ}\tilde{\omega}|a_1\rangle - \frac{1}{2}\epsilon_{IJ}u|a_2\rangle, \tag{25}$$

assuming that the realization is linear in the fields and that the $SO(2)$ covariance of the fermionic degrees of freedom is respected. The unknown quantities are the spinors ω and $\tilde{\omega}$. Closure of the algebra on a_1 requires $\omega^{\{\alpha}\tilde{\omega}^{\beta\}} = -2P^{\alpha\beta}$. The solution to this equation is given by $\{\omega,\tilde{\omega}\} = \{u, v\}$ or $\{\omega,\tilde{\omega}\} = \{v, u\}$. Furthermore, requiring that there be no $U(1)$ extension to the algebra requires $\omega = u$ and $\tilde{\omega} = v$. Thus, given a convention of the oscillator expansion of the fermion fields, the on-shell algebra is unambiguously determined.

Comparing this with (14), we see that the main difference between the two sets of transformations is that the spinors appearing on the right-hand side of the transformation laws of the scalars above are conjugates of each other. The two spinors were the same in the transformation laws for the CSM theory. The differences in the two realizations have to do with whether or not the algebra is mass-deformed.

Rather than rely on the argument above alone, it is instructive to derive (24) using the methods of canonical quantization. To this end we revert to a Hamiltonian framework and set $A_0 = 0$. We define the complex combination $A = \frac{1}{2}(A_1 + iA_2)$ (and its conjugate relation) for the gauge potentials. Due to the noncommutativity induced by the Chern-Simons term on the components of the electric field, the Gauss law constraints can be shown to be satisfied by wave functions of the form [50]

$$\Omega = \exp\left[\frac{k}{2}(S_{WZW}(M^\dagger) - S_{WZW}(M))\right]\Lambda, \tag{26}$$

where M is a complex matrix related to the gauge potential as $A = -\partial MM^{-1}$ that transforms under time-independent local gauge transformations as $M \to UM$, where U is an element of the gauge group. S_{WZW} is a Wess-Zumino-Witten functional defined over the spatial manifold [50]. The Gauss law constraint can be translated into the following condition on Λ

$$\left(D\frac{\delta}{\delta A}+\bar{D}\frac{\delta}{\delta\bar{A}}\right)^a+f^{amn}\left(-i\bar{\Psi}_I^m\Psi_I^n+\Phi^m\frac{\delta}{\delta\Phi^n}\right)\Lambda=0. \tag{27}$$

Clearly any wave functional Λ that is a gauge invariant combination of the gauge and matter fields satisfies this constraint.

Now, to derive the on-shell supersymmetry transformation law, our strategy would be to express the quadratic part of the supercharges in terms of the canonical variables followed by a dualization of the gauge field into a scalar. We can then read off the on-shell supersymmetry transformation by looking at the action of the dualized supercharge on the dynamical fields in momentum space. To avoid the ambiguity associated with the fermionic fields and their canonical momenta in a real representation for the three-dimensional Dirac matrices, we take the gamma matrices to be $\gamma^\mu=\{i\sigma^3,\sigma^1,\sigma^2\}$ for the purposes of this discussion (everywhere else in the paper we shall continue to use the real γ matrices mentioned previously). The fermions can be taken to be $\Psi=\binom{\psi}{\psi^*}$ with ψ and ψ^* being canonically conjugate. The quadratic part of the top component of the $\mathcal{N}=2$ supercharge in this notation (the bottom component can simply be obtained by Hermitian conjugation) can be written as

$$q_I=ie\int\psi_I^{a\dagger}\frac{\delta}{\delta A^a}-\frac{1}{e}\int\psi_I^aB^a+e\epsilon_{IJ}\int\psi_J^a\left(\Pi_\Phi^a+\frac{im}{e^2}\Phi^a\right)-\frac{2i}{e}\epsilon_{IJ}\int\psi_J^{a\dagger}(\bar{D}\Phi)^a. \tag{28}$$

This charge, derived from the action, acts on the wave function Ω. Ω and Λ differ by a pure phase, so their norms are the same. However, the physical observables acting on Λ differ from those acting on Ω by a unitary transformation. The charge acting on $\Lambda=q_I'=\Omega^\dagger q\Omega$ [50]. The effect of this unitary transformation is to replace

$$\frac{\delta}{\delta A^a}\to\frac{\delta}{\delta A^a}+\frac{m}{e^2}(\bar{A}^a-\bar{a}^a),\qquad \bar{a}=\bar{\partial}MM^{-1}. \tag{29}$$

This extra term generated by the unitary transformation is what generates an effective mass-term for the gauge field in the Hamiltonian obtained from the supercharge above [50].

We can now dualize the gauge field by expressing $M=e^\theta$ and retaining terms to linear order in θ. This gives (after dropping the color indices, as we are only interested in the Abelian theory) $A=-\partial\theta$, $\bar{A}=+\bar{\partial}\,\bar{\theta}$, $a=\partial\bar{\theta}$ and $\bar{a}=-\bar{\partial}\theta$. The real part of θ is related to the physical gauge-invariant on-shell degree of freedom Φ_H as [34,50,51]

$$\theta+\bar{\theta}=\frac{1}{\sqrt{-\partial\bar{\partial}}}\Phi_H. \tag{30}$$

On gauge invariant wave functionals, the dualized supercharge can now be written as

$$q_I'=ie\omega_I\left(\frac{\delta}{\delta\Phi_H}+\frac{im}{e^2}\Phi_H\right)+\frac{2i}{e}\omega_I^\dagger\bar{\partial}\Phi_H+e\epsilon_{IJ}\int\psi_J\left(\Pi_\Phi+\frac{im}{e^2}\Phi\right)-\frac{2i}{e}\epsilon_{IJ}\int\psi_J^\dagger(\bar{\partial}\Phi), \tag{31}$$

where

$$\omega_I=-ie^{i\alpha}\psi^\dagger,\qquad e^{i\alpha}=\sqrt{\bar{\partial}/\partial}\equiv\sqrt{\bar{p}/p}. \tag{32}$$

The momenta p and $\bar{p}$ appearing in the last term above are the complex combinations of the spatial components of the three-momentum. It is important to note that the fermionic variable multiplying the momentum for the dual scalar has to be identified as the top component of a fermionic field (in our case ω) so that the SUSY variation of Φ_H can be written in a Lorentz invariant form in the two component notation i.e. $\delta\Phi_H\sim\bar{\epsilon}\rho$ for some Majorana fermion ρ. In our case the dualization dictates that the top component of ρ is ω. Crucially for our purposes, it can be readily seen from the Hamiltonian obtained from q_I' that the Hamiltonian for ω has the opposite sign for the mass term as that of ψ. Or in other words, the spinors appearing with the positive (negative) frequency parts of the mode expansion of ψ can be identified with those associated with the negative (positive) parts of ω. Since the SUSY variations involving the on-shell fields $a_1\equiv\Phi_H$ and $a_2\equiv\Phi$ involve fermions with the opposite mass terms, the spinors appearing in the momentum space realization of these transformations are conjugate to each other. Reverting back to our real conventions for the γ matrices, we see that (24) can now be justified based on the grounds of canonical quantization.

III. MASS-DEFORMED CHERN-SIMONS AMPLITUDES

In this section, we will describe the scattering amplitudes of the CSM theory (7). Since the Chern-Simons gauge field has no propagating degrees of freedom, scattering amplitudes with at least one external gauge field vanish. This implies that all odd-point amplitudes vanish, so the first nontrivial amplitudes occur at four point. In the next two subsections, we will compute the 4-point amplitudes and show that they can be encoded in superamplitudes. We also describe the symmetries of these superamplitudes.

A. 4-point amplitudes

All of the 4-point amplitudes of the CSM theory are related to each other by the supersymmetry algebra in (16). Hence, there are only two independent amplitudes involving four legs. With the definitions of the complex combinations of the real degrees of freedom described in (15), we get the following relations between the color ordered four particle amplitudes

$$\langle\chi_+\chi_-a_+a_-\rangle = +\frac{\langle\bar{4}1\rangle}{\langle\bar{2}\,\bar{4}\rangle}\langle a_+a_-a_+a_-\rangle, \qquad \langle\chi_+\chi_-a_-a_+\rangle = +\frac{\langle\bar{3}1\rangle}{\langle\bar{2}\,\bar{3}\rangle}\langle a_+a_-a_-a_+\rangle,$$
$$\langle a_+\chi_-\chi_+a_-\rangle = +\frac{\langle 3\bar{4}\rangle}{\langle\bar{2}\,\bar{4}\rangle}\langle a_+a_-a_+a_-\rangle, \qquad \langle a_+\chi_-a_-\chi_+\rangle = +\frac{\langle\bar{3}4\rangle}{\langle\bar{3}\,\bar{2}\rangle}\langle a_+a_-a_-a_+\rangle,$$
$$\langle\chi_+a_+\chi_-a_-\rangle = +\frac{\langle\bar{4}1\rangle}{\langle\bar{3}\,\bar{4}\rangle}\langle a_+a_+a_-a_-\rangle, \qquad \langle a_+\chi_+\chi_-a_-\rangle = +\frac{\langle\bar{4}2\rangle}{\langle\bar{3}\,\bar{4}\rangle}\langle a_+a_+a_-a_-\rangle,$$
$$\langle\chi_+a_-\chi_-a_+\rangle = +\frac{\langle 1\bar{2}\rangle}{\langle\bar{2}\,\bar{3}\rangle}\langle a_+a_-a_-a_+\rangle, \qquad \langle a_+a_-\chi_-\chi_+\rangle = +\frac{\langle\bar{2}4\rangle}{\langle\bar{2}\,\bar{3}\rangle}\langle a_+a_-a_-a_+\rangle,$$
$$\langle a_+a_-\chi_+\chi_-\rangle = +\frac{\langle 3\bar{2}\rangle}{\langle\bar{2}\,\bar{4}\rangle}\langle a_+a_-a_+a_-\rangle, \qquad \langle\chi_+a_-a_+\chi_-\rangle = +\frac{\langle 1\bar{2}\rangle}{\langle\bar{2}\,\bar{4}\rangle}\langle a_+a_-a_+a_-\rangle,$$
$$\langle a_+\chi_+a_-\chi_-\rangle = +\frac{\langle 2\bar{3}\rangle}{\langle\bar{3}\,\bar{4}\rangle}\langle a_+a_+a_-a_-\rangle, \qquad \langle\chi_+a_+a_-\chi_-\rangle = +\frac{\langle 1\bar{3}\rangle}{\langle\bar{3}\,\bar{4}\rangle}\langle a_+a_+a_-a_-\rangle. \tag{33}$$

The three independent four-fermion amplitudes are related to the other amplitudes as

$$\langle\chi_+\chi_+\chi_-\chi_-\rangle = +\frac{\langle 21\rangle}{\langle\bar{4}2\rangle}\langle a_+\chi_+\chi_-a_-\rangle = +\frac{\langle 21\rangle}{\langle\bar{3}\,\bar{4}\rangle}\langle a_+a_+a_-a_-\rangle,$$
$$\langle\chi_+\chi_-\chi_+\chi_-\rangle = +\frac{\langle 13\rangle}{\langle 3\bar{2}\rangle}\langle a_+a_-\chi_+\chi_-\rangle = +\frac{\langle 13\rangle}{\langle\bar{2}\,\bar{4}\rangle}\langle a_+a_-a_+a_-\rangle,$$
$$\langle\chi_+\chi_-\chi_-\chi_+\rangle = +\frac{\langle 41\rangle}{\langle\bar{2}4\rangle}\langle a_+a_-\chi_-\chi_+\rangle = +\frac{\langle 41\rangle}{\langle\bar{2}\,\bar{3}\rangle}\langle a_+a_-a_-a_+\rangle. \tag{34}$$

In Appendix B we compute 4-fermion amplitudes and find

$$\langle\chi_+\chi_+\chi_-\chi_-\rangle_{\rm CSM} = 2i\frac{\langle 43\rangle\langle 42\rangle}{\langle 4\bar{1}\rangle}, \langle\chi_+\chi_-\chi_+\chi_-\rangle_{\rm CSM}$$
$$= 2i\langle 24\rangle\left(\frac{\langle 41\rangle\langle 4\bar{1}\rangle - \langle 43\rangle\langle 4\bar{3}\rangle}{\langle 4\bar{1}\rangle\langle 4\bar{3}\rangle}\right). \tag{35}$$

Using these amplitudes, all the other 4-point amplitudes are determined from the above relations and cyclic permutations.

B. Superamplitudes

The natural question to ask is if these relations among the 4-point amplitudes obtained in the previous subsection can be derived from superamplitudes. To the best of our knowledge no superamplitude is known for any massive CSM model so far. However, since the superalgebra and the kinematics constraining the S-matrix of the massive three-dimensional theories can be thought of as dimensional reductions of the four-dimensional quantities, it is natural to expect that some of the known results for four-dimensional SYM theories can be reduced to get massive three-dimensional superamplitudes. In fact, one can define two types of superamplitudes, which are analogous to the "$\Phi - \Psi$" formalism and the "$\Phi - \Phi^\dagger$" formalisms used to obtain superamplitudes for 4d super-Yang-Mills theories with $\mathcal{N} < 4$ supersymmetry [52]. In the "$\Phi - \Psi$" formalism, the superamplitudes can be expressed in terms of supercharges so one can in principle apply super-BCFW recursion relations to these amplitudes. On the other hand, the $SO(2)$ R-symmetry of the theory is not manifest in the "$\Phi - \Psi$" formalism, so the on-shell superalgebra obtained from the supercharges which act on the superamplitudes does not have a central extension. In the "$\Phi - \Phi^\dagger$" formalism, the superamplitudes are not expressed in terms of supercharges, but the superamplitudes and superfields transform covariantly under the $U(1)$ R-symmetry. We describe the "$\Phi - \Psi$" and "$\Phi - \Phi^\dagger$" formalisms in greater detail below, and we describe super-BCFW for 3d mass-deformed theories in Sec. V.

1. $\Phi - \Psi$ formalism

We introduce the on-shell superfields

$$\Phi = a_+ + \bar{\eta}\chi_+, \qquad \Psi = \chi_- + \eta a_-, \tag{36}$$

where η is a complex Grassmann variable. The 4-point superamplitudes can then be written in terms of a supermomentum delta function

$$\mathcal{A}_4 = \Omega\delta^3(P)\delta^2(Q), \tag{37}$$

where Ω is a prefactor and

$$P^{\alpha\beta} = \sum_{i=1}^{4}\lambda_i^{(\alpha}\bar{\lambda}_i^{\beta)}, \qquad Q^\alpha = \sum_{i=1}^{4}\left(\lambda_i^\alpha\bar{\eta}_i + \bar{\lambda}_i^\alpha\eta_i\right),$$
$$\delta^2(Q) = Q^\alpha Q_\alpha. \tag{38}$$

The 4-point superamplitudes of the CSM theory are given by

$$\mathcal{A}_{\Phi\Phi\Psi\Psi} = \frac{\langle 24\rangle}{\langle \bar{3}2\rangle}\delta^3(P)\delta^2(Q),$$
$$\mathcal{A}_{\Phi\Psi\Phi\Psi} = \frac{\langle 41\rangle\langle 4\bar{1}\rangle - \langle 43\rangle\langle 4\bar{3}\rangle}{\langle 1\bar{2}\rangle\langle 4\bar{1}\rangle}\delta^3(P)\delta^2(Q), \quad (39)$$

where we have ignored the numerical prefactor $2i$. Note that the superamplitudes encode the scattering of all component fields. In particular, the component amplitudes correspond to the coefficients of the Taylor expansions of the superamplitudes in the fermionic variables. For example, the coefficient of $\bar{\eta}_1\bar{\eta}_2$ in the Taylor expansion of $A_{\Phi\Phi\Psi\Psi}$ is

$$\langle 12\rangle \frac{\langle 24\rangle}{\langle \bar{3}2\rangle} = \langle 12\rangle \frac{\langle 34\rangle\langle 24\rangle}{\langle 34\rangle\langle \bar{3}2\rangle} = \frac{\langle 34\rangle\langle 42\rangle}{\langle \bar{1}4\rangle}$$

where we noted that $\langle 34\rangle\langle \bar{3}2\rangle = -\langle \bar{1}4\rangle\langle 12\rangle$. This indeed matches our result for the $\langle \chi_+\chi_+\chi_-\chi_-\rangle$ amplitude in (35). Using a similar analysis, one sees that the $\langle \chi_+\chi_-\chi_+\chi_-\rangle$ amplitude in (35) corresponds to the $\bar{\eta}_1\bar{\eta}_3$ component of $A_{\Phi\Psi\Phi\Psi}$. Furthermore, it is easy to reproduce the relations in (33) and (34). Hence, the superamplitudes $A_{\Phi\Phi\Psi\Psi}$ and $A_{\Phi\Psi\Phi\Psi}$ encode all the 4-point component amplitudes in the mass-deformed Chern-Simons theory.

In addition to the supercharge defined in (38), we can also define the following supercharge which annihilates the 4-point superamplitudes

$$\bar{Q}^\alpha = \sum_{i=1}^n \left(\bar{\lambda}_i^\alpha \frac{\partial}{\partial \bar{\eta}_i} + \lambda_i^\alpha \frac{\partial}{\partial \eta_i}\right). \quad (40)$$

The superalgebra which acts on the superamplitudes of the CSM theory is given by

$$\{\bar{Q}^\alpha, Q^\beta\} \propto P^{\alpha\beta}.$$

Note that the superalgebra does not have a central extension, like the one in (13). This is because the $SO(2) \sim U(1)$ R-symmetry of the theory is not manifest in the "$\Phi - \Psi$" formalism. The charges acting on the superamplitude and superfields can be regarded as the subset of charges in (13) that carry the same $SO(2)$ index, and thus do not have the central term in their anti-commutator. In particular, the $U(1)$ R-symmetry acts on the on the (λ, η) variables as follows:

$$(\lambda, \eta) \to \alpha(\lambda, \eta), \qquad (\bar{\lambda}, \bar{\eta}) \to \alpha^{-1}(\bar{\lambda}, \bar{\eta}),$$

where $\alpha \in U(1)$. Under this symmetry, fields of helicity h should be multiplied by α^{2h}, however the superfields in (36) do not respect this symmetry. Hence, in the "$\Phi - \Psi$" formalism, the $U(1)$ R-symmetry is broken to $\mathbb{Z}_2$, which corresponds to the little group in three dimensions. This corresponds to multiplying bosons by $+1$ and fermions by -1.

The $SO(2)$ R-symmetry of the theory is realized by the superamplitudes as follows. It is easy to see that the 4-point superamplitude is an eigenfunction of the R-symmetry generator

$$R = \sum_{i=1}^n \left(\eta_i \frac{\partial}{\partial \eta_i} + \bar{\eta}_i \frac{\partial}{\partial \bar{\eta}_i}\right),$$

with eigenvalue 2. Hence, $R - 2$ is a symmetry of the 4-point superamplitude. This corresponds to a $U(1) = SO(2)$ R-symmetry. We expect this symmetry to persist for higher point amplitudes. In particular, we expect that the n-point amplitude will be annihilated by $R - n/2$.

2. $\Phi - \Phi^\dagger$ *formalism*

To make contact with the $\Phi - \Phi^\dagger$ formalism for four-dimensional SYM theories, we first recall that in the notation of [52] four-dimensional *mhv* amplitudes (with negative helicity particles in the i and j slots of the color ordered amplitude) correspond to

$$\mathcal{A}_{i,j}^{mhv} = \langle \ldots \Phi_i^\dagger \ldots \Phi_j^\dagger \ldots \rangle \quad (41)$$

where the other entries correspond to Φ. For 4-point amplitudes

$$\mathcal{A}_{i,j}^{mhv} = \tilde{\Omega}(i,j)\left(\langle ij\rangle + \langle ik\rangle \bar{\eta}_j\eta_k - \langle jk\rangle \bar{\eta}_i\eta_k - \frac{1}{2}\langle kl\rangle \bar{\eta}_i\bar{\eta}_k\eta_k\eta_l\right) \quad (42)$$

where for the four-dimensional theory—as well as the massless three-dimensional SYM theory—the prefactor Ω is given by the famous Parke-Taylor relation:

$$\tilde{\Omega}(i,j) = \frac{\langle ij\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \quad (43)$$

Even though the massive SCS theory is not known to be obtainable as a dimensional reduction of a higher dimensional gauge theory the massive supersymmetry algebra (16) can be regarded as a dimensional reduction of the four-dimensional massless supersymmetry algebra [where the two $SU(2)$'s of the $d = 4$ Lorentz group are identified and fourth components of all the physical momenta are fixed to be m]. It is thus expected that the kinematic constraints relating the different components of the superamplitudes for the massive CSM theory can be cast in a $\Phi - \Phi^\dagger$ form as in $d = 4$. Indeed, after defining the adjoint superfield $\Phi^\dagger = a_- + \eta\chi_-$ in our notation, it is readily seen that

$$\mathcal{A}_{i,j}^{\mathrm{CSM}} = \Omega(i,j)\Big(\langle \bar{i}\,\bar{j}\rangle + \langle \bar{i}k\rangle \eta_j\bar{\eta}_k - \langle \bar{j}k\rangle \eta_i\bar{\eta}_k - \frac{1}{2}\langle kl\rangle \eta_i\eta_k\bar{\eta}_k\bar{\eta}_l\Big) \quad (44)$$

correctly reproduces all the relations between the massive amplitudes given above. The prefactor Ω can be read off once any of the known 4-point component amplitudes are known. For our present purposes, they can be determined in terms of the four-fermion amplitudes computed in Appendix B.

IV. YANG-MILLS-CHERN-SIMONS AMPLITUDES

In this section, we will describe various 3- and 4-point tree-level color-ordered amplitudes of the $\mathcal{N}=2$ YMCS theory. In particular, we compute all of the 3- and 4-point amplitudes without external gauge fields, and obtain the remaining 4-point amplitudes using the on-shell superalgebra (24)[5]. In the end of this section, we describe the difficulties associated with computing on-shell YMCS amplitudes with external gauge fields using Feynman diagrams.

A. 3-point amplitudes

The color ordered 3-point amplitudes are defined for completely general fields $\phi_{\mathcal{A}_i}$ by the expression

$$\begin{aligned}&\langle\phi_{\mathcal{A}_1}^{a_1\,\dagger}(p_1)\phi_{\mathcal{A}_2}^{a_2\,\dagger}(p_2)\phi_{\mathcal{A}_3}^{a_3\,\dagger}(p_3)\rangle\\&\quad=2ie\langle\phi_{\mathcal{A}_1}\phi_{\mathcal{A}_2}\phi_{\mathcal{A}_3}\rangle\mathrm{Tr}[T^{a_1}T^{a_2}T^{a_3}]+\ldots,\end{aligned}\tag{45}$$

where the momenta are all ingoing and $\phi_{\mathcal{A}}^{a\,\dagger}(p)$ is the creation operator for the associated field.

The only 3-point amplitude which does not have external gauge fields is

$$\langle\Psi_{A_1}\Psi_{A_2}\Phi\rangle=-\epsilon_{A_1A_2}\bar{v}(p_2)u(p_1)=-\epsilon_{A_1A_2}\langle 12\rangle.\tag{46}$$

Rearrangement of the fields is achieved using

$$\begin{aligned}\langle\phi_{\mathcal{A}_1}\phi_{\mathcal{A}_2}\phi_{\mathcal{A}_3}\rangle&=-\langle\phi_{\mathcal{A}_2}\phi_{\mathcal{A}_1}\phi_{\mathcal{A}_3}\rangle\\\langle\phi_{\mathcal{A}_1}\phi_{\mathcal{A}_2}\phi_{\mathcal{A}_3}\rangle&=\langle\phi_{\mathcal{A}_2}\phi_{\mathcal{A}_3}\phi_{\mathcal{A}_1}\rangle.\end{aligned}\tag{47}$$

The SUSY algebra does not help us determine the remaining 3-point amplitudes from (46).

Note that the amplitude in (46) has $SO(2)$ R-symmetry which rotates the two fermions. This symmetry follows from the $SO(2)$ R-symmetry in the fermionic sector of the Lagrangian and should therefore hold for higher-point amplitudes, as we will demonstrate at 4-point. The form of this amplitude will be useful for deducing whether or not the BCFW recursion relations are applicable to this massive gauge theory. We shall return to this issue later.

[5]Note that the on-shell superalgebra implies constraints on the 4-point amplitudes but not on the 3-point amplitudes. This has to do with the fact that the algebra is only valid when the external momenta are real. In the case of 3-point amplitudes one necessarily needs to continue the amplitudes to complex momenta.

B. 4-point amplitudes

In this section, we compute various tree-level 4-point amplitudes of the YMCS theory. One may determine the remaining amplitudes using the following rearrangement rules

$$\begin{aligned}\langle\phi_{\mathcal{D}}\phi_{\mathcal{C}}\phi_{\mathcal{B}}\phi_{\mathcal{A}}\rangle&=(-1)^{\text{f.e.}}\langle\phi_{\mathcal{A}}\phi_{\mathcal{B}}\phi_{\mathcal{C}}\phi_{\mathcal{D}}\rangle\quad\text{with}\\&\qquad p_1\leftrightarrow p_4,\,p_2\leftrightarrow p_3,\\\langle\phi_{\mathcal{B}}\phi_{\mathcal{C}}\phi_{\mathcal{D}}\phi_{\mathcal{A}}\rangle&=(-1)^{\text{f.e.}}\langle\phi_{\mathcal{A}}\phi_{\mathcal{B}}\phi_{\mathcal{C}}\phi_{\mathcal{D}}\rangle\quad\text{with}\\&\qquad p_1\to p_4,\,p_2\to p_1,\,p_3\to p_2,\,p_4\to p_3,\\\langle\phi_{\mathcal{A}}\phi_{\mathcal{C}}\phi_{\mathcal{B}}\phi_{\mathcal{D}}\rangle&=-(-1)^{\text{f.e.}}\langle\phi_{\mathcal{A}}\phi_{\mathcal{B}}\phi_{\mathcal{C}}\phi_{\mathcal{D}}\rangle\quad\text{with}\\&\qquad p_2\leftrightarrow p_3-(-1)^{\text{f.e.}}\langle\phi_{\mathcal{A}}\phi_{\mathcal{C}}\phi_{\mathcal{D}}\phi_{\mathcal{B}}\rangle\quad\text{with}\\&\qquad p_3\leftrightarrow p_4,\end{aligned}\tag{48}$$

where $\phi_{\mathcal{A}}$ indicates a general field and "f.e." means the number of times fermions (if present) are exchanged in the reordering.

We begin by computing the 4-fermion amplitudes. Then we compute two fermion–two boson amplitudes, followed by 4-boson amplitudes.

1. Four fermion amplitudes

The calculation of the 4-fermion amplitudes of the YMCS theory is described in Appendix B. We obtain

$$\begin{aligned}\langle\chi_+\chi_+\chi_-\chi_-\rangle&=\langle\chi_-\chi_-\chi_+\chi_+\rangle=-\frac{2\langle 34\rangle}{u+m^2}\left[\langle 12\rangle+im\frac{\langle 42\rangle}{\langle 4\bar{1}\rangle}\right],\\\langle\chi_+\chi_-\chi_-\chi_+\rangle&=\langle\chi_-\chi_+\chi_+\chi_-\rangle=\frac{2\langle 41\rangle}{s+m^2}\left[\langle 23\rangle+im\frac{\langle 13\rangle}{\langle 1\bar{2}\rangle}\right],\\\langle\chi_+\chi_-\chi_+\chi_-\rangle&=\langle\chi_-\chi_+\chi_-\chi_+\rangle=\frac{2\langle 13\rangle}{s+m^2}\left[\langle 42\rangle-im\frac{\langle 14\rangle}{\langle 1\bar{2}\rangle}\right]\\&\quad-\frac{2\langle 42\rangle}{u+m^2}\left[\langle 31\rangle-im\frac{\langle 43\rangle}{\langle 4\bar{1}\rangle}\right],\end{aligned}\tag{49}$$

where $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$, $u=(p_1+p_4)^2$.

It is interesting to consider the massless limit of the four-fermion amplitude. In the strict $m=0$ limit, we should recover the $\mathcal{N}=2$ SYM amplitude computed in Eq. (3.20) of [3], and indeed that is what is found here. At the next order, $\mathcal{O}(m)$, we find that the massive spinor products may not be expressed using massless spinor products. Using the first amplitude above as an example, we find that the massless limit is

$$\langle\chi_+\chi_+\chi_-\chi_-\rangle=-2\frac{\langle 12\rangle^2}{\langle 23\rangle\langle 41\rangle}+\mathcal{O}(m),\tag{50}$$

where the spinor brackets are massless.

2. Two fermion–two boson amplitudes

We continue with the two fermion–two boson amplitudes. In what follows, the subscripts appearing on the spinors u and v refer to particle (i.e. leg) number. Note that perturbation theory using the mode expansions (A3) is consistent with the on-shell algebra for amplitudes without external gauge fields and can therefore be used to compute $\langle\chi_+\chi_-\Phi\Phi\rangle$. In particular, we obtain

$$\begin{aligned}\langle\chi_+\chi_-\Phi\Phi\rangle &= \langle\chi_-\chi_+\Phi\Phi\rangle\\ &= \frac{\bar{v}_1\not{p}_4u_2}{u+m^2} - \frac{1}{s(s+m^2)}\left(-2s\bar{v}_1\not{p}_4u_2 + 2im\epsilon_{\mu\nu\rho}p_4^\mu p_3^\nu\bar{v}_1\gamma^\rho u_2\right)\\ &= -\frac{1}{2(u+m^2)}(\langle 14\rangle\langle\bar{4}2\rangle + \langle 1\bar{4}\rangle\langle 42\rangle) - \frac{2(2m^2+s)}{s+m^2}\frac{\langle 2\bar{3}\rangle\langle 3\bar{1}\rangle}{\langle\bar{1}\,\bar{2}\rangle\langle\bar{1}2\rangle} + \frac{im}{s+m^2}\frac{\langle 1\bar{3}\rangle\langle\bar{1}3\rangle - \langle 2\bar{3}\rangle\langle\bar{2}3\rangle}{\langle\bar{1}\,\bar{2}\rangle}. \end{aligned} \tag{51}$$

The two fermion–two boson amplitudes with an external gauge field may be determined using the algebra (24). Specifically one finds

$$\begin{aligned}\langle\Phi\chi_+A\chi_-\rangle &= i\frac{\langle\bar{4}\,\bar{1}\rangle}{\langle 4\bar{3}\rangle}\langle\Psi_2\Psi_2\Psi_1\Psi_1\rangle + i\frac{\langle 2\bar{4}\rangle}{\langle 4\bar{3}\rangle}\langle\Phi\Phi\chi_+\chi_-\rangle\\ &= i\frac{\langle\bar{4}\,\bar{1}\rangle}{\langle 4\bar{3}\rangle}\left[\frac{\langle 41\rangle\langle 23\rangle}{u+m^2} + \frac{1}{s+m^2}\left(2\langle 23\rangle\langle 41\rangle - \langle 12\rangle\langle 34\rangle - 2im\frac{\langle 13\rangle\langle 14\rangle}{\langle 1\bar{2}\rangle}\right)\right]\\ &\quad + i\frac{\langle 2\bar{4}\rangle}{\langle\bar{4}\bar{3}\rangle}\left[-\frac{1}{2(u+m^2)}(\langle 32\rangle\langle\bar{2}4\rangle + \langle 3\bar{2}\rangle\langle 24\rangle) - \frac{2(2m^2+s)}{s+m^2}\frac{\langle 4\bar{1}\rangle\langle 1\bar{3}\rangle}{\langle\bar{3}\,\bar{4}\rangle\langle\bar{3}4\rangle} + \frac{im}{s+m^2}\frac{\langle 3\bar{1}\rangle\langle\bar{3}1\rangle - \langle 4\bar{1}\rangle\langle\bar{4}1\rangle}{\langle\bar{3}\,\bar{4}\rangle}\right]. \end{aligned} \tag{52}$$

$$\begin{aligned}\langle\chi_+AA\chi_-\rangle &= -\frac{\langle 41\rangle\langle\bar{4}\,\bar{1}\rangle}{\langle\bar{2}4\rangle\langle\bar{4}\,\bar{3}\rangle}\langle\Psi_2\Psi_2\Psi_1\Psi_1\rangle + \frac{\langle 43\rangle}{\langle\bar{2}4\rangle}\langle\Psi_1\Psi_2\Psi_2\Psi_1\rangle - \frac{\langle 41\rangle\langle 2\bar{4}\rangle}{\langle\bar{2}4\rangle\langle\bar{4}\,\bar{3}\rangle}\langle\Phi\Phi\chi_+\chi_-\rangle\\ &= \frac{1}{\langle\bar{2}\,\bar{1}\rangle}\left[-2\frac{\langle 41\rangle\langle 23\rangle}{\langle\bar{3}1\rangle}(s+2m^2) + 2\frac{\langle 12\rangle\langle 34\rangle}{\langle\bar{3}1\rangle}(s+4m^2) - im\left(\langle 32\rangle\langle 34\rangle - 2\frac{\langle 42\rangle\langle 34\rangle}{\langle\bar{3}1\rangle\langle 4\bar{1}\rangle}(s+4m^2)\right)\right]\frac{1}{u+m^2}\\ &\quad + \frac{1}{\langle\bar{4}\,\bar{3}\rangle}\left[\frac{\langle 12\rangle\langle 34\rangle}{\langle\bar{2}4\rangle}(s-u) - 2\frac{\langle 23\rangle\langle 41\rangle}{\langle\bar{2}4\rangle}(t+s) - 2\frac{\langle 23\rangle\langle 4\bar{1}\rangle\langle 1\bar{3}\rangle}{\langle\bar{1}\,\bar{2}\rangle\langle\bar{3}4\rangle}(s+2m^2) + 2im\frac{\langle 13\rangle\langle 14\rangle}{\langle\bar{2}4\rangle\langle 1\bar{2}\rangle}(t+s)\right.\\ &\quad \left. + im\frac{\langle 23\rangle}{\langle\bar{1}\,\bar{2}\rangle}(u-t)\right]\frac{1}{s+m^2}. \end{aligned} \tag{53}$$

3. Four boson amplitudes

The four Φ amplitude may be computed using perturbation theory and one finds

$$\begin{aligned}\langle\Phi\Phi\Phi\Phi\rangle &= \frac{(t-u)s - 4im\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{s(s+m^2)} + \frac{(t-s)u + 4im\epsilon_{\mu\nu\rho}p_1^\mu p_2^\nu p_3^\rho}{u(u+m^2)}\\ &= \frac{\langle 1\bar{4}\rangle\langle\bar{1}4\rangle - \langle 1\bar{3}\rangle\langle\bar{1}3\rangle}{s+m^2} + \frac{2im\langle 1\bar{2}\rangle\langle 2\bar{3}\rangle\langle 3\bar{1}\rangle}{s(s+m^2)} + \frac{\langle 1\bar{2}\rangle\langle\bar{1}2\rangle - \langle 1\bar{3}\rangle\langle\bar{1}3\rangle}{u+m^2} - \frac{2im\langle 1\bar{2}\rangle\langle 2\bar{3}\rangle\langle 3\bar{1}\rangle}{u(u+m^2)}. \end{aligned} \tag{54}$$

The four boson amplitudes with external gauge fields may be gotten using the algebra in (24). One finds

$$\langle\Phi\Phi AA\rangle = -\frac{\langle\bar{1}\,\bar{2}\rangle}{\langle\bar{3}\,\bar{4}\rangle}\langle\Psi_2\Psi_2\Psi_1\Psi_1\rangle = -\frac{\langle\bar{1}\,\bar{2}\rangle}{\langle\bar{3}\,\bar{4}\rangle}\left[\frac{\langle 41\rangle\langle 23\rangle}{u+m^2} + \frac{1}{s+m^2}\left(2\langle 23\rangle\langle 41\rangle - \langle 12\rangle\langle 34\rangle - 2im\frac{\langle 13\rangle\langle 14\rangle}{\langle 1\bar{2}\rangle}\right)\right]. \tag{55}$$

$$\langle AAAA\rangle = \frac{\langle 32\rangle}{\langle\bar{1}3\rangle}\langle\chi_+\chi_-AA\rangle + \frac{\langle 34\rangle}{\langle\bar{1}3\rangle}\langle\chi_+AA\chi_-\rangle = -\frac{\langle 32\rangle}{\langle\bar{1}3\rangle}\langle\chi_+AA\chi_-\rangle_{i\to i+1} + \frac{\langle 34\rangle}{\langle\bar{1}3\rangle}\langle\chi_+AA\chi_-\rangle, \tag{56}$$

where $i\to i+1$ indicates momentum relabeling, and $\langle\chi_+AA\chi_-\rangle$ is given in (53).

We end this section by pointing out that the computation of amplitudes involving external gauge fields using Feynman diagrams is significantly more complicated than the other computations presented in this paper. The complications have to do with defining a mode expansion for the gauge fields that is compatible with both the noncommutativity of the spatial components of the vector potential as well as the Gauss law constraints mentioned in Sec. II B. In particular, it has been argued in [53,54] that the canonical commutation relations of the gauge field cannot be satisfied if the mode expansion of the gauge field only contains the modes of an on-shell massive scalar field. Auxiliary fields must also appear in the mode expansion in order for the theory to be consistently quantized and for the noncommutativity of the gauge fields to be respected, making the use of Feynman diagrams extremely unwieldy. We are able to circumvent this difficulty in the results presented above by using the on-shell SUSY algebra—the algebra was shown to be consistent with both the off-shell superalgebra as well as the canonical quantization procedure in Sec. II B—to determine the amplitudes containing the external gauge fields.

It would be extremely desirable to have a spinor helicity framework for the computations of gauge field amplitudes in YMCS theories (with and without supersymmetry) using Feynman diagrams efficiently. We hope to analyze this issue in further detail elsewhere.

V. BCFW FOR MASS-DEFORMED THREE-DIMENSIONAL THEORIES

A very useful tool for computing scattering amplitudes are the BCFW recursion relations, which allow one to construct higher point on-shell amplitudes from lower-point on-shell amplitudes [8]. The BCFW recursion relations in $d \geq 4$ do not hold in 3d, however, even in the mass-deformed case. In $d \geq 4$ one derives the recursion relations by deforming two external legs of an on-shell amplitude as follows:

$$p_i \to p_i + zq, \qquad p_j \to p_j - zq,$$

where z is a complex number and q is some vector. In order for the momenta to remain on-shell for general z, we must impose the following conditions on q:

$$q \cdot p_i = q \cdot p_j = q^2 = 0.$$

In 3d, the only solution is $q = 0$. Hence, the usual BCFW deformation does not apply in 3d, even in the mass-deformed case. In order to define a two-line deformation, we must allow the deformation to be nonlinear. The BCFW recursion relations for massless 3d theories were derived in [10]. In this section, we will propose BCFW recursion relations for massive 3d theories.

A. Two-line deformation

The BCFW recursion relations follow from deforming the momenta of two external legs of an on-shell amplitude. Suppose we deform legs i and j. The deformation must preserve the total momentum

$$(p_i + p_j)^{\alpha\beta} = \lambda_i^{(\alpha}\bar{\lambda}_i^{\beta)} + \lambda_j^{(\alpha}\bar{\lambda}_j^{\beta)}. \tag{57}$$

The deformation must also preserve the following two conditions

$$\langle i\bar{i}\rangle^2 = -4m^2, \qquad \langle j\bar{j}\rangle^2 = -4m^2. \tag{58}$$

We will assume that all external particles of an on-shell amplitude have the same mass.

If the external particles are massless, then the momentum is given by

$$(p_i + p_j)^{\alpha\beta} = \lambda_i^{\alpha}\lambda_i^{\beta} + \lambda_j^{\alpha}\lambda_j^{\beta}.$$

In this case, the BCFW deformation is given by [10]

$$\begin{pmatrix} \lambda_i \\ \lambda_j \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}(z+z^{-1}) & \frac{i}{2}(z-z^{-1}) \\ -\frac{i}{2}(z-z^{-1}) & \frac{1}{2}(z+z^{-1}) \end{pmatrix} \begin{pmatrix} \lambda_i \\ \lambda_j \end{pmatrix}, \tag{59}$$

where z is an arbitrary complex number. The deformation above clearly conserves momentum since it is an orthogonal transformation. For the mass deformed case, there is a natural generalization. We simply deform the antiholomorphic spinors in the same way as the holomorphic ones in (59)

$$\begin{pmatrix} \bar{\lambda}_i \\ \bar{\lambda}_j \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}(z+z^{-1}) & \frac{i}{2}(z-z^{-1}) \\ -\frac{i}{2}(z-z^{-1}) & \frac{1}{2}(z+z^{-1}) \end{pmatrix} \begin{pmatrix} \bar{\lambda}_i \\ \bar{\lambda}_j \end{pmatrix}. \tag{60}$$

It is easy to see that (59) and (60) preserve momentum in (57). Furthermore, after these transformations we see that

$$\langle i\bar{i}\rangle \to \langle i\bar{i}\rangle + (\langle i\bar{j}\rangle - \langle \bar{i}j\rangle)\frac{i}{4}(z^2 - z^{-2}),$$
$$\langle j\bar{j}\rangle \to \langle j\bar{j}\rangle - (\langle i\bar{j}\rangle - \langle \bar{i}j\rangle)\frac{i}{4}(z^2 - z^{-2}).$$

Note that

$$\langle i\bar{j}\rangle = e^{i\kappa}\sqrt{(p_i + p_j)^2}, \qquad \langle \bar{i}j\rangle = e^{-i\kappa}\sqrt{(p_i + p_j)^2},$$

where $e^{i\kappa}$ is some $U(1)$ phase. Also note that we can we can redefine λ_i and $\bar{\lambda}_i$ by a phase since $p_i^{\alpha\beta} = \lambda_i^{(\alpha}\bar{\lambda}_i^{\beta)}$ is invariant under $(\lambda_i, \bar{\lambda}_i) \to (e^{i\omega}\lambda_i, e^{-i\omega}\bar{\lambda}_i)$. Hence, by taking $(\lambda_i, \bar{\lambda}_i) \to (e^{-i\kappa}\lambda_i, e^{i\kappa}\bar{\lambda}_i)$, this will set $\langle i\bar{j}\rangle = \langle \bar{i}j\rangle$ and the mass-shell conditions in (58) will be preserved. After fixing the phases of the (λ, η) variables, there is still a residual $U(1)$ symmetry which rotates all the (λ, η) variables in the same way. In the mass-deformed Chern-Simons theory, this $U(1)$ phase is then fixed once we define the superfields, as explained in Sec. III B 1. Note that the deformations in (59) and (60) also preserve $\langle ij\rangle$ and $\langle\bar{i}\,\bar{j}\rangle$.

To generalize this to a super-BCFW shift, consider the definition of supermomentum in (38)

$$q = \lambda\bar{\eta} + \bar{\lambda}\eta.$$

Then the sum of the supermomenta of the particles which are being shifted is given by

$$q_i + q_j = \lambda_i\bar{\eta}_i + \bar{\lambda}_i\eta_i + \lambda_j\bar{\eta}_j + \bar{\lambda}_j\eta_j.$$

The supermomentum will be preserved if we apply the same BCFW deformation to the fermionic coordinates as we do to the bosonic coordinates of the on-shell superspace

$$\begin{pmatrix}\eta_i\\ \eta_j\end{pmatrix} \to \begin{pmatrix}\frac{1}{2}(z+z^{-1}) & \frac{i}{2}(z-z^{-1})\\ -\frac{i}{2}(z-z^{-1}) & \frac{1}{2}(z+z^{-1})\end{pmatrix}\begin{pmatrix}\eta_i\\ \eta_j\end{pmatrix},$$
$$\begin{pmatrix}\bar{\eta}_i\\ \bar{\eta}_j\end{pmatrix} \to \begin{pmatrix}\frac{1}{2}(z+z^{-1}) & \frac{i}{2}(z-z^{-1})\\ -\frac{i}{2}(z-z^{-1}) & \frac{1}{2}(z+z^{-1})\end{pmatrix}\begin{pmatrix}\bar{\eta}_i\\ \bar{\eta}_j\end{pmatrix}. \tag{61}$$

B. Recursion relation

After performing the BCFW deformation, the amplitude becomes a function of z. Assuming the amplitude vanishes when $z\to\infty$, we have the following

$$\oint_{|z|=\infty}\frac{A(z)dz}{z-1} = 0. \tag{62}$$

On the other hand, this contour integral must also be equal to the sum of the residues of the integrand in the complex plane, which occur at $z=1$ and the poles of $A(z)$. Near its poles, $A(z)$ factorizes into two on-shell amplitudes (denoted A_L and A_R) multiplied by a propagator. Hence, we find that

$$A(z=1) = -\frac{1}{2\pi i}\sum_{f,j}\int d\eta\oint_{z_{f,j}}\frac{A_L(z,\eta)A_R(z,i\eta)}{\hat{p}_f(z)^2+m^2}\frac{1}{z-1}, \tag{63}$$

where the factorization channels are labeled by f, and $z_{f,j}$ corresponds to the jth root of $\hat{p}_f(z)^2+m^2$. In obtaining this formula, we assumed that all the external legs of the on-shell scattering amplitudes have the same mass, m. The integral $\int d\eta$ takes into account all the fields in the supermultiplet which can appear in the propagator. Note that $A(z=1)$ corresponds to the undeformed on-shell amplitude. Using (63), we can compute higher-point on-shell amplitudes from lower-point on-shell amplitudes.

From the deformation in (59) and (60), one can see that in any channel, $\hat{p}_f(z)^2+m^2$ has the following form

$$\hat{p}_f(z)^2 + m^2 = a_f z^{-2} + b_f + c_f z^2.$$

Hence the roots are obtained by solving a quadratic equation in z^2, see Appendix C.

C. Large-z behavior

In order for the recursion relation described in the previous section to be applicable, the on-shell amplitudes must vanish after performing the BCFW deformations in (59), (60), and (61) and taking the deformation parameter z to infinity.

The amplitudes of the YMCS theory do not generally have good large-z behavior. Furthermore, it does not appear to be possible to combine them into superamplitudes (which could in principle have better large-z behavior). Hence, our proposed BCFW recursion relation does not appear to be applicable to the $\mathcal{N}=2$ YMCS theory. The situation may improve for YMCS theories with more supersymmetry however.

Although the 4-point component amplitudes of the CSM theory also do not generally have good large-z behavior, our proposed recursion relation may be applicable to the superamplitudes of the CSM theory. In particular, the first 4-point superamplitude in (39) is $\mathcal{O}(1/z)$ when legs (1, 3) or (2, 4) are shifted. In order to test this, one should use the recursion relation to compute a 6-point superamplitude of the CSM theory, and match various components of the superamplitude with Feynman diagram calculations.

D. Factorization of YMCS amplitudes from BCFW shift

It is interesting to investigate the 4-point amplitudes of the YMCS theory in the vicinity of their poles, and to look for simple factorization into two 3-point amplitudes. As an illustrative example, we will look at one of the four-fermion amplitudes

$$\mathcal{A}(1) = \langle\chi_+\chi_+\chi_-\chi_-\rangle = \frac{2\langle 43\rangle}{s_{23}+m^2}\left[\langle 12\rangle + im\frac{\langle 42\rangle}{\langle 4\bar{1}\rangle}\right], \tag{64}$$

and perform the BCFW shift on legs 1 and 2. We find

$$\mathcal{A}(z) = \frac{2z^2}{m^2(s_- + \sqrt{s_-^2 - s_+^2})}\frac{2\langle 43\rangle}{(z^2-z_1^2)(z^2-z_2^2)} \times\left[\langle 12\rangle + im\frac{(z^2+1)\langle 42\rangle - i(z^2-1)\langle 41\rangle}{(z^2+1)\langle 4\bar{1}\rangle + i(z^2-1)\langle 4\bar{2}\rangle}\right], \tag{65}$$

where (see Appendix C for details)

$$s_\pm = \frac{1}{2m^2}(s_{13}\pm s_{23}), \tag{66}$$

and where the massive poles corresponding to $s_{23}=-m^2$ are found at $z=\pm z_1$ and $z=\pm z_2$ where

$$\{z_1^2, z_2^2\} = \left\{\frac{1}{2}\frac{(\sqrt{2s_+ + 1}+1)^2}{s_- + \sqrt{s_-^2 - s_+^2}}, \frac{1}{2}\frac{(\sqrt{2s_+ + 1}-1)^2}{s_- + \sqrt{s_-^2 - s_+^2}}\right\}. \tag{67}$$

Note that the massless pole corresponding to $s_{23} = 0$ has a vanishing residue[6]; this is consistent with the fact that the YMCS gauge field has only a single, massive degree of freedom.

In order to understand the factorization, we should associate the residues at the massive poles with the product of "left" and "right" 3-point amplitudes $\mathcal{A}_L$ and $\mathcal{A}_R$. We should be able to see both a contribution from two fermion-fermion-scalar amplitudes and also one from two fermion-fermion-gauge field amplitudes. We therefore write the deformed amplitude in the following way

$$\begin{aligned}\mathcal{A}(z) = {}& \frac{2z^2}{m^2(s_- + \sqrt{s_-^2 - s_+^2})}\frac{1}{(z^2 - z_1^2)(z^2 - z_2^2)} \\ & \times [(\langle 1'4\rangle\langle 2'3\rangle) + (-2\langle 12\rangle\langle 34\rangle - \langle 1'4\rangle\langle 2'3\rangle \\ & + 2im\langle 42'\rangle\langle 43\rangle/\langle 4\bar{1}'\rangle)],\end{aligned} \tag{68}$$

where the prime denotes the BCFW rotated spinor. The first term (enclosed in rounded parentheses) is the scalar exchange and the following factorization

$$\mathcal{A}_L(z_1) = \langle 1'4\rangle, \quad \mathcal{A}_R(z_1) = \langle 2'3\rangle, \tag{69}$$

matches with the 3-point amplitudes calculated for fermion-fermion-scalar scattering in Sec. IV A.

The second rounded-parentheses term in (68) corresponds to the gauge field exchange and so obviously the product of the two fermion-fermion-gauge field three-point functions yield

$$\mathcal{A}_L\mathcal{A}_R = -2\langle 12\rangle\langle 34\rangle - \langle 1'4\rangle\langle 2'3\rangle + 2im\langle 42'\rangle\langle 43\rangle/\langle 4\bar{1}'\rangle. \tag{70}$$

There is some freedom in how to factorize this expression into left and right components—computing the relevant 3-point functions using the techniques of [53,54] would allow one to determine this factorization, and we leave this issue as further work.

VI. CONCLUSION

In this paper, we study scattering amplitudes of mass-deformed three-dimensional gauge theories. In particular, we focus on mass-deformed Chern-Simons and Yang-Mills-Chern-Simons theories with $\mathcal{N} = 2$ supersymmetry. Note that the mass deformations in these theories preserve locality, Lorentz invariance, and gauge invariance. We derive the superalgebras which underlie the scattering matrices of the $\mathcal{N} = 2$ mass-deformed CSM theory and YMCS theory and show that the on-shell supersymmetry algebras for the two theories are fundamentally different. In particular, the algebra for YMCS contains no mass-deformation.

Using perturbative techniques and on-shell superalgebras, we compute 3- and 4-point tree-level color-ordered amplitudes in these theories (note that the odd point amplitudes of the CSM theory vanish). For the CS theory, we find that perturbation theory gives results that are consistent with the mass-deformed on-shell superalgebra. Further, we find that the 4-point amplitudes of the CSM theory can be encoded in very simple superamplitudes. On the other hand, for the YMCS theory we are able to deduce all the 4-point amplitudes using a combination of perturbative techniques and algebraic relations. Namely, we compute all the amplitudes without external gluons perturbatively (and show that they are consistent with the on-shell algebra) and deduce the remaining 4-point amplitudes using the on-shell superalgebra in (24).

We also propose a BCFW recursion relation for mass-deformed three-dimensional gauge theories which reduces to the BCFW recursion relation proposed in [10] in the massless limit. This recursion relation involves deforming the supermomenta of two external legs of an on-shell amplitude by a complex parameter z and is only applicable if the amplitude vanishes as $z \to \infty$. Although the component amplitudes of the $\mathcal{N} = 2$ CSM and YMCS theories do not generally have good large-z behavior, we find that one of the 4-point superamplitudes of the CSM theory exhibits good large-z behavior, which suggests that the recursion relation may be applicable to this theory.

There are a number of open questions that would be interesting to address. First of all, it would be very desirable to understand how to compute amplitudes with external gauge fields in the YMCS theory using Feynman diagrams. In particular, it would be desirable to use Feynman diagrams to compute the 3-point amplitudes with external gauge fields and confirm the 4-point amplitudes with external gauge fields which we deduced using the on-shell superalgebra. It would also be interesting to test our BCFW proposal by using it to compute a 6-point superamplitude of the CSM theory and then compare it to a Feynman diagram calculation.

Another interesting direction would be to extend our analysis to loop amplitudes. Note that IR divergences of loop amplitudes are more severe in three-dimensions than in four. On the other hand, we expect that mass-deformations will lead to better IR behavior. It would also be interesting to extend our analysis to mass-deformed theories with more supersymmetry, like the mass-deformed ABJM theory, which has $\mathcal{N} = 6$ supersymmetry. If the amplitudes of

[6]This can be seen by noting that the $\langle 4\bar{1}\rangle$ appearing in the second term in (64) is proportional to $\sqrt{s_{23}}$.

YMCS theories with $\mathcal{N} > 2$ supersymmetry can be encoded in superamplitudes, then the BCFW recursion relation proposed in this paper may be applicable to these theories since superamplitudes generally have better large-z behavior than component amplitudes.

The techniques developed in this paper may also be useful for studying the scattering amplitudes of three-dimensional gauge theories with spontaneously broken gauge symmetry. In this case, masses are acquired via the Higgs mechanism. In particular, it would be interesting to study scattering amplitudes in the Coulomb branch of the ABJM theory and see if they can be related to the amplitudes of maximal three-dimensional SYM theory in some limit. There is already some evidence that the amplitudes of three-dimensional SYM and ABJM theory can be related order by order in perturbation theory in a certain limit [4,31].

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APPENDIX A: CONVENTIONS, PROPAGATORS AND FEYNMAN RULES

We work in $(-++)$ signature and use the following gamma matrices:

$$\gamma^\mu = \{i\sigma^2, \sigma^1, \sigma^3\}. \tag{A1}$$

The $SU(N)$ generators t^a obey the following relations

$$t^a t^a = \frac{N^2-1}{2N}\mathbf{1}, \quad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}, \quad [t^a, t^b] = i f^{abc} t^c,$$
$$f^{abc} f^{abd} = N\delta^{cd}, \quad \{t^a, t^b\} = \frac{1}{N}\delta^{ab}\mathbf{1} + d^{abc} t^c. \tag{A2}$$

The scalar and fermionic fields in this paper have mode expansions given by

$$\Psi_\alpha(x) = \int \frac{d^2p}{(2\pi)^2}\frac{1}{\sqrt{2p^0}}(v_\alpha(p) b^\dagger(p) e^{ip\cdot x} + u_\alpha(p) b(p) e^{-ip\cdot x}),$$
$$\Phi(x) = \int \frac{d^2p}{(2\pi)^2}\frac{1}{\sqrt{2p^0}}(a_2^\dagger(p) e^{ip\cdot x} + a_2(p) e^{-ip\cdot x}), \tag{A3}$$

where

$$v(p) = \frac{1}{\sqrt{p_0 - p_1}}\begin{pmatrix} p_2 + im \\ p_1 - p_0 \end{pmatrix},$$
$$u(p) = \frac{1}{\sqrt{p_0 - p_1}}\begin{pmatrix} p_2 + im \\ p_1 - p_0 \end{pmatrix}, \tag{A4}$$

and we have neglected color and R-symmetry indices. There are many useful formulae involving spinors and gamma matrices

$$\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu\rho}\gamma_\rho, \qquad \epsilon^{\mu\nu\rho}\epsilon_{\gamma\delta\rho} = -\delta^\mu_\gamma\delta^\nu_\delta + \delta^\mu_\delta\delta^\nu_\gamma, \qquad (\gamma^\mu)_{\alpha\beta}(\gamma_\mu)_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta},$$
$$\epsilon^{\rho\mu\nu}(\gamma_\mu)_{\sigma\tau}(\gamma_\nu)_{\alpha\delta} = 2(\gamma^\rho)_{\alpha\tau}\delta_{\sigma\delta} - (\gamma^\rho)_{\alpha\delta}\delta_{\sigma\tau} - (\gamma^\rho)_{\sigma\tau}\delta_{\alpha\delta}, \qquad u^*(p) = v(p), \qquad \bar{v} = u^T C, \qquad \bar{u} = v^T C,$$
$$\bar{v}(p)v(p) = 2im, \qquad \bar{u}(p)u(p) = -2im, \qquad \bar{v}(p)u(p) = 0 = \bar{u}(p)v(p), \qquad \not{p}v = imv, \qquad \bar{v}\not{p} = im\bar{v},$$
$$\not{p}u = -imu, \quad \bar{u}\not{p} = -im\bar{u}, \qquad \bar{v}(k)\gamma^\mu u(p) = \bar{v}(k)u(p)\frac{im(p-k)^\mu + \epsilon^{\mu\nu\rho}p_\nu k_\rho}{m^2 + p\cdot k},$$
$$\bar{u}(k)\gamma^\mu u(p) = \bar{u}(k)u(p)\frac{im(p+k)^\mu - \epsilon^{\mu\nu\rho}p_\nu k_\rho}{m^2 - p\cdot k}, \qquad |\bar{u}(p)u(k)|^2 = |\bar{v}(p)v(k)|^2 = -(p+k)^2,$$
$$|\bar{u}(p)v(k)|^2 = |\bar{v}(p)u(k)|^2 = (p-k)^2, \bar{v}(p_i)v(p_j) = \langle\bar{j}i\rangle, \qquad \bar{u}(p_i)u(p_j) = \langle j\bar{i}\rangle,$$
$$\bar{u}(p_i)v(p_j) = \langle\bar{i}\,\bar{j}\rangle, \qquad \bar{v}(p_i)u(p_j) = \langle ij\rangle, \qquad -\sqrt{-\frac{st}{u}} = 2im - \frac{\langle 13\rangle\langle\bar{1}\,\bar{2}\rangle}{\langle\bar{2}3\rangle} = -\frac{\langle\bar{1}3\rangle\langle 1\bar{2}\rangle}{\langle\bar{2}3\rangle}. \tag{A5}$$

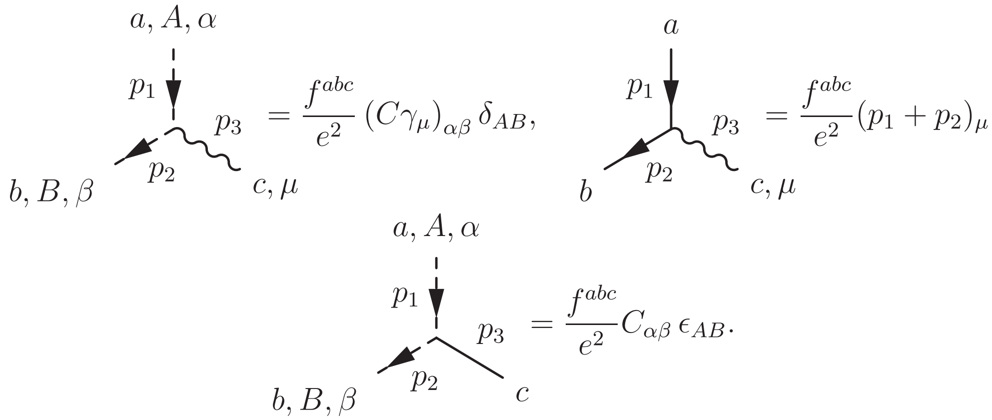
For the $\mathcal{N} = 2$ YMCS theory, the propagators are given by the following expressions

$$\langle A^a_\mu(p)A^b_\nu(-p)\rangle = -ie^2\delta^{ab}\Delta_{\mu\nu}(p) = \frac{-ie^2\delta^{ab}}{p^2(p^2+m^2)}(p^2\eta_{\mu\nu} - p_\mu p_\nu + im\epsilon_{\mu\nu\rho}p^\rho),$$
$$\langle\Phi^a(p)\Phi^b(-p)\rangle = \frac{-ie^2\delta^{ab}}{p^2+m^2},$$
$$\langle\Psi^a_{A\alpha}(p)\Psi^b_{B\beta}(-p)\rangle = \frac{-ie^2\delta^{ab}\delta_{AB}}{p^2+m^2}[(\not{p} + im)C^{-1}]_{\alpha\beta}, \tag{A6}$$

where $C = \gamma^0$ is the charge conjugation matrix. In obtaining the gauge field propagator, we have chosen Landau gauge. For the $\mathcal{N} = 2$ massive Chern-Simons theory, the scalar and fermion propagators are given by similar expressions and the gauge field propagator may be read off from the $m \to \infty$ limit of the YMCS gauge field propagator

$$\langle A^a_\mu(p) A^b_\nu(-p)\rangle_{\rm CS} = \frac{1}{\kappa}\frac{\epsilon_{\mu\nu\rho}p^\rho}{p^2}. \tag{A7}$$

Finally, we have made use of the following Feynman rules, where all momenta are ingoing unless explicitly indicated via an arrow, and where gluons, fermions, and scalars are represented by wiggly, dashed, and solid lines, respectively.



$$= \frac{f^{abc}}{e^2}(C\gamma_\mu)_{\alpha\beta}\,\delta_{AB}, \qquad = \frac{f^{abc}}{e^2}(p_1 + p_2)_\mu,$$

$$= \frac{f^{abc}}{e^2} C_{\alpha\beta}\,\epsilon_{AB}.$$

APPENDIX B: CALCULATIONAL DETAILS

In this appendix, we will compute the 4-fermion amplitudes of the YMCS theory and the CSM theory. We first compute the 4-fermion YMCS amplitudes, since the corresponding result in the CSM theory will then follow straightforwardly.

In the YMCS theory, the basic building blocks for the four-fermion amplitudes are the gluon and scalar exchange, given by

$$\mathcal{A}(1,2,3,4) = \bar{v}(p_1)\gamma^\mu u(p_2)\Delta_{\mu\nu}(p_1+p_2)\bar{v}(p_3)\gamma^\nu u(p_4), \tag{B1}$$

where $\Delta_{\mu\nu}$ is the YMCS gauge field propagator [see (A6)], and

$$\mathcal{B}(1,2,3,4) = \bar{v}(p_1)u(p_2)\frac{1}{(p_1+p_2)^2+m^2}\bar{v}(p_3)u(p_4), \tag{B2}$$

respectively. Defining the color-ordered amplitudes $\langle \phi_{\mathcal{A}_1}\phi_{\mathcal{A}_2}\phi_{\mathcal{A}_3}\phi_{\mathcal{A}_4}\rangle$ of completely general fields $\phi_{\mathcal{A}_i}$ as

$$\langle \phi^{a_1}_{\mathcal{A}_1}{}^\dagger(p_1)\phi^{a_2}_{\mathcal{A}_2}{}^\dagger(p_2)\phi^{a_3}_{\mathcal{A}_3}{}^\dagger(p_3)\phi^{a_4}_{\mathcal{A}_4}{}^\dagger(p_4)\rangle = 2ie^2\langle \phi_{\mathcal{A}_1}\phi_{\mathcal{A}_2}\phi_{\mathcal{A}_3}\phi_{\mathcal{A}_4}\rangle {\rm Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}] + \cdots, \tag{B3}$$

we find that

$$\begin{aligned}\langle \Psi_{A_1}\Psi_{A_2}\Psi_{A_3}\Psi_{A_4}\rangle = {}& \delta_{A_1A_2}\delta_{A_3A_4}\Big(\mathcal{B}(4,1,2,3)+\mathcal{A}(1,2,3,4)\Big)\\ &-\delta_{A_1A_3}\delta_{A_2A_4}\Big(\mathcal{B}(4,1,2,3)-\mathcal{B}(1,2,3,4)\Big)\\ &-\delta_{A_1A_4}\delta_{A_2A_3}\Big(\mathcal{A}(4,1,2,3)+\mathcal{B}(1,2,3,4)\Big).\end{aligned} \tag{B4}$$

The expressions for the gluon and scalar exchange may be compactly expressed as follows

$$\begin{aligned}\mathcal{B}(1,2,3,4) &= \frac{\langle 12\rangle\langle 34\rangle}{(p_1+p_2)^2+m^2},\\ \mathcal{A}(1,2,3,4) &= \frac{1}{(p_1+p_2)^2+m^2}\bigg(2\langle 23\rangle\langle 41\rangle - \langle 12\rangle\langle 34\rangle\\ &\quad - 2im\frac{\langle 13\rangle\langle 14\rangle}{\langle 1\bar{2}\rangle}\bigg).\end{aligned} \tag{B5}$$

Because of the inherent $SO(2)$ symmetry enjoyed by the fermions, it is useful to make the combinations

$$\chi_\pm = \frac{1}{\sqrt{2}}(\Psi_1 \pm i\Psi_2), \tag{B6}$$

which gives rise to the following amplitudes[7]

[7]We define $s = (p_1+p_2)^2$, $t = (p_1+p_3)^2$, $u = (p_1+p_4)^2$.

$$\begin{aligned}\langle\chi_+\chi_+\chi_-\chi_-\rangle &= \langle\chi_-\chi_-\chi_+\chi_+\rangle = -\mathcal{A}(4,1,2,3) - \mathcal{B}(4,1,2,3)\\ &= -\frac{2\langle 34\rangle}{u+m^2}\left[\langle 12\rangle + im\frac{\langle 42\rangle}{\langle 4\bar{1}\rangle}\right],\\ \langle\chi_+\chi_-\chi_-\chi_+\rangle &= \langle\chi_-\chi_+\chi_+\chi_-\rangle = \mathcal{A}(1,2,3,4) + \mathcal{B}(1,2,3,4)\\ &= \frac{2\langle 41\rangle}{s+m^2}\left[\langle 23\rangle + im\frac{\langle 13\rangle}{\langle 1\bar{2}\rangle}\right],\\ \langle\chi_+\chi_-\chi_+\chi_-\rangle &= \langle\chi_-\chi_+\chi_-\chi_+\rangle = \mathcal{A}(1,2,3,4) - \mathcal{B}(1,2,3,4)\\ &\quad - \mathcal{A}(4,1,2,3) + \mathcal{B}(4,1,2,3)\\ &= \frac{2\langle 13\rangle}{s+m^2}\left[\langle 42\rangle - im\frac{\langle 14\rangle}{\langle 1\bar{2}\rangle}\right]\\ &\quad - \frac{2\langle 42\rangle}{u+m^2}\left[\langle 31\rangle - im\frac{\langle 43\rangle}{\langle 4\bar{1}\rangle}\right]. \qquad \text{(B7)}\end{aligned}$$

The calculation of the color-ordered four-fermion amplitudes of the CSM theory is similar to the one we carried out for the YMCS theory. In fact the $\langle\chi_+\chi_+\chi_-\chi_-\rangle$ amplitude may be read off from (B5). There is no Yukawa coupling in the CSM theory, thus the tree-level four-fermion amplitudes are given only by the exchange of the gauge field. Thus we can take $\mathcal{B}$ to zero, and take the $m \to \infty$ limit in $\mathcal{A}$, in order to single out the pure CS term in the YMCS gauge field propagator. This corresponds to keeping only the last term in $\mathcal{A}$, and replacing $(p_1 + p_2)^2 + m^2 \to m^2$ in the factor outside the rounded brackets. Multiplying by e^2 and noting that $e^2/m = \kappa$, where $\kappa = k/4\pi$, one then finds that the 4-fermion amplitude of the CSM theory is given by

$$\langle\chi_+\chi_+\chi_-\chi_-\rangle_{\mathrm{CSM}} = -2i\frac{\langle 34\rangle\langle 42\rangle}{\langle 4\bar{1}\rangle}, \qquad \text{(B8)}$$

where we absorbed κ into the normalization of the fields.

APPENDIX C: BCFW DETAILS

We note that the BCFW shift has the following form on momenta:

$$p_{ij} \to \frac{1}{2}(p_i + p_j) \pm z^2 q \pm z^{-2}\tilde{q}, \qquad \text{(C1)}$$

so that $p_i + p_j \to p_i + p_j$. We find that q and $\tilde{q}$ may be parametrized as follows:

$$\begin{aligned} q &= \frac{1}{4}\left(p_i - p_j + \frac{2}{\sqrt{s_{ij}}} p_i \wedge p_j\right),\\ \tilde{q} &= \frac{1}{4}\left(p_i - p_j - \frac{2}{\sqrt{s_{ij}}} p_i \wedge p_j\right), \qquad \text{(C2)}\end{aligned}$$

where $(a\wedge b)^\mu = \varepsilon^{\mu\nu\rho} a_\nu b_\rho$, $s_{ij} = (p_i + p_j)^2$, and

$$\begin{aligned} q\cdot(p_i + p_j) &= \tilde{q}\cdot(p_i + p_j) = 0 = q^2 = \tilde{q}^2,\\ q + \tilde{q} &= \frac{1}{2}(p_i - p_j). \qquad \text{(C3)}\end{aligned}$$

We will be interested in the deformation of the remaining Mandelstam invariants s_{ik} and s_{jk}, where[8] $i \neq j \neq k \in 1, 2, 3$. We find

$$\begin{aligned} s_{ik} \to \frac{1}{2}(s_{ik} + s_{jk}) &+ \frac{z^2}{2}\left(\frac{1}{2}(s_{ik} - s_{jk}) + \varepsilon_{ijk}\sqrt{-s_{ik}s_{jk}}\right)\\ &+ \frac{z^{-2}}{2}\left(\frac{1}{2}(s_{ik} - s_{jk}) - \varepsilon_{ijk}\sqrt{-s_{ik}s_{jk}}\right), \qquad \text{(C4)}\end{aligned}$$

$$\begin{aligned} s_{jk} \to \frac{1}{2}(s_{ik} + s_{jk}) &- \frac{z^2}{2}\left(\frac{1}{2}(s_{ik} - s_{jk}) + \varepsilon_{ijk}\sqrt{-s_{ik}s_{jk}}\right)\\ &- \frac{z^{-2}}{2}\left(\frac{1}{2}(s_{ik} - s_{jk}) - \varepsilon_{ijk}\sqrt{-s_{ik}s_{jk}}\right), \qquad \text{(C5)}\end{aligned}$$

where we have used $p_i \cdot p_j \wedge p_k = \varepsilon_{ijk}\sqrt{-stu}$. We will be looking for poles in the massive channels $s_{ik} + m^2$ and $s_{jk} + m^2$; these correspond to the following equations

$$\frac{z^4}{2}\left(s_- + \sqrt{s_-^2 - s_+^2}\right) \pm z^2(s_+ + 1) + \frac{1}{2}\left(s_- - \sqrt{s_-^2 - s_+^2}\right) = 0, \qquad \text{(C6)}$$

where the upper sign corresponds to s_{ik} and the lower to s_{jk}, and where

$$s_\pm = \frac{1}{2m^2}(s_{ik} \pm s_{jk}). \qquad \text{(C7)}$$

The roots of these two equations are

$$\{z_1^2, z_2^2\} = \left\{\mp\frac{1}{2}\frac{(\sqrt{2s_+ + 1} \mp 1)^2}{s_- + \sqrt{s_-^2 - s_+^2}}, \mp\frac{1}{2}\frac{(\sqrt{2s_+ + 1} \pm 1)^2}{s_- + \sqrt{s_-^2 - s_+^2}}\right\}. \qquad \text{(C8)}$$

[8]The remaining momentum p_4 in the four-particle process is equal to $-p_i - p_j - p_k$.

[1] A. Agarwal, N. Beisert, and T. McLoughlin, J. High Energy Phys. 06 (**2009**) 045.
[2] D.-W. Chiou, O. J. Ganor, Y. P. Hong, B. S. Kim, and I. Mitra, Phys. Rev. D **71**, 125016 (2005).
[3] A. Agarwal and D. Young, J. High Energy Phys. 05 (**2011**) 100.
[4] A. E. Lipstein and L. Mason, arXiv:1207.6176.
[5] J. Bagger and N. Lambert, Phys. Rev. D **77**, 065008 (2008).
[6] A. Gustavsson, Nucl. Phys. **B811**, 66 (2009).
[7] Y.-t. Huang and A. E. Lipstein, J. High Energy Phys. 10 (**2010**) 007.
[8] R. Britto, F. Cachazo, B. Feng, and E. Witten, Phys. Rev. Lett. **94**, 181602 (2005).
[9] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, S. Caron-Huot, and J. Trnka, J. High Energy Phys. 01 (**2011**) 041.
[10] D. Gang, Y.-t. Huang, E. Koh, S. Lee, and A. E. Lipstein, J. High Energy Phys. 03 (**2011**) 116.
[11] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, J. High Energy Phys. 10 (**2008**) 091.
[12] T. Bargheer, F. Loebbert, and C. Meneghelli, Phys. Rev. D **82**, 045016 (2010).
[13] Y.-t. Huang and A. E. Lipstein, J. High Energy Phys. 11 (**2010**) 076.
[14] J. M. Drummond, J. Henn, V. A. Smirnov, and E. Sokatchev, J. High Energy Phys. 01 (**2007**) 064.
[15] A. Brandhuber, P. Heslop, and G. Travaglini, Phys. Rev. D **78**, 125005 (2008).
[16] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B828**, 317 (2010).
[17] L. Dolan, C. R. Nappi, and E. Witten, arXiv:hep-th/0401243.
[18] L. F. Alday and J. M. Maldacena, J. High Energy Phys. 06 (**2007**) 064.
[19] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B795**, 52 (2008).
[20] G. P. Korchemsky, J. M. Drummond, and E. Sokatchev, Nucl. Phys. **B795**, 385 (2008).
[21] L. J. Mason and D. Skinner, J. High Energy Phys. 12 (**2010**) 018.
[22] S. Caron-Huot, J. High Energy Phys. 07 (**2011**) 058.
[23] N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, J. High Energy Phys. 03 (**2010**) 020.
[24] S. Lee, Phys. Rev. Lett. **105**, 151603 (2010).
[25] J. M. Henn, J. Plefka, and K. Wiegandt, J. High Energy Phys. 08 (**2010**) 032; 11 (**2011**) 053(E).
[26] W.-M. Chen and Y.-t. Huang, J. High Energy Phys. 11 (**2011**) 057.
[27] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, and A. Santambrogio, J. High Energy Phys. 01 (**2012**) 056.
[28] M. S. Bianchi, M. Leoni, A. Mauri, S. Penati, and A. Santambrogio, J. High Energy Phys. 07 (**2012**) 029.
[29] T. Bargheer, N. Beisert, F. Loebbert, and T. McLoughlin, J. Phys. A **45**, 475402 (2012).
[30] A. Brandhuber, G. Travaglini, and C. Wen, J. High Energy Phys. 07 (**2012**) 160.
[31] M. S. Bianchi and M. Leoni, J. High Energy Phys. 03 (**2013**) 101.
[32] T. Bargheer, S. He, and T. McLoughlin, Phys. Rev. Lett. **108**, 231601 (2012).
[33] Y.-t. Huang and H. Johansson, Phys. Rev. Lett. **110**, 171601 (2013).
[34] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982); **185**, 406(E) (1988); **281**, 409 (2000).
[35] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982).
[36] H. Lin and J. M. Maldacena, Phys. Rev. D **74**, 084014 (2006).
[37] A. Agarwal and D. Young, Phys. Rev. D **82**, 045024 (2010).
[38] O. Aharony, Phys. Lett. B **404**, 71 (1997).
[39] A. Giveon and D. Kutasov, Nucl. Phys. **B812**, 1 (2009).
[40] F. Benini, C. Closset, and S. Cremonesi, J. High Energy Phys. 10 (**2011**) 075.
[41] D. L. Jafferis, J. High Energy Phys. 05 (**2012**) 159.
[42] D. L. Jafferis, I. R. Klebanov, S. S. Pufu, and B. R. Safdi, J. High Energy Phys. 06 (**2011**) 102.
[43] D. Martelli and J. Sparks, Phys. Rev. D **84**, 046008 (2011).
[44] S. Cheon, H. Kim, and N. Kim, J. High Energy Phys. 05 (**2011**) 134.
[45] T. Dimofte, D. Gaiotto, and S. Gukov, arXiv:1108.4389.
[46] T. Dimofte, D. Gaiotto, and S. Gukov, arXiv:1112.5179.
[47] T. Dimofte, M. Gabella, and A. B. Goncharov, arXiv:1301.0192.
[48] V. P. Nair, Phys. Lett. B **352**, 117 (1995).
[49] A. Agarwal and A. Fayyazuddin, J. High Energy Phys. 05 (**2012**) 153.
[50] A. Agarwal and V. P. Nair, Phys. Rev. D **85**, 085011 (2012).
[51] D. Karabali, C. j. Kim, and V. P. Nair, Nucl. Phys. **B566**, 331 (2000).
[52] H. Elvang, Y.-t. Huang, and C. Peng, J. High Energy Phys. 09 (**2011**) 031.
[53] K. Haller and E. Lim-Lombridas, Ann. Phys. (N.Y.) **246**, 1 (1996); **257**, 205(E) (1997).
[54] L.-s. Chen, G. V. Dunne, K. Haller, and E. Lim-Lombridas, J. Math. Phys. (N.Y.) **37**, 2602 (1996).