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# Generally covariant formulation of relative locality in curved spacetime

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We construct a theory of particles moving in both curved momentum space and spacetime, being a generalization of relative locality. We find that in order to construct such a theory, with desired symmetries, including the general coordinate invariance, we have to use nonlocal position variables. It turns out that free particles move on geodesics and momentum dependent translations of relative locality are replaced with momentum dependent geodesic deviations.

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## I. INTRODUCTION

In recent years the largely forgotten idea that momentum space may have a nontrivial geometric structure, known under the name of Born reciprocity [1], has been revived in many different guises in the context of quantum gravity. It was noticed in [2] that there is a one-to-one correspondence between spacetime noncommutativity, expected to be one of the features of quantum gravity, and nontrivial geometric structures in momentum space. This general observation is supported by explicit calculations done in the context of gravity in $2 + 1$ dimensions [3], [4]. A few years later it was realized that many nontrivial features of the doubly special relativity class of theories [5–7] can be conveniently described in terms of the geometry of de Sitter momentum space [8]. Recently, Born reciprocity has been also explored in the context of string theory [9].

Relative locality [10–12] is a theoretical framework that has its roots in Born reciprocity. In this framework the momentum space is brought to the foreground. It is first observed that most, if not all, physical measurements correspond, in fact, to momentum-space data. Second, it is noticed that the emergence of a nontrivial geometry in momentum space requires, as a prerequisite, the presence of a mass scale. Such a scale must be provided by a fundamental theory, and it was assumed that there exists a regime of quantum gravity, in which the length scale, the Planck length, is negligibly small, while the mass scale, the Planck mass, remains finite.

In the couple of years that passed since relative locality was first proposed the bulk of research investigated systems defined on flat Minkowski spacetime. The question arises, however, if curved momentum space could coexists with a nontrivial geometry of spacetime. This possibility is particularly intriguing from the phenomenological perspectives of doubly special relativity (DSR) and relative locality, since many of the opportunities that have been proposed in recent years rely on tests of (Planck-scale) deformation of kinematics of particles coming from cosmological distances [13,14]. The incredibly small size of such effects ($1/M_p \sim 10^{-19}$ GeV) could be within the reach of present observations thanks to the huge amplification provided by the cosmological distances. In such a context it is clear that the effects of spacetime curvature cannot be ignored. Some preliminary results on the interplay between spacetime expansion and relativity of locality have been presented in paper [15] for the case of a de Sitter-like spacetime expansion.

Recently two of the present authors proposed the action of a particle moving in curved spacetime whose geometry is given by the tetrad $e^a_\mu(x)$, with curved momentum space, provided by the tetrad $E^\alpha_a(p)$ [16]. While the framework presented in [16] reproduces the correct action in both the complementary limits of flat spacetime/curved momentum space and curved spacetime/flat momentum space, the theory described by the action proposed in [16] is not manifestly invariant under general coordinate transformations (at least in their classical form). This, by itself, may not be very problematic, because the scale governing the effects related to this loss of invariance would be, in this theory, presumably of order of the Planck scale, and therefore they may not lead to relevant phenomenological consequences. However, from the theoretical perspective, one may find annoying the fact that the violation of the general coordinate invariance would lead to the presence of a preferred spacetime coordinate system, in which the particle action has the form proposed in [16]. The problem would be then to find out what this coordinate system is: is it indeed the system of Cartesian coordinates in Minkowski space, in the case of flat spacetime, as implicitly assumed in the construction of relative locality? And, more importantly, how do we find such system, and the form of the particle action, in the case of an arbitrary spacetime?

One possibility to avoid this conflict could be to look for a generalized class of coordinate transformations under which the action presented in [16] is still invariant. This would lead inevitably to coordinate transformations mixing

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spacetime and momentum space. Taking into account that the existence of such kinds of transformations is not guaranteed, it may also be conceptually compelling to explore the implications of having a theory of both curved spacetime and momentum space, in which invariance under general coordinate transformation is lost at the Planck scale. While we postpone these analyses to future studies, we feel, however, that the option of a coordinate invariant theory is still the most desirable road to pursue in the search of generalizing relative locality to curved spacetime.

Thus, in this paper we present a novel formulation of the action for particles in both curved spacetime and momentum space, which is manifestly invariant under general coordinate invariance. We also show that in this theory the particles are described to move along worldlines coinciding with the standard spacetime geodesics. In the next section we show how one can construct such an action for free particles. In Sec. III we discuss symmetries of that so-defined theory. In Sec. IV we show how, starting from this action, one can also introduce particle interactions in the spirit of relative locality. The final section is devoted to discussion.

When this work was being completed we learned about an interesting complementary results presented in [17].

## II. CONSTRUCTION OF THE ACTION

The action of a free relativistic particle with curved momentum space has the form reciprocal, in a sense, to the one of the standard free relativistic particle moving in curved spacetime, with flat momentum space. The Lagrangian of the latter reads

$$L = \dot{x}^\mu(\tau) e^a_\mu(x(\tau)) p_a(\tau) - N(\eta^{ab} p_a p_b - m^2), \quad (1)$$

where $x^\mu(\tau)$ is the position of the particle at time $\tau$, $p_a(\tau)$ is the particle momentum, and $e^a_\mu$ is the tetrad, characterizing the geometry of spacetime

$$e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}, \qquad e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab}. \quad (2)$$

Finally, $N$ is the Lagrange multiplier enforcing the mass-shell constraint $p^2 = m^2$. To write the Lagrangian (1) we use two kind of indices: the curved spacetime index $\mu$ and the index $a$ related to the orthonormal coordinate system in the ambient Minkowski space, to which the tetrad $e^a_\mu$ maps. It can be checked by direct calculation that the Euler-Lagrange equations following from (1) reduce, after solving for $p$, to the standard geodesic equation.

The action (1) is the first order form of the better known second order action. The latter can be obtain from (1) by solving the momentum equation of motion and substituting it back to (1); as a result one obtains

$$L \sim g_{\mu\nu}(x(\tau))\dot{x}^\mu(\tau)\dot{x}^\nu(\tau).$$

Before we proceed, let us recall the basic properties of the tetrads. It follows from the defining equation that tetrads are spacetime vectors, transforming under diffeomorphisms as

$$\delta_\xi e^a_\mu(x) = \xi^\nu \partial_\nu e^a_\mu + e^a_\nu \partial_\mu \xi^\nu. \quad (3)$$

Moreover the relations (2) are invariant under infinitesimal local Lorentz transformations

$$\delta_\lambda e^a_\mu(x) = \lambda^a{}_b e^b_\mu, \qquad \lambda^{ab} = -\lambda^{ba}. \quad (4)$$

These two symmetries commute $[\delta_\xi, \delta_\lambda] e^a_\mu = 0$. In addition, assuming vanishing torsion, the tetrad satisfies

$$\partial_{[\mu} e^a_{\nu]} + \omega_{[\mu}{}^a{}_b e^b_{\nu]} = 0, \quad (5)$$

where $\omega$ is a gauge field for local Lorentz symmetry, transforming as

$$\omega'_\mu = \Lambda^{-1}\partial_\mu \Lambda + \Lambda^{-1}\omega_\mu \Lambda, \quad (6)$$

where $\Lambda$ is the matrix of a finite Lorentz transformation $\Lambda = \exp(\lambda^{ab} T_{ab})$, with $T_{ab}$ being the matrix generators of the Lorentz group. Finally, the tetrads satisfy the tetrad postulate, according to which they are covariantly constant

$$\partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu - \Gamma^\alpha_{\nu\mu} e^a_\alpha = 0. \quad (7)$$

It follows from the Born reciprocity idea that the kinetic part of the relativistic particle Lagrangian in the case of curved momentum space should look like (1) with the roles of $x$ and $p$ exchanged. One also has to replace the mass-shell condition with its curved momentum-space counterpart $\mathcal{C}(p) - m^2 = 0$, where $\mathcal{C}(p)$ is the square of the geodesic distance between the point with coordinates $p^\alpha$ and the origin of the momentum space, obtaining as a result [11,19]

$$L^{RL} = \dot{p}_\alpha(\tau) E^\alpha_a(p(\tau)) x^a(\tau) + N(\mathcal{C}(p) - m^2). \quad (8)$$

It should be recalled at this point that although the action (8) looks much more complex than the one of the standard relativistic particle moving in flat space, the equations of motion following from it are remarkably similar: they say that momentum $p_\alpha$ and velocity $\dot{x}^a$ are both constant. What makes the action (8) different from its standard counterpart is the relation between momentum and velocity, which becomes in the case of (8) highly nonlinear. Moreover, when interactions between particles are introduced, the effects of relative locality starts being visible.

Generalizing these considerations, we want now to construct an action for a particle with momentum space and spacetime both possessing nontrivial geometries. Of course, we want the action to reproduce the two limiting cases of flat momentum space/flat spacetime discussed above. We require moreover the new action to be still manifestly invariant under general coordinate transformations. In order to meet these requirements, we introduce nonlocal variables which we denote $X^a$. We will describe their construction in the following subsection.

### A. Nonlocal variables

Let us denote by $\Gamma$ the $C^\infty$ curve $x = x(\tau)$ for $\tau \in [t_1, t_2]$ and consider the subcurves $\Gamma_\tau : x = x(\sigma)$ with $\sigma \in [t_1, \tau]$ which coincide with $\Gamma$ up to $x(\tau)$.

As discussed above, given the background spacetime metric $g_{\mu\nu}$, there exists a whole family of tetrads satisfying Eq. (2), which differ from one another by the action of local Lorentz transformations. We can now gauge fix the local Lorentz transformations in such a way that the Lorentz connection $\omega_{\mu}{}^{a}{}_{b}$ vanishes along a given curve $x = x(\tau)$ and then it follows from (5) that on this curve we can construct a tetrad $\bar{e}^a_\mu$ with $\bar{e}^a_\mu \bar{e}^b_\nu \eta_{ab} = g_{\mu\nu}$ such that

$$(\partial_\nu \bar{e}^a_\mu(x) - \partial_\mu \bar{e}^a_\nu(x))|_{x=x(\tau)} = 0. \tag{9}$$

The existence of the Lorentz connection with these properties can be proved as follows. We first show that the component of $\omega_\mu$ along the worldline $\Gamma$, $\omega_{\tau}{}^{a}{}_{b} \equiv \dot{x}^\mu \omega_{\mu}{}^{a}{}_{b}$ can be gauge fixed to zero. To this end we have to solve the equation [cf. (6)] $0 = \Lambda^{-1}\dot{\Lambda} + \Lambda^{-1}\omega_\tau \Lambda$. But this equation is solved by a time ordered Wilson line (holonomy), $\Lambda = T\exp\left(\int d\tau \omega_\tau\right)$. Having fixed $\omega_\tau = 0$ we are left with the gauge transformations that are constant along $\Gamma$. Let us now consider an arbitrary constant time surface, corresponding to some particular value of the parameter $\tau$ on $\Gamma$.[1] Then in the vicinity of the point in which the worldline crosses the surface we have $\Lambda(\tau, x^i) = \Lambda^{(0)} + \Lambda^{(1)}_i(\tau)x^i + O(x^2)$. This is sufficient freedom to gauge fix to zero the spacial components of Lorentz connection $\Lambda^{(0)-1}\Lambda^{(1)}_i(\tau) = -\Lambda^{(0)-1}\omega_i(\tau)\Lambda^{(0)}$. This can be done for any $\tau$ and therefore all the components of Lorentz connection can be gauge fixed to zero along a curve. After gauge fixing $\omega_i = 0$ on $\Gamma$, we are left with the gauge freedom $\Lambda(\tau, x^i) = \Lambda^{(0)} + O(x^2)$. Therefore, once we gauge fix the connection along one curve, in general, we cannot do the same for another curve in its small neighborhood.

The tetrads $\bar{e}^a_\mu$ (9) are determined modulo a global Lorentz transformation $\Lambda^{(0)}$, which can be used to fix them equal to an arbitrary tetrad $e^a_\mu$ at one point of $\Gamma$,

$$\bar{e}^a_\mu(x(\bar{\tau})) = e^a_\mu(x(\bar{\tau})). \tag{10}$$

Since the construction of the tetrads $\bar{e}^a_\mu$ recalls the one of Fermi coordinates, in what follows we will call them Fermi tetrads.

[1]We assume that the worldline is timelike, but an analogous construction works in the case of null worldlines.

With these prerequisites we are ready to define the nonlocal variable $X^a$ as

$$X^a(\Gamma; x(\tau)) = \int_{\Gamma_\tau} d\sigma \bar{e}^a_\mu(x(\sigma))\frac{dx^\mu}{d\sigma} = \int_0^\tau d\sigma \bar{e}^a_\mu(x(\sigma))\dot{x}^\mu. \tag{11}$$

The variable $X^a(\Gamma; x(\tau))$ depends, in general, on the curve $\Gamma$ along which it is calculated.

In order to calculate the variation of the action below, we will have to evaluate the difference between variables $X^a$ calculated along different curves $\Gamma$ and $\Gamma'$ lying infinitesimally close to each other, with appropriate Fermi tetrads associated with each of them. Thus (see Fig. 1)

$$\delta X^a = \int_{\Gamma'} \bar{e}^a_{\mu\Gamma'}(x + \delta x)(\dot{x}^\mu + \delta\dot{x}^\mu)d\sigma - \int_{\Gamma} \bar{e}^a_{\mu\Gamma}(x)\dot{x}^\mu d\sigma, \tag{12}$$

where $\bar{e}^a_{\mu\Gamma}$ denotes a tetrad field, defined in spacetime, which becomes a Fermi tetrad on the curve $\Gamma$. The variation of the tetrads $\bar{e}^a_\mu$ can be decomposed into two parts:

$$\delta\bar{e}^a_{\mu\Gamma}(x) = \delta_1\bar{e}^a_\mu(x) + \delta_2\bar{e}^a_\mu(x), \tag{13}$$

with

$$\delta_1\bar{e}^a_{\mu\Gamma}(x) = \bar{e}^a_{\delta\mu\Gamma'}(x + \delta x) - \bar{e}^a_{\mu\Gamma}(x + \delta x), \tag{14}$$

and

$$\delta_2\bar{e}^a_{\mu\Gamma}(x) = \bar{e}^a_{\mu\Gamma}(x + \delta x) - \bar{e}^a_{\mu\Gamma}(x) = \delta x^\nu \bar{e}^a_{\mu,\nu\Gamma}. \tag{15}$$

In order to evaluate the expression (14) we need to compare the Fermi tetrads associated with different curves. Since they are both tetrads of the same spacetime metric, there exists a local Lorentz transformation $\Lambda$ relating them

$$\bar{e}^a_{\mu\Gamma'}(x) = \Lambda^a_b(x)\bar{e}^b_{\mu\Gamma}(x), \tag{16}$$

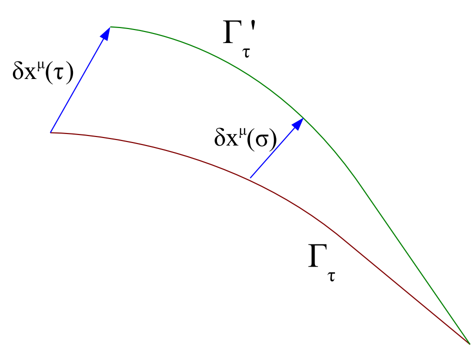


FIG. 1 (color online). The curves $\Gamma : x^\mu(\sigma)$ and $\Gamma' : x^\mu(\sigma) + \xi^\mu(\sigma)$.

and the associated Lorentz connections are related by the local Lorentz gauge transformation

$$\bar{\omega}_{\mu\Gamma'}(x) = \Lambda^{-1}(x)\bar{\omega}_{\mu\Gamma}(x)\Lambda(x) + \Lambda^{-1}(x)\partial_\mu\Lambda(x). \quad (17)$$

We know that the Lorentz connection $\bar{\omega}_{\Gamma'}$ vanishes on $\Gamma'$ and $\bar{\omega}_\Gamma$ vanishes on $\Gamma$; thus, we have

$$\begin{aligned} 0 &= \bar{\omega}_{\mu\Gamma'}(x+\delta x) = \Lambda^{-1}\bar{\omega}_{\mu\Gamma}\Lambda(x+\delta x) + \Lambda^{-1}\partial_\mu\Lambda(x+\delta x) \\ &= \Lambda^{-1}(x)(\delta x^\nu \bar{\omega}_{\mu,\nu\Gamma}(x))\Lambda(x) + \Lambda^{-1}\partial_\mu\Lambda(x+\delta x). \quad (18) \end{aligned}$$

For an infinitesimal local Lorentz gauge transformation $\Lambda(x) \simeq I + \lambda(x)$, keeping the leading order terms we obtain

$$0 = \delta x^\nu \bar{\omega}_{\mu,\nu\Gamma}(x) + \partial_\mu \lambda(x), \quad (19)$$

from which, multiplying by $\dot{x}^\mu$ and adding a term proportional to $\bar{\omega}^{ab}_{\mu\Gamma} = 0$, we get

$$\begin{aligned} \frac{d\lambda^{ab}}{d\sigma} &= -\delta x^\nu \bar{\omega}^{ab}_{\mu,\nu\Gamma}\dot{x}^\mu - \bar{\omega}^{ab}_{\mu\Gamma}\delta\dot{x}^\mu \\ &= -\frac{d}{d\sigma}(\delta x^\mu \bar{\omega}^{ab}_{\mu\Gamma}) - R^{ab}_{\nu\mu}\delta x^\mu \dot{x}^\nu, \quad (20) \end{aligned}$$

where in the last line we used an expression for the curvature tensor that holds on $\Gamma$

$$R^{ab}_{\nu\mu}(\sigma) = \bar{\omega}_{\nu,\mu\Gamma} - \bar{\omega}_{\mu,\nu\Gamma}.$$

Equation (20) is solved by

$$\lambda^{ab} = -\delta x^\mu \bar{\omega}^{ab}_{\mu\Gamma} + \tilde{\lambda}^{ab} = \tilde{\lambda}^{ab},$$

where we used again the fact that $\bar{\omega}^{ab}_{\mu\Gamma}$ vanishes on $\Gamma$ and

$$\tilde{\lambda}^{ab}(\sigma) = \int^\sigma R^{ab}_{\nu\mu}(\sigma')\delta x^\mu(\sigma')\dot{x}^\mu d\sigma'.$$

Hence, the total variation reads

$$\begin{aligned} \delta X^a(\tau) &= \int_\Gamma d\sigma \tilde{\lambda}^a_b \bar{e}^b_\mu \dot{x}^\mu + \int_\Gamma d\sigma(\bar{e}_{\mu,\nu}\delta x^\nu + \bar{e}^a_\nu \delta x^\nu{}_{,\mu})\dot{x}^\mu \\ &= \int_\Gamma d\sigma \tilde{\lambda}^a_b \bar{e}^b_\mu \dot{x}^\mu + \int_\Gamma d\sigma(\bar{e}_{\nu,\mu}\delta x^\nu + \bar{e}^a_\nu \delta x^\nu{}_{,\mu})\dot{x}^\mu \\ &= \int_\Gamma d\sigma \tilde{\lambda}^a_b \bar{e}^b_\mu \dot{x}^\mu + \int_\Gamma d\sigma \frac{d}{d\sigma}(\bar{e}^a_\nu \delta x^\nu) \\ &= \bar{e}^a_\nu(x(\tau))\delta x^\nu(x(\tau)) + \int_\Gamma d\sigma \tilde{\lambda}^a_b \bar{e}^b_\mu \dot{x}^\mu, \quad (21) \end{aligned}$$

where we used $\delta x^\mu(t_1) = 0$.

Notice that when $\delta x^\mu(\tau) = x^\mu(\tau + d\tau) - x^\mu(\tau) = \dot{x}^\mu d\tau$, $\tilde{\lambda}$ vanishes and one gets

$$\frac{dX^a}{d\tau} = \bar{e}^a_\mu(x(\tau))\dot{x}^\mu. \quad (22)$$

The total variation can be rewritten via an integration by part as

$$\begin{aligned} \delta X^a &= \bar{e}^a_\nu(x(\tau))\delta x^\nu(x(\tau)) + \int_\Gamma d\sigma \tilde{\lambda}^a_b(\sigma)\dot{X}^b(\sigma) \\ &= \bar{e}^a_\nu(x(\tau))\delta x^\nu(x(\tau)) + \tilde{\lambda}^a_b(\tau)X^b(\tau) \\ &\quad - \int_\Gamma d\sigma X^b(\sigma)\frac{d}{d\sigma}\tilde{\lambda}^a_b \\ &= \bar{e}^a_\nu(x(\tau))\delta x^\nu(x(\tau)) \\ &\quad + \int_0^\tau d\sigma(X_b(\tau) - X_b(\sigma))R^{ab}_{\mu\nu}\delta x^\mu(\sigma)\dot{x}^\mu. \quad (23) \end{aligned}$$

This expression provides a linear map between the variations $\delta x^\mu$ and $\delta X^a$. Since the basic physical variable of the particle model is the position of the worldline $x^\mu(\tau)$, and because the expression (23) is very complex and nonlocal, to make sure that the equations of motion following from varying $X^a(\tau)$ and $x^\mu(\tau)$ are identical, we must show that the linear mapping (23) is invertible. This is done in Appendix A. Equations (22) and (23) contain all the information we need to construct the action of a free particle moving in curved spacetime and momentum space, and compute the corresponding equations of motion from the variational principle.

### B. The action and equations of motion

Using the nonlocal variable $X^a$ discussed in the preceding subsection we define the action of a particle moving in curved spacetime and momentum spaces as follows:

$$S = \int_{t_1}^{t_2} d\tau\{X^a[x(\tau)]E^\alpha_a \dot{p}_\alpha + N(\mathcal{C}(p) - m^2)\}, \quad (24)$$

where $X^a[x(\tau)]$ is calculated along the same curve $\Gamma$ as the integral in (24). Before turning to the discussion of the properties of this action it is worth checking if it acquires the desired form in the limiting cases of flat spacetime/momentum space, respectively.

In the case of flat spacetime, to evaluate (11) we have to find the form of the associated Fermi tetrad. It follows from the tetrad postulate (7) that since both ${\omega_\mu}^a{}_b$ and $\Gamma^\alpha_{\mu\nu}$ vanish such a tetrad must satisfy $\partial_\mu \bar{e}^a_\nu = 0$, and thus $\bar{e}^a_\mu = \delta^a_\mu$ (up to a global Lorentz transformation). Then

$$X^a = x^\mu(\tau)\delta^a_\mu - x^\mu(t_1)\delta^a_\mu.$$

It can be checked that the second (constant) term produces neither a contribution to the equation of motion nor a boundary term, and thus for flat spacetime the action reproduces the one of relative locality.

In the opposite case, when the momentum space is flat, $E^\alpha_a = \delta^\alpha_a$. We integrate (24) by parts and use (22) to obtain the standard curved spacetime particle action (up to a boundary term), with the only difference being that now we have to do with the Fermi tetrad instead of the generic one. However, the equations of motion are the same in both cases, so we may conclude that the actions are equivalent, the only difference being that the Lagrangian (1) is invariant under local Lorentz symmetry, while in the action (24) only the global Lorentz symmetry remains.[2]

Now we can turn to the equations of motion following from the action (24). Its variation reads

$$\delta S = \int_{t_1}^{t_2} d\tau \left( \delta X^a E^\alpha_a \dot{p}_\alpha + X^a \delta(E^\alpha_a \dot{p}_\alpha) + N \frac{\partial \mathcal{C}}{\partial p_\alpha} \delta p_\alpha + \delta N(\mathcal{C}(p) - m^2) \right) = 0. \tag{25}$$

Since we demonstrated that the map (23) is invertible, we know that the equations of motion we get from the stationarity of the action ($\delta S = 0$) with respect to arbitrary variations $\delta x^\mu$ are equivalent to the ones obtained by considering arbitrary variations $\delta X^a$. Hence, we get

$$E^\alpha_a \dot{p}_\alpha = 0, \qquad \dot{X}^a E^\alpha_a = N \frac{\partial \mathcal{C}}{\partial p_\alpha}, \qquad \mathcal{C}(p) - m^2 = 0, \tag{26}$$

which are equivalent to (for constant $N$)

$$\dot{p}_\alpha = 0, \qquad \ddot{X}^a = 0. \tag{27}$$

In particular, from the second relation in (27) one finds

$$\frac{d}{d\tau}(\dot{x}^\mu \bar{e}^a_\mu) = 0 \to \ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0, \tag{28}$$

where we used the expression of the Christoffel symbols in terms of the tetrads $\bar{e}^a_\mu$, that follows from the tetrad postulate (7)

$$\Gamma^\mu_{\nu\rho} = \bar{e}^\mu_a \partial_{(\rho} \bar{e}^a_{\nu)}. \tag{29}$$

Therefore, the trajectory in spacetime is a geodesic, independent of the geometry in momentum space.

## III. SYMMETRIES OF THE ACTION

Having discussed the form of the action, let us now consider its symmetries. First of all, the action is manifestly invariant under general coordinate transformations, in both momentum space[3] and spacetime, since $X^a$ is a spacetime scalar. Second, it is invariant under residual, global Lorentz transformations that leave invariant the condition that the connection vanishes along the curve $\omega|_\Gamma = 0$.

From the relative locality perspective we are especially interested in translational symmetries: in the case of the model of a particle moving in flat spacetime, the main features of relativity of spacetime locality are encoded in that the fact that the translations become momentum dependent [11,18,19]. As we will see, an analogous effect takes place in the case of curved spacetime.

Like in the case of relative locality in flat spacetime [11,18,19], the action (24) is invariant (up to a boundary term) under the translation

$$\delta X^a = E^a_\alpha(p) \xi^\alpha, \qquad \dot{\xi}^\alpha = 0. \tag{30}$$

In the flat-spacetime relative locality, this symmetry translates rigidly a (straight) particle worldline by an amount that depends on the momentum carried by the particle. As we will see, an analogous effect takes place in curved spacetime.

To see this we must find out what is the infinitesimal shift of the particle trajectory $\delta x^\mu$ corresponding to the translation (30). Since we are interested in the effect that the transformation (30) has on trajectories, we assume that the equations of motion are satisfied.

We start with (23)

$$E^a_\alpha(p)\xi^\alpha(\tau) = \bar{e}^a_\nu(x(\tau))\delta x^\nu(x(\tau)) + \int_{t_1}^{\tau} d\sigma (X_b(\tau) - X_b(\sigma)) R^{ab}_{\mu\nu} \delta x^\mu(\sigma) \dot{x}^\nu. \tag{31}$$

At $\tau = t_1$ we have

$$E^a_\alpha(p)\xi^\alpha = \bar{e}^a_\nu(x(t_1))\delta x^\nu(x(t_1)). \tag{32}$$

This defines the first initial condition for $\delta x^\nu(x(t_1))$. Next, let us differentiate (31) over $\tau$

$$0 = \frac{d}{d\tau}(\bar{e}^a_\nu(\tau)\delta x^\nu(\tau)) + \dot{X}_b(\tau) \int_{t_1}^{\tau} d\sigma R^{ab}_{\mu\nu} \delta x^\mu(\sigma) \dot{x}^\nu, \tag{33}$$

so that at $\tau = t_1$,

$$\left[\frac{d}{d\tau}(\bar{e}^a_\nu(\tau)\delta x^\nu(\tau))\right]_{t_1} = 0. \tag{34}$$

[2]Notice also that, with the same caveat, the action (24) differs, up to a boundary term, from the action presented in [16] (written for Fermi tetrads) by the term $(\int \dot{x}^\mu \bar{e}^a_\mu - x^\mu \bar{e}^a_\mu)\dot{E}^\alpha_a p_\alpha$.

[3]While the invariance under spacetime diffeomorphisms has a clear physical interpretation already in the flat momentum-space limit of action (24), the invariance under momentum-space diffeomorphisms, which could be a mere formal invariance, has no classical counterpart, and we postpone its discussion to future studies.

Taking the second derivative of (31) over $\tau$ we find

$$0 = \frac{d^2}{d\tau^2}(\bar{e}^a_\nu(\tau)\delta x^\nu(\tau)) + \ddot{X}_b(\tau)\int_{t_1}^{\tau} d\sigma R^{ab}_{\mu\nu}\delta x^\mu(\sigma)\dot{x}^\nu + \dot{X}_b(\tau)R^{ab}_{\mu\nu}(\tau)\delta x^\mu(\tau)\dot{x}^\nu(\tau). \tag{35}$$

Since $\ddot{X}^a$ is zero by equations of motion, the second term in the above expression disappears and we are left with

$$\bar{e}_{a\rho}\frac{d^2}{d\tau^2}(\bar{e}^a_\nu \delta x^\nu) + R_{\mu\nu\rho\sigma}\delta x^\mu \dot{x}^\nu \dot{x}^\sigma = 0. \tag{36}$$

As shown in Appendix B, this equation can be rewritten as

$$\frac{D^2}{D\tau^2}\delta x^\mu - R^\mu_{\nu\rho\sigma}\dot{x}^\nu \dot{x}^\rho \delta x^\sigma = 0, \tag{37}$$

where $D/D\tau \equiv \dot{x}^\mu \nabla_\mu$ is the covariant derivative projected along the worldline, subject to the initial conditions

$$\delta x^\mu(t_1) = \bar{e}^\mu_a(x(t_1))E^a_\alpha(p)\xi^\alpha, \qquad \frac{D}{D\tau}\delta x^\mu|_{t_1} = 0. \tag{38}$$

Equation (37) is an equation of geodesic deviation and therefore we see that the translational symmetry (30) maps the original geodesic, being the particle worldline, to another one, with the magnitude of translation depending on the momentum carried by the particle. This is exactly the effect one could foresee from the flat-spacetime relative locality, where straight lines (geodesics) are translated by a constant, momentum dependent amount.

It follows from (38) that $\delta x^\mu$ has the momentum dependence encoded by the initial condition. Let us define another variable $\zeta^\alpha$, which describes the momentum independent translation

$$\delta x^\mu = \bar{e}^\mu_a(x)E^a_\alpha(p)\zeta^\alpha. \tag{39}$$

Since, as shown in Appendix B, both first and second covariant derivatives of the tetrad $\bar{e}$ along the worldline vanish, we can rewrite (37) as

$$\frac{D^2}{D\tau^2}\zeta^\alpha - (\bar{e}^a_\mu(x)E^\alpha_a(p)R^\mu_{\nu\rho\sigma}\bar{e}^\sigma_b(x)E^b_\beta(p))\dot{x}^\nu \dot{x}^\rho \zeta^\beta = 0. \tag{40}$$

This equation describes a congruence of particle worldlines in the spacetime whose curvature is momentum dependent. It might serve as a starting point of more phenomenologically oriented investigations.

## IV. INTERACTIONS

In the spirit of the relative locality framework introduced in [11] (see also [18] for an extensive discussion of the properties of these boundary terms), in order to describe particle processes (at a semiclassical, nonquantum, level), we introduce in the action (9) boundary terms enforcing constraints on the endpoints of the particles worldlines. To illustrate how such a constraint may be introduced in our framework, suppose that we want to describe an idealized process depicted in Fig. 2, with two incoming particles labeled respectively by momenta and coordinates $p_\alpha$, $x^\mu$, $q_\alpha$, $y^\mu$ and the outgoing one labeled by $r_\alpha$, $z^\mu$.

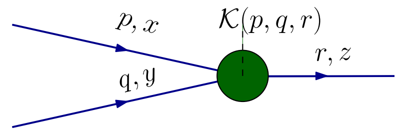


FIG. 2 (color online). A process with two incoming particles of momenta $p$, $q$ and one outgoing particle of momentum $r$. $\mathcal{K}(q,p,r)$ is a function of the particles' momenta enforcing energy-momentum conservation at the vertex. The graphic must not be intended as a spacetime representation but just as a qualitative picture illustrating the combination of momenta in the process.

The action for this process is

$$S = \int_{-\infty}^{t} d\tau[X^a E^\alpha_a \dot{p}_\alpha + N_p(\mathcal{C}(p) - m_p^2)] + \int_{-\infty}^{t} d\tau[Y^a E^\alpha_a \dot{q}_\alpha + N_q(\mathcal{C}(q) - m_q^2)] + \int_{t}^{\infty} d\tau[Z^a E^\alpha_a \dot{r}_\alpha + N_r(\mathcal{C}(r) - m_r^2)] - k^\alpha \mathcal{K}_\alpha(p,q,r)|_{\bar{\tau}}. \tag{41}$$

In this formula $k^\alpha$ is a Lagrange multiplier enforcing the constraint $\mathcal{K}_\alpha(p,q,r)$ on the worldlines endpoints, which plays a role of a (in general) deformed law of energy-momentum conservation at the vertex. Typically, $\mathcal{K}_\alpha(p,q,r) = (p \oplus q \oplus (\ominus r))_\alpha$, with the symbol $\oplus$ ($\ominus$) encoding the connection on the momentum-space geometry, characterizing the (in general nonlinear) law of summation for momenta [11]. In addition to the equations of motion for the bulk part, Eqs. (26), (28), the boundary term contributes with the constraints

$$\mathcal{K}_\alpha(p,q,r)|_t = (p \oplus q \oplus (\ominus r))_\alpha|_t = 0, \tag{42}$$

$$X^a(t) = k^\beta E^a_\alpha(p)\frac{\partial \mathcal{K}_\beta}{\partial p_\alpha}\Big|_t, \tag{43}$$

$$Y^a(t) = k^\beta E^a_\alpha(q)\frac{\partial \mathcal{K}_\beta}{\partial q_\alpha}\Big|_t. \tag{44}$$

$$Z^a(t) = k^\beta E^a_\alpha(r)\frac{\partial \mathcal{K}_\beta}{\partial r_\alpha}\Big|_t. \tag{45}$$

When $k^\alpha$ changes, the $X^a$ transforms as

$$\delta X^a|_t = E^a_\alpha(p)\frac{\partial(p \oplus q \oplus (\ominus r))_\beta}{\partial p_\alpha}\delta k^\beta|_t, \qquad (46)$$

with analogous relations holding for the other particles. Assuming that we take the initial condition for the geodesic deviation equation (38) at the interaction point, we find that

$$\delta x^\mu|_t = \bar{e}^\mu_a E^a_\alpha(p)\frac{\partial(p \oplus q \oplus (\ominus r))_\beta}{\partial p_\alpha}\delta k^\beta|_t. \qquad (47)$$

We see therefore that the structure of the interaction vertex in the case of curved spacetime is essentially the same as in the flat-spacetime case of relative locality [11].

## V. CONCLUSIONS

In this paper we generalized relative locality, originally defined [11] in flat spacetime, to the case of an arbitrary, curved background spacetime, preserving invariance under general coordinate transformations. It turns out that on the formal level this latter theory is a natural generalization of the former one: free particle trajectories are now geodesics instead of flat-spacetime straight lines, and rigid, momentum dependent translations of the flat case are replaced with geodesic deviations, sensitive to curvature.

In spite of the apparent similarities there are, however, some major differences between the flat and curved spacetime cases. In the latter we were forced to use nonlocal variables $X^a$ to define the action that had desired symmetry properties. This might be just a technical artifact, but it may also signal a presence of some deeper layer present in theories with a nontrivial geometry in both momentum space and spacetime. Furthermore, in flat spacetime the symmetries of the action are associated with some transformations (rigid translations) defined in the whole spacetime manifold. On the contrary, in the curved case we have to solve the equation of geodesic deviation on a given geodesic to find the symmetry. This implies that the transformation $\delta x^\mu$ which leaves the action invariant (up to a boundary term) depends on each particular solution of equations of motion and generically, because of curvature, it cannot be extended to the whole spacetime.

Aside from its conceptual relevance, our result opens some interesting phenomenological perspectives. Indeed most of the opportunities to test Planck-scale deformation effects on particle kinematics, that have been proposed in the recent literature, rely on some source of amplification of the relevant effects due to cosmological distance of astrophysical sources. Most of the results[4] for theories with curved momentum space (and earlier of the DSR theories) have been discussed in the context of flat spacetime, while in the proposed scenarios relevant for Planck-scale phenomenology the effect of spacetime curvature cannot be neglected. Our result can be taken then as a starting point for further studies of relative locality effects in the presence of spacetime curvature. Moreover, using the results of the present work one may try to investigate the relative locality effects in the case of a strong gravitational field, for example, in the context of black hole physics (see [20]).

[4]In [15], a first investigation of the interplay between the spacetime expansion and relativity of locality has been presented for the case of de Sitter-like spacetime expansion.

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## APPENDIX A: INVERTIBILITY OF $\delta X^a$

To show that the map (23) is invertible we must show that its kernel contains only $\delta x^\mu = 0$, to wit

$$\delta X^a = 0 \Rightarrow \delta x^\mu = 0. \qquad (A1)$$

In order to prove this let us note that since we assumed $x^\mu = x^\mu(\tau)$ to be a $C^\infty$ function of $\tau$, $X^a(\tau)$ and $\delta X^a(\tau)$ are $C^\infty$ too and the condition $\delta X^a(\tau) = 0$ for each $\tau$ is equivalent to

$$\left[\frac{d^n}{d\tau^n}\delta X^a(\tau)\right]|_0 = 0, \qquad \forall n \in \mathbb{N}. \qquad (A2)$$

From the expression (23) we have

$$\frac{d}{d\tau}\delta X^a(\tau) = \frac{d}{d\tau}(\bar{e}^a_\nu(x(\tau))\delta x^\nu(x(\tau))) + \frac{dX_b}{d\tau}\int_0^\tau d\sigma R^{ab}_{\mu\nu}\delta x^\mu(\sigma)\dot{x}^\mu, \qquad (A3)$$

and for $\tau = 0$ we get

$$0 = \left[\frac{d}{d\tau}\delta X^a(\tau)\right]\Big|_0 = \left[\frac{d}{d\tau}(\bar{e}^a_\nu \delta x^\nu)\right]\Big|_0, \qquad (A4)$$

which implies [we recall that $\delta x^\mu(0) = 0$]

$$\left[\frac{d}{d\tau}\delta x^\mu\right]\Big|_0 = 0. \qquad (A5)$$

Similarly, one can show that

$$\left[\frac{d^2}{d\tau^2}\delta X^a(\tau)\right]\Big|_0 = 0 \Rightarrow \left[\frac{d^2}{d\tau^2}\delta x^\mu\right]\Big|_0 = 0, \qquad (A6)$$

and by iterating one gets

$$\left[\frac{d^n}{d\tau^n}\delta x^\mu(\tau)\right]\Bigg|_0 = 0, \qquad \forall n \in \mathbb{N}, \tag{A7}$$

which is equivalent to $\delta x^\mu(\tau) = 0$ for each $\tau \in [t_1, t_2]$. Therefore we proved (A1) and the map (23) is invertible.

## APPENDIX B: DERIVATION OF GEODESIC DEVIATION

In this Appendix we show that Eq. (36) is equivalent to the equation of geodesic deviation (37). To see this notice that

$$\delta\dot{x}^a = \frac{d}{d\tau}(\delta x^\mu \bar{e}^a_\mu) = \left(\frac{D}{D\tau}\delta x^\mu\right)\bar{e}^a_\mu + \delta x^\mu \frac{D}{D\tau}\bar{e}^a_\mu, \tag{B1}$$

and

$$\delta\ddot{x}^a = \left(\frac{D^2}{D\tau^2}\delta x^\mu\right)\bar{e}^a_\mu + \delta x^\mu \frac{D^2}{D\tau^2}\bar{e}^a_\mu + 2\left(\frac{D}{D\tau}\delta x^\mu\right)\frac{D}{D\tau}\bar{e}^a_\mu. \tag{B2}$$

It follows from the tetrad postulate (7) and the properties of Fermi tetrads that

$$\frac{D}{D\tau}\bar{e}^a_\mu = 0. \tag{B3}$$

As for the second covariant derivative we get

$$\frac{D^2}{D\tau^2}\bar{e}^a_\mu = \frac{D}{D\tau}(\dot{x}^\nu \bar{e}^b_\mu)\omega^a_{b\nu} + \dot{x}^\nu \bar{e}^b_\mu \frac{D}{D\tau}\omega^a_{b\nu}.$$

This expression is again zero for a Fermi tetrad, because connection $\omega$ is zero everywhere on the worldline and thus its derivative along it vanishes; therefore,

$$\frac{D^2}{D\tau^2}\bar{e}^a_\mu = 0. \tag{B4}$$

Using (B3) and (B4), one straightforwardly derives (37).


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