Higher-derivative $\mathcal{N} = 4$ superparticle in three-dimensional spacetime

Nikolay Kozyrev,[1] Sergey Krivonos,[1] Olaf Lechtenfeld,[2] and Armen Nersessian[3]
[1]*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia*
[2]*Leibniz Universität Hannover, Appelstrasse 2, 30167 Hannover, Germany*
[3]*Yerevan State University, 1 Alex Manoogian Street, Yerevan 0025, Armenia*
(Received 23 November 2013; published 24 February 2014)

Using the coset approach (nonlinear realization) we construct component actions for a superparticle in three-dimensional spacetime with $\mathcal{N} = 4$ supersymmetry partially broken to $\mathcal{N} = 2$. These actions may contain an anyonic term and the square of the first extrinsic worldline curvature. We present the supercharges for the unbroken and broken supersymmetries as well as the Hamiltonian for the supersymmetric anyon. In terms of the nonlinear realization superfields, the superspace actions take a simple form in all cases.

DOI: 10.1103/PhysRevD.89.045013 PACS numbers: 11.30.Pb, 11.30.Na, 11.30.Qc

I. INTRODUCTION

In a relativistic theory, any particlelike configuration spontaneously breaks the target-space Poincaré invariance to the stability group of the worldline. This breakdown is accompanied by the appearance of Goldstone bosons associated with the spontaneously broken translations and Lorentz boosts. The most appropriate method to construct low-energy effective actions for these Goldstone modes is the nonlinear-realization (or coset) approach [1], suitably modified for the case of supersymmetric spacetime symmetries [2].

Toward the construction of particle actions in D-dimensional spacetime, the coset approach works as follows. Let P, Z_i with $i = 1, \ldots, D-1$ be the generators of the target spacetime translations, M_{ij} be the generators of the $SO(D-1)$ subgroup of the Lorentz group $SO(1, D-1)$ rotating the spatial coordinates Z_i among each other, and K_i generate the coset $SO(1, D-1)/SO(D-1)$. All transformations of the full Poincaré group may be realized by a left action on the coset element

$$g = e^{tP} e^{q_i(t) Z_i} e^{\Lambda_i(t) K_i}. \tag{1.1}$$

The dependence of the coset coordinates $q_i(t)$ and $\Lambda_i(t)$ on the time t signals that the Z and K symmetries are spontaneously broken.

According to the general theorem [3], not all of the above Goldstone fields have to be treated as independent. In the present case, the fields $\Lambda_i(t)$ can be covariantly expressed through time derivatives of $q_i(t)$ by imposing the constraint

$$\Omega_Z^i = 0, \tag{1.2}$$

where the Cartan forms Ω are defined in a standard way,

$$g^{-1} dg = \Omega_P P + \Omega_M^{ij} M_{ij} + \Omega_Z^i Z_i + \Omega_K^i K_i. \tag{1.3}$$

Thus, we are dealing with the fields $q_i(t)$ only. The form Ω_P defines the einbein E, which connects the covariant world-volume form Ω_P and the differential dt via

$$\Omega_P = E dt. \tag{1.4}$$

Observing that the form Ω_P is invariant under all symmetries, one may immediately write an invariant action [4–7],

$$S_0 = \int dt E. \tag{1.5}$$

This action describes a free particle moving in D-dimensional spacetime in the static gauge.

The Cartan forms Ω_K^i pertaining to the coset may be used for constructing actions with higher time derivatives [5,6,8–10]. Moreover, in three spacetime dimensions, $D = 3$, there exists an additional possibility: the form Ω_M allows for a Wess-Zumino-like term in the action,

$$S_{\text{anyon}} = \frac{\alpha}{2} \int \Omega_M, \tag{1.6}$$

which provides the system with a nonzero (anyonic) spin [11]. The above integrand Ω_M is only quasi-invariant under the three-dimensional Poincaré transformations [12]; i.e. it shifts by a full time derivative under K_i transformations.

The supersymmetric generalization of particle actions within the coset approach requires spinor generators Q and S, which extend the Poincaré group to the super-Poincaré one,

$$\{Q, Q\} \sim P, \qquad \{S, S\} \sim P, \qquad \{Q, S\} \sim Z. \tag{1.7}$$

All symmetries can then be realized by group elements acting on the coset element

$$g = e^{tP} e^{\theta^a Q_a} e^{q^i(t,\theta) Z_i} e^{\psi^a(t,\theta) S_a} e^{\Lambda^i(t,\theta) K_i}. \tag{1.8}$$

One obtains a collection $q^i(t,\theta), \psi^a(t,\theta), \Lambda^i(t,\theta)$ of Goldstone superfields that depend on the worldline superspace coordinates t, θ. The appearance of the Goldstone fermions $\psi^a(t)$ is crucial for ensuring the symmetry with respect to spontaneously broken (S) supersymmetry. The rest of the coset approach machinery works as before: one may construct the Cartan forms $g^{-1}dg$ for the coset element (1.8) (and obtain new forms Ω_Q and Ω_S), and one may find the supersymmetric einbein and the corresponding bosonic and spinor covariant derivatives ∇_P and ∇_Q, respectively. One may even invent proper generalizations of the covariant constraints (1.2) as

$$\Omega_Z = 0, \qquad \Omega_S| = 0, \tag{1.9}$$

where $|$ denotes the $d\theta$ projection of a form (see e.g., [13] and references therein). The structure of the coset element (1.8) implies that Q supersymmetry is kept unbroken while S supersymmetry is spontaneously broken.[1]

The constraints (1.9) leave the lowest components of the superfields $q^i(t,\theta)$ and $\psi^a(t,\theta)$ as the only independent component fields of the theory. Unfortunately, as it happened in (1.6), any superparticle Lagrangian is only *quasi-invariant* with respect to the super-Poincaré group. For this reason, the corresponding action cannot be built from the Cartan forms. Commonly adopted alternatives for constructing supersymmetric particle (or brane) actions are

(i) to construct a linear realization of target-space Poincaré supersymmetry, in which the superfield Lagrangian appears as a supermultiplet component [14–16],
(ii) to perform a reduction from higher-dimensional component actions,
(iii) to make a superfield ansatz for the action (manifestly invariant under Q supersymmetry) and then impose the spontaneously broken S supersymmetry invariance.

Clearly, in all these approaches the coset method is not too helpful. The method working perfectly in bosonic models seems to be almost useless in the supersymmetric case. This shortcoming is caused by our concentrating on unbroken Q supersymmetry and on the superspace action. If instead we focus on the component action with broken S supersymmetry being manifest, the coset approach will again be quite useful. It has indeed been demonstrated in [17,18] that, with the coset parametrization (1.8), it is easy to produce an ansatz for the component action manifestly invariant with respect to the broken S supersymmetry. To this end, the following properties are important:

(i) with the chosen parametrization (1.8) of the coset element, the superspace coordinates θ are inert under S supersymmetry. Therefore, all superfield components transform *independently* with respect to S supersymmetry,
(ii) the $\theta = 0$ projection of the covariant derivative ∇_P is invariant under the broken S supersymmetry,
(iii) all physical fermionic components are just $\theta = 0$ projections of the superfields $\psi^a(t,\theta)$, and these components transform as the fermions of the Volkov-Akulov model [19] with respect to the broken S supersymmetry.

Thus, an ansatz for the component action with the smallest number of time derivatives can be written down immediately, because the physical fermionic components can enter the action only through the $\theta = 0$ projection of the einbein E or through the spacetime derivatives ∇_P of the "matter fields" $q^i(t)$. This ansatz will contain some arbitrary functions that can be determined by two additional requirements:

(i) the supersymmetric action should have a proper bosonic limit,
(ii) the supersymmetric action has to be invariant under unbroken supersymmetry.

These conditions completely fix the component action. Actions for $D = 2 + 1$ superparticles realizing an $\mathcal{N} = 2^{k+1} \to \mathcal{N} = 2^k$ pattern of supersymmetry breaking have been constructed in such a way [18].

The situation becomes more interesting if we admit terms with a nonminimal number of time derivatives in the action. The main goal of the present paper is to demonstrate how the corresponding component actions can be constructed for a three-dimensional superparticle with $\mathcal{N} = 4$ supersymmetry partially broken to $\mathcal{N} = 2$ and how an anyonic term (1.6) and the first extrinsic curvature ("rigidity") come to appear in the action. It should be clear from our exposition that the choice of the physical fermionic components is very important: it is the choice of the coset element as in (1.8) that forces the $\psi|_{\theta=0}$ components to be Volkov-Akulov goldstini. In terms of these fermions all the actions we will construct have a clear geometric interpretation. For the super anyonic case we will provide the Hamiltonian description as well. For completeness, for all cases considered we will also present the superspace actions that, in terms of the superfields $q^i(t,\theta), \psi^a(t,\theta)$, take a simple form. We shall conclude with a few comments and remarks.

II. SPONTANEOUS BREAKDOWN OF $D = 2 + 1$ POINCARÉ SYMMETRY

A. Coset approach: Kinematics

The commutation relations of the $D = 2 + 1$ Poincaré algebra read

$$[M_{ab}, P_{cd}] = \epsilon_{ac}P_{bd} + \epsilon_{bd}P_{ac} + \epsilon_{ad}P_{bc} + \epsilon_{bc}P_{ad},$$
$$[M_{ab}, M_{cd}] = \epsilon_{ac}M_{bd} + \epsilon_{bd}M_{ac} + \epsilon_{ad}M_{bc} + \epsilon_{bc}M_{ad}. \tag{2.1}$$

To get a convenient $d = 1$ form let us define the following generators:

[1] In this paper we shall only consider the case where $\#Q = \#S$, i.e. a half-breaking of global supersymmetry.

$$\begin{aligned} P &= P_{11}+P_{22}, \\ Z &= P_{11}-P_{22}-2\mathrm{i}P_{12}, \\ \bar{Z} &= P_{11}-P_{22}+2\mathrm{i}P_{12}, \\ J &= \frac{\mathrm{i}}{4}(M_{11}+M_{22}), \\ T &= \frac{\mathrm{i}}{4}(M_{11}-M_{22}-2\mathrm{i}M_{12}), \\ \bar{T} &= \frac{\mathrm{i}}{4}(M_{11}-M_{22}+2\mathrm{i}M_{12}). \end{aligned} \tag{2.2}$$

Being rewritten in terms of these generators (2.2) the algebra (2.1) acquires the familiar $d=1$ form,

$$\begin{aligned} &[J,T]=T, \qquad [J,\bar{T}]=-\bar{T}, \qquad [T,\bar{T}]=-2J, \\ &[J,Z]=Z, \qquad [T,P]=-Z, \qquad [\bar{T},P]=\bar{Z}, \\ &[J,\bar{Z}]=-\bar{Z}, \qquad [T,\bar{Z}]=-2P, \qquad [\bar{T},Z]=2P. \end{aligned} \tag{2.3}$$

From the $d=1$ point of view the generators $Z,\bar{Z}$ are the central charge generators.

We are going to consider the spontaneous breakdown of $D=2+1$ Poincaré symmetry down to $d=1$ Poincaré, generated by P and $U(1)$ rotations, generated by J. Therefore, we will put the generator J in the stability subgroup and choose the parametrization of our coset as

$$g=e^{\mathrm{i}tP}e^{\mathrm{i}(qZ+\bar{q}\bar{Z})}e^{\mathrm{i}(\Lambda T+\bar{\Lambda}\bar{T})}. \tag{2.4}$$

Here, $q(t),\bar{q}(t),\Lambda(t),\bar{\Lambda}(t)$ are Goldstone fields depending on the time t.

The local geometric properties of the system are specified by the left-invariant Cartan forms

$$g^{-1}dg=\mathrm{i}\omega_P P+\mathrm{i}\omega_Z Z+\mathrm{i}\bar{\omega}_Z\bar{Z}+\mathrm{i}\omega_T T+\mathrm{i}\bar{\omega}_T\bar{T}+\mathrm{i}\omega_J J, \tag{2.5}$$

which look extremely simple,

$$\begin{aligned} \omega_P &= \frac{1}{1-\lambda\bar{\lambda}}[(1+\lambda\bar{\lambda})dt+2\mathrm{i}(\lambda d\bar{q}-\bar{\lambda}dq)], \\ \omega_Z &= \frac{1}{1-\lambda\bar{\lambda}}[dq-\lambda^2 d\bar{q}+\mathrm{i}\lambda dt], \\ \bar{\omega}_Z &= \frac{1}{1-\lambda\bar{\lambda}}[d\bar{q}-\bar{\lambda}^2 dq-\mathrm{i}\bar{\lambda}dt], \\ \omega_T &= \frac{d\lambda}{1-\lambda\bar{\lambda}}, \\ \bar{\omega}_T &= \frac{d\bar{\lambda}}{1-\lambda\bar{\lambda}}, \\ \omega_J &= \mathrm{i}\frac{\lambda d\bar{\lambda}-d\lambda\bar{\lambda}}{1-\lambda\bar{\lambda}}, \end{aligned} \tag{2.6}$$

where

$$\lambda=\frac{\tanh(\sqrt{\Lambda\bar{\Lambda}})}{\sqrt{\Lambda\bar{\Lambda}}}\Lambda \quad \text{and} \quad \bar{\lambda}=\frac{\tanh(\sqrt{\Lambda\bar{\Lambda}})}{\sqrt{\Lambda\bar{\Lambda}}}\bar{\Lambda}. \tag{2.7}$$

The transformation properties of the coordinates and fields are induced by the left multiplications of the coset element (2.5),

$$g_0 g=g'h, \tag{2.8}$$

where $h\in U(1)$ belong to the stability subgroup. Thus, for the mostly interesting transformations with $g_0=e^{i(\alpha T+\bar{\alpha}\bar{T})}$ one gets

$$\begin{aligned} \delta t &= -2\mathrm{i}(\alpha\bar{q}-\bar{\alpha}q), \\ \delta q &= -\mathrm{i}\alpha t, \\ \delta\bar{q} &= \mathrm{i}\bar{\alpha}t, \\ \delta\lambda &= \alpha-\bar{\alpha}\lambda^2, \\ \delta\bar{\lambda} &= \bar{\alpha}-\alpha\bar{\lambda}^2. \end{aligned} \tag{2.9}$$

Finally, one may reduce the number of independent Goldstone fields by imposing the following conditions on the Cartan forms ω_Z and $\bar{\omega}_Z$ (inverse Higgs phenomenon [3]):

$$\begin{aligned} &\omega_Z=0\Rightarrow \dot{q}=-\mathrm{i}\frac{\lambda}{1+\lambda\bar{\lambda}} \quad \text{and} \\ &\bar{\omega}_Z=0\Rightarrow \dot{\bar{q}}=\mathrm{i}\frac{\bar{\lambda}}{1+\lambda\bar{\lambda}}, \end{aligned} \tag{2.10}$$

and therefore,

$$\lambda=2\mathrm{i}\frac{\dot{q}}{1+\sqrt{1-4\dot{q}\dot{\bar{q}}}}, \qquad \bar{\lambda}=-2\mathrm{i}\frac{\dot{\bar{q}}}{1+\sqrt{1-4\dot{q}\dot{\bar{q}}}}.$$

These constraints are purely kinematic ones. Thus, to realize this spontaneous breaking of $D=2+1$ Poincaré symmetry we need two scalar fields, $q(t)$ and $\bar{q}(t)$.

Using the constraints (2.10), one may further simplify the Cartan forms (2.6) to be

$$\begin{aligned} &\omega_P=\frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}}dt, \qquad \omega_T=\frac{d\lambda}{1-\lambda\bar{\lambda}}, \\ &\bar{\omega}_T=\frac{d\bar{\lambda}}{1-\lambda\bar{\lambda}}, \qquad \omega_J=\mathrm{i}\frac{\lambda d\bar{\lambda}-d\lambda\bar{\lambda}}{1-\lambda\bar{\lambda}}. \end{aligned} \tag{2.11}$$

B. Actions

(i) The simplest action, invariant under full $D=2+1$ Poincaré symmetry, is

$$S_0=-m_0\int\omega_P=-m_0\int dt\sqrt{1-4\dot{q}\dot{\bar{q}}}. \tag{2.12}$$

It can easily be represented in Poincaré- and reparametrization-invariant form as

$$S_0 = -m_0 \int d\tau \sqrt{\frac{dq^a}{d\tau}\frac{dq_a}{d\tau}} = -m_0 \int d\tau \left|\frac{dq}{d\tau}\right| \equiv -m_0 \int ds,$$
$$q^0 \equiv t, \qquad \frac{q^1 + \mathrm{i}q^2}{2} \equiv q, \tag{2.13}$$

and for the summation we have used the Minkowski metric $g_{ab} = \mathrm{diag}(+,-,-)$. This is the action of a massive particle in $D = 2 + 1$ spacetime.

(ii) A less trivial action can be constructed as

$$S_{\text{anyon}} = -\frac{\alpha}{2}\int \omega_J = -\frac{\mathrm{i}\alpha}{2}\int dt \frac{\dot{\bar\lambda}\lambda - \bar\lambda\dot\lambda}{1-\lambda\bar\lambda} = \mathrm{i}\alpha \int dt \frac{\ddot q\dot{\bar q} - \dot q\ddot{\bar q}}{\sqrt{1-4\dot q\dot{\bar q}}(1+\sqrt{1-4\dot q\dot{\bar q}})}. \tag{2.14}$$

In reparametrization-invariant form it reads

$$S_{\text{anyon}} = \mathrm{i}\alpha \int \frac{(d^2q^1/d\tau^2)(dq^2/d\tau) - (d^2q^2/d\tau^2)(dq^1/d\tau)}{|dq/d\tau|(dq^0/d\tau + |dq/d\tau|)} d\tau,$$
$$q^0 \equiv t. \tag{2.15}$$

It is seen that this defines the vector potential of a Dirac monopole in three-dimensional Minkowski space, parametrized by the velocities $v^a \equiv dq^a/d\tau$. Hence, we arrive at an action defining anyonic spin (see, e.g., [11]).

(iii) Finally, one may consider the action

$$S_{\text{rigid}} = \beta \int \frac{\omega_T \bar\omega_T}{\omega_P} = \beta \int dt \frac{1+\lambda\bar\lambda}{(1-\lambda\bar\lambda)^3}\dot\lambda\dot{\bar\lambda} = \beta \int dt \frac{(\dot{\bar q}\ddot q + \dot q\ddot{\bar q})^2 + (1-4\dot q\dot{\bar q})\ddot q\ddot{\bar q}}{(1-4\dot q\dot{\bar q})^{5/2}}. \tag{2.16}$$

Representing this action in Poincaré- and reparametrization-invariant form, we get

$$S_{\text{rigid}} = \beta \int k_1^2(\dot q, \ddot q) ds, \tag{2.17}$$

where

$$k_1^2(\dot q, \ddot q) \equiv \frac{(\ddot q^a \dot q_a)^2 - (\ddot q^a \ddot q_a)(\dot q^b \dot q_b)}{(\dot q^c \dot q_c)^3} \quad \text{with}$$
$$\dot q^a = \frac{dq^a}{d\tau}, \quad \ddot q^a = \frac{d^2 q^a}{(d\tau)^2} \tag{2.18}$$

is the square of the first extrinsic curvature ("rigidity") of the worldline in $\mathbb{R}^{1,2}$. Note that systems defined by the sum of (2.12) and (2.16) have been studied by various authors (see, e.g., [9]).

(iv) The most general action depending on $\lambda, \bar\lambda$ and $\dot\lambda, \dot{\bar\lambda}$ only (i.e. depending on up to second derivatives of q and $\bar q$) has the form

$$S_{\text{gen}} = \int \frac{1-\lambda\bar\lambda}{1+\lambda\bar\lambda}\mathcal{F}\left[\frac{(1+\lambda\bar\lambda)^2\dot\lambda\dot{\bar\lambda}}{(1-\lambda\bar\lambda)^4}\right]dt = \int \mathcal{F}(k_1^2)ds, \tag{2.19}$$

where $\mathcal{F}$ is an arbitrary function. For the Hamiltonian analyses of such systems we refer to [20,21]. The most interesting case corresponds to the choice $\mathcal{F}(x) = c_0 + c_1\sqrt{x}$, i.e. to a Lagrangian linear in the curvature, which has been studied extensively [10].

We remark that S_0 and S_{rigid} as well as S_{gen} define Poincaré-invariant actions, while S_2 is only weakly invariant under $D = 2 + 1$ Poincaré transformations.

C. Hamiltonian formulations

In this subsection we shall consider the Hamiltonian formulation of the actions (2.12), (2.14), and (2.16) introduced in the previous subsection.

The Hamiltonian formulation of the action (2.12) is a textbook exercise. In the static-gauge parametrization it is defined by the symplectic structure $dp\wedge dq + d\bar p\wedge d\bar q$ and by the Hamiltonian $p_0 = \sqrt{m_0^2 + p\bar p}$ and, obviously, it describes a $(2+1)$-dimensional scalar relativistic particle with mass m_0.

1. Majorana anyon

Adding to (2.12) the Wess-Zumino term (2.14) provides the system with a nonzero spin but relaxes, at the classical level, the mass-shell condition. So let us give the Hamiltonian formulation of $S = S_0 + S_{\text{anyon}}$, in the static-gauge parametrization

$$S = -m_0 \int \omega_p - \frac{\alpha}{2}\int \omega_J = -m_0 \int dt \sqrt{1-4\dot q\dot{\bar q}} + \mathrm{i}\alpha \int dt \frac{\ddot q\dot{\bar q} - \dot q\ddot{\bar q}}{\sqrt{1-4\dot q\dot{\bar q}}(1+\sqrt{1-4\dot q\dot{\bar q}})}. \tag{2.20}$$

Taking into account the relations (2.10) we rewrite its Lagrangian in a first-order form,

$$\tilde{\mathcal{L}} = -m_0\frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}} - \frac{\mathrm{i}\alpha}{2}\frac{\lambda\dot{\bar{\lambda}}-\bar{\lambda}\dot{\lambda}}{1-\lambda\bar{\lambda}} + p\left(\dot{q}+\frac{\mathrm{i}\lambda}{1+\lambda\bar{\lambda}}\right) + \bar{p}\left(\dot{\bar{q}}-\frac{\mathrm{i}\bar{\lambda}}{1+\lambda\bar{\lambda}}\right). \tag{2.21}$$

This expression is of the form $\tilde{\mathcal{L}} = \mathcal{A}_{(1)A}(x)\dot{x}^A - \mathcal{H}(x)$, where $x^A \in \{p,\bar{p},\lambda,\bar{\lambda},q,\bar{q}\}$ are independent variables,

$$\mathcal{H} = p_0 = \frac{\mathrm{i}\bar{p}\,\bar{\lambda} - \mathrm{i}p\lambda + m_0(1-\lambda\bar{\lambda})}{1+\lambda\bar{\lambda}} \tag{2.22}$$

is the Hamiltonian, and

$$\mathcal{A}_{(1)} = pdq + \bar{p}d\bar{q} - \frac{\mathrm{i}\alpha}{2}\frac{\lambda d\bar{\lambda}-\bar{\lambda}d\lambda}{1-\lambda\bar{\lambda}} \tag{2.23}$$

is a one-form defining the symplectic structure

$$\omega = d\mathcal{A}_{(1)} = dp\wedge dq + d\bar{p}\wedge d\bar{q} - \mathrm{i}\alpha\frac{d\lambda\wedge d\bar{\lambda}}{(1-\lambda\bar{\lambda})^2}. \tag{2.24}$$

This symplectic structure defines Poisson brackets given by the nonzero relations

$$\{p,q\}=1, \qquad \{\bar{p},\bar{q}\}=1, \qquad \{\lambda,\bar{\lambda}\} = \frac{\mathrm{i}}{\alpha}(1-\lambda\bar{\lambda})^2. \tag{2.25}$$

One can easily check that the generators of $so(1,2)$ are defined by

$$J_0 = 2\mathrm{i}(\bar{p}\bar{q}-pq) + \alpha\frac{1+\lambda\bar{\lambda}}{1-\lambda\bar{\lambda}},$$
$$J_+ = \bar{p} + q^2 p - \mathrm{i}\alpha\frac{\lambda}{1-\lambda\bar{\lambda}}: \{J_\pm, J_0\} = 2\mathrm{i}J_\pm, \quad \{J_+, J_-\} = \mathrm{i}J_0. \tag{2.26}$$

Together with $p_0 \equiv \mathcal{H}$, $p = (p_1 + \mathrm{i}p_2)/2$, they form the $(2+1)$-dimensional Poincaré algebra. The Casimirs of this algebra, $p_a p^a =: m^2$ and $p_a J^a =: ms$, define the spin s and mass m of the particle. Thus, we have the so-called Majorana condition

$$ms = m_0\alpha = \mathrm{const}; \tag{2.27}$$

i.e. we deal with a reducible representation of the Poincaré group. This $(2+1)$-dimensional system has been studied in detail in [22], where it was called a "Majorana anyon." We remark that the Lagrangian of [22] featured a linear dependence on the second extrinsic curvature (torsion) κ_2 and thus included third-order time derivatives as well. A Majorana anyon can also be described by a simple second-order action on null curves [23].

2. Rigid particle

Let us give a Hamiltonian formulation for the action containing a rigidity term quadratic in the first extrinsic curvature,

$$S = S_0 + S_{\mathrm{anyon}} + S_{\mathrm{rigid}} = \int \mathcal{L}dt,$$
$$\mathcal{L} = -m_0\omega_P - \frac{\alpha}{2}\Omega_J + \beta\frac{\omega_T\bar{\omega}_T}{\omega_P}. \tag{2.28}$$

Its Poincaré-covariant formulation (in the absence of an anyonic term, i.e. for $\alpha = 0$) is well known and has been considered by many authors [9,21]. Here, we restrict ourselves to the Hamiltonian formulation in the static gauge. In complete analogy with the previous case, we replace the Lagrangian by an equivalent first-order one,

$$\tilde{\mathcal{L}} = -m_0\frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}} - \frac{\alpha}{2}\frac{\mathrm{i}(\lambda\dot{\bar{\lambda}}-\bar{\lambda}\dot{\lambda})}{1-\lambda\bar{\lambda}} + \Pi\dot{\lambda} + \bar{\Pi}\dot{\bar{\lambda}} - \frac{1}{\beta}\frac{(1-\lambda\bar{\lambda})^3}{1+\lambda\bar{\lambda}}\Pi\bar{\Pi} + p\left(\dot{q}+\frac{\mathrm{i}\lambda}{1+\lambda\bar{\lambda}}\right) + \bar{p}\left(\dot{\bar{q}}-\frac{\mathrm{i}\bar{\lambda}}{1+\lambda\bar{\lambda}}\right). \tag{2.29}$$

Hence, the system is described by the Hamiltonian

$$\mathcal{H}_{\mathrm{rigid}} = p_0 = \frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}}\left(\frac{1}{\beta}(1-\lambda\bar{\lambda})^2\Pi\bar{\Pi} - \frac{\mathrm{i}(p\lambda-\bar{p}\,\bar{\lambda})}{1-\lambda\bar{\lambda}} + m_0\right) \tag{2.30}$$

and by the symplectic one-form

$$\mathcal{A}_{(1)} = pdq + \bar{p}d\bar{q} + \Pi d\lambda + \bar{\Pi}d\bar{\lambda} - \frac{\mathrm{i}\alpha}{2}\frac{\lambda d\bar{\lambda}-\bar{\lambda}d\lambda}{1-\lambda\bar{\lambda}}. \tag{2.31}$$

The latter yields the symplectic structure

$$\omega = d\mathcal{A}_{(1)} = dp\wedge dq + d\bar{p}\wedge d\bar{q} + d\Pi\wedge d\lambda + d\bar{\Pi}\wedge d\bar{\lambda} - \mathrm{i}\alpha\frac{d\lambda\wedge d\bar{\lambda}}{(1-\lambda\bar{\lambda})^2}, \tag{2.32}$$

and the corresponding nonzero Poisson brackets

$$\{p,q\}=1, \qquad \{\bar{p},\bar{q}\}=1, \qquad \{\Pi,\lambda\}=1,$$
$$\{\bar{\Pi},\bar{\lambda}\}=1, \qquad \{\Pi,\bar{\Pi}\} = -\mathrm{i}\alpha(1-\lambda\bar{\lambda})^2. \tag{2.33}$$

The Lorentz generators read

$$J_0 = 2\mathrm{i}(\bar p\,\bar q - pq) + 2\mathrm{i}(\bar\Pi\,\bar\lambda - \Pi\lambda) + \alpha\frac{1+\lambda\bar\lambda}{1-\lambda\bar\lambda} \quad \text{and}$$

$$J_+ = \bar p + q^2 p + \bar\Pi + \lambda^2\Pi - \mathrm{i}\alpha\frac{\lambda}{1-\lambda\bar\lambda}, \tag{2.34}$$

while the translation generators are given, as before, by $p_0 = \mathcal{H}_{\rm rigid}, p, \bar p$. It is easy to check that neither spin nor mass are fixed in this model.

III. SUPERSYMMETRIC GENERALIZATION

In this section we turn to $\mathcal{N}=4$ supersymmetric extensions of the actions given above. Two of the four supercharges are assumed to be spontaneously broken, leaving us with $\mathcal{N}=2$ unbroken supersymmetry.

A. Coset approach: Kinematics

We begin with the $\mathcal{N}=2, D=2+1$ super-Poincaré algebra, which in $d=1$ notation appears as $\mathcal{N}=4, d=1$ super-Poincaré algebra with two central charges. The basic (anti)commutation relations extend the previous relations (2.3) by

$$\{Q,\bar Q\}=2P,\qquad \{S,\bar S\}=2P,\qquad \{Q,S\}=2Z,$$
$$\{\bar Q,\bar S\}=2\bar Z,\qquad [J,Q]=\frac{1}{2}Q,\qquad [J,\bar Q]=-\frac{1}{2}\bar Q,$$
$$[T,\bar Q]=-S,\qquad [\bar T,Q]=\bar S,\qquad [J,S]=\frac{1}{2}S,$$
$$[J,\bar S]=-\frac{1}{2}\bar S,\qquad [T,\bar S]=-Q,\qquad [\bar T,S]=\bar Q. \tag{3.1}$$

Here, $Q,\bar Q$ and $S,\bar S$ are the generators of the unbroken and spontaneously broken supersymmetries, respectively. P is the generator of translation, $Z,\bar Z$ are the central charge generators, while $T,\bar T,J$ are the generators of the $D=2+1$ Lorentz group, as before.

In the coset approach [1,2], the breakdown of S supersymmetry and $Z,\bar Z$ translations is reflected in the structure of the coset element

$$g = e^{\mathrm{i}tP}e^{\theta Q+\bar\theta\bar Q}e^{\boldsymbol{\psi} S+\bar{\boldsymbol{\psi}}\bar S}e^{\mathrm{i}(\mathbf{q}Z+\bar{\mathbf{q}}\bar Z)}e^{\mathrm{i}(\Lambda T+\bar\Lambda\bar T)}. \tag{3.2}$$

The $\mathcal{N}=2$ superfields $\mathbf{q}(t,\theta,\bar\theta),\boldsymbol{\psi}(t,\theta,\bar\theta)$, and $\Lambda(t,\theta,\bar\theta)$ are Goldstone superfields accompanying the $\mathcal{N}=2, D=2+1$ super-Poincaré to $\mathcal{N}=2, d=1$ super-Poincaré breaking.

The transformation properties of the coordinates and superfields are induced by the left multiplications of the coset element (3.2),

$$g_0 g = g' h, \qquad h\sim e^{fJ}. \tag{3.3}$$

The most important transformations read

(i) Unbroken SUSY $(g_0=e^{\epsilon Q+\bar\epsilon\bar Q})$: $\delta\theta=\epsilon$, $\delta t = \mathrm{i}(\epsilon\bar\theta+\bar\epsilon\theta)$;

(ii) Broken SUSY $(g_0=e^{\varepsilon S+\bar\varepsilon\bar S})$: $\delta t=\mathrm{i}(\varepsilon\bar{\boldsymbol{\psi}}+\bar\varepsilon\boldsymbol{\psi})$, $\delta\boldsymbol{\psi}=\varepsilon$, $\delta\mathbf{q}=2\mathrm{i}\varepsilon\theta$,

(iii) Automorphism group $(g_0=e^{\mathrm{i}(\alpha T+\bar\alpha\bar T)})$:

$$\begin{cases}\delta t=-2\mathrm{i}(\alpha\bar{\mathbf{q}}-\bar\alpha\mathbf{q})+2\alpha\bar\theta\bar{\boldsymbol{\psi}}-2\bar\alpha\theta\boldsymbol{\psi}, & \delta\theta=-\mathrm{i}\alpha\bar{\boldsymbol{\psi}},\\ \delta\mathbf{q}=\alpha(-\mathrm{i}t-\theta\bar\theta+\boldsymbol{\psi}\bar{\boldsymbol{\psi}}), \quad \delta\boldsymbol{\psi}=-\mathrm{i}\alpha\bar\theta, & \delta\boldsymbol{\lambda}=\alpha-\bar\alpha\boldsymbol{\lambda}^2,\end{cases}$$

where, as in (2.7) before,

$$\boldsymbol{\lambda}=\frac{\tanh(\sqrt{\Lambda\bar\Lambda})}{\sqrt{\Lambda\bar\Lambda}}\Lambda. \tag{3.4}$$

The left invariant Cartan forms read

$$\omega_P=\frac{1}{1-\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}}[(1+\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}})\Delta t+2\mathrm{i}(\boldsymbol{\lambda}\Delta\bar{\mathbf{q}}-\bar{\boldsymbol{\lambda}}\Delta\mathbf{q})],$$
$$\omega_Z=\frac{1}{1-\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}}[\Delta\mathbf{q}-\boldsymbol{\lambda}^2\Delta\bar{\mathbf{q}}+\mathrm{i}\boldsymbol{\lambda}\Delta t],$$
$$\omega_T=\frac{d\boldsymbol{\lambda}}{1-\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}},\qquad \omega_J=\mathrm{i}\frac{\boldsymbol{\lambda}d\bar{\boldsymbol{\lambda}}-d\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}}{1-\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}}.$$
$$\omega_Q=\frac{1}{\sqrt{1-\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}}}[d\theta+\mathrm{i}\boldsymbol{\lambda}d\bar{\boldsymbol{\psi}}],\qquad \omega_S=\frac{1}{\sqrt{1-\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}}}[d\boldsymbol{\psi}+\mathrm{i}\boldsymbol{\lambda}d\bar\theta]. \tag{3.5}$$

Here,

$$\Delta t = dt-\mathrm{i}(\theta d\bar\theta+\bar\theta d\theta+\boldsymbol{\psi}d\bar{\boldsymbol{\psi}}+\bar{\boldsymbol{\psi}}d\boldsymbol{\psi}) \quad \text{and}$$
$$\Delta\mathbf{q}=d\mathbf{q}-2\mathrm{i}\boldsymbol{\psi}d\theta. \tag{3.6}$$

Having at hand the Cartan forms, one may construct "semicovariant" derivatives (covariant with respect to P,J, broken and unbroken supersymmetries, only) via

$$\Delta t\nabla_t+d\theta\nabla_\theta+d\bar\theta\bar\nabla_\theta=dt\frac{\partial}{\partial t}+d\theta\frac{\partial}{\partial\theta}+d\bar\theta\frac{\partial}{\partial\bar\theta}. \tag{3.7}$$

Explicitly, they read

$$\nabla_t=E^{-1}\partial_t,$$
$$\nabla_\theta=D-\mathrm{i}(\bar{\boldsymbol{\psi}}D\boldsymbol{\psi}+\boldsymbol{\psi}D\bar{\boldsymbol{\psi}})\nabla_t,$$
$$\bar\nabla_\theta=\bar D-\mathrm{i}(\bar{\boldsymbol{\psi}}\bar D\boldsymbol{\psi}+\boldsymbol{\psi}\bar D\bar{\boldsymbol{\psi}})\nabla_t, \tag{3.8}$$

where

$$E=1+\mathrm{i}(\dot{\boldsymbol{\psi}}\bar{\boldsymbol{\psi}}+\dot{\bar{\boldsymbol{\psi}}}\boldsymbol{\psi}),$$
$$D=\frac{\partial}{\partial\theta}-\mathrm{i}\bar\theta\partial_t,$$
$$\bar D=\frac{\partial}{\partial\bar\theta}-\mathrm{i}\theta\partial_t:\ \{D,\bar D\}=-2\mathrm{i}\partial_t. \tag{3.9}$$

These derivatives obey the following algebra:

$$\begin{aligned} \{\nabla_\theta,\bar\nabla_\theta\} &= -2\mathrm{i}(1+\nabla_\theta\boldsymbol{\psi}\bar\nabla_\theta\bar{\boldsymbol{\psi}}+\bar\nabla_\theta\boldsymbol{\psi}\nabla_\theta\bar{\boldsymbol{\psi}})\nabla_t,\\ \{\nabla_\theta,\nabla_\theta\} &= -4\mathrm{i}\nabla_\theta\bar{\boldsymbol{\psi}}\nabla_\theta\boldsymbol{\psi}\nabla_t,\\ \{\bar\nabla_\theta,\bar\nabla_\theta\} &= -4\mathrm{i}\bar\nabla_\theta\bar{\boldsymbol{\psi}}\bar\nabla_\theta\boldsymbol{\psi}\nabla_t,\\ [\nabla_t,\nabla_\theta] &= -2\mathrm{i}(\nabla_\theta\bar{\boldsymbol{\psi}}\nabla_t\boldsymbol{\psi}+\nabla_\theta\boldsymbol{\psi}\nabla_t\bar{\boldsymbol{\psi}})\nabla_t,\\ [\nabla_t,\bar\nabla_\theta] &= -2\mathrm{i}(\bar\nabla_\theta\bar{\boldsymbol{\psi}}\nabla_t\boldsymbol{\psi}+\bar\nabla_\theta\boldsymbol{\psi}\nabla_t\bar{\boldsymbol{\psi}})\nabla_t. \end{aligned} \tag{3.10}$$

Finally, imposing the same constraints (2.10) as in the bosonic case, one may reduce the number of independent Goldstone superfields,

$$\begin{aligned} \omega_Z=0 \Rightarrow \nabla_t\mathbf{q}=-\mathrm{i}\frac{\boldsymbol{\lambda}}{1+\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}},\quad \nabla_\theta\mathbf{q}+2\mathrm{i}\boldsymbol{\psi}=0,\quad \bar\nabla_\theta\mathbf{q}=0,\\ \bar\omega_Z=0 \Rightarrow \nabla_t\bar{\mathbf{q}}=\mathrm{i}\frac{\bar{\boldsymbol{\lambda}}}{1+\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}},\quad \bar\nabla_\theta\bar{\mathbf{q}}+2\mathrm{i}\bar{\boldsymbol{\psi}}=0,\quad \nabla_\theta\bar{\mathbf{q}}=0. \end{aligned} \tag{3.11}$$

These constraints impose covariant chirality conditions on the superfields $\mathbf{q}$ and $\bar{\mathbf{q}}$ and, in addition, they express the Goldstone superfields $\boldsymbol{\psi},\bar{\boldsymbol{\psi}},\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}$ as the derivatives of the $\mathbf{q}$ and $\bar{\mathbf{q}}$, thereby realizing the inverse Higgs effect [3]. Thus, we have in the system only one, covariantly chiral, $\mathcal{N}=2$ complex bosonic superfield $\mathbf{q}(t,\theta,\bar\theta)$.

The constraints (3.11) imply some further restrictions. For example, if we act by ∇_θ on the constraint $\nabla_\theta\mathbf{q}+2\mathrm{i}\boldsymbol{\psi}=0$, we will get

$$\nabla_\theta^2\mathbf{q}+2\mathrm{i}\nabla_\theta\boldsymbol{\psi}=0 \Rightarrow 2\mathrm{i}\nabla_\theta\boldsymbol{\psi}(1-\nabla_\theta\bar{\boldsymbol{\psi}}\nabla_t\mathbf{q})=0. \tag{3.12}$$

Thus, we have to conclude that

$$\nabla_\theta\boldsymbol{\psi}=0. \tag{3.13}$$

Moreover, on the constraint surface given by (3.11) and (3.13) the algebra of covariant derivatives slightly simplifies,

$$\begin{aligned} \nabla_\theta{}^2 &= \bar\nabla_\theta^2=0,\\ \{\nabla_\theta,\bar\nabla_\theta\} &= -2\mathrm{i}(1+\bar\nabla_\theta\boldsymbol{\psi}\nabla_\theta\bar{\boldsymbol{\psi}})\nabla_t,\\ [\nabla_t,\nabla_\theta] &= -2\mathrm{i}\nabla_\theta\bar{\boldsymbol{\psi}}\nabla_t\boldsymbol{\psi}\nabla_t. \end{aligned} \tag{3.14}$$

B. Component transformation laws

As we are going to define component actions, we need transformation laws for the components. Let us first denote the components of superfields in the following way:

$$\begin{aligned} \mathbf{q}|_{\theta=0}=q,\qquad \bar{\mathbf{q}}|_{\theta=0}=\bar q,\qquad \boldsymbol{\psi}|_{\theta=0}=\psi,\\ \bar{\boldsymbol{\psi}}|_{\theta=0}=\bar\psi,\qquad \boldsymbol{\lambda}|_{\theta=0}=\lambda,\qquad \bar{\boldsymbol{\lambda}}|_{\theta=0}=\bar\lambda. \end{aligned} \tag{3.15}$$

It appears to be convenient to introduce also the quantity

$$\mathcal{E}=E|_{\theta=0}=1+\mathrm{i}(\dot\psi\,\bar\psi+\dot{\bar\psi}\psi) \tag{3.16}$$

and to define a new time derivative,

$$\mathcal{D}_t=\mathcal{E}^{-1}\partial_t. \tag{3.17}$$

We list the active transformation laws (at fixed t) for these components under the broken and unbroken supersymmetries.

Broken supersymmetry:

$$\begin{aligned} \delta_S^\star q &= -\mathrm{i}(\varepsilon\bar\psi+\bar\varepsilon\psi)\dot q,\\ \delta_S^\star\psi &= \varepsilon-\mathrm{i}(\varepsilon\bar\psi+\bar\varepsilon\psi)\dot\psi,\\ \delta_S^\star\mathcal{E} &= -\mathrm{i}\partial_t[\mathcal{E}(\varepsilon\bar\psi+\bar\varepsilon\psi)]. \end{aligned} \tag{3.18}$$

Unbroken supersymmetry $\delta_Q^\star f|_{\theta=0}=(\epsilon Df+\bar\epsilon\bar Df)|_{\theta=0}$:

$$\begin{aligned} \delta_Q^\star q &= -2\mathrm{i}\epsilon\psi+(\bar\epsilon\,\bar\psi\,\lambda-\epsilon\psi\bar\lambda)\dot q,\\ \delta_Q^\star\psi &= -\mathrm{i}\bar\epsilon\lambda+(\bar\epsilon\,\bar\psi\,\lambda-\epsilon\psi\bar\lambda)\dot\psi,\\ \delta_Q^\star\mathcal{E} &= \partial_t[\mathcal{E}(\bar\epsilon\,\bar\psi\,\lambda-\epsilon\psi\bar\lambda)]+2(\epsilon\dot\psi\,\bar\lambda-\bar\epsilon\,\dot{\bar\psi}\,\lambda). \end{aligned} \tag{3.19}$$

Finally, we stress that the relations between the components λ and q are given by the following expressions:

$$\mathcal{D}_tq=-\mathrm{i}\frac{\lambda}{1+\lambda\bar\lambda},\Leftrightarrow\lambda=2\mathrm{i}\frac{\mathcal{D}_tq}{1+\sqrt{1-4\mathcal{D}_tq\mathcal{D}_t\bar q}}. \tag{3.20}$$

C. Actions

We are ready to construct the supersymmetric generalization of the actions (2.12), (2.14), and (2.16). As they have different dimensions, these actions must be invariant separately.

1. Superparticle

It is easy to check that the evident ansatz

$$\int dt\mathcal{E}F_1(\lambda\bar\lambda) \tag{3.21}$$

for the supersymmetric extension of the particle action (2.12) is perfectly invariant with respect to the broken supersymmetry (3.18), because

$$\begin{aligned} \delta_S^\star(\mathcal{E}F_1(\lambda\bar\lambda)) &= -\mathrm{i}\partial_t[\mathcal{E}(\varepsilon\bar\psi+\bar\varepsilon\psi)]F_1\\ &\quad -\mathrm{i}\mathcal{E}(\varepsilon\bar\psi+\bar\varepsilon\psi)(\lambda\dot{\bar\lambda}+\dot\lambda\,\bar\lambda)F_1'\\ &= -\mathrm{i}\partial_t[\mathcal{E}F_1(\varepsilon\bar\psi+\bar\varepsilon\psi)]. \end{aligned} \tag{3.22}$$

To determine the function $F_1(\lambda\bar\lambda)$, we impose invariance under the unbroken supersymmetry (3.19). The corresponding variation of $\mathcal{E}F_1(\lambda\bar\lambda)$ computes to

$$\begin{aligned} \delta^\star(\mathcal{E}F_1) &= -\partial_t[\mathcal{E}(\epsilon\psi\bar\lambda-\bar\epsilon\,\bar\psi\,\lambda)F_1]\\ &\quad +2(\epsilon\dot\psi\,\bar\lambda-\bar\epsilon\,\dot{\bar\psi}\,\lambda)(F_1+(1+\lambda\bar\lambda)F_1'). \end{aligned} \tag{3.23}$$

The first term of this variation is a total time derivative, while the second one is not. It is absent, however, for $F_1 \sim (1+\lambda\bar{\lambda})^{-1}$. So, choosing $F_1 = -\frac{2m_0}{1+\lambda\bar{\lambda}}$, our ansatz (3.21) produces a supersymmetic action.

Then, we directly get the invariant supersymmetric extension of the action (2.12) as

$$\mathcal{S}_0 = m_0 \int dt - 2m_0 \int \frac{\mathcal{E}dt}{1+\lambda\bar{\lambda}} = -m_0 \int dt[\mathcal{E}\sqrt{1-4\mathcal{D}_t q \mathcal{D}_t \bar{q}} + \mathcal{E} - 1]. \tag{3.24}$$

This is just the action of the $\mathcal{N}=2, D=2+1$ superparticle in the form considered in [18]. Having in mind the relations (3.20), one may rewrite the Lagrangian in the form

$$\mathcal{L}_0 = -m_0\sqrt{\mathcal{E}^2 - 4\dot{q}\,\dot{\bar{q}}} - m_0(\mathcal{E}-1). \tag{3.25}$$

Let us give the Hamiltonian formulation of this system. The momenta p, π conjugate to q, ψ read

$$p = \frac{2m_0\dot{\bar{q}}}{\sqrt{\mathcal{E}^2 - 4\dot{q}\,\dot{\bar{q}}}} \quad \text{and} \quad \pi = \mathrm{i}m_0\left(\frac{\mathcal{E}}{\sqrt{\mathcal{E}^2-4\dot{q}\,\dot{\bar{q}}}}+1\right)\bar{\psi}, \tag{3.26}$$

from where we immediately get the Hamiltonian

$$\mathcal{H}_0 = \sqrt{m_0^2 + p\bar{p}} \tag{3.27}$$

and fermionic constraints

$$\pi = \mathrm{i}\left(m_0 + \sqrt{m_0^2+p\bar{p}}\right)\bar{\psi} \quad \text{and} \quad \bar{\pi} = -\mathrm{i}\left(m_0 + \sqrt{m_0^2+p\bar{p}}\right)\psi. \tag{3.28}$$

Substituting these expressions into the symplectic one-form $\mathcal{A}_1 = pdq + \bar{p}d\bar{q} + \pi d\psi - \bar{\pi}d\bar{\psi}$, it reduces to

$$\mathcal{A}_{\rm red} = pdq + \bar{p}d\bar{q} + \mathrm{i}(m_0 + \sqrt{m_0^2+p\bar{p}})(\psi d\bar{\psi} + \bar{\psi}d\psi). \tag{3.29}$$

From the symplectic structure $d\mathcal{A}_{\rm red}$, we read off the Poisson brackets defined by the nonzero relations

$$\{p,q\}=1, \quad \{\psi,\bar{\psi}\} = \frac{\mathrm{i}}{2(m_0+\mathcal{H}_0)}, \quad \{\psi,q\} = -\frac{\psi\bar{p}}{4(m_0+\mathcal{H}_0)\mathcal{H}_0}, \quad \{\psi,\bar{q}\} = -\frac{\psi p}{4(m_0+\mathcal{H}_0)\mathcal{H}_0}. \tag{3.30}$$

The transformation properties (3.18) and (3.19) then tell us the supercharges

$$Q = 2p\psi, \qquad S = 2(m_0+\mathcal{H}_0)\bar{\psi}. \tag{3.31}$$

Indeed, these forms of Q and S produce the proper shifts of q and ψ, respectively,

$$\delta^\star_Q q = -\mathrm{i}\epsilon\{Q,q\} \sim -2\mathrm{i}\epsilon\psi + \cdots \quad \text{and} \quad \delta^\star_S \psi = -\mathrm{i}\varepsilon\{S,\psi\} = \varepsilon. \tag{3.32}$$

It is matter of straightforward calculations to check that the remaining terms in (3.18) and (3.19) are also reproduced.

The supercharges (3.31) form centrally extended $\mathcal{N}=4, d=1$ super-Poincaré algebra,

$$\{Q,\bar{Q}\} = 2\mathrm{i}(\mathcal{H}_0 - m_0), \qquad \{S,\bar{S}\} = 2\mathrm{i}(\mathcal{H}_0+m_0), \qquad \{Q,S\} = 2\mathrm{i}p. \tag{3.33}$$

The appearance of the central charge m_0 in the algebra is a signal that the supersymmetry is partially broken and that the vacuum cannot be annihilated simultaneously by both Q and S.

From (3.29) we can readily deduce the canonical coordinates p and

$$\chi = \sqrt{m_0+\mathcal{H}_0}\psi, \quad \tilde{q} = q - \mathrm{i}\frac{\bar{p}}{\sqrt{m_0+\mathcal{H}_0}}(\psi\bar{\psi}): \quad \{p,\tilde{q}\}=1, \quad \{\chi,\bar{\chi}\} = -\frac{\mathrm{i}}{2}. \tag{3.34}$$

In these coordinates, the supercharges read

$$Q = 2\frac{p\chi}{\sqrt{m_0+\mathcal{H}_0}} \quad \text{and} \quad S = 2\sqrt{m_0+\mathcal{H}_0}\bar{\chi}. \tag{3.35}$$

Finally, we note that the action (3.24) can be written in terms of superfields as

$$\mathcal{S}_0 = 2m_0\int dtd\theta d\bar{\theta}\frac{\boldsymbol{\psi}\bar{\boldsymbol{\psi}}}{1+\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}}. \tag{3.36}$$

2. Supersymmetric anyon

The supersymmetrization of the anyonic action (2.14) is more involved. The most general ansatz with the proper bosonic limit reads[2]

$$\mathcal{S}_{\rm anyon} = \frac{\mathrm{i}\alpha}{2}\int dt\mathcal{E}\frac{\mathcal{D}_t\lambda\bar{\lambda} - \lambda\mathcal{D}_t\bar{\lambda}}{1-\lambda\bar{\lambda}} + \int dt\mathcal{E}F_2(\lambda\bar{\lambda})\mathcal{D}_t\psi\mathcal{D}_t\bar{\psi}. \tag{3.37}$$

This action is invariant with respect to the broken supersymmetry (3.18) because

[2]The second term in (3.37) is of the proper dimension but disappears in the bosonic limit.

$$\delta_S^\star\left[\mathrm{i}\mathcal{E}\frac{\mathcal{D}_t\lambda\bar\lambda-\lambda\mathcal{D}_t\bar\lambda}{1-\lambda\bar\lambda}\right]=\partial_t\left[(\varepsilon\bar\psi+\bar\varepsilon\psi)\mathcal{E}\frac{\mathcal{D}_t\lambda\bar\lambda-\mathcal{D}_t\bar\lambda\lambda}{1-\lambda\bar\lambda}\right] \tag{3.38}$$

and

$$\delta_S^\star[\mathcal{E}\mathcal{D}_t\psi\mathcal{D}_t\bar\psi F_2]=-\mathrm{i}\partial_t[(\varepsilon\bar\psi+\bar\varepsilon\psi)\mathcal{E}\mathcal{D}_t\psi\mathcal{D}_t\bar\psi F_2]. \tag{3.39}$$

A straightforward calculation shows that invariance under unbroken supersymmetry fixes F_2 to

$$F_2=-2\alpha\frac{1+\lambda\bar\lambda}{(1-\lambda\bar\lambda)^2}, \tag{3.40}$$

and the full supersymmetric anyonic action acquires the form

$$\begin{aligned}\mathcal{S}_{\rm anyon}&=\frac{\mathrm{i}\alpha}{2}\int dt\mathcal{E}\frac{\bar\lambda\mathcal{D}_t\lambda-\lambda\mathcal{D}_t\bar\lambda}{1-\lambda\bar\lambda}\\&\quad-2\alpha\int dt\mathcal{E}\frac{1+\lambda\bar\lambda}{(1-\lambda\bar\lambda)^2}\mathcal{D}_t\psi\mathcal{D}_t\bar\psi\\&=\mathrm{i}\alpha\int dt\mathcal{E}\frac{\mathcal{D}_t(\mathcal{D}_tq)\mathcal{D}_t\bar q-\mathcal{D}_t(\mathcal{D}_t\bar q)\mathcal{D}_tq}{\sqrt{1-4\mathcal{D}_tq\mathcal{D}_t\bar q}(1+\sqrt{1-4\mathcal{D}_tq\mathcal{D}_t\bar q})}\\&\quad-\alpha\int dt\mathcal{E}\frac{1+\sqrt{1-4\mathcal{D}_tq\mathcal{D}_t\bar q}}{1-4\mathcal{D}_tq\mathcal{D}_t\bar q}\mathcal{D}_t\psi\mathcal{D}_t\bar\psi.\end{aligned} \tag{3.41}$$

Two notes are in order:

(i) The forms ω_S and $\bar\omega_S$ can be evaluated on the superfield constraints (3.14), which removes the $d\theta$ and $d\bar\theta$ projections. We find that the $\dot\psi\dot{\bar\psi}$ term can be represented as

$$-2\int\frac{\omega_S|\cdot\bar\omega_S|}{\omega_P|}. \tag{3.42}$$

(ii) The superfield expression for the action (3.41) takes the simple form

$$\mathcal{S}_{\rm anyon}=\frac{\mathrm{i}\alpha}{2}\int dtd\theta d\bar\theta\frac{\dot{\boldsymbol\psi}\bar{\boldsymbol\psi}+\dot{\bar{\boldsymbol\psi}}\boldsymbol\psi}{1-\boldsymbol\lambda\bar{\boldsymbol\lambda}}. \tag{3.43}$$

We are ready to give a Hamiltonian formulation of the supersymmetric extension of the anyonic system. It is defined as the sum of the particular actions (3.24) and (3.41), $\mathcal{S}=\mathcal{S}_0+\mathcal{S}_{\rm anyon}$. Introducing fermionic momenta η and $\bar\eta$ conjugate to the Grassmann variables ψ and $\bar\psi$, the first-order Lagrangian reads

$$\begin{aligned}\tilde{\mathcal{L}}&=m_0-\frac{2m_0\mathcal{E}}{1+\lambda\bar\lambda}-\frac{\mathrm{i}\alpha}{2}\frac{\lambda\dot{\bar\lambda}-\bar\lambda\dot\lambda}{1-\lambda\bar\lambda}+\eta\dot\psi\\&\quad-\bar\eta\dot{\bar\psi}-\frac{1}{2\alpha}\frac{(1-\lambda\bar\lambda)^2\eta\bar\eta}{1+\lambda\bar\lambda}+p\left(\dot q+\frac{\mathrm{i}\mathcal{E}\lambda}{1+\lambda\bar\lambda}\right)\\&\quad+\bar p\left(\dot{\bar q}-\frac{\mathrm{i}\mathcal{E}\bar\lambda}{1+\lambda\bar\lambda}\right).\end{aligned} \tag{3.44}$$

Hence, the Hamiltonian is given by the expression

$$\mathcal{H}_{\rm SUSY}=\mathcal{H}+\frac{1}{2\alpha}\frac{(1-\lambda\bar\lambda)^2\eta\bar\eta}{1+\lambda\bar\lambda}, \tag{3.45}$$

where $\mathcal{H}$ is defined in (2.22) as

$$\mathcal{H}=\mathrm{i}\frac{\bar p\,\bar\lambda-p\lambda}{1-\lambda\bar\lambda}+m_0\frac{1-\lambda\bar\lambda}{1+\lambda\bar\lambda}. \tag{3.46}$$

The symplectic structure follows from the one-form:

$$\mathcal{A}_1=pdq+\bar pd\bar q-\frac{\mathrm{i}\alpha}{2}\frac{d\bar\lambda\lambda-d\lambda\bar\lambda}{1-\lambda\bar\lambda}+\pi d\psi-\bar\pi d\bar\psi, \tag{3.47}$$

where

$$\pi=\eta-\mathrm{i}(\mathcal{H}+m_0)\bar\psi\quad\text{and}\quad\bar\pi=\bar\eta+\mathrm{i}(\mathcal{H}+m_0)\psi. \tag{3.48}$$

Therefore, the Poisson brackets are defined by the relations

$$\begin{aligned}&\{p,q\}=1,\qquad\{\lambda,\bar\lambda\}=\frac{\mathrm{i}}{\alpha}(1-\lambda\bar\lambda)^2,\\&\{\pi,\psi\}=1,\qquad\{\bar\pi,\bar\psi\}=-1.\end{aligned} \tag{3.49}$$

In these terms the Hamiltonian and supercharges read

$$\begin{aligned}\mathcal{H}_{\rm SUSY}&=\mathcal{H}+\frac{1}{2\alpha}\frac{(1-\lambda\bar\lambda)^2}{1+\lambda\bar\lambda}(\pi+\mathrm{i}(\mathcal{H}+m_0)\bar\psi)\\&\quad\times(\bar\pi-\mathrm{i}(\mathcal{H}+m_0)\psi),\end{aligned} \tag{3.50}$$

$$\begin{aligned}Q&=2p\psi+\bar\lambda(\bar\pi-\mathrm{i}(\mathcal{H}_{\rm SUSY}+m_0)\psi),\\S&=\mathrm{i}\pi+\bar\psi(\mathcal{H}_{\rm SUSY}+m_0).\end{aligned} \tag{3.51}$$

They form the superalgebra

$$\begin{aligned}&\{Q,\bar Q\}=2\mathrm{i}(\mathcal{H}_{\rm SUSY}-m_0),\\&\{S,\bar S\}=2\mathrm{i}(\mathcal{H}_{\rm SUSY}+m_0),\qquad\{Q,S\}=2\mathrm{i}p.\end{aligned} \tag{3.52}$$

3. Rigid superparticle

The supersymmetric extension of the bosonic term

$$\int dt\frac{1+\lambda\bar\lambda}{(1-\lambda\bar\lambda)^3}\dot\lambda\dot{\bar\lambda}=:\int dtG_1(\lambda\bar\lambda)\dot\lambda\dot{\bar\lambda} \tag{3.53}$$

from (2.16) is a more complicated task, due to the existence of two further expressions of the proper dimension, which, however, vanish in the bosonic limit, namely

$$\mathrm{i}G_2(\lambda\bar{\lambda})(\ddot{\psi}\,\dot{\bar{\psi}}+\ddot{\bar{\psi}}\,\dot{\psi})\quad\text{and}\quad \mathrm{i}G_3(\lambda\bar{\lambda})(\dot{\lambda}\,\bar{\lambda}-\dot{\bar{\lambda}}\lambda)\dot{\psi}\,\dot{\bar{\psi}}\,. \tag{3.54}$$

All three terms can be immediately promoted to be invariant under the broken supersymmetry, giving

$$\mathcal{S}_{\text{rigid}} = \int \mathcal{E}dt[G_1\mathcal{D}_t\lambda\mathcal{D}_t\bar{\lambda} + \mathrm{i}G_2(\mathcal{D}_t^2\psi\mathcal{D}_t\bar{\psi} + \mathcal{D}_t^2\bar{\psi}\mathcal{D}_t\psi) + \mathrm{i}G_3(\mathcal{D}_t\lambda\bar{\lambda} - \mathcal{D}_t\lambda\lambda)\mathcal{D}_t\psi\mathcal{D}_t\bar{\psi}], \tag{3.55}$$

where we temporarily unfix the function G_1. We expect the three functions G_1, G_2, and G_3 to be constrained by invariance under unbroken supersymmetry.

After quite lengthy calculations, we find that our action

$$\mathcal{S}_{\text{rigid}} = \int dt[G_1\mathcal{E}^{-1}\dot{\lambda}\,\dot{\bar{\lambda}} + \mathrm{i}G_2\mathcal{E}^{-2}(\ddot{\psi}\,\dot{\bar{\psi}}+\ddot{\bar{\psi}}\,\dot{\psi}) + \mathrm{i}G_3\mathcal{E}^{-2}(\dot{\lambda}\,\bar{\lambda}-\dot{\bar{\lambda}}\lambda)\dot{\psi}\,\dot{\bar{\psi}}] \tag{3.56}$$

is invariant under unbroken supersymmetry if the equations

$$-G_3 + G_2' + 2G_1 = 0, \qquad G_3 + G_2' + 2(1+\lambda\bar{\lambda})G_1' = 0, \qquad G_2 + (1+\lambda\bar{\lambda})G_1 = 0 \tag{3.57}$$

hold, where the prime denotes a derivative with respect to the single argument $\lambda\bar{\lambda}$ of these functions. These equations are not independent, because the sum of the first two reduces to the derivative of the third. The solution of this system reads

$$G_2 = -(1+\lambda\bar{\lambda})G_1 \quad\text{and}\quad G_3 = G_1 - (1+\lambda\bar{\lambda})G_1'. \tag{3.58}$$

Thus, invariance with respect to both $\mathcal{N}=2$ supersymmetries determines the action up to one arbitrary function $G_1(\lambda\bar{\lambda})$. The prescribed bosonic limit fixes this function to

$$G_1 = \frac{1+\lambda\bar{\lambda}}{(1-\lambda\bar{\lambda})^3}, \tag{3.59}$$

and thus the complete $\mathcal{N}=4$ supersymmetric generalization of the rigid-particle action has the form

$$\mathcal{S}_{\text{rigid}} = \int dt\left[\frac{1+\lambda\bar{\lambda}}{(1-\lambda\bar{\lambda})^3}\mathcal{E}^{-1}\dot{\lambda}\,\dot{\bar{\lambda}} - \mathrm{i}\frac{(1+\lambda\bar{\lambda})^2}{(1-\lambda\bar{\lambda})^3}\mathcal{E}^{-2}(\ddot{\psi}\,\dot{\bar{\psi}}+\ddot{\bar{\psi}}\,\dot{\psi}) - 3\mathrm{i}\frac{(1+\lambda\bar{\lambda})^3}{(1-\lambda\bar{\lambda})^4}(\dot{\lambda}\,\bar{\lambda}-\dot{\bar{\lambda}}\lambda)\mathcal{E}^{-2}\dot{\psi}\,\dot{\bar{\psi}}\right]. \tag{3.60}$$

In superfield language this action can be written in the much more compact form

$$\mathcal{S}_{\text{rigid}} = \int dtd\theta d\bar{\theta}\frac{1+\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}}{(1-\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}})^3}\dot{\boldsymbol{\psi}}\,\dot{\bar{\boldsymbol{\psi}}}\,. \tag{3.61}$$

The Hamiltonian formulation of the supersymmetric rigid particle will be considered elsewhere.

IV. DISCUSSION AND OUTLOOK

We have applied the coset approach to the construction of component actions describing a superparticle in $D = 2+1$ spacetime, with $\mathcal{N}=4$ supersymmetry partially broken to $\mathcal{N}=2$, and with the bosonic action containing higher time derivatives, in the forms of an anyonic term and the square of the first extrinsic curvature. We presented the supercharges for the unbroken and broken supersymmetries as well as the Hamiltonian for the supersymmetric anyon and provided the superspace actions for all cases.

Our main goal was to find out whether it is possible to apply the approach, previously developed for the construction of supersymmetric actions with a minimal number of time derivatives [17,18], also to systems with higher time derivatives in the bosonic sector. We are aware that the simple $\mathcal{N}=4\to\mathcal{N}=2$ pattern of supersymmetry breaking drastically simplifies the analysis (for example, by the absence of auxiliary components). Clearly, the analysis of more involved systems with higher supersymmetries or higher target-space dimensions is desired. Using the fermions of the nonlinear realization as the physical fermionic components renders the constructed actions quite compact and involves only geometric objects such as the einbein and covariant derivatives of the bosonic “matter” fields and the fermions.

Because of the fact that our actions are just gauge-fixed forms of the standard ones (modulo a proper redefinition of the fermions), interactions with background fields (including electromagnetism) may be introduced in a standard way. An interesting further question is whether also p-brane actions (with $p\geq 1$) containing higher derivatives can be supersymmetrized in a similar way. Such a generalization is not obvious, however, due to the presence of auxiliary fields, which have to be excluded by their, *a priori* unknown, equations of motion. Other issues not discussed here are the physical properties of the rigid-particle model or the quantization of our systems. We hope to treat them elsewhere.

ACKNOWLEDGMENTS

S. K. and A. N. are grateful to Leibniz Universität Hannover for warm hospitality. This work was partially supported by RFBR Grants No. 12-02-00517-a, No. 13-02-91330-NNIO-a, and No. 13-02-90602 Apm-a, by Grants of Armenian State Committee of Science No. 13RF-018, No. 13-1C114, and by the Volkswagen Foundation under Grant No. I/84 496.

[1] S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. G. Callan, S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969).

[2] D. V. Volkov, Sov. J. Part. Nucl. **4**, 3 (1973); V. I. Ogievetsky, in Proceedings of the Xth Winter School of Theoretical Physics in Karpacz (Wroclaw, 1974), Vol. **1**, p. 117.

[3] E. A. Ivanov and V. I. Ogievetsky, Theor. Math. Phys. **25**, 1050 (1975).

[4] J. P. Gauntlett, J. Gomis, and P. K. Townsend, Phys. Lett. B **249**, 255 (1990).

[5] J. P. Gauntlett, K. Itoh, and P. K. Townsend, Phys. Lett. B **238**, 65 (1990).

[6] J. Gomis, K. Kamimura, and J. M. Pons, Nucl. Phys. **B871**, 420 (2013).

[7] J. Gomis, K. Kamimura, and P. C. West, Classical Quantum Gravity **23**, 7369 (2006).

[8] J. A. de Azcárraga and J. Lukierski, Phys. Lett. **113B**, 170 (1982).

[9] M. Pavšič, Phys. Lett. B **205**, 231 (1988); **221**, 264 (1989); A. A. Kapustnikov, A. Pashnev, and A. Pichugin, Phys. Rev. D **55**, 2257 (1997); M. S. Plyushchay, Electron. J. Theor. Phys. **3N10**, 17 (2006).

[10] R. D. Pisarski, Phys. Rev. D **34**, 670 (1986); M. S. Plyushchay, Int. J. Mod. Phys. A **04**, 3851 (1989); Phys. Lett. B **248**, 107 (1990); V. V. Nesterenko, J. Phys. A **22**, 1673 (1989); Int. J. Mod. Phys. A **06**, 3989 (1991); J. Isberg, U. Lindström, H. Nordström, and J. Grundberg, Mod. Phys. Lett. A **05**, 2491 (1990); A. Deriglazov and A. Nersessian, arXiv:1303.0483.

[11] R. Jackiw and V. P. Nair, Phys. Rev. D **43**, 1933 (1991); C.-H. Chou, V. P. Nair, and A. P. Polychronakos, Phys. Lett. B **304**, 105 (1993); J. L. Cortes and M. S. Plyushchay, Int. J. Mod. Phys. A **11**, 3331 (1996); I. V. Gorbunov, S. M. Kuzenko, and S. L. Lyakhovich, Int. J. Mod. Phys. A **12**, 4199 (1997); I. V. Gorbunov, S. M. Kuzenko, and S. L. Lyakhovich, Phys. Rev. D **56**, 3744 (1997); A. Nersessian, Mod. Phys. Lett. A **12**, 1783 (1997).

[12] J. A. de Azcárraga, J. P. Gauntlett, J. M. Izquierdo, and P. K. Townsend, Phys. Rev. Lett. **63**, 2443 (1989).

[13] S. Bellucci, E. Ivanov, and S. Krivonos, Nucl. Phys. B, Proc. Suppl. **102–103**, 26 (2001).

[14] J. Bagger and A. Galperin, Phys. Rev. D **55**, 1091 (1997).

[15] M. Roček and A. A. Tseytlin, Phys. Rev. D **59**, 106001 (1999).

[16] E. Ivanov and S. Krivonos, Phys. Lett. B **453**, 237 (1999).

[17] S. Bellucci, S. Krivonos, and A. Sutulin, Phys. Lett. B **726**, 497 (2013).

[18] S. Bellucci, N. Kozyrev, S. Krivonos, and A. Sutulin, arXiv:1309.3902.

[19] D. V. Volkov and V. P. Akulov, JETP Lett. **16**, 438 (1972); Phys. Lett. **46B**, 109 (1973).

[20] M. S. Plyushchay, Phys. Lett. B **253**, 50 (1991).

[21] A. Nersessian, Theor. Math. Phys. **117**, 1214 (1998).

[22] M. S. Plyushchay, Phys. Lett. B **262**, 71 (1991); Nucl. Phys. **B362**, 54 (1991).

[23] A. Nersessian and E. Ramos, Phys. Lett. B **445**, 123 (1998); A. Nersessian and E. Ramos, Mod. Phys. Lett. A **14**, 2033 (1999).