PHYSICAL REVIEW D **89,** 045003 (2014)

Action for reaction in general dimension

Ofek Birnholtz* and Shahar Hadar†

Racah Institute of Physics, Hebrew University, Jerusalem 91904, Israel

(Received 25 November 2013; published 5 February 2014)

We present an effective field theory study of radiation and radiation reaction effects for scalar and electromagnetic fields in general spacetime dimensions. Our method unifies the treatment of outgoing radiation and its reaction force within a single action principle. Central ingredients are the field doubling method, which is the classical version of the closed time path formalism and allows a treatment of nonconservative effects within an action, action level matching of system and radiation zones, and the use of fields which are adapted to the enhanced symmetries of each zone. New results include compact expressions for radiative multipoles, radiation, and radiation reaction effective action in any spacetime dimension. We emphasize dimension-dependent features such as the difference between electric and magnetic multipoles in higher dimensions and the temporal nonlocality nature of the effective action for odd spacetime dimensions, which is a reflection of indirect propagation ("tail effect") in the full theory. This work generalizes the method and results developed for four dimensions in Phys. Rev. D **88**, 104037 (2013) and prepares the way for a treatment of the gravitational case.

DOI: 10.1103/PhysRevD.89.045003 PACS numbers: 03.50.De, 04.25.Nx, 04.30.Nk, 04.50.-h

I. INTRODUCTION

While extensive work has been done on the post-Newtonian (PN) two-body problem in general relativity (GR) in four spacetime dimensions (see Sec. I of paper I [1] for a comprehensive review of existing literature), much less is known in general spacetime dimensions (d). In this paper we study PN radiative effects, namely radiation and radiation reaction (RR), for (massless) scalar and electromagnetic (EM) theories in general d. We work within an effective field theory (EFT) approach, which was introduced in [2] to the PN limit of GR and applied to the study of dissipative effects in four-dimensional GR also in [3–8]. Our study of nonconservative effects in scalar and EM theories gives new results and insights into the problem already in these physically important cases and lays foundations for a comprehensive treatment of these effects in higher d GR (see [9] for the first treatment of gravitational radiation in higher d within the EFT approach), where the main additional complication is the theory's nonlinearity.

The well known $4d$ Abraham-Lorentz-Dirac (ALD) formula [10–13] gives the EM self-force on a point charge in flat spacetime in a covariant manner. There has been an extensive effort to generalize ALD to any d undertaken by Kosyakov, Gal'tsov, Kazinski and others [14–22]. Joint treatment of self-force under scalar, vector and tensor fields has been given by [23–26]. As is well known, in even d waves exhibit direct propagation, that is the Green's function is supported only on the light cone, while in noneven d waves propagate also *indirectly*—exhibiting the so-called tail effect. These features appear also in the above treatments of RR, but to our knowledge a closed-form formula for generalized ALD is available currently only for EM in even d [16]. The very regularizability of RR in high d is sometimes put to question in the above treatments.

Our method divides the problem into two zones, the system and radiation zones (Fig. 1). Each zone has an enhanced symmetry, namely a symmetry that is not a symmetry of the full problem. Different symmetries emerge when zooming out to the radiation zone or in to the system zone. The system zone is approximately stationary (time independent) since by assumption (of the PN approximation) all velocities are nonrelativistic. The radiation zone is approximately spherically symmetric since the system shrinks to a point at the origin and hence rotations leave it invariant. Identifying symmetries is central to making fitting choices for the formulation of a perturbation theory in each zone, including the choice of how to divide the action into a dominant part and a perturbation, the choice of field variables and—when relevant—the choice of gauge. Hence we insist on using spherical field variables (in fact, symmetric trace-free tensors), following paper I and [27], and unlike the more common plane-wave and wave-vectors approach used in many EFT works [3–7].

As is well known, Hamilton's traditional action formalism is not compatible with dissipative effects. In order to account for nonconservative effects within the action we use the method of field doubling (see [8] for a crisp general formulation) which is the classical version of the closed time path or in-in formalism [28] introduced in the quantum field theory context in the 1960s. In this method each degree of freedom in the system is *doubled* and a generalized action principle for the whole system (including doubled fields) is constructed from the original action.

*ofek.birnholtz@mail.huji.ac.il
†shaharhadar@phys.huji.ac.il

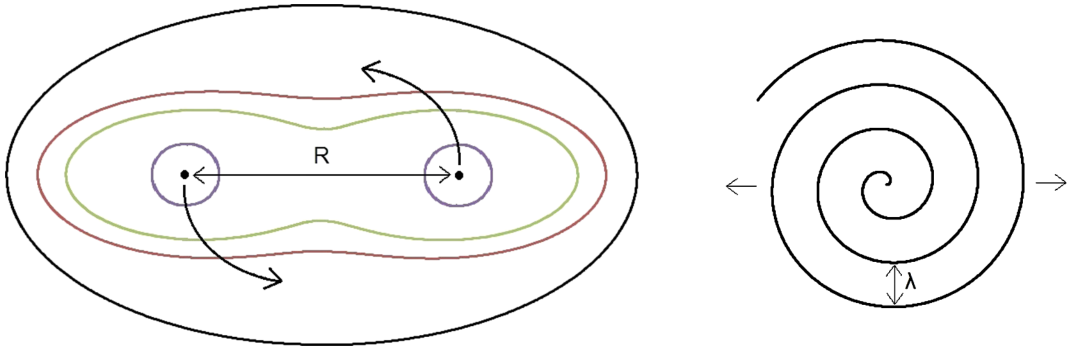


FIG. 1 (color online). Schematic sketch of the two relevant zones. On the left side is the system zone, with a typical stationarylike field configuration. On the right side is the radiation zone with its typical outspiraling waves.

Only after deriving equations of motion (EOM) from this generalized action, one enforces the so-called physical condition, identifying each field with its doubled counterpart. The resulting forces in the EOM can be dissipative.

In order to formulate the whole problem within a single action principle, we introduce new matching fields which couple to system and radiation zone fields at an appropriate boundary. We call these fields "two-way multipoles." We physically interpret them as the multipole moments of the system (living at the origin of the radiation zone and at infinity of the system zone), but they are treated as any other field in the theory. Thereby, matching is lifted to the level of the action.

The main goal of this paper is to compute, in general d, the radiative multipoles, outgoing radiation, RR effective action, dissipated energy and RR force in the cases of scalar and electromagnetic fields, thereby generalizing the method devised in paper I to general d. We comment that this paper is not intended to be self-contained: the main ideas and ingredients of our formulation are thoroughly elaborated on in paper I, and here we generalize and apply them to obtain new results in higher dimensions. The ability to generalize the machinery set up in paper I in order to obtain new results demonstrates the method's efficiency. Moreover, this generalization is natural since a central step in the method is a reduction to one dimension (corresponding to the radial coordinate)—and from there on the treatment is very similar to the one done in $4d$.

This paper is organized as follows: In Sec. 2 we treat the scalar field case, which already contains most of our formulation's main ingredients. In particular this section contains a thorough discussion of the tail effect in noneven spacetime dimensions. In Sec. 3 we treat the EM case, where one must account for the additional polarizations available for general dimensional vector fields. In both sections we also show explicitly successful comparisons to known results in $4d$ and $6d$. Sections 2 and 3 contain detailed derivations of our results, intended to maximally clarify our computations for interested readers. In Sec. 4 we briefly summarize our main results and definitions. In Sec. 5 we discuss these results and elaborate on future directions.

A. Conventions and nomenclature

We use the mostly plus signature for the flat d-dimensional spacetime metric $\eta_{\mu\nu}$, as well as $c = 1$. We denote $D := d - 1$, $\hat{d} := d - 3$, and $\Omega_{\hat{d}+1}$ is the volume of a unit $\hat{d} + 1$-dimensional sphere. Lowercase Greek letters denote $\{0, 1, \ldots, D\}$ spacetime indices, lowercase Latin letters denote $\{1 \ldots D\}$ spatial indices, uppercase Greek letters denote $\{1 \ldots (\hat{d} + 1)\}$ indices on the sphere, uppercase Latin letters are spatial multi-indices, and Hebrew letters ($\aleph$) enumerate different vectorial harmonics on the sphere.

B. Field doubling conventions

We work in the Keldysh [28] representation of the field doubling formalism, where for every field ϕ (and source function) in the original action, we introduce a counterpart $\hat{\phi}$, interpreted as the difference between the doubled degrees of freedom, and with it a new EOM:

$$\frac{\delta \hat{S}}{\delta \hat{\phi}} = 0. \tag{1.1}$$

$\hat{S}$ is always linear in the hatted fields, so enforcing the so-called physical condition $\hat{\varphi} = 0$ (after the derivation of EOM) is trivial.

We write the RR effective action $\hat{S}$ in terms of the system's multipoles and their doubled counterparts, which are defined as

$$\hat{Q} = \frac{\delta Q}{\delta \rho} \hat{\rho}. \tag{1.2}$$

In particular, for a source composed of point particles the doubled multipoles are given by

$$\hat{Q} = \sum_A \frac{\delta Q}{\delta x_A} \hat{x}_A, \tag{1.3}$$

and the EOM for the Ath point particle is $\frac{\delta \hat{S}}{\delta \hat{x}_A} = 0$. For a detailed survey of the doubling formalism used in this paper, see Sec. II D of paper I.

II. SCALAR CASE

A. Spherical waves and double-field action

We start with a massless scalar field coupled to some charge distribution in arbitrary spacetime dimension d. We take the action to be

$$S_\Phi = +\frac{1}{2\Omega_{\hat{d}+1}G}\int (\partial_\mu \Phi)^2 r^{\hat{d}+1} dr d\Omega_{\hat{d}+1} dt - \int \rho \Phi r^{\hat{d}+1} dr d\Omega_{\hat{d}+1} dt, \tag{2.1}$$

which leads to the usual field equation

$$\Box \phi = -\Omega_{\hat{d}+1} G \rho. \tag{2.2}$$

1. Spherical waves: Conventions

We shall work in the frequency domain and use a basis of spherical decomposition to multipoles. We note that any symmetric trace-free tensor ϕ_{L_ℓ} (ℓ being given) can be represented equivalently using the standard (scalar) spherical harmonic representation $\phi_{\ell m}$ [which form a basis of dimension $D_\ell(\hat{d}+1,0)$[1]] or using functions on the unit sphere $\phi_\ell(\Omega_{\hat{d}+1})$, with the different forms related by

$$\phi_\ell(\Omega_{\hat{d}+1}) = \phi_{L_\ell}\frac{x^{L_\ell}}{r^\ell} = \sum_m \varphi_{\ell m} Y_{\ell m}(\Omega_{\hat{d}+1}). \tag{2.3}$$

We found it convenient to use the ϕ_L decomposition, which is similar to the Maxwell Cartesian spherical multipoles [31–34]. We thus decompose the field and the sources as

$$\Phi(\vec{r},t) = \int \frac{d\omega}{2\pi}\sum_L e^{-i\omega t}\Phi_{L\omega}(r) x^L, \tag{2.4}$$

$$\rho(\vec{r},t) = \int \frac{d\omega}{2\pi}\sum_L e^{-i\omega t}\rho_{L\omega}(r) x^L, \tag{2.5}$$

where $L = (k_1 k_2 \ldots k_\ell)$ is a multi-index and x^L is the corresponding symmetric-trace-free (STF) multipole

$$x^L = (x^{k_1} x^{k_2} \ldots x^{k_\ell})^{\mathrm{STF}} \equiv r^\ell n^L. \tag{2.6}$$

With $g^{\Omega\Omega'}$ the metric on the $\hat{d}+1$-dimensional unit sphere, the x^L are eigenfunctions of the Laplacian operator on that sphere:

$$\Delta_{\Omega_{\hat{d}+1}} x^L = -c_s x^L = -\ell(\ell+\hat{d}) x^L. \tag{2.7}$$

For any dimension d, an orthogonal basis can be constructed from the multipoles, satisfying

$$\int x_{L_\ell}(r,\Omega_{\hat{d}+1}) x^{L'}{}_{\ell'}(r,\Omega_{\hat{d}+1}) d\Omega_{\hat{d}+1} = N_{\ell,\hat{d}} r^{2\ell}\Omega_{\hat{d}+1}\delta_{\ell\ell'}\delta_{L_\ell L'_{\ell'}}, \tag{2.8}$$

$$\int g^{\Omega\Omega'}\partial_\Omega x_{L_\ell}(r,\Omega_{\hat{d}+1})\partial_{\Omega'} x^{L'_{\ell'}}(r,\Omega_{\hat{d}+1}) d\Omega_{\hat{d}+1} = c_s \cdot N_{\ell,\hat{d}} r^{2\ell}\Omega_{\hat{d}+1}\delta_{\ell\ell'}\delta_{L_\ell L'_{\ell'}}, \tag{2.9}$$

$$N_{\ell,\hat{d}} = \frac{\Gamma(1+\hat{d}/2)}{2^\ell \Gamma(\ell+1+\hat{d}/2)} = \frac{\hat{d}!!}{(2\ell+\hat{d})!!}. \tag{2.10}$$

We use the summation conventions and definitions following [1] (see Appendix B.1) and note that $d=4$ implies $\hat{d}=1$, $N_{\ell,1} = (2\ell+1)!!^{-1}$. We also use the inverse transformation

$$\rho_{L\omega}(r) = \int \rho_\omega(\vec{r}) x_L \frac{d\Omega_{\hat{d}+1}}{N_{\ell,\hat{d}}\Omega_{\hat{d}+1} r^{2\ell}} = \int\int dt e^{i\omega t}\rho(\vec{r},t) x_L \frac{d\Omega_{\hat{d}+1}}{N_{\ell,\hat{d}}\Omega_{\hat{d}+1} r^{2\ell}}. \tag{2.11}$$

2. Spherical waves: Dynamics

In the new notation, using $\Phi_{L-\omega} = \Phi^*_{L\omega}$, the action (2.1) becomes

$$S_\Phi = \frac{1}{2}\int \frac{d\omega}{2\pi}\sum_L \int dr\left[\frac{r^{2\ell+\hat{d}+1}N_{\ell,\hat{d}}}{G}\Phi^*_{L\omega}\left(\omega^2 + \partial_r^2 + \frac{2\ell+\hat{d}+1}{r}\partial_r\right)\Phi_{L\omega} - (\rho^\Phi_{L\omega}\Phi^*_{L\omega} + \mathrm{c.c.})\right], \tag{2.12}$$

with the source term defined as

$$\rho^\Phi_{L\omega}(r) = N_{\ell,\hat{d}}\Omega_{\hat{d}+1} r^{2\ell+\hat{d}+1}\rho_{L\omega}(r) = r^{\hat{d}+1}\int d\Omega_{\hat{d}+1}\rho_\omega(\vec{r}) x_L. \tag{2.13}$$

From (2.12) we derive the EOM

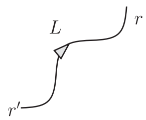
$$0 = \frac{\delta S}{\delta \Phi^*_{L\omega}} = \frac{N_{\ell,\hat{d}} r^{2\ell+\hat{d}+1}}{G}\left(\omega^2 + \partial_r^2 + \frac{2\ell+\hat{d}+1}{r}\partial_r\right)\Phi_{L\omega} - \rho^\Phi_{L\omega}. \tag{2.14}$$

Defining $x \coloneqq \omega r$ the homogenous part of this equation is

$$\left[\partial_x^2 + \frac{2\ell+\hat{d}+1}{x}\partial_x + 1\right]\psi_{\ell,\hat{d}} = 0, \tag{2.15}$$

[1] $D_\ell(n,s)$ is the number of independent spherical harmonics of degree ℓ and spin s on the n sphere. We use $D_\ell(\hat{d}+1,0) = \frac{(2\ell+\hat{d})(\ell+\hat{d}-1)!}{\ell!\hat{d}!}$ [29,30].

and its solutions $\psi_{\ell,\hat{d}} = \psi_\alpha = \tilde{j}_\alpha, \tilde{y}_\alpha, \tilde{h}^\pm_\alpha$ (with $\alpha = \ell + \hat{d}/2$) are Bessel functions up to normalization (see Appendix B.2). Thus the propagator for spherical waves is



$$\equiv G^\Phi_{ret}(r', r) = -i\omega^{2\ell+\hat{d}} M_{\ell,\hat{d}}\, G\, \tilde{j}_\alpha(\omega r_1)\, \tilde{h}^\pm_\alpha(\omega r_2)\,;$$

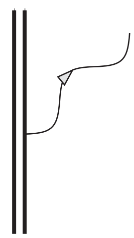
$$r_1 := \min\{r', r\},\ \ r_2 := \max\{r', r\}\,, \tag{2.16}$$

where

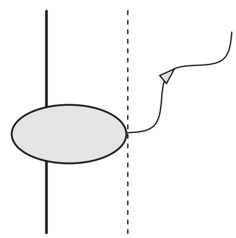
$$M_{\ell,\hat{d}} = \frac{\pi}{2^{2\alpha+1} N_{\ell,\hat{d}} \Gamma^2(\alpha+1)} = \frac{\pi}{2^{\ell+1+\hat{d}}\Gamma(1+\hat{d}/2)\Gamma(\ell+1+\hat{d}/2)}, \tag{2.17}$$

which we notice is equal to $[\hat{d}!!(2\ell+\hat{d})!!]^{-1}$ for odd $\hat{d}$ and to $\frac{\pi}{2}[\hat{d}!!(2\ell+\hat{d})!!]^{-1}$ for even $\hat{d}$. We turn to derive the source terms (vertices) in the radiation zone. This is done through matching with the system zone according to the diagrammatic definition

$$-Q^{s,\epsilon}_{L\omega} :=$$



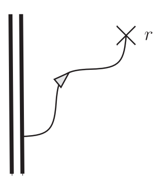
$$:=$$



$$. \tag{2.18}$$

From the radiation zone point of view the sources $Q_{L\omega}$ are located at the origin or $r = 0$. Hence the radiation zone field can be written as

$$\Phi^{EFT}_{L\omega}(r) =$$



$$= -Q_{L\omega}\left(-iG\omega^{2\ell+\hat{d}} M_{\ell,\hat{d}}\right)\tilde{h}^+_\alpha(\omega r)\ . \tag{2.19}$$

In the full theory (or equivalently in the system zone), on the other hand, we can also use spherical waves to obtain the field outside the source as

$$\begin{aligned}\Phi_{L\omega}(r) &= -\int dr' \rho^\Phi_{L'\omega}(r') G^\Phi_{\rm ret}(r', r) = -\left[\int dr' \tilde{j}_\alpha(\omega r')\rho^\Phi_{L\omega}(r')\right](-iG\omega^{2\ell+\hat{d}} M_{\ell,\hat{d}})\tilde{h}^+_\alpha(\omega r)\\ &= -\left[\int d^D x' \tilde{j}_\alpha(\omega r')\rho_\omega(\vec{r}') x'_L\right](-iG\omega^{2\ell+\hat{d}} M_{\ell,\hat{d}})\tilde{h}^+_\alpha(\omega r).\end{aligned} \tag{2.20}$$

By comparing the above expressions for the field (2.19) and (2.20) and using (2.11) to return to the time domain we find that the radiation source multipoles are

$$Q_L = \int d^D x \tilde{j}_\alpha(ir\partial_t) x^{\rm STF}_L \rho(\vec{r}, t). \tag{2.21}$$

We note that from the series expansion of $\tilde{j}_\alpha$ (B8) it can be seen that Q^L includes only even powers of $i\partial_t$, for every ℓ, $\hat{d}$, and is thus well defined and real. A simple test of substituting $d = 4$ shows these multipoles coincide with the multipoles given by Ross [35] and by paper I. We can think of this process as a "zoom out balayage" (French for sweeping or scanning—see paper I) of the original charge distribution $\rho(\vec{r})$ into Q_L carried out through propagation with $\tilde{j}_\alpha(\omega r)$. A useful representation of this result is

$$Q_L = \int d^D x\, x_L^{\mathrm{STF}} \int_{-1}^{1} dz \delta_{\ell,\hat{d}}(z) \rho(\vec{r}, u + zr), \tag{2.22}$$

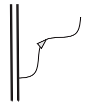
where (inspired by [36,37]) we have implicitly defined the $\hat{d}$-dimensional generating time-weighted function

$$\int_{-1}^{1} dz \delta_{\ell,\hat{d}}(z) f(\vec{r}, u + zr) = \sum_{p=0}^{+\infty} \frac{(2\ell + \hat{d})!!}{(2p)!!(2\ell + 2p + \hat{d})!!} (r\partial_u)^{2p} f(\vec{r}, u), \tag{2.23}$$

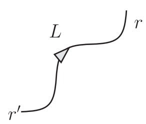
remarking that for odd $\hat{d}$ (even spacetime dimension),

$$\delta_{\ell,\hat{d}}(z) = \frac{(2\ell + \hat{d})!!}{2(2\ell + \hat{d} - 1)!!} (1 - z^2)^{\ell + (\hat{d}-1)/2}. \tag{2.24}$$

Altogether *the Feynman rules in the radiation zone* for the propagator and vertices are


$$= -Q^{s,\epsilon}_{L\omega},$$

$$= -\hat{Q}^{s,\epsilon *}_{L\omega}, \tag{2.25}$$



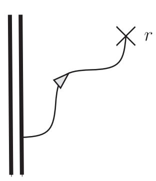
$$= G^{s,\epsilon}_{ret}(r', r) = -i\omega^{2\ell+\hat{d}} M_{\ell,\hat{d}} R^{\epsilon}_s G\, \tilde{j}_\alpha(\omega r_1)\, \tilde{h}^+_\alpha(\omega r_2) \qquad r_1 \le r_2 \;, \tag{2.26}$$

where for future use we allowed for a possible polarization label ϵ and a rational ℓ, $\hat{d}$-dependent factor R^{ϵ}_s which is absent in the scalar case, namely $R|_{s=0} = 1$.

B. Outgoing radiation and the RR effective action

We can now use these Feynman rules to compute central quantities.

Outgoing radiation can now be found diagrammatically as

$$\Phi_{L\omega}(r) =$$


$$= -Q_{L'\omega} G^{\Phi}_{ret}(0, r) = \sqrt{\frac{\pi}{2^{\hat{d}+1}}} \frac{G}{\Gamma(1 + \hat{d}/2)} (-i\omega)^{\ell + \frac{\hat{d}-1}{2}} \frac{Q_{L\omega}}{r^{\ell}} \frac{e^{i\omega r}}{r^{\frac{\hat{d}+1}{2}}}, \tag{2.27}$$

where we used the asymptotic forms of $\tilde{h}_\alpha(x)$ (B9) and $\tilde{j}_\alpha(\omega r')|_{r'=0} = 1$ for the source at $r' = 0$, and (2.17). In odd $\hat{d}$ we can use (2.5) to find that (as $r \to \infty$) the outgoing radiation is

$$\Phi(\vec{r}, t) \sim \frac{G}{\hat{d}!!} r^{-\frac{\hat{d}+1}{2}} \sum_L n^L \partial_t^{\ell + \frac{\hat{d}-1}{2}} Q_L(t - r). \tag{2.28}$$

For $d = 4$ this coincides with paper I [Eq. (3.20)] and with [35] [Eq. (21); note our normalizations differ by 4π]. While (2.28) is valid and local for all odd $\hat{d}$, for nonodd $\hat{d}$ (noneven spacetime dimension), the corresponding expression in the time domain would include a noninteger number of time derivatives, implying nonlocality (see more below). Equation (2.27) is, of course, valid in every dimension.

1. Dissipated power

The power carried away by the radiation field (2.27) is

$$\begin{aligned}\dot{E} &= \frac{1}{G}\int \dot{\Phi}^2 r^{\hat{d}+1} d\Omega_{\hat{d}+1} = \frac{1}{G}\int \frac{d\omega}{2\pi}\sum_{L,L'}\int r^{\hat{d}+1} d\Omega_{\hat{d}+1}\omega^2 \Phi^*_{L\omega} x^L x_{L'} \Phi^{L'\omega} \\ &= \sum_L \int \frac{d\omega}{2\pi} \frac{\pi G N_{\ell,\hat{d}}}{2^{\hat{d}+1}\Gamma^2(1+\hat{d}/2)} \omega^{2\ell+\hat{d}+1} Q^{L\omega} Q^*_{L\omega} \\ &= \sum_L \int \frac{d\omega}{2\pi} G\omega^{2\ell+\hat{d}+1} |Q_{L\omega}|^2 \cdot \begin{cases} \frac{\pi}{2}[\hat{d}!!(2\ell+\hat{d})!!]^{-1} & d \text{ odd,} \\ [\hat{d}!!(2\ell+\hat{d})!!]^{-1} & d \text{ even,} \\ \pi\left[2^{\ell+\hat{d}+1}\Gamma\left(\ell+1+\frac{\hat{d}}{2}\right)\Gamma\left(1+\frac{\hat{d}}{2}\right)\right]^{-1} & d \text{ noninteger,} \end{cases} \end{aligned} \tag{2.29}$$

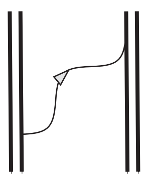
where we have used (2.4), (2.8), and (2.10). We note in particular that in even dimension d we can find the symmetric time-domain form:

$$\dot{E} = \sum_L \frac{G}{\hat{d}!!(2\ell+\hat{d})!!} \langle (\partial_t^{\ell+\frac{\hat{d}+1}{2}} Q^L)^2 \rangle = \sum_L \frac{G}{\ell!\hat{d}!!(2\ell+\hat{d})!!} \langle (\partial_t^{\ell+\frac{\hat{d}+1}{2}} Q^{\rm STF}_{k_1k_2\ldots k_\ell})^2 \rangle. \tag{2.30}$$

We remark that in $4d$ this matches (3.22) of [1] and (15) of [35] (note normalization and summation conventions).

Radiation reaction effective action encodes the RR force and is given formally by

$$\hat{S}_\Phi =$$



$$\begin{aligned} &= \frac{1}{2}\int \frac{d\omega}{2\pi} \sum_{L,L'} (-Q_{L\omega})\, G^\Phi_{ret}(0,0) \left(-\hat{Q}^*_{L'\omega}\right) + c.c. \\ &= \frac{G}{2}\int \frac{d\omega}{2\pi} \sum_L -i\omega^{2\ell+\hat{d}} M_{\ell,\hat{d}} \cdot \mathbb{G}_\alpha \cdot Q^{L\omega} \hat{Q}^*_{L\omega} + c.c \quad , \end{aligned} \tag{2.31}$$

$$\mathbb{G}_\alpha = \tilde{j}_\alpha(\omega r)|_{r\to 0} \cdot [\tilde{j}_\alpha(\omega r') + i\tilde{y}_\alpha(\omega r')]|_{r'\to 0}, \tag{2.32}$$

where now both vertices are taken at $r = r' = 0$. For noneven $\hat{d}$ (odd or noninteger), we use $\tilde{j}_\alpha(0) = 1$, and (B8) and (B10) for $\tilde{j}_\alpha$, $\tilde{y}_\alpha$, to find

$$\begin{aligned} \mathbb{G}_\alpha &= \tilde{j}_\alpha(0)\cdot\tilde{j}_\alpha(0) + \left\{ \frac{i\Gamma(\alpha+1)2^\alpha \tilde{j}_\alpha(x)[J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)]}{\sin(\alpha\pi)x^\alpha} \right\}\Bigg|_{x\to 0} \\ &= 1 + \frac{i\Gamma(\alpha+1)2^\alpha}{\sin(\alpha\pi)} \left\{ \sum_{p,q=0}^{\infty} \frac{(-)^{p+q}\Gamma(\alpha+1)x^{2p+2q}}{2^{p+q}(2p)!!(2q)!!\Gamma(q+\alpha+1)} \left[\frac{\cos(\alpha\pi)}{2^\alpha\Gamma(p+\alpha+1)} - \frac{2^\alpha x^{-2\alpha}}{\Gamma(p-\alpha+1)} \right] \right\}\Bigg|_{x\to 0}. \end{aligned} \tag{2.33}$$

The contribution from the $\tilde{j}^2_\alpha(0) = 1$ term is simple and independent of dimension. The $\tilde{j}_\alpha\tilde{y}_\alpha$ contributes both a Taylor series x^{2p+2q} and a Laurent series $x^{2p+2q-2\alpha}$. We note that for every $\hat{d}$, the number of terms divergent as $x \to 0$ is finite ($\ell + \lfloor \frac{\hat{d}+1}{2} \rfloor$), and these amount to renormalizations of the different multipole moments. The Taylor series consists of an infinite number of zeros ($p + q > 0$), and a single ($p = q = 0$) imaginary numerical correction, $i\cot(\alpha\pi)$. Therefore for noneven $\hat{d}$ we are left with

$$\mathbb{G}_\alpha = 1 + i\cot(\pi\alpha) = i\sin^{-1}(\pi\alpha)[\cos(\pi\alpha) - i\sin(\pi\alpha)] = [i^{2\ell+\hat{d}-1}\sin(\pi\alpha)]^{-1}. \tag{2.34}$$

Substituting in (2.31), we find

$$\hat{S}_\Phi = \frac{G}{2}\int\frac{d\omega}{2\pi}\sum_L \frac{(-i\omega)^{2\ell+\hat{d}}}{\sin(\ell\pi+\frac{\hat{d}\pi}{2})} M_{\ell,\hat{d}} Q^{L\omega}\hat{Q}^*_{L\omega} + \text{c.c.}, \tag{2.35}$$

in all noneven $\hat{d}$. In odd integer $\hat{d}$, using (2.11) and (2.17) we return to the time domain, finding the radiation reaction effective action to be

$$\hat{S}_\Phi = G\int dt \sum_L \frac{(-)^{\ell+\frac{\hat{d}+1}{2}}}{\hat{d}!!(2\ell+\hat{d})!!}\hat{Q}^L\partial_t^{2\ell+\hat{d}}Q_L, \tag{2.36}$$

where Q_L was given by (2.21) and

$$\hat{Q}^L = \frac{\delta Q^L}{\delta\rho}\hat{\rho} = \frac{\delta Q^L}{\delta x^i}\hat{x}^i = \frac{\partial Q^L}{\partial x^i}\hat{x}^i + \frac{\partial Q^L}{\partial v^i}\hat{v}^i + \frac{\partial Q^L}{\partial a^i}\hat{a}^i + \cdots. \tag{2.37}$$

We remark especially that this action, and as we shall see the dissipated energy and the self-force, are given by regular, local and real expressions for all odd $\hat{d}$. The computation itself reduces to mere multiplication: vertex-propagator-vertex. The expression (4.6) contains an odd number of time derivatives, matching our expectation of a time-asymmetric dissipative term.

In fractional dimensions, transforming (2.35) back to the time domain introduces the Fourier transform of a fractional derivative, yielding

$$\begin{aligned}
\hat{S}_\Phi &= \frac{G}{2}\sum_L \frac{M_{\ell,\hat{d}}}{\sin(\ell\pi+\frac{\hat{d}\pi}{2})}\int\frac{d\omega}{2\pi}(-i\omega)^{2\ell+\hat{d}}\int dt e^{-i\omega t}\hat{Q}^L(t)\int dt' e^{i\omega t'}Q_L(t') + \text{c.c.}\\
&= \frac{G}{2}\sum_L \frac{M_{\ell,\hat{d}}}{\sin(\ell\pi+\frac{\hat{d}\pi}{2})}\int dt\hat{Q}^L(t)\int dt' Q_L(t')\partial_t^n\int\frac{d\omega}{2\pi}(-i\omega)^{-b}e^{i\omega(t'-t)} + \text{c.c.}\\
&= G\sum_L \frac{M_{\ell,\hat{d}}}{\Gamma(b)\sin(\ell\pi+\frac{\hat{d}\pi}{2})}\int dt\hat{Q}^L(t)\int dt' Q_L(t')\partial_t^n\frac{\Theta(t-t')}{(t-t')^{1-b}}\\
&= \frac{G}{\Gamma(b)}\sum_L \frac{\pi}{2^{\ell+1+\hat{d}}\Gamma(1+\frac{\hat{d}}{2})\Gamma(\ell+1+\frac{\hat{d}}{2})\sin(\ell\pi+\frac{\hat{d}\pi}{2})}\int_{-\infty}^{\infty}dt\hat{Q}^L(t)\int_{-\infty}^{t}dt' Q_L(t')\partial_t^n(t-t')^{b-1},
\end{aligned} \tag{2.38}$$

where $n = 2\ell + \lceil\hat{d}\rceil$, $b = \lceil\hat{d}\rceil - \hat{d}$. Thus in fractional dimensions, the self-force has a nonlocal (tail) contribution, originating from transforming $(-i\omega)^{-b}$. We note that this tail integral converges at $t' \to t$ and under reasonable assumptions regarding $Q^L(t)$ at very early times converges at $t' \to -\infty$ as well. As shown earlier, in even integer dimensions $b = 0$ and the transformation produces a delta function in time, indicating locality. However, the same form does not apply at odd integer d, because the Bessel functions and $\mathbb{G}_\alpha$ assume a different form. Using a limiting process over $d' = d + \epsilon$ for $\epsilon \to 0^+$, corresponding to $n = 2\ell + \hat{d} + 1$, $b = 1 - \epsilon$, we expect a ln term to appear as $b \to 1^-$; we treat it separately.

C. Odd spacetime dimensions

Radiation in odd spacetime dimensions—as in any dimension—is given in the frequency domain by Eq. (2.27). However, radiation in the time domain will look essentially different, as we know that in odd d an additional effect of indirect (off the light cone) propagation comes into play. In our analysis, this effect appears in (2.27) via nonanalyticity of the frequency domain solution for the field, in the form of a branch cut. An inverse Fourier transform of the solution can now have two distinct contributions (see Fig. 2): one from the integral along the branch cut, which gives rise to a tail term, and the other from the arc at $|\omega| \to -\infty(\Im\omega < 0)$, which may give rise to a local term. Fourier transforming as described gives the radiation at infinity

$$\Phi(\vec{r},t) = \frac{-G}{\sqrt{2\pi}\hat{d}!!}r^{-\frac{\hat{d}+1}{2}}\sum_L n^L\partial_t^{\ell+\frac{\hat{d}}{2}}\int_{-\infty}^{t-r}\frac{Q_L(t')}{|t-r-t'|^{1/2}}dt', \tag{2.39}$$

which converges at both integration limits under reasonable assumptions for $Q_L(t)$.

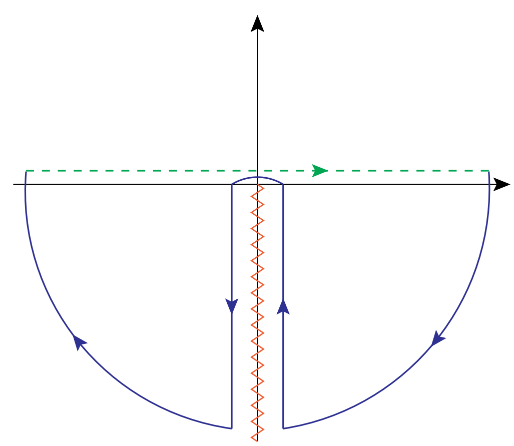


FIG. 2 (color online). Contour integration in the complex ω plane, avoiding the branch cut along $\omega = -i\sigma$. Different possible contours are represented by dashed (green) and solid (blue) lines. The branch cut is represented by the zigzag (red) line.

In the case of odd spacetime dimensions (even $\hat{d}$), $\alpha = \ell + \frac{\hat{d}}{2} = n$ is an integer, and (B10) for $\tilde{y}_\alpha$ must be replaced with (B11). This is of importance in the analysis of the RR effective action (2.31,2.32). Multiplying the Bessel functions $\tilde{j}_n(\omega r)$, $\tilde{j}_n(\omega r') + i\tilde{y}_n(\omega r')$ and taking the limit r, $r' \to 0$ at the same rate,[2] we again regularize the finite number of terms of joint negative degree, ignore the infinite number of positive degree terms (each giving zero), and are left only with a ln term and with constant terms:

$$\mathbb{G}_n = 1 + \frac{i}{\pi}\left[2\ \ln\left(\frac{\omega r'}{2}\right)\bigg|_{r'\to 0} - (\psi(1) + \psi(n+1)) - \sum_{m=1}^{n} \frac{(-)^m n!^2}{m(n+m)!(n-m)!}\right] = 1 + \frac{i}{\pi}\left[2\ \ln\left(\frac{\omega r'}{2}\right)\bigg|_{r'\to 0} + 2\gamma + H(2n) - 2H(n)\right], \tag{2.40}$$

where $H(n)$ is the nth harmonic number. In the frequency domain, therefore, the RR effective action is given by (2.31) with $\mathbb{G}_\alpha$ given in (2.40). To obtain the time-domain representation of the RR effective action, we integrate over ω. Again, we are presented with the branch-cut discontinuity of the ln term, absent from the even-dimensional case. Placing the branch cut along the negative imaginary w axis ($\omega = -i\sigma$, with $\sigma \in \mathbb{R}$, $\sigma \geq 0$), we use contour integration (as shown in Fig. 2) to find the (dimensionally regularized) identity

$$\int \frac{d\omega}{2\pi}\ \ln(\omega r') e^{-i\omega(t-t')} = -\frac{\Theta(t-t')}{t-t'} + \left(i\frac{\pi}{2} - \gamma + \ln\left(\frac{r'}{\mu}\right)\right)\delta(t-t'), \tag{2.41}$$

where μ is some arbitrary time scale of the system. Using the definition of the Fourier transform $Q_{L\omega} = \int_{-\infty}^{+\infty} e^{i\omega t} Q_L(t) dt$, the integral (2.41) and the expressions (2.40) and (2.17), from (2.31) we obtain the odd-dimensional radiation-reaction effective action in the time domain:

$$\hat{S}_\Phi = G\int_{-\infty}^{\infty} dt \sum_L \frac{(-)^{\ell+\frac{\hat{d}}{2}}}{\hat{d}!!(2\ell+\hat{d})!!}\hat{Q}^L(t)\left[\left(\frac{1}{2}H(2\ell+\hat{d}) - H\left(\ell+\frac{\hat{d}}{2}\right) + \ln\left(\frac{r'}{\mu}\right)\bigg|_{r'\to 0}\right)\partial_t^{2\ell+\hat{d}} Q_L(t) - \int_{-\infty}^{t} dt'\left(\frac{1}{t-t'}\partial_{t'}^{2\ell+\hat{d}} Q_L(t')\right)\right]. \tag{2.42}$$

[2] This makes sense from the point of view of the effective radiation zone theory, since the "zoom out" procedure out to the radiation zone affects both hatted and unhatted sources in the same manner. From the point of view of the full theory, for a single point source and its hatted counterpart obviously $r = r'$. For a composite source, however, there is no unique radial position of the source. This is strongly related to the fact that in general the local term in the RR effective action depends on the source's short-distance details.

The ln (r'/μ) term acts as a regulator for the integral—it takes care of its divergence near the coincidence limit $t' \to t$, as the size of the system ($c = 1$) is also roughly the natural cutoff for the t' integration. However, it also generically adds a finite local term of the same form as the local terms already present in (2.42). The value of this additional term depends on the short-scale structure of the sources (as, for example, in [6]). In principle it should be determined through matching to the full theory (cf. [2]). It would be interesting to start by analyzing the simple case of a single point particle in the full theory, which would also require Detweiler-Whiting [38] regularization.

One can write the RR effective action, therefore, implicitly as

$$\hat{S}_\Phi = G\int_{-\infty}^{\infty} dt \sum_L \frac{(-)^{\ell+\frac{\hat{d}}{2}}}{\hat{d}!!(2\ell+\hat{d})!!}\hat{Q}^L(t)\left[\left(\frac{1}{2}H(2\ell+\hat{d}) - H\left(\ell+\frac{\hat{d}}{2}\right)\right)\partial_t^{2\ell+\hat{d}} Q_L(t) - \int_{-\infty}^{t} dt'\left(\frac{1}{t-t'}\partial_{t'}^{2\ell+\hat{d}} Q_L(t')\right)\bigg|_{\text{regularized}}\right]. \tag{2.43}$$

Thus we find that in odd spacetime dimensions the action includes both a local (purely conservative) term and a nonlocal term, which is responsible for dissipative effects. The nonlocal (but causal) term is a manifestation, in our effective theory of multipoles, of field propagation inside the light cone (the so-called tail effect) in the spacetime picture. It is known that integrating out a massless degree of freedom generically leads to a nonlocal effective action [22,39,40], as was shown to be the case in odd spacetime dimensions. The even-dimensional case turns out to be the special case where the effective action is *still local* after this elimination.

D. Applications and tests

1. Perturbative expansion of the RR force

Consider the case of a single charged body with a prescribed trajectory ("being held and waved at the tip of a wand") interacting with a scalar field. In this case the notion of RR force and self-force coincide. For $4d$, the fully relativistic force expression is given analogously to the ALD self-force [11,12] familiar from electromagnetism:

$$F^\mu_{\text{ALD}} \equiv \frac{dp^\mu}{d\tau} = \frac{1}{3}Gq^2\left(\frac{d^3x^\mu}{d\tau^3} - \frac{d^3x^\nu}{d\tau^3}\frac{dx_\nu}{d\tau}\frac{dx^\mu}{d\tau}\right). \tag{2.44}$$

Expansion of this equation to leading and +1PN orders yields

$$\vec{F}_{\text{ALD}} = Gq^2\left[\frac{1}{3}\dot{\vec{a}} + \frac{1}{3}v^2\dot{\vec{a}} + (\vec{v}\cdot\vec{a})\vec{a} + \frac{1}{3}(\vec{v}\cdot\dot{\vec{a}})\vec{v}\right] \tag{2.45}$$

Upon substituting $\hat{d}=1$, all the expressions given in Secs. 2.1 and 2.2 can be seen to yield expressions identical to those given in paper I (Section III A 2) for the scalar field; in particular, the 4d ALD force at leading and next-to-leading order (2.45) is recovered immediately. We derive the formula for the RR force in general even dimensions, and also check explicitly $d=6$, for which an ALD-like radiation-reaction 4-force on a scalar charge was developed by Galt'sov [25,26]. Our method derives the RR force from the action and multipoles (2.21,4.6)) in three stages: by using a source term of a point particle for ρ, by matching the appropriate $\hat{\rho}$, and by finally calculating the contribution from generalized Euler-Lagrange (E-L) equation $\delta S/\delta \hat{x}^i$. The source term corresponding to a scalar-charged point particle with a trajectory $\vec{x}(t)$ in d spacetime dimensions is

$$\rho(\vec{x}',t)=q\int\delta^{(d)}(x'-x)d\tau=\frac{q}{\gamma}\delta^{(D)}(\vec{x}'-\vec{x})=\sum_{s=0}^{\infty}\frac{-(2s-3)!!v^{2s}}{(2s)!!}\delta^{(D)}(\vec{x}'-\vec{x}). \tag{2.46}$$

Thus we find

$$Q^L=-(2\ell+\hat{d})!!\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\frac{(2s-3)!!}{(2p)!!(2\ell+2p+\hat{d})!!(2s)!!}\partial_t^{2p}(v^{2s}r^{2p}x^L_{TF}),$$
$$\hat{Q}^L=-(2\ell+\hat{d})!!\sum_{\hat{p}=0}^{\infty}\sum_{\hat{s}=0}^{\infty}\frac{(2\hat{s}-3)!!}{(2\hat{p})!!(2\ell+2\hat{p}+\hat{d})!!(2\hat{s})!!}\partial_t^{2\hat{p}}\frac{\delta}{\delta x^i}(v^{2\hat{s}}r^{2\hat{p}}x^L_{TF})\hat{x}^i. \tag{2.47}$$

Accordingly we obtain the Lagrangian, which after moving $2\hat{p}$ time derivatives from the $\hat{x}^L$ multipoles to the x^L multipoles by partial integration becomes

$$\hat{L}_\Phi=Gq^2\sum_L(-)^{\ell+\frac{\hat{d}+1}{2}}\frac{(2\ell+\hat{d})!!}{\hat{d}!!}\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\hat{p}=0}^{\infty}\sum_{\hat{s}=0}^{\infty}\frac{(2s-3)!!}{(2p)!!(2\ell+2p+\hat{d})!!(2s)!!}\frac{(2\hat{s}-3)!!}{(2\hat{p})!!(2\ell+2\hat{p}+\hat{d})!!(2\hat{s})!!}$$
$$\times\hat{x}^i\frac{\delta}{\delta x^i}(v^{2\hat{s}}r^{2\hat{p}}x^L_{TF})\partial_t^{2(\ell+p+\hat{p})+\hat{d}}(v^{2s}r^{2p}x_L). \tag{2.48}$$

In the EOM, the RR force contribution is given by the variation by $\hat{x}^j$:

$$F^j=\frac{\delta\hat{S}}{\delta\hat{x}^j}=\left[\frac{\partial\hat{L}}{\partial\hat{x}^j}-\frac{d}{dt}\left(\frac{\partial\hat{L}}{\partial\dot{\hat{x}}^j}\right)\right]. \tag{2.49}$$

The leading order arises from the same term (the leading dipole term $\ell=1$, $p=\hat{p}=s=\hat{s}=0$) in every (even) dimension and is given by

$$\vec{F}^{(d)}_{LO}=\frac{(-)^{\frac{d}{2}}Gq^2}{(d-1)!!(d-3)!!}\partial_t^{d-1}\vec{x}. \tag{2.50}$$

The term for $d=6$ ($\hat{d}=3$) is shown in Table I. Out of the 15 possible action terms in the next-to-leading order (for different $\ell, p, \hat{p}, s, \hat{s}$), we find using the E-L equation nine nonzero contributions to the force (recorded in Table II for $d=6$). Adding these contributions we find

$$F^i_{RR}=-Gq^2\left[\frac{1}{45}\ddot{a}^i+\frac{2}{45}v^2\ddot{a}^i+\frac{2}{9}(\vec{v}\cdot\vec{a})\ddot{a}^i+\frac{2}{9}(\vec{v}\cdot\dot{\vec{a}})\dot{a}^i\right.$$
$$\left.+\frac{1}{9}(\vec{v}\cdot\ddot{\vec{a}})a^i+\frac{1}{45}(\vec{v}\cdot\dddot{\vec{a}})v^i+\frac{1}{9}(\vec{a}\cdot\dot{\vec{a}})a^i+\frac{1}{9}a^2\dot{a}^i\right]. \tag{2.51}$$

We compared this result (2.51) to Galt'sov's expression for the six-dimensional scalar d force in flat space. We seem to find sign mismatches between his results from 2007 [25] and 2011 [26]; we assume an expression identical to his up to the signs of two subleading terms ($(\ddot{x}\ddot{x})\ddot{x}^\mu$, $\ddot{x}^2\ddot{x}_v$),

$$f^\mu_{\text{flat}}=-(\eta^{\mu\nu}+\dot{x}^\mu\dot{x}^\nu)\left[\frac{1}{45}x^{(5)}_\nu-\frac{1}{9}\ddot{x}^2\dddot{x}_v\right]+\frac{2}{9}(\ddot{x}\dddot{x})\ddot{x}^\mu, \tag{2.52}$$

where $\eta_{\mu\nu}$ is the flat (mostly plus) metric and with the derivatives taken with respect to proper time. Expansion of this expression up to 1PN (next-to-leading) order to find the D force gives

$$F^i_{Galt'sov}=-\left[\frac{1}{45}\ddot{a}^i+\frac{2}{45}v^2\ddot{a}^i+\frac{2}{9}(\vec{v}\cdot\vec{a})\ddot{a}^i+\frac{2}{9}(\vec{v}\cdot\dot{\vec{a}})\dot{a}^i\right.$$
$$\left.+\frac{1}{9}(\vec{v}\cdot\ddot{\vec{a}})a^i+\frac{1}{45}(\vec{v}\cdot\dddot{\vec{a}})v^i+\frac{1}{9}(\dot{\vec{a}}\cdot\vec{a})a^i+\frac{1}{9}a^2\dot{a}^i\right], \tag{2.53}$$

TABLE I. Leading-order contribution to the scalar self-force.

ℓ p $\hat{p}$ s $\hat{s}$	$\hat{L}/(Gq^2)$	$F^j/(Gq^2)$
1 0 0 0 0	$-\frac{1}{45}\hat{x}^i\partial_t^5x_i$	$-\frac{1}{45}\dddot{\alpha}^j$

TABLE II. Next-to-leading-order contribution to the scalar self-force.

ℓ p $\hat{p}$ s $\hat{s}$	$\hat{L}/(Gq^2)$	$F^j/(Gq^2)$
2 0 0 0 0	$\frac{1}{630}\hat{x}^k\frac{\delta}{\delta x^k}[x^ix^j-\frac{1}{5}x^2\delta^{ij}]\partial_t^7(x_ix_j)$	$\frac{1}{315}[\partial_t^7(x^ix^j)x_i-\frac{1}{5}\partial_t^7(x^2)x^j]$
1 1 0 0 0	$-\frac{1}{630}\hat{x}^i\partial_t^7(x^2x_i)$	$-\frac{1}{630}\partial_t^7(x^2x^j)$
1 0 1 0 0	$-\frac{1}{630}\hat{x}^k\frac{\delta}{\delta x^k}[x^ix^2]\partial_t^7x_i$	$-\frac{1}{630}[x^2\partial_t^7x^j+2x^jx_i\partial_t^7x^i]$
1 0 0 1 0	$+\frac{1}{90}\hat{x}^i\partial_t^5(v^2x_i)$	$+\frac{1}{90}\partial_t^5(v^2x^j)$
1 0 0 0 1	$+\frac{1}{90}\hat{x}^k\frac{\delta}{\delta x^k}[v^2x^i]\partial_t^5x_i$	$+\frac{1}{90}v^2\partial_t^5x^j-\frac{1}{45}\partial_t[v^jx^i\partial_t^5x_i]$
0 1 1 0 0	$+\frac{1}{450}\hat{x}^jx_j\partial_t^7x^2$	$+\frac{1}{450}x^j\partial_t^7x^2$
0 1 0 0 1	$-\frac{1}{90}\hat{v}^jv_j\partial_t^5x^2$	$+\frac{1}{90}\partial_t[v^j\partial_t^5x^2]$
0 0 1 1 0	$-\frac{1}{90}\hat{x}^jx_j\partial_t^5v^2$	$-\frac{1}{90}x^j\partial_t^5v^2$
0 0 0 1 1	$+\frac{1}{18}\hat{v}^jv_j\partial_t^3v^2$	$-\frac{1}{18}\partial_t[v^j\partial_t^3v^2]$

where all derivatives are now with respect to t, namely $v^i \coloneqq dx^i/dt$, $a^i \coloneqq d^2x^i/dt^2$, etc. This result coincides exactly with our result (2.51) to +1PN order.

We can also use the multipole formulation [(4.6), (2.37), (2.48), and (2.49)] to calculate the power dissipated by the RR force in even dimension d:

$$P_{RR} = -\vec{v}\cdot\vec{F} = -\frac{dx^i}{dt}\frac{\delta\hat{L}}{\delta\hat{x}^i}\Big|_{\hat{\vec{x}}\to\vec{x}} = G\sum_L \frac{(-)^{\ell+\frac{\hat{d}-1}{2}}}{\hat{d}!!(2\ell+\hat{d})!!}\frac{\delta Q^L}{\delta x^i}\frac{dx^i}{dt}\partial_t^{2\ell+\hat{d}}Q_L. \quad (2.54)$$

The time-averaged power is found using

$$\int dt\frac{dx^i}{dt}\frac{\delta Q^L}{\delta x^i} = \int dt\frac{dQ^L}{dt}, \quad (2.55)$$

followed by $\ell+\frac{\hat{d}-1}{2}$ integrations by parts, to be

$$\dot{E} = \langle P_{RR}\rangle = \sum_L \frac{G}{\hat{d}!!(2\ell+\hat{d})!!}\langle(\partial_t^{\ell+\frac{\hat{d}+1}{2}}Q^L)^2\rangle, \quad (2.56)$$

which agrees with (2.30). In $6d$, we record the radiated power to +1PN order:

$$\dot{E} = Gq^2\left[\frac{1}{45}\dot{a}^2+\frac{1}{15}v^2(\vec{v}\cdot\dddot{\vec{a}})+\frac{1}{3}(\vec{v}\cdot\vec{a})(\vec{v}\cdot\ddot{\vec{a}}) +\frac{1}{9}(\vec{a}\cdot\dot{\vec{a}})(\vec{v}\cdot\vec{a})+\frac{1}{9}a^2(\vec{v}\cdot\dot{\vec{a}})+\frac{2}{9}(\vec{v}\cdot\dot{\vec{a}})^2\right]. \quad (2.57)$$

We note the special case of $2d$, where the only STF multipoles are $\{1,x\}$; the leading order remains (2.50), while at higher orders the sum over L trivializes.

III. ELECTROMAGNETISM

The EM action is given by

$$S = -\frac{1}{4\Omega_{\hat{d}+1}}\int F_{\mu\nu}F^{\mu\nu}r^{\hat{d}+1}drd\Omega_{\hat{d}+1}dt - \int A_\mu J^\mu r^{\hat{d}+1}drd\Omega_{\hat{d}+1}dt. \quad (3.1)$$

Working in spherical coordinates (t,r,Ω), we reduce over the sphere as in (2.4) and (2.5). The EM field and sources are decomposed as

$$A_{t/r} = \int\frac{d\omega}{2\pi}\sum_L A^{L\omega}_{t/r}x_Le^{-i\omega t},$$
$$A_\Omega = \int\frac{d\omega}{2\pi}\sum_L(A^{L\omega}_S\partial_\Omega x_L + A^{L\omega}_{V\aleph}x^L_{\aleph\Omega})e^{-i\omega t},$$
$$J^{t/r} = \int\frac{d\omega}{2\pi}\sum_L J^{t/r}_{L\omega}x^Le^{-i\omega t},$$
$$J^\Omega = \int\frac{d\omega}{2\pi}\sum_L(J^S_{L\omega}\partial^\Omega x^L + J^{V\aleph}_{L\omega}x^{\aleph\Omega}_L)e^{-i\omega t}, \quad (3.2)$$

where the scalar multipoles x^L [(2.8) and (2.9)] are now supplemented by the divergenceless vector multipoles $x^L_{\aleph\Omega}$, enumerated by an antisymmetric multi-index $\aleph$ taken from the Hebrew alphabet, representing $D-3$ spherical indices:

$$x^L_{\aleph\Omega} = \epsilon^{(\hat{d}+1)}_{\aleph\Omega\Omega'}\partial^{\Omega'}x^L = \epsilon^{(D)}_{\aleph\Omega bc}x^b\partial^cx^L = \delta^a_\Omega\epsilon^{(D)}_{\aleph abc}x^b\partial^cx^L = (*(\vec{r}\wedge\nabla^\rightarrow))_{\aleph\Omega}x^L, \quad (3.3)$$

where $\epsilon^{(\hat{d}+1)}_{\Omega_1\ldots\Omega_{\hat{d}+1}}$ ($\epsilon^{(D)}_{a_1\ldots a_D}$) is the complete antisymmetric tensor on the $\Omega_{\hat{d}+1}$ sphere (in D spatial dimensions), $\wedge$ is the exterior product and $*$ is the Hodge duality operator [41–47]. The dimension of this independent vectorial basis is $D_\ell(\hat{d}+1,1) = \frac{\ell(\ell+\hat{d})(2\ell+\hat{d})(\ell+\hat{d}-2)!}{(\hat{d}-1)!(\ell+1)!}$ [29,30][3]. The complete normalization conditions in d dimensions are [29,30]

[3]These are generalizations of the single $4d$ ($\hat{d}=1$) multipole family $x^L_\Omega = \epsilon_{\Omega\Omega'}\partial^{\Omega'}x^L = (\vec{r}\times\nabla^\rightarrow x^L)_\Omega$. In $4d$, $D_\ell(\hat{d}+1,1) = 2\ell+1 = D_\ell(\hat{d}+1,0)$, and thus these indeed form a single family; $\aleph$ is then an empty string.

$$\int x_{L_\ell} x^{L'_{\ell'}} d\Omega_{\hat{d}+1} = N_{\ell\hat{d}} r^{2\ell} \Omega_{\hat{d}+1} \delta_{\ell\ell'} \delta_{L_\ell L'_{\ell'}},$$

$$\int g^{\Omega\Omega'} \partial_\Omega x_{L_\ell} \partial_{\Omega'} x^{L'_{\ell'}} d\Omega_{\hat{d}+1} = c_s \cdot N_{\ell,\hat{d}} r^{2\ell} \Omega_{\hat{d}+1} \delta_{\ell\ell'} \delta_{L_\ell L'_{\ell'}},$$

$$\int g^{\Omega\Omega'} x^{L_\ell}_{\aleph\Omega} x^{L'_{\ell'}}_{\aleph'\Omega'} d\Omega_{\hat{d}+1} = c_s \cdot N_{\ell,\hat{d}} r^{2\ell} \Omega_{\hat{d}+1} \delta_{\ell\ell'} \delta_{L_\ell L'_{\ell'}} \delta_{\aleph\aleph'},$$

$$\int g^{\Omega\Omega'} g^{\Psi\Psi'} x^{L_\ell}_{\aleph[\Psi;\Omega]} x^{L'_{\ell'}}_{\aleph'[\Psi';\Omega']} d\Omega_{\hat{d}+1} = 2 c_v c_s \cdot N_{\ell\hat{d}} \Omega_{\hat{d}+1} r^{2\ell} \delta_{\ell\ell'} \delta_{L_\ell L'_{\ell'}} \delta_{\aleph\aleph'}, \tag{3.4}$$

where $c_s \coloneqq \ell(\ell + \hat{d})$, $c_v = c_s + \hat{d} - 1$ and the semicolon ";" represents the covariant derivative on the sphere. The inverse transformations are given by

$$\begin{aligned}
J^t_{L\omega}(r) &= (N_{\ell,\hat{d}} r^{2\ell} \Omega_{\hat{d}+1})^{-1} \int \rho_\omega(\vec{r}) x_L d\Omega_{\hat{d}+1} = (N_{\ell,\hat{d}} r^{2\ell} \Omega_{\hat{d}+1})^{-1} \int\int dt e^{i\omega t} \rho(\vec{r}, t) x_L d\Omega_{\hat{d}+1}, \\
J^r_{L\omega}(r) &= (N_{\ell,\hat{d}} r^{2\ell} \Omega_{\hat{d}+1})^{-1} \int \vec{J}_w(\vec{r}) \cdot \vec{n} x_L d\Omega_{\hat{d}+1} = (N_{\ell,\hat{d}} r^{2\ell} \Omega_{\hat{d}+1})^{-1} \int\int dt e^{i\omega t} \vec{J}(\vec{r}, t) \cdot \vec{n} x_L d\Omega_{\hat{d}+1}, \\
J^{V\aleph}_{L\omega}(r) &= (c_s N_{\ell,\hat{d}} r^{2\ell} \Omega_{\hat{d}+1})^{-1} \int \vec{J}_w(\vec{r}) \cdot (*(\vec{r} \wedge \vec{\nabla}))_\aleph x^L d\Omega_{\hat{d}+1} \\
&= (c_s N_{\ell,\hat{d}} r^{2\ell} \Omega_{\hat{d}+1})^{-1} \int\int dt e^{i\omega t} \vec{J}(\vec{r}, t) \cdot (*(\vec{r} \wedge \vec{\nabla}))_\aleph x^L d\Omega_{\hat{d}+1}.
\end{aligned} \tag{3.5}$$

We note that only three inverse transformations are required, as we henceforth replace $J^S_{L\omega}$ using current conservation

$$0 = D_\mu J^\mu_{L\omega} = i\omega J^t_{L\omega} + \left(\partial_r + \frac{\ell + \hat{d} + 1}{r}\right) J^r_{L\omega} - c_s J^S_{L\omega}. \tag{3.6}$$

We plug (3.2) into the action (3.1) to obtain

$$\begin{aligned}
S &= \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_L N_{\ell,\hat{d}} S_{L\omega}, \\
S_{L\omega} &= \int dr r^{2\ell+\hat{d}+1} \left\{ \left[\left| i\omega A^{L\omega}_r - \frac{1}{r^\ell}(r^\ell A^{L\omega}_t)' \right|^2 + \frac{c_s}{r^2} |i\omega A^{L\omega}_S - A^{L\omega}_t|^2 - \frac{c_s}{r^2} \left| \frac{1}{r^\ell}(r^\ell A^{L\omega}_S)' - A^{L\omega}_r \right|^2 + c_s \left(\frac{\omega^2}{r^2} - \frac{c_v}{r^4} \right) |A^{L\omega}_{V\aleph}|^2 \right.\right. \\
&\quad \left.\left. - \frac{c_s}{r^2} \left| \frac{1}{r^\ell}(r^\ell A^{L\omega}_{V\aleph})' \right|^2 \right] - \Omega_{\hat{d}+1} [A^{L\omega}_r J^{r*}_{L\omega} + A^{L\omega}_t J^{t*}_{L\omega} + c_s A^{L\omega}_S J^{S*}_{L\omega} + c_s A^{L\omega}_{V\aleph} J^{V\aleph*}_{L\omega} + \text{c.c.}] \right\},
\end{aligned} \tag{3.7}$$

where $' \coloneqq \frac{d}{dr}$, and we use $A_{L-\omega} = A^*_{L\omega}$, $J_{L-\omega} = J^*_{L\omega}$ since $A_\mu(x)$, $J^\mu(x)$ are real. In the spirit of [27], we notice that $A^{L\omega}_r$ is an auxiliary field, a field whose derivative A'_r does not appear in (3.7). Therefore, its EOM is algebraic and is solved to yield

$$A^r_{L\omega} = -\frac{1}{\omega^2 - \frac{c_s}{r^2}} \left[\frac{i\omega}{r^\ell}(r^\ell A^t_{L\omega})' + \frac{c_s}{r^{\ell+2}}(r^\ell A^S_{L\omega})' - \Omega_{\hat{d}+1} J^r_{L\omega} \right]. \tag{3.8}$$

Substituting the solution into the action, it is seen that the action can be separated into independent fields, corresponding to the different possible polarizations. We distinguish between the vectorial $A^{L\omega}_{V\aleph}$ fields, as they appear already in (3.7), coupled to the vector source terms

$$\rho^{V\aleph}_{L\omega} \coloneqq J^{V\aleph}_{L\omega}, \tag{3.9}$$

and the scalar fields

$$\tilde{A}^{L\omega}_S \coloneqq A^{L\omega}_t - i\omega A^{L\omega}_S, \tag{3.10}$$

which are coupled to the corresponding source terms

$$\rho^S_{L\omega} := -J^t_{L\omega} + \frac{i}{\omega r^{\ell+\hat{d}+1}}\left(r^{\ell+\hat{d}+1}\frac{\Lambda}{\Lambda-1}J^r_{L\omega}\right)', \tag{3.11}$$

where $\Lambda := \frac{\omega^2 r^2}{c_s}$ and we have used (3.6). The action can now be concisely decoupled to scalar and vector parts [omitting hereafter the indices $(L\omega)$ for brevity]:

$$S_{\rm EM} = \frac{1}{2}\int\frac{d\omega}{2\pi}\sum_L\left[S^{L\omega}_S + \sum_{\aleph}S^{L\omega}_{V\aleph}\right], \tag{3.12}$$

with

$$S^{L\omega}_S = N_{\ell,\hat{d}}\int r^{2\ell+\hat{d}+1}dr\left[\frac{1}{1-\Lambda}\left|\frac{1}{r^\ell}(r^\ell\tilde{A}_S)'\right|^2 + \frac{c_s}{r^2}|\tilde{A}_S|^2 + \Omega_{\hat{d}+1}(\tilde{A}_S\rho^{S*}_{L\omega} + \text{c.c.})\right], \tag{3.13}$$

$$S^{L\omega}_{V\aleph} = N_{\ell,\hat{d}}\int r^{2\ell+\hat{d}+1}drc_s\left[\left(\frac{\omega^2}{r^2} - \frac{c_v}{r^4}\right)|A_{V\aleph}|^2 - \left|\frac{1}{r^{\ell+1}}(r^\ell\tilde{A}_{V\aleph})'\right|^2 - \Omega_{\hat{d}+1}(\tilde{A}_{V\aleph}\rho^{V\aleph}_{L\omega} + \text{c.c.})\right], \tag{3.14}$$

and we treat them separately. Note that for $\ell = 0$ we have $\rho^S_{L\omega} = 0$ [see Eqs. (3.6) and (3.11)] as well as $S_V = 0$; thus, only $\ell \geq 1$ need be considered.

The scalar part of the EM action

We derive the equation of motion for the scalar action from $S^{L\omega}_S$ (3.13) by treating $(r^\ell\tilde{A}_S)$ as the field and finding equations for its conjugate momentum $(N_{\ell,\hat{d}}r^\ell\Pi_S)$:

$$(N_{\ell,\hat{d}}r^\ell\Pi_S) := \frac{\partial L}{\partial(r^\ell\tilde{A}^*_S)'} = \frac{N_{\ell,\hat{d}}}{1-\Lambda}r^{\hat{d}+1}(r^\ell\tilde{A}_S)', \tag{3.15}$$

$$(N_{\ell,\hat{d}}r^\ell\Pi_S)' := \frac{\partial L}{\partial(r^\ell\tilde{A}^*_S)} = N_{\ell,\hat{d}}c_s r^{\hat{d}-1}(r^\ell\tilde{A}_S) + N_{\ell,\hat{d}}\Omega_{\hat{d}+1}r^{\ell+\hat{d}+1}\rho^S_{L\omega}. \tag{3.16}$$

Differentiating (3.16) with respect to r, substituting (3.15), and renaming the field A_E and source term $\rho^{A_E}_{L\omega}$ [recalling (3.5) and (3.11)] as

$$A_E = (\ell r^{\hat{d}})^{-1}\Pi_S,$$
$$\rho^{A_E}_{L\omega} = N_{\ell,\hat{d}}\Omega_{\hat{d}+1}\frac{r^{\ell+\hat{d}}(r^{\ell+2}\rho^S_{L\omega})'}{\ell+\hat{d}} = \int d\Omega_{\hat{d}+1}\frac{r^{\hat{d}}x_L}{\ell+\hat{d}}\left[\frac{i}{\omega r^{\hat{d}-1}}\left(r^{\hat{d}+1}\frac{\Lambda}{\Lambda-1}\vec{J}_w(\vec{r})\cdot\vec{n}\right)' - r^2\rho_\omega(\vec{r})\right]', \tag{3.17}$$

we find the equation

$$0 = N_{\ell,\hat{d}}r^{2\ell+\hat{d}+1}\frac{\ell}{\ell+\hat{d}}\left(\omega^2 + \partial^2_r + \frac{2\ell+\hat{d}+1}{r}\partial_r\right)A_E - \rho^{A_E}_{L\omega}. \tag{3.18}$$

This equation is of the same form as (2.14), up to the replacement of G by

$$R^+_1 = \frac{\ell+\hat{d}}{\ell}, \tag{3.19}$$

and thus we find a propagator similar to (2.16):

$$G^{A_E}_{\rm ret}(r',r) = R^+_1G^\Phi_{\rm ret}(r',r) = -i\omega^{2\ell+\hat{d}}M_{\ell,\hat{d}}R^+_1\tilde{j}_\alpha(\omega r_1)\tilde{h}^+_\alpha(\omega r_2)\delta_{LL'};\, r_1 := \min\{r',r\}, \qquad r_2 := \max\{r',r\}. \tag{3.20}$$

We again present the EFT Feynman rules following the steps (2.19–2.21). In the radiation zone, the field can be written as

$$A^{L\omega\text{EFT}}_E(r) = (-Q^E_{L\omega})(-i\omega^{2\ell+\hat{d}}M_{\ell,\hat{d}}R^+_1\tilde{h}^+_\ell(\omega r)), \tag{3.21}$$

where $Q^E_{L\omega}$ are the sources [Eq. (2.18)]. In the full theory the solution outside the sources is given (see (3.18)) by

$$A_E^{L\omega}(r) = \int dr' \rho_{L\omega}^{A_E}(r') G_{\rm ret}^{A_E}(r',r) = \left[-\int dr' \tilde{j}_\alpha(\omega r')\rho_{L\omega}^{A_E}(r')\right][-i\omega^{2\ell+\hat{d}} M_{\ell,\hat{d}} R_1^+ \tilde{h}_\ell^+(\omega r)], \tag{3.22}$$

and the sources can be read off and identified [using (3.5) and (3.17), integrations by parts and (2.15)] to be

$$\begin{aligned} Q_{L\omega}^{(E)} &= \int dr' \tilde{j}_\alpha(\omega r')\rho_{L\omega}^{A_E}(r') \\ &= \frac{1}{\ell+\hat{d}} \int dr' \tilde{j}_\alpha(\omega r') \int d\Omega'_{\hat{d}+1} r'^{\hat{d}} x'_L \left[-r'^2 \rho_\omega(\vec{x}') + \frac{i}{\omega r'^{\hat{d}-1}} \left(r'^{\hat{d}+1} \frac{\Lambda}{\Lambda-1} \vec{J}_\omega(\vec{x}')\cdot \vec{n}'\right)'\right]' \\ &= \frac{1}{\ell+\hat{d}} \int d^D x' x'_L \left[\frac{1}{r'^{\ell+\hat{d}-1}} (r'^{\ell+\hat{d}} \tilde{j}_\alpha(\omega r'))' \rho_\omega(\vec{x}') - i\omega \tilde{j}_\alpha(\omega r') \vec{J}_\omega(\vec{x}')\cdot\vec{x}'\right]. \end{aligned} \tag{3.23}$$

We return to the time domain using (3.5) to find the electric type radiation source multipoles [compare (2.21)]

$$Q_{(E)}^L = \frac{1}{\ell+\hat{d}} \int d^D x x_{TF}^L \left[\frac{1}{r^{\ell+\hat{d}-1}} (r^{\ell+\hat{d}} \tilde{j}_\alpha(ir\partial_t))' \rho(\vec{x}) - \tilde{j}_\alpha(ir\partial_t)\partial_t \vec{J}(\vec{x})\cdot\vec{x}\right], \tag{3.24}$$

$$\hat{Q}_{(E)}^L = \frac{\delta Q_E^L}{\delta J^i} \hat{J}^i = \frac{\delta Q_E^L}{\delta x^i} \hat{x}^i. \tag{3.25}$$

We note here that an expansion of $\tilde{j}_\alpha$ according to (B8) reproduces, for $d=4$, Eq. (47) of [35] [after using current conservation, Eq. (49) there].

The vector part of the EM action

For the vector sector of the action, we rewrite (3.14) in a form similar to (2.12):

$$S_{V\aleph}^{L\omega} = \int dr \left[\frac{N_{\ell,\hat{d}} r^{2\ell+\hat{d}+1}}{R_1^-} A_{M\aleph}^* \left(\omega^2 + \partial_r^2 + \frac{2\ell+\hat{d}+1}{r}\partial_r\right) A_{M\aleph} - (\rho_{L\omega}^{A_{M\aleph}} A_{M\aleph}^* + \text{c.c.})\right], \tag{3.26}$$

where we have defined [recalling (3.5)] as

$$R_1^- = \frac{\ell}{(\ell+\hat{d})}, \tag{3.27}$$

$$A_{M\aleph} = \frac{\ell A_{V\aleph}}{r}, \quad \rho_{L\omega}^{A_{M\aleph}} = \ell N_{\ell,\hat{d}} \Omega_{\hat{d}+1} r^{2\ell+\hat{d}+2} \rho_{L\omega}^V(r) = \ell r^{\hat{d}+1} \int \vec{J}_w(\vec{r}) \cdot (*(\vec{r}\wedge\vec{\nabla}))_\aleph x_L d\Omega_{\hat{d}+1}. \tag{3.28}$$

The action (3.26) is again identical to (2.12) up to a prefactor of R_1^-, with a source similar to (2.13). The propagator, therefore, is [compare (2.16) and (3.20)]

$$G_{\rm ret}^{A_{M\aleph}}(r',r) = -i\omega^{2\ell+\hat{d}} M_{\ell,\hat{d}} R_1^- \tilde{j}_\alpha(\omega r_1)\tilde{h}_\alpha^+(\omega r_2)\delta_{LL'}; \qquad r_1 := \min\{r',r\}, \qquad r_2 := \max\{r',r\}. \tag{3.29}$$

We find these sources $Q_{L\omega}^{(M,\aleph)}$ by again matching $\Phi_{L\omega}^{A_{M\aleph}}(r)$ for large r and from the diagrammatic representation [in analogy with (2.19), (2.20), (2.21), (3.21), (3.22), and (3.24)], to find

$$Q_{L\omega}^{(M,\aleph)} = \int d^D x \tilde{j}_\alpha(\omega r)(*(\vec{r}\wedge\vec{J}_w(\vec{r})))^\aleph_{(k_\ell x_{L-1})}, \tag{3.30}$$

where we have used (3.5) and (3.9). In the time domain, we find the magnetic radiation source multipoles [compare (2.21) and (3.24)]

$$Q_{(M,\aleph)}^L = \int d^D x \tilde{j}_\alpha(ir\partial_t)[(*(\vec{r}\wedge\vec{J}(\vec{r})))_\aleph^{k_\ell x^{L-1}}]^{\rm STF}, \tag{3.31}$$

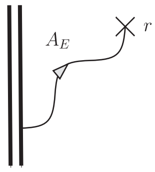
$$\hat{Q}_{(M,\aleph)}^L = \frac{\delta Q_{(M,\aleph)}^L}{\delta J^i} \hat{J}^i = \frac{\delta Q_{(M,\aleph)}^L}{\delta x^i} \hat{x}^i. \tag{3.32}$$

Presenting $\tilde{j}_\alpha$ as a series expansion using (B8), these coincide with Eq. (48) of [35].

A. Outgoing EM radiation and the RR effective action

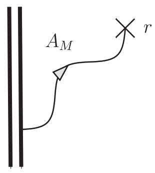
Outgoing EM radiation can now be found diagrammatically [compare (2.27)] as

$$A_E^{L\omega}(r) =$$



$$= -Q_{(E)}^{L'\omega} G_{ret}^{A_E}(0,r) = \sqrt{\frac{\pi}{2^{\hat{d}+1}}} \frac{R_1^+}{\Gamma(1+\hat{d}/2)} (-i\omega)^{\ell+\frac{\hat{d}-1}{2}} \frac{Q_{(E)}^{L\omega}}{r^\ell} \frac{e^{i\omega r}}{r^{\frac{\hat{d}+1}{2}}} ,$$

$$A_{M\aleph}^{L\omega}(r) =$$



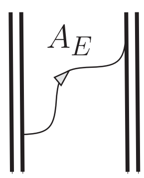
$$= -Q_{(M,\aleph)}^{L'\omega} G_{ret}^{A_M\aleph}(0,r) = \sqrt{\frac{\pi}{2^{\hat{d}+1}}} \frac{R_1^-}{\Gamma(1+\hat{d}/2)} (-i\omega)^{\ell+\frac{\hat{d}-1}{2}} \frac{Q_{(M,\aleph)}^{L\omega}}{r^\ell} \frac{e^{i\omega r}}{r^{\frac{\hat{d}+1}{2}}} . \tag{3.33}$$

In the time domain, for even d, we find [where ϵ is $(+)/(-,\aleph)$ for the electric/magnetic part/s]

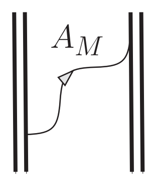
$$A_\epsilon(\vec{r},t) = \frac{1}{\hat{d}!!} r^{-\frac{\hat{d}+1}{2}} \sum_L R_1^\epsilon n^L \partial_t^{\ell+\frac{\hat{d}-1}{2}} Q_L^\epsilon(t-r). \tag{3.34}$$

The EM double field effective action is a sum of the scalar and vector action diagrams and can be written using our Feynman rules, similarly to (2.31) and (2.33),

$$\hat{S}_{EM} =$$



$$+$$



$$= \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_{L,L'} \left[\left(-Q_{L\omega}^{(E)}\right) G_{ret}^{A_E}(0,0) \left(-\hat{Q}_{L'\omega}^{(E)*}\right) + \left(-Q_{L\omega}^{(M,\aleph)}\right) G_{ret}^{A_M\aleph}(0,0) \left(-\hat{Q}_{L'\omega}^{(M,\aleph)*}\right) \right] + c.c.$$

$$= \int \frac{d\omega}{2\pi} \sum_L \frac{-i\omega^{2\ell+\hat{d}} M_{\ell,\hat{d}} \cdot \mathbb{G}_\alpha(\omega)}{2} \left[R_1^+ Q_{(E)}^{L\omega} \hat{Q}_{L\omega}^{(E)*} + R_1^- Q_{(M\aleph)}^{L\omega} \hat{Q}_{L\omega}^{(M\aleph)*} \right] + c.c. , \tag{3.35}$$

where $Q_{(E)}^L, \hat{Q}_{(E)}^L, Q_{(M)}^L, \hat{Q}_{(M)}^L$ were given by (3.24), (3.25), (3.31), and (3.32), respectively. In even d, we can transform to the time domain and use (2.17), (3.19), (3.27), and (2.33) explicitly to find

$$\hat{S}_{EM} = \int dt \sum_L \frac{(-)^{\ell+\frac{\hat{d}+1}{2}}}{\hat{d}!!(2\ell+\hat{d})!!} \left[\frac{\ell+\hat{d}}{\ell} \hat{Q}_L^{(E)} \cdot \partial_t^{2\ell+\hat{d}} Q_{(E)}^L + \frac{\ell^2 \hat{d}}{(\ell+1)(\ell+\hat{d}-1)} \hat{Q}_L^{(M)} \cdot \partial_t^{2\ell+\hat{d}} Q_{(M)}^L \right], \tag{3.36}$$

where we have also summed over the $\aleph$ indices, defining the bivector multipoles

$$Q_{(M)}^L = \int d^D x \tilde{j}_\alpha(ir\partial_t)(\vec{r}\wedge\vec{J}(\vec{r}))^{(k_\ell} x^{L-1)}, \tag{3.37}$$

defining $\hat{Q}_{(M)}^L$ accordingly, and incurring a factor of

$$\frac{D_\ell(\hat{d}+1,1)}{D_\ell(\hat{d}+1,0)} = \frac{\ell\hat{d}(\ell+\hat{d})}{(\ell+1)(\ell+\hat{d}-1)} \tag{3.38}$$

from summing over the different $\aleph$ combinations.

The case of the EM field in noneven (and in particular odd) spacetime dimensions is treated in a similar manner to the scalar case (Sec. 2.3), and similar nonlocal tail expressions appear [compare with (2.43)]:

$$\hat{S}_{EM} = \int dt \sum_L \frac{(-)^{\ell+\frac{\hat{d}+1}{2}}}{\hat{d}!!(2\ell+\hat{d})!!}\left[\frac{\ell+\hat{d}}{\ell}S^{(E)}(t) + \frac{\ell^2\hat{d}}{(\ell+1)(\ell+\hat{d}-1)}S^{(M)}(t)\right], \tag{3.39}$$

$$S^{(E/M)}(t) = \hat{Q}_L^{(E/M)}(t)\left[\left(\frac{1}{2}H(2\ell+\hat{d}) - H\left(\ell+\frac{\hat{d}}{2}\right)\right)\partial_t^{2\ell+\hat{d}}Q^L_{(E/M)}(t) - \int_{-\infty}^t dt'\left(\frac{1}{t-t'}\partial_{t'}^{2\ell+\hat{d}}Q^L_{(E/M)}(t')\right)\Big|_{\text{regularized}}\right]. \tag{3.40}$$

B. Applications and tests

1. Perturbative expansion of the RR force and comparison with ALD

For the RR force on a single accelerating electric charge we have the ALD formula [11] in $4d$,

$$F^\mu_{\text{ALD}} \equiv \frac{dp^\mu}{d\tau} = \frac{2}{3}q^2\left(\frac{d^3x^\mu}{d\tau^3} - \frac{d^3x^\nu}{d\tau^3}\frac{dx_\nu}{d\tau}\frac{dx^\mu}{d\tau}\right). \tag{3.41}$$

Our expressions, specialized to $d=4$, can be seen to be identical to those given in paper I (Sec. III B 2), already shown to reproduce the ALD result. We test here for $d=6$, comparing the force with Galt'sov's result [25]:

$$f^\mu_{\text{flat}} = -(\eta^{\mu\nu} + \dot{x}^\mu\dot{x}^\nu)\left(\frac{4}{45}x_\nu^{(5)} - \frac{2}{9}\ddot{x}^2\dddot{x}_\nu\right) + \frac{2}{3}(\dddot{x}\ddot{x})\ddot{x}^\mu. \tag{3.42}$$

Expanded to leading and next-to-leading order, we find

$$F^i_{Galt'sov} = -\left[\frac{4}{45}a^{i'''} + \frac{8}{45}v^2a^{i'''} + \frac{8}{9}(\vec{v}\cdot\vec{a})a^{i''} + \frac{8}{9}(\vec{a}'\cdot\vec{v})a^{i'} + \frac{4}{45}(\vec{a}'''\cdot\vec{v})v^i + \frac{4}{9}(\vec{a}''\cdot\vec{v})a^i + \frac{2}{3}a^2a^{i'} + \frac{2}{3}(\vec{a}'\cdot\vec{a})a^i\right], \tag{3.43}$$

from which we also find an expression for the emitted power:

$$\begin{aligned} P_{Galt'sov} &= -\vec{v}\cdot\vec{F}_{Galt'sov} \\ &= \frac{4}{45}(\vec{v}\cdot\vec{a}''') + \frac{4}{15}\vec{v}^2(\vec{v}\cdot\vec{a}''') + \frac{4}{3}(\vec{v}\cdot\vec{a})(\vec{v}\cdot\vec{a}'') + \frac{8}{9}(\vec{v}\cdot\vec{a}')^2 + \frac{2}{3}\vec{a}^2(\vec{a}'\cdot\vec{v}) + \frac{2}{3}(\vec{a}'\cdot\vec{a})(\vec{v}\cdot\vec{a}). \end{aligned} \tag{3.44}$$

For our RR force calculation on a point charge q along a path $\vec{x}(t)$, we rewrite the action

$$\hat{S}_{EM} = \int dt\hat{L}_{EM}, \qquad \hat{L}_{EM} = \hat{L}^S_{EM} + \hat{L}^V_{EM}, \tag{3.45}$$

as a PN series expansion. With (3.24), (3.31), (3.35), and (B8) we find for every (even) dimension

$$\begin{aligned} \hat{L}^S_{EM} = q^2\sum_L \frac{(-)^{\ell+\frac{\hat{d}+1}{2}}(2\ell+\hat{d})!!}{\ell(\ell+\hat{d})\hat{d}!!}\cdot\sum_{\hat{p}=0}^\infty \frac{\partial_t^{2\hat{p}}}{(2\hat{p})!!(2\ell+2\hat{p}+\hat{d})!!}\frac{\delta}{\delta x^i}[(2\hat{p}+\ell+\hat{d})r^{2\hat{p}}x_L - \partial_t(r^{2\hat{p}}x_L\vec{v}\cdot\vec{r})]\hat{x}^i \\ \cdot\,\partial_t^{2\ell+\hat{d}}\sum_{p=0}^\infty \frac{\partial_t^{2p}}{(2p)!!(2\ell+2p+\hat{d})!!}[(2p+\ell+\hat{d})r^{2p}x^L - \partial_t(r^{2p}x^L\vec{v}\cdot\vec{r})]^{\text{STF}}, \end{aligned} \tag{3.46}$$

for the scalar part and

$$\begin{aligned} \hat{L}^V_{EM} = q^2\sum_L \frac{(-)^{\ell+\frac{\hat{d}+1}{2}}\ell^2(2\ell+\hat{d})!!}{(\hat{d}-1)!!(\ell+1)(\ell+\hat{d}-1)}\sum_{\hat{p}=0}^\infty \frac{\partial_t^{2\hat{p}}}{(2\hat{p})!!(2\ell+2\hat{p}+\hat{d})!!}\hat{x}^i\frac{\delta}{\delta x^i}[r^{2\hat{p}}(\vec{r}\wedge\vec{v})^{(k_\ell}x^{L-1)}] \\ \cdot\,\partial_t^{2\ell+\hat{d}}\sum_{p=0}^\infty \frac{\partial_t^{2p}}{(2p)!!(2\ell+2p+\hat{d})!!}[r^{2p}(\vec{r}\wedge\vec{v})^{(k_\ell}x^{L-1)}]^{\text{STF}}, \end{aligned} \tag{3.47}$$

for the vector part.

TABLE III. Leading-order contribution to the $6d$ EM self-force (only electric $\epsilon = +$).

ℓ p $\hat{p}$	Src	$\hat{L}/q^2$	F^j/q^2
1 0 0	ρ $\hat{\rho}$	$-\frac{4}{45}\hat{x}^i\partial_t^5 x_i$	$-\frac{4}{45}\dddot{a}^j$

Similarly to the scalar RR computation, we integrate by parts to move the $2p$ (or $2p+1$) time derivatives from the $\hat{x}^L$ multipoles to the x^L multipoles, and use the EOM (2.49) found from variation with respect to $\hat{x}^j$. We thus find the leading RR force, arising from the electric dipole term ($\ell = 1$, $p = \hat{p} = 0$, sources ρ, $\hat{\rho}$) at every (even) dimension d to be

$$\vec{F}^{(d)}_{LO} = q^2 \frac{(-)^{\frac{d}{2}}(d-2)}{(d-1)!!(d-3)!!}\partial_t^{d-1}\vec{x}. \tag{3.48}$$

The term for $d = 4$ of course matches ALD; the term for $d = 6$ is recorded in Table III and matches that expected from Galt'sov (3.43). In fact, in every even dimension this expression matches exactly the leading-order PN ($\frac{v}{c} \ll 1$) which can be derived from Kazinski, Lyakhovich and Sharapov [[16], Eqs. (26) and (30)], found in a very different method [17,18]. We find the exact match to significantly support the validity of both our method and Kazinski, Lyakhovich and Sharapov's.

The next-to-leading order includes five contributions to the scalar sector, summarized for $6d$ in Table IV, as well as the leading vector contribution (Table V). Their sum is identical with Galt'sov's result (3.43)[4]. We have also tested these expressions using MATHEMATICA code [48] for dimensions (8, 10, 12, 14, 16), finding in every case nontrivial cancellations of all nonphysical terms (the terms involving explicit position coordinates, breaking translation invariance).

We remark also that at $d = 2$ the only STF tensor is x ($\ell = 1$), and we reproduce the expected null result for the action, radiation and self-force identically to any order.

2. Dissipated power

Similarly to (2.54), we compute the power of the RR force on the accelerating charge, now using (3.35):

$$\begin{aligned} P_{RR} &= -\vec{v}\cdot\vec{F} = -\frac{dx^i}{dt}\frac{\delta\hat{L}}{\delta x^i}\bigg|_{\hat{\vec{x}}\to\vec{x}} \\ &= \sum_L \frac{(-)^{\ell+\frac{d}{2}}}{(2\ell+\hat{d})!!\hat{d}!!}\left[\frac{\ell+\hat{d}}{\ell}\frac{dx^i}{dt}\frac{\delta Q^L_{(E)}}{dx^i}\cdot\partial_t^{2\ell+\hat{d}}Q^{(E)}_L\right. \\ &\quad \left.+\frac{\ell^2\hat{d}}{(\ell+1)(\ell+\hat{d}-1)}\frac{dx^i}{dt}\frac{\delta Q^L_{(M)}}{dx^i}\cdot\partial_t^{2\ell+\hat{d}}Q^{(M)}_L\right]\bigg|_{\hat{\vec{x}}\to\vec{x}}. \end{aligned} \tag{3.49}$$

[4]Kosyakov [21] gives a result for the power at $d = 6$ that seems to agree at LO but disagree at NLO with our and with Galt'sov's result.

The time-averaged power is found using

$$\begin{aligned} \int dt\frac{dx^i}{dt}\frac{\partial Q^L_{(E)}}{\partial x^i} &= \int dt\frac{dQ^L_{(E)}}{dt}, \\ \int dt\frac{dx^i}{dt}\frac{\delta Q^L_{(M)}}{\delta x^i} &= \int dt\frac{dQ^L_{(M)}}{dt}, \end{aligned} \tag{3.50}$$

followed by $\ell + \frac{\hat{d}-1}{2}$ integrations by parts (and recalling $\hat{d}$ is odd), to be

$$\begin{aligned} \langle P_{RR}\rangle &= \sum_L \frac{1}{(2\ell+\hat{d})!!\hat{d}!!}\left\langle R_1^+(\partial_t^{\ell+\frac{\hat{d}+1}{2}}Q^L_{(E)})^2\right. \\ &\quad \left.+R_1^-\frac{D_\ell(\hat{d}+1,1)}{D_\ell(\hat{d}+1,0)}(\partial_t^{\ell+\frac{\hat{d}+1}{2}}Q^L_{(M)})^2\right\rangle \\ &= \sum_L \frac{(\ell+\hat{d})}{\ell(2\ell+\hat{d})!!\hat{d}!!}\langle(\partial_t^{\ell+\frac{\hat{d}+1}{2}}Q^L_{(E)})^2\rangle \\ &\quad +\sum_L \frac{\ell^2}{(\ell+1)(\ell+\hat{d}-1)(2\ell+\hat{d})!!(\hat{d}-1)!!} \\ &\quad \times\langle(\partial_t^{\ell+\frac{\hat{d}+1}{2}}Q^L_{(M)})^2\rangle \\ &= P_{\mathrm{rad}}. \end{aligned} \tag{3.51}$$

In the $4d$ case, we recognize this result as Ross' Eq. (52) [35] (with a 4π normalization factor, and reintroducing the $\frac{1}{\ell!}$ factor for comparison, see Appendix (B1)).

IV. SUMMARY OF RESULTS

In this section we summarize the essential definitions and main results obtained in the paper. We use Bessel-like functions defined (B8) as

$$\tilde{j}_\alpha \coloneqq \Gamma(\alpha+1)2^\alpha\frac{J_\alpha(x)}{x^\alpha} = 1+\cdots, \tag{4.1}$$

where $J_\alpha(x)$ is the Bessel function of the first kind. $x_L^{TF} = [x^{k_1}\ldots x^{k_\ell}]^{TF}$ are trace-free tensor products of spatial position vectors (2.6), using the summation conventions of (B1). We write the RR effective action $\hat{S}$ in terms of the system's multipoles Q^L and their doubled counterparts $\hat{Q} = \frac{\delta Q}{\delta\rho}\hat{\rho}$ (1.2) and find the EOM by variation of $\hat{S}$ with respect to hatted fields (1.1).

For a massless scalar field coupled to sources

$$S_\Phi = \frac{1}{2\Omega_{\hat{d}+1}G}\int(\partial_\mu\Phi)^2 d^dx - \int\rho\Phi d^dx, \tag{4.2}$$

we find the radiative multipoles to be

TABLE IV. Next-to-leading-order contribution to the $6d$ EM self-force, scalar (electric $\epsilon = +$) sector.

ℓ p $\hat{p}$	Src	$\hat{L}/q^2$	F^j/q^2
2 0 0	ρ $\hat{\rho}$	$\frac{1}{252}\hat{x}^j \frac{\delta}{\delta x^j}[x_i x_k]\partial_t^7[x^i x^k - \frac{1}{5}x^2\delta^{ik}]$	$\frac{1}{126}[x_i\partial_t^7(x^i x^j) - \frac{1}{5}x^j\partial_t^7 x^2]$
1 1 0	ρ $\hat{\rho}$	$-\frac{1}{105}\hat{x}^j \frac{\delta}{\delta x^j}[x^2 x^i]\partial_t^7 x_i$	$-\frac{1}{105}[x^2\partial_t^7 x^j + 2x^j x_i \partial_t^7 x^i]$
1 0 1	ρ $\hat{\rho}$	$-\frac{1}{105}\hat{x}_i\partial_t^7(x^i x^2)$	$-\frac{1}{105}\partial_t^7(x^2 x^j)$
1 0 0	j_r $\hat{\rho}$	$\frac{1}{45}\hat{x}^i\partial_t^6[v_k x^k x_i]$	$\frac{1}{45}\partial_t^6(x^j x_i v^i)$
1 0 0	ρ $\hat{j}_r$	$-\frac{1}{45}\hat{x}^j \frac{\delta}{\delta x^j}[v_k x^k x^i]\partial_t^6 x_i$	$-\frac{1}{45}[x_i v^i \partial_t^6 x^j + v^j x_i \partial_t^6 x^i - \frac{d}{dt}(x^j x^i \partial_t^6 x_i)]$

TABLE V. Next-to-leading-order contribution to the $6d$ EM self-force, from vector (magnetic $\epsilon = -$) sector.

ℓ p $\hat{p}$	$\hat{L}/q^2$	F^j/q^2
1 0 0	$-\frac{1}{90}\hat{x}^i \frac{\delta}{\delta x^i}[r_j v_k]\partial_t^5(r^j v^k - r^k v^j)$	$-\frac{1}{90}(2v_i\partial_t^5[x^j v^i - x^i v^j] + x_i\partial_t^5[x^j a^i - x^i a^j])$

$$Q_L = \int d^D x \tilde{j}_{\ell+\frac{\hat{d}}{2}}(ir\partial_t) x_L^{TF}\rho(\vec{r},t). \tag{4.3}$$

In even-dimensional spacetime, outgoing radiation is given by

$$\Phi(\vec{r},t) = \frac{G}{\hat{d}!!} r^{-\frac{\hat{d}+1}{2}} \sum_L n^L \partial_t^{\ell+\frac{\hat{d}-1}{2}} Q_L(t-r), \tag{4.4}$$

and the power dissipated through it is

$$\begin{aligned}\dot{E} &= \sum_L \frac{G}{\hat{d}!!(2\ell+\hat{d})!!}\langle(\partial_t^{\ell+\frac{\hat{d}+1}{2}} Q^L)^2\rangle \\ &= \sum_L \frac{G}{\ell!\hat{d}!!(2\ell+\hat{d})!!}\langle(\partial_t^{\ell+\frac{\hat{d}+1}{2}} Q^{\mathrm{STF}}_{k_1 k_2 \ldots k_\ell})^2\rangle. \end{aligned}\tag{4.5}$$

Elimination of system and radiation zone fields gives the RR effective action

$$\hat{S}_\Phi = G\int dt \sum_L \frac{(-)^{\ell+\frac{\hat{d}+1}{2}}}{\hat{d}!!(2\ell+\hat{d})!!}\hat{Q}^L \partial_t^{2\ell+\hat{d}} Q_L. \tag{4.6}$$

In odd-dimensional spacetime, frequency domain results are similar to those of the even-dimensional case. In the time domain, outgoing radiation is given by

$$\Phi(\vec{r},t) = \frac{-G}{\sqrt{2\pi}\hat{d}!!} r^{-\frac{\hat{d}+1}{2}} \sum_L n^L \partial_t^{\ell+\frac{\hat{d}}{2}} \int_{-\infty}^{t-r} \frac{Q_L(t')}{|t-r-t'|^{1/2}} dt', \tag{4.7}$$

and the RR effective action by

$$\hat{S}_\Phi = G\int_{-\infty}^{\infty} dt \sum_L \frac{(-)^{\ell+\frac{\hat{d}}{2}}}{\hat{d}!!(2\ell+\hat{d})!!}\hat{Q}^L(t)\left[\left(\frac{1}{2}H(2\ell+\hat{d}) - H\left(\ell+\frac{\hat{d}}{2}\right)\right)\partial_t^{2\ell+\hat{d}} Q_L(t) - \int_{-\infty}^{t} dt' \left(\frac{1}{t-t'}\partial_{t'}^{2\ell+\hat{d}} Q_L(t')\right)\Big|_{\mathrm{regularized}}\right], \tag{4.8}$$

where the regularization is discussed in Sec. 2.3. The coefficient of the local term in (4.8) above is not universal but depends on the short-distance details of the system.

For an EM field coupled to sources

$$S = -\frac{1}{4\Omega_{\hat{d}+1}}\int F_{\mu\nu}F^{\mu\nu} r^{\hat{d}+1} d^d x - \int A_\mu J^\mu r^{\hat{d}+1} d^d x, \tag{4.9}$$

we find the following electric and magnetic multipoles:

$$\begin{aligned} Q^L_{(E)} &= \frac{1}{\ell+\hat{d}}\int d^D x x^L_{TF}\left[\frac{1}{r^{\ell+\hat{d}-1}}(r^{\ell+\hat{d}}\tilde{j}_\alpha(ir\partial_t))'\rho(\vec{x}) - \tilde{j}_\alpha(ir\partial_t)\partial_t \vec{J}(\vec{x})\cdot\vec{x}\right], \\ Q^{L\aleph}_{(M)} &= \int d^D x \tilde{j}_{\ell+\frac{\hat{d}-1}{2}}(ir\partial_t)[\epsilon^{k_\ell \aleph}_{a\,b} r^a J^b x^{L-1}]^{\mathrm{STF}}, \end{aligned}\tag{4.10}$$

where $\aleph$ is an antisymmetric multi-index (3.3). Radiation in even spacetime dimensions is given by

$$A_\epsilon(\vec{r},t) = \frac{1}{\hat{d}!!} r^{-\frac{\hat{d}+1}{2}} \sum_L R_1^\epsilon n^L \partial_t^{\ell+\frac{\hat{d}-1}{2}} Q_L^\epsilon(t-r), \tag{4.11}$$

where ϵ is $(+)$ for the electric sector and $(-,\aleph)$ for the magnetic sectors and

$$R_1^+ = \frac{\ell+\hat{d}}{\ell}; \qquad R_1^- = \frac{\ell}{\ell+\hat{d}}. \tag{4.12}$$

The RR effective action in even spacetime dimensions is

$$\hat{S}_{EM} = \int dt \sum_L \frac{(-)^{\ell+\frac{\hat{d}+1}{2}}}{\hat{d}!!(2\ell+\hat{d})!!} \left[\frac{\ell+\hat{d}}{\ell} \hat{Q}_L^{(E)} \cdot \partial_t^{2\ell+\hat{d}} Q^L_{(E)} + \frac{\ell^2 \hat{d}}{(\ell+1)(\ell+\hat{d}-1)} \hat{Q}_L^{(M)} \cdot \partial_t^{2\ell+\hat{d}} Q^L_{(M)} \right]. \tag{4.13}$$

The case of the EM field in noneven (and in particular odd) spacetime dimensions is treated in a similar manner to the scalar case, and similar nonlocal tail expressions appear:

$$\hat{S}_{EM} = \int dt \sum_L \frac{(-)^{\ell+\frac{\hat{d}+1}{2}}}{\hat{d}!!(2\ell+\hat{d})!!} \left[\frac{\ell+\hat{d}}{\ell} S^{(E)}(t) + \frac{\ell^2 \hat{d}}{(\ell+1)(\ell+\hat{d}-1)} S^{(M)}(t) \right], \tag{4.14}$$

$$S^{(E/M)}(t) = \hat{Q}_L^{(E/M)}(t) \left[\left(\frac{1}{2} H(2\ell+\hat{d}) - H\left(\ell + \frac{\hat{d}}{2}\right) \right) \partial_t^{2\ell+\hat{d}} Q^L_{(E/M)}(t) - \int_{-\infty}^t dt' \left(\frac{1}{t-t'} \partial_{t'}^{2\ell+\hat{d}} Q^L_{(E/M)}(t') \right) \Big|_{\text{regularized}} \right]. \tag{4.15}$$

V. DISCUSSION

In this paper we formulated an EFT describing radiative effects in scalar and EM theories in general spacetime dimensions and applied it to solve for the radiation and radiation reaction effective action in these cases, thereby generalizing the $4d$ treatment of paper I. We found that the method devised there naturally generalizes to higher dimensions, providing new results even in these linear, well studied theories (see Sec. 4) and laying a solid foundation for the study of such effects in higher dimensional GR.

Some dimension-dependent issues that need to be handled with care appeared. One of them is the tail effect in odd spacetime dimensions (and, formally, in all noneven d), which is due to indirect propagation—propagation not restricted to the light cone. We found that while frequency domain analyses of any spacetime dimensions are similar, time-domain results are substantially different. From our analysis' point of view, the difference is all due to different analytic properties of the fields and effective actions in the complex frequency plane. It appears only when transforming the results into the time domain. In particular, we find that in odd d the RR effective action is composed of a nonlocal part, which contains all the dissipative effects, and a local conservative part, while in even d the RR effective action contains only a local part which is purely dissipative.

We remark that there has been debate [14–19,21,22] over the very possibility of defining and regularizing self-force and radiation reaction even in general even dimensions. We hope our independent method and results for scalar and EM fields in any dimension help shed new light on the matter.

Another important issue appeared when treating fields with nonzero spin, namely the issue of having multiple fields, and the associated question of gauges. Here these arose in the case of the EM field of spin 1. One of the main ideas of our method is the use of gauge-invariant spherical fields and the reduction of the problem to $1d$. This was done with a vector spherical harmonic decomposition. In $4d$, the electric field (which behaves like a scalar on the sphere) and magnetic field (which behaves like a vector on the sphere) are very similar—one is a vector field and the other an axial vector field. However, the magnetic field is more generally a two-form field. This is more visible when working in $d > 4$, where for example one obtains essentially different multipoles for these two sectors, which live in different representations of the rotation group. Going to higher spins in higher dimensions, more sectors appear—a rank-2 tensor sector in GR, etc.

The *main* complication that arises in the gravitational case is the theory's nonlinearity. Although for the leading order the linearized theory suffices, just as in $4d$, going to higher order one would need to include nonlinear interactions—a +1PN correction from first system zone nonlinearities, +1.5PN from first radiation zone nonlinearities, and so on. Also, there is a PN order hierarchy between the different sectors: only the scalar sector is needed for leading order, for next-to-leading (+1PN) order the vector sector must also be accounted for, and at next-to-next-to-leading (+2PN) order the tensor sector also begins to contribute. We treat the general d gravitational two-body problem and its nonlinearities separately in [49].

It would be interesting to use and enhance our method in the study of higher order radiation zone effects (first in $4d$). For example, one can replace the Bessel functions in our propagators with Regge-Wheeler or Zerilli functions (solutions for gravitational perturbations around nonrotating black holes) and compare results for an observable—e.g., emitted radiation—to those obtained by the usual EFT sum of diagrams for scattering of waves off the total mass of the system. This could both improve the EFT and give insight on the problem from a new angle.

ACKNOWLEDGMENTS

We thank the organizers of the 16th CAPRA meeting hosted in Dublin in July 2013 during which this work was initiated. We thank Chad Galley, Tomer Shacham and Barry Wardell for helpful discussions. We especially thank Barak Kol for many very useful discussions and encouragement and for comments on the manuscript. This research was supported by the Israel Science Foundation Grant No. 812/11 and was part of the Einstein Research Project "Gravitation and High Energy Physics," which is funded by the Einstein Foundation Berlin. O. B. was partly supported by an ERC advanced grant to T. Piran.

APPENDIX A: SOLUTION WITHOUT ZONE SEPARATION

1. Scalar field

Looking back at the field equation for a scalar field Φ (2.2) using the spherical decomposition (2.4), (2.5), (2.10), (2.11), and (2.13), we see that it can be solved using the propagator (2.16), yielding ($r > r'$)

$$\begin{aligned}\Phi_\omega(\vec{r}) &= \sum_L x^L \Phi_{Lw}(r) = \sum_L x^L \int dr' G^{\Phi}_{\rm ret}(r,r')\rho^{\Phi}_{L\omega}(r') \\ &= -iG\sum_L \omega^{2\ell+\hat{d}} M_{\ell,\hat{d}} x^L \tilde{h}^+_\alpha(\omega r) \\ &\quad\times \int d^D x' \tilde{j}_\alpha(\omega r') x'_L \rho_\omega(\vec{r}'). \end{aligned} \tag{A1}$$

Restricting to even integer spacetimes d (odd $\hat{d}$), the self-force arises only from the time-asymmetric part of the propagator. Since $\tilde{y}_\alpha$ contains only odd powers of ω and $\tilde{j}_\alpha$ only even powers, by expanding the spherical Bessel functions as series we find the time-asymmetric propagator is given by

$$\begin{aligned}G^{\rm odd}_\omega(\vec{r},\vec{r}') &= -iG\sum_L \omega^{2\ell+\hat{d}} M_{\ell,\hat{d}} x^L \tilde{j}_\alpha(\omega r)\tilde{j}_\alpha(\omega r') x'_L \\ &= G\sum_L \frac{(-)^{\ell+\frac{\hat{d}-1}{2}}(2\ell+\hat{d})!!}{\hat{d}!!}\sum_{p=0}^{\infty}\sum_{\hat{p}=0}^{\infty}\frac{(-i\omega)^{2\ell+2p+2\hat{p}+\hat{d}} r^{2\hat{p}} x^L r'^{2p} x'_L}{(2\hat{p})!!(2\ell+2\hat{p}+\hat{d})!!(2p)!!(2\ell+2p+\hat{d})!!},\end{aligned} \tag{A2}$$

and

$$\Phi_\omega(\vec{r}) = \int d^D x' G^{\rm odd}_\omega(\vec{r},\vec{r}')\rho_\omega(\vec{r}'). \tag{A3}$$

Here we find it more direct to use the $\{r, r'\}$ basis rather than the Keldysh basis. The Lagrangian for the self-interactions of a scalar-charged point particle q described by (2.46) is given by

$$\begin{aligned}L_\omega &= \int d^D x \hat{\rho}_w(\vec{r})\Phi_w(\vec{r}) = \int d^D x d^D x' \hat{\rho}_w(\vec{r}) G^{\rm odd}_w(\vec{r},\vec{r}')\rho_w(\vec{r}) \\ &= Gq^2\sum_L \frac{(-)^{\ell+\frac{\hat{d}-1}{2}}(2\ell+\hat{d})!!}{\hat{d}!!}\sum_{p=0}^{\infty}\sum_{\hat{p}=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\hat{s}=0}^{\infty} C^{p\hat{p}s\hat{s}}_{\ell,\hat{d}}(-i\omega)^{2\ell+2p+2\hat{p}+\hat{d}}(r^{2p}v^{2s}x^L)(r'^{2\hat{p}}v'^{2\hat{s}}x'_L),\end{aligned} \tag{A4}$$

$$C^{p\hat{p}s\hat{s}}_{\ell,\hat{d}} = \frac{(2s-3)!!(2\hat{s}-3)!!}{(2\hat{p})!!(2\ell+2\hat{p}+\hat{d})!!(2p)!!(2\ell+2p+\hat{d})!!(2\hat{s})!!(2s)!!}, \tag{A5}$$

similarly to the one given by (2.48), and where an $\ell!^{-1}$ is implied by the summation convention. The self-force on the particle is found using the E-L equation by first varying according to the difference in trajectories $r - r'$ (equivalent to the hatted trajectory of the Keldysh basis) and then setting $r' \to r$; we see immediately that the force found in (2.49) is recovered exactly.

2. Electromagnetic self-force in the Lorentz gauge

We examine the EM action for the field A^μ in d spacetime dimensions (3.1), with a source d current:

$$j^\mu = (\rho, \vec{j}), \tag{A6}$$

TABLE VI. Leading-order contributions from A^0 and A^k.

ℓ	p	$\hat{p}$	$A^0/q = \phi/q$	F^j/q^2
1	0	0	$(-)^{\frac{d}{2}} \frac{1}{D!!\hat{d}!!} x_i (x_p^i)^{(D)}$	$(-)^{\frac{d}{2}+1} \frac{1}{D!!\hat{d}!!} (x_p^j)^{(D)}$
			A^k/q	F^j/q^2
0	0	0	$(-)^{\frac{d}{2}-1} \frac{1}{\hat{d}!!^2} v_p^{k\,(\hat{d})}$	$(-)^{\frac{d}{2}} \frac{1}{\hat{d}!!^2} (x_p^j)^{(D)}$

TABLE VII. Next-to-leading-order contributions from A^0.

ℓ	p	$\hat{p}$	$A^0/q = \phi/q$	F^j/q^2
0	1	1	$\frac{(-)^{\frac{d}{2}-1}}{4D!!^2} x^2 (x_p^2)^{(d+1)}$	$\frac{(-)^{\frac{d}{2}}}{2D!!^2} x_p^j (x_p^2)^{(d+1)}$
1	0	1	$\frac{(-)^{\frac{d}{2}}}{2\hat{d}!!(d+1)!!} x^2 x_i (x_p^i)^{(d+1)}$	$\frac{(-)^{\frac{d}{2}-1}}{2\hat{d}!!(d+1)!!} [2x_p^j x_{p\,i} (x_p^i)^{(d+1)} + x_p^2 (x_p^j)^{(d+1)}]$
1	1	0	$\frac{(-)^{\frac{d}{2}}}{2\hat{d}!!(d+1)!!} x_i (x_p^2 x_p^i)^{(d+1)}$	$\frac{(-)^{\frac{d}{2}-1}}{2\hat{d}!!(d+1)!!} (x_p^j x_p^2)^{(d+1)}$
2	0	0	$\frac{(-)^{\frac{d}{2}-1}}{2\hat{d}!!(d+1)!!} [x_i x_k (x_p^i x_p^k)^{(d+1)} - \frac{1}{D} x^2 (x_p^2)^{(d+1)}]$	$\frac{(-)^{\frac{d}{2}}}{\hat{d}!!(d+1)!!} [x_{p\,i} (x_p^i x_p^j)^{(d+1)} - \frac{1}{D} x_p^j (x_p^2)^{(d+1)}]$

TABLE VIII. Next-to-leading-order contributions from A^k.

ℓ	p	$\hat{p}$	A^k/q	F^j/q^2
0	0	1	$\frac{(-)^{\frac{d}{2}-1}}{2D!!\hat{d}!!} x^2 (v_p^k)^{(D)}$	$\frac{(-)^{\frac{d}{2}-1}}{2D!!\hat{d}!!} [2x_p^j v_{p\,k} v_p^{k\,(D)} - x_p^2 v_p^{j\,(d)} - 2(\vec{r}_p \cdot \vec{v}_p) v_p^{j\,(D)}]$
0	1	0	$\frac{(-)^{\frac{d}{2}-1}}{2D!!\hat{d}!!} (x_p^2 v_p^k)^{(D)}$	$\frac{(-)^{\frac{d}{2}}}{2D!!\hat{d}!!} (x_p^2 v_p^j)^{(d)}$
1	0	0	$\frac{(-)^{\frac{d}{2}}}{D!!\hat{d}!!} x_i (x_p^i v_p^k)^{(D)}$	$\frac{(-)^{\frac{d}{2}}}{D!!\hat{d}!!} [v_{p\,k} (v_p^k x_p^j)^{(D)} - x_{p\,i} (v_p^j x_p^i)^{(d)} - v_{p\,i} (v_p^j x_p^i)^{(D)}]$

$$\rho = q\delta^{(D)}(x - x_p(t)), \tag{A7}$$

$$\vec{j} = q\vec{v}_p \delta^{(D)}(x - x_p(t)). \tag{A8}$$

Under the Lorenz gauge condition, the wave equation (2.2) is replaced for the electromagnetic field $A^\mu = (\phi, \vec{A})$ by

$$\Box A^\mu = \Omega_{\hat{d}+1} j^\mu. \tag{A9}$$

These d independent equations are each of the form (2.2), except for an opposite overall sign (and $G = 1$). We can thus use the same propagator (A2), and we solve the d equations using the same method as in the scalar case: we Fourier transform each equation along with its source, integrate using this propagator $G^{\rm odd}$ in the frequency domain, and transform back to find the d EM components, analogously to (A3):

$$A^\mu_\omega(\vec{r}) = \int d^D x' G^{\rm odd}_\omega(\vec{r}, \vec{r}') j^\mu_w(\vec{x}'). \tag{A10}$$

We note that for the scalar potential term $A^0 = \phi$, this is just Eq. (A3) again, where the only difference from the scalar case is the simpler source term (A7) instead of (2.46), or in other words the absence of the γ term. This means we will find the same contributions to this potential term as we have found for the scalar potential, but without $s > 0$ corrections. For A^k $(k = 1 \ldots D)$, solving (A10) is similar to solving (A3), but with an additional v'^k present in j^k:

$$\phi_\omega(\vec{r}) = q \sum_L \frac{(-)^{\ell + \frac{\hat{d}+1}{2}} (2\ell + \hat{d})!!}{\hat{d}!!} \sum_{p=0}^\infty \sum_{\hat{p}=0}^\infty \frac{(-i\omega)^{2\ell+2p+2\hat{p}+\hat{d}} r^{2\hat{p}} x_L r_p^{2p} x_p^L}{(2\hat{p})!!(2\ell + 2\hat{p} + \hat{d})!!(2p)!!(2\ell + 2p + \hat{d})!!}, \tag{A11}$$

$$A^k_\omega(\vec{r}) = q \sum_L \frac{(-)^{\ell + \frac{\hat{d}+1}{2}} (2\ell + \hat{d})!!}{\hat{d}!!} \sum_{p=0}^\infty \sum_{\hat{p}=0}^\infty \frac{(-i\omega)^{2\ell+2p+2\hat{p}+\hat{d}} r^{2\hat{p}} x_L r_p^{2p} x_p^L v_p^k}{(2\hat{p})!!(2\ell + 2\hat{p} + \hat{d})!!(2p)!!(2\ell + 2p + \hat{d})!!}, \tag{A12}$$

and an $\ell!^{-1}$ is implied by the summation convention.

We again solve order by order. The leading and next-to-leading orders are recorded in Tables VI–VIII.

We derive the EM force from the action for the particle trajectory in the EM field:

$$S = S_{\rm worldline} + S_{EM} = -m \int d\tau - \int d\tau j_\mu A^\mu. \quad \text{(A13)}$$

When deriving the E-L equation, the first part introduces the $m\ddot{\vec{x}}$ term, while

$$S_{EM} = \int dt q(\vec{A}\cdot\vec{v} - \phi) = \int dt L_{EM} \quad \text{(A14)}$$

gives the EM force. We thus find

$$F^j_{EM} = q\left(v_k \frac{\partial A^k}{\partial x_j} - \dot{A}^j - \frac{\partial \phi}{\partial x_j}\right) = F^j_A + F^j_\phi. \quad \text{(A15)}$$

The leading and next-to-leading contributions to the self-force are recorded in Tables VI–VIII. Summed together, we find the two leading orders of the total self-force on an EM charge q in d spacetime dimensions to be

$$\vec{F}_{LO} = (-)^{\frac{d}{2}} \frac{D-1}{(D-2)!!D!!} q^2 \vec{x}_p^{(D)}, \quad \text{(A16)}$$

$$\vec{F}^{d=4}_{\rm NLO} = +q^2\left[\frac{2}{3} v_p^2 \dot{\vec{a}}_p + 2(\vec{v}_p\cdot\vec{a}_p)\vec{a}_p + \frac{2}{3}(\vec{v}_p\cdot\dot{\vec{a}}_p)\vec{v}_p\right], \quad \text{(A17)}$$

$$\vec{F}^{d=6}_{\rm NLO} = -q^2\left[\frac{2}{3}(\vec{a}^2)\dot{\vec{a}} + \frac{2}{3}(\vec{a}\cdot\dot{\vec{a}})\vec{a} + \frac{8}{9}(\vec{v}\cdot\vec{a})\ddot{\vec{a}} + \frac{8}{9}(\vec{v}\cdot\dot{\vec{a}})\dot{\vec{a}} + \frac{4}{9}(\vec{v}\cdot\ddot{\vec{a}})\vec{a} + \frac{8}{45}(\vec{v}^2)\dddot{\vec{a}} + \frac{4}{45}(\vec{v}\cdot\dddot{\vec{a}})\vec{v}\right]. \quad \text{(A18)}$$

We see that the leading order matches ALD for $d = 4$ and Galt'sov for $d = 6$ (the numerical factors are $+\frac{2}{3}$ and $-\frac{4}{45}$, correspondingly) and that the next-to-leading order also exactly matches the expected ALD result for $d = 4$ and Galt'sov's result for $d = 6$ (3.43) [25]. We also record the emitted power for $d = 6$ (LO and NLO):

$$\begin{aligned} P^{d=6} &= -\vec{v}\cdot(\vec{F}^{d=6}_{\rm LO} + \vec{F}^{d=6}_{\rm NLO}) \\ &= q^2\left[\frac{4}{45}(\vec{v}\cdot\dddot{\vec{a}}) + \frac{2}{3}\vec{a}^2(\vec{v}\cdot\dot{\vec{a}}) + \frac{2}{3}(\vec{v}\cdot\vec{a})(\vec{a}\cdot\dot{\vec{a}}) \right. \\ &\quad \left. + \frac{4}{3}(\vec{v}\cdot\vec{a})(\vec{v}\cdot\ddot{\vec{a}}) + \frac{8}{9}(\vec{v}\cdot\dot{\vec{a}})^2 + \frac{4}{15}\vec{v}^2(\vec{v}\cdot\dddot{\vec{a}})\right]. \end{aligned} \quad \text{(A19)}$$

The code for computing the self-force for any even dimension is available on our Web server [48].

APPENDIX B: USEFUL DEFINITIONS AND CONVENTIONS

In this Appendix we collect several definitions and conventions used in this paper.

1. Multi-index summation convention

Multi-indices are denoted by uppercase Latin letters:

$$I \equiv I_\ell := (i_1 \ldots i_\ell). \quad \text{(B1)}$$

Here each $i_k = 1, \ldots, D$ is an ordinary spatial index, and ℓ is the total number of indices. We define a slightly modified summation convention for multi-indices by

$$P_I Q_J := \sum_l P_{I_\ell} Q_{J_\ell} := \sum_\ell \frac{1}{\ell!} P_{i_1\ldots i_\ell} Q_{j_1\ldots j_\ell}, \quad \text{(B2)}$$

so repeated multi-indices are summed over as in the standard summation convention, but an additional division by $\ell!$ is implied. When ℓ is unspecified the summation is over all ℓ.

In addition a multi-index deltalike function is defined through

$$\delta_{I_\ell J_\ell} := \ell! \delta_{i_1 j_1} \ldots \delta_{i_\ell j_\ell}. \quad \text{(B3)}$$

These definitions are such that factors of $\ell!$ are accounted for automatically.

2. Normalizations of Bessel functions

We find it convenient to define an origin-biased normalization of Bessel functions. We start with conventionally normalized solutions of the Bessel equation

$$\left[\partial_x^2 + \frac{1}{x}\partial_x + 1 - \frac{\nu^2}{x^2}\right] B_\nu(x) = 0, \quad \text{(B4)}$$

where $B \equiv \{J, Y, H^\pm\}$, namely B represents Bessel (of the first or second kind) and Hankel functions, and ν denotes their order. Given α, we define

$$\tilde{b}_\alpha := \Gamma(\alpha+1) 2^\alpha \frac{B_\alpha(x)}{x^\alpha}, \quad \text{(B5)}$$

the "origin normalized" Bessel functions $\tilde{b}_\alpha$, which satisfy the equation [compare (2.15)]

$$\left[\partial_x^2 + \frac{2\alpha+1}{x}\partial_x + 1\right] \tilde{b}_\alpha(x) = 0. \quad \text{(B6)}$$

The purpose of this definition is to have a simple behavior of $\tilde{j}_\alpha$ in the vicinity of the origin $x = 0$:

$$\tilde{j}_\alpha(x) = 1 + \mathcal{O}(x^2). \quad \text{(B7)}$$

More precisely, the series expansion for $\tilde{j}_\alpha(x)$ around $x = 0$ is given by the even series

$$\tilde{j}_\alpha(x) = \sum_{p=0}^{\infty} \frac{(-)^p (2\alpha)!!}{(2p)!!(2p+2\alpha)!!} x^{2p}. \tag{B8}$$

The asymptotic form of our solutions for $x \to \infty$ is best stated in terms of Hankel functions $\tilde{h}^\pm \coloneqq \tilde{j} \pm i\tilde{y}$:

$$\tilde{h}^\pm_\alpha(x) \sim (\mp i)^{\alpha+1/2} \frac{2^{\alpha+1/2}\Gamma(\alpha+1)}{\sqrt{\pi}} \frac{e^{\pm ix}}{x^{\alpha+1/2}}. \tag{B9}$$

Around $x = 0$ the Bessel function of the second kind for α noninteger is

$$\tilde{y}_\alpha(x) = \frac{\Gamma(\alpha+1)2^\alpha}{x^\alpha} \frac{\cos(\alpha\pi)J_\alpha(x) - J_{-\alpha}(x)}{\sin(\alpha\pi)} = \frac{\Gamma(\alpha+1)2^\alpha}{\sin(\alpha\pi)} \sum_{p=0}^{\infty} \frac{(-)^p}{(2p)!!} \frac{x^{2p}}{2^p} \left[\frac{\cos(\alpha\pi)}{2^\alpha\Gamma(p+\alpha+1)} - \frac{2^\alpha x^{-2\alpha}}{\Gamma(p-\alpha+1)}\right], \tag{B10}$$

while for integer $\alpha = n$, using [cf. Eq. (9.1.11) of [50]] it is given by

$$\tilde{y}_n(x) = -\frac{2(2n)!!}{\pi} \sum_{m=1}^{n} \frac{(2m-2)!!}{(2n-2m)!!} x^{-2m} + \frac{2}{\pi} \ln\left(\frac{x}{2}\right) \tilde{j}_n(x) - \frac{(2n)!!}{\pi} \sum_{k=0}^{\infty} [\psi(k+1) + \psi(n+k+1)] \frac{(-)^k}{(2k)!!(2n+2k)!!} x^{2k}, \tag{B11}$$

where ψ is the digamma function, defined for integers using the harmonic numbers $H(N)$ and the Euler-Mascheroni constant γ:

$$\psi(N+1) = H(N) - \gamma. \tag{B12}$$

[1] O. Birnholtz, S. Hadar, and B. Kol, Phys. Rev. D **88**, 104037 (2013).
[2] W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D **73**, 104029 (2006); W. D. Goldberger, arXiv:hep-ph/0701129.
[3] W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D **73**, 104030 (2006).
[4] W. D. Goldberger and A. Ross, Phys. Rev. D **81**, 124015 (2010).
[5] C. R. Galley and M. Tiglio, Phys. Rev. D **79**, 124027 (2009).
[6] S. Foffa and R. Sturani, Phys. Rev. D **87**, 044056 (2013).
[7] C. R. Galley and A. K. Leibovich, Phys. Rev. D **86**, 044029 (2012).
[8] C. R. Galley, Phys. Rev. Lett. **110**, 174301 (2013).
[9] V. Cardoso, O. J. C. Dias, and P. Figueras, Phys. Rev. D **78**, 105010 (2008).
[10] M. Abraham, *Theorie der Elektrizität* (Springer, Leipzig, 1905), Vol. II.
[11] P. A. M. Dirac, Proc. R. Soc. A **167**, 148 (1938).
[12] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1998), 3rd ed.
[13] E. Poisson, arXiv:gr-qc/9912045.
[14] B. P. Kosyakov, Teor. Mat. Fiz. **119**, 119 (1999) [Theor. Math. Phys. **119**, 493 (1999)].
[15] D. V. Gal'tsov, Phys. Rev. D **66**, 025016 (2002).
[16] P. O. Kazinski, S. L. Lyakhovich, and A. A. Sharapov, Phys. Rev. D **66**, 025017 (2002).
[17] D. V. Gal'tsov and P. Spirin, Gravitation Cosmol. **12**, 1 (2004).
[18] P. O. Kazinski, S. L. Lyakhovich, and A. A. Sharapov, arXiv:hep-th/0405287.
[19] D. Galakhov, JETP Lett. **87**, 452 (2008).
[20] B. P. Kosyakov, *Introduction to the Classical Theory of Particles and Fields* (Springer, Berlin, 2007).
[21] B. P. Kosyakov, Int. J. Mod. Phys. A **23**, 4695 (2008).
[22] E. Shuryak, H.-U. Yee, and I. Zahed, Phys. Rev. D **85**, 104007 (2012).
[23] A. Mironov and A. Morozov, Theor. Math. Phys. **156**, 1209 (2008).
[24] A. Mironov and A. Morozov, Int. J. Mod. Phys. A **23**, 4677 (2008).
[25] D. V. Gal'tsov and P. A. Spirin, Gravitation Cosmol. **13**, 241 (2007).
[26] D. Gal'tsov, Fundam. Theor. Phys. **162**, 367 (2011).
[27] V. Asnin and B. Kol, Classical Quantum Gravity **24**, 4915 (2007).
[28] J. S. Schwinger, J. Math. Phys. (N.Y.) **2**, 407 (1961); K. T. Mahanthappa, Phys. Rev. **126**, 329 (1962); L. V. Keldysh,

Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].

[29] M. A. Rubin and C. R. Ordonez, J. Math. Phys. (N.Y.) **25**, 2888 (1984); **26**, 65 (1985).

[30] A. Higuchi, J. Math. Phys. (N.Y.) **28**, 1553 (1987); **43**, 6385 (E) (2002).

[31] J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Clarendon, Oxford, 1873), Vol. 1; *The Scientific Papers of James Clerk Maxwell* (Cambridge University Press, Cambridge, England, 1890).

[32] C. Frye and C. J. Efthimiou, arXiv:1205.3548.

[33] H. Kalf, Bull. Belg. Math. Soc. Simon Stevin **2**, 361 (1995).

[34] J. Applequist, J. Phys. A **22**, 4303 (1989).

[35] A. Ross, Phys. Rev. D **85**, 125033 (2012).

[36] L. Blanchet and T. Damour, Ann. I. H. P.: Phys. Theor. **50**, 377 (1989).

[37] T. Damour and B. R. Iyer, Phys. Rev. D **43**, 3259 (1991).

[38] S. L. Detweiler and B. F. Whiting, Phys. Rev. D **67**, 024025 (2003).

[39] N. L. Balazs, Proc. Phys. Soc. London Sect. A **68**, 521 (1955).

[40] E. S. C. Ching, P. T. Leung, W. M. Suen, and K. Young, Phys. Rev. D **52**, 2118 (1995).

[41] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics* (Reidel, Dordrecht, 1984).

[42] D. Hestenes, *New Foundations for Classical Mechanics* (Kluwer Academic, Dordrecht, 1990).

[43] W. E. Baylis, *Electrodynamics, A Modern Geometric Approach* (Birkhäuser, Boston, 1999).

[44] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, England, 2003).

[45] D. Hestenes, Am. J. Phys. **71**, 104 (2003).

[46] D. R. Rowland, Am. J. Physiol. **78**, 187 (2010).

[47] J. M. Chappell, A. Iqbal, and D. Abbott, arXiv:1010.4947.

[48] http://phys.huji.ac.il/~ofek/PNRR/.

[49] O. Birnholtz and S. Hadar (to be published).

[50] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, Mineola, NY, 1972).