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Thermodynamics of magnetized Kerr-Newman black holes

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The thermodynamics of a magnetized Kerr-Newman black hole is studied to all orders in the appended magnetic field B. The asymptotic properties of the metric and other fields are dominated by the magnetic flux that extends to infinity along the axis, leading to subtleties in the calculation of conserved quantities such as the angular momentum and the mass. We present a detailed discussion of the implementation of a Wald-type procedure to calculate the angular momentum, showing how ambiguities that are absent in the usual asymptotically flat case may be resolved by the requirement of gauge invariance. We also present a formalism from which we are able to obtain an expression for the mass of the magnetized black holes. The expressions for the mass and the angular momentum are shown to be compatible with the first law of thermodynamics and a Smarr-type relation. Allowing the appended magnetic field B to vary results in an extra term in the first law of the form $-\mu dB$ where μ is interpreted as an induced magnetic moment. Minimizing the total energy with respect to the total charge Q at fixed values of the angular momentum and energy of the seed metric allows an investigation of Wald's process. The Meissner effect is shown to hold for electrically neutral extreme black holes. We also present a derivation of the angular momentum for black holes in the four-dimensional STU model, which is $\mathcal{N} = 2$ supergravity coupled to three vector multiplets.

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I. INTRODUCTION

Understanding the behavior of rotating black holes and their ergoregions when immersed in magnetic fields forms a central part of astrophysical theories of quasars, active galactic nuclei and other objects containing black holes (see, for example, [1–3]). Typically the magnetic fields, which contribute a negligible amount of energy density near the black hole, are treated as test fields on the background of the Kerr solution of the vacuum Einstein equations, as, for example, in the work of Wald [4]. While perfectly justified astrophysically, it is not without interest to treat the energy exchange between black hole and magnetic field at the fully nonlinear level, and in particular to ask to what extent the ideas of black hole thermodynamics, which have proved so useful in the study of quantum processes near black holes, may be extended to this more general setting. This seems especially appropriate since rotating black holes are believed to drag magnetic field lines, inducing electric fields to flow and hence currents to flow. For sufficiently strong magnetic fields this may lead to the breakdown of the vacuum due to pair creation [5,6].

These thoughts motivated a recent study [7] of the exact metric and electromagnetic field of a magnetized Kerr-Newman black hole, constructed using solution-generating methods pioneered by Ernst [8]. Contrary to the widespread belief that the asymptotic metric was approximately static and Melvin-like, it was found that generically the metric has ergoregions that extend all the way to infinity. The complicated nature of the metric at infinity presented difficulties in evaluating the total energy and angular momentum of the system and in the treatment of the thermodynamics. In this paper, we are able partially to overcome these problems and to present expressions for the total angular momentum J and total energy E of the system, together with a form of the relevant Smarr relation and first law in which variations of the appended magnetic field B are fully taken into account. (The thermodynamics of the Schwarzschild-Melvin black hole were discussed in [9].)

The plan of the paper is as follows. In Sec. II we discuss the Wald procedure for evaluating the total angular momentum J, and some subtleties that can arise in cases such as the magnetized black holes that were not present for the asymptotically flat geometries considered by Wald. These lead to potential ambiguities in the definition of the angular momentum. We argue that these may be resolved by a careful consideration of the behavior of the conserved angular momentum under gauge transformations. In Sec. III we show how the electric charge and the angular momentum may be conveniently evaluated by first performing a Kaluza-Klein reduction on the azimuthal coordinate ϕ, and then expressing the conserved Wald charge in terms of three-dimensional quantities.

The definition of the mass of a black hole in an external magnetic field is also somewhat problematical, on account of the unusual asymptotic behavior of the metric and the other fields. We discuss this in Sec. IV, where we present a formalism, again based on the Kaluza-Klein reduction to three dimensions, within which we are able to obtain an expression for the mass. In Sec. V we evaluate our general expressions for the angular momentum and the mass in the case of the magnetized Kerr-Newman black holes. We show that these results are consistent with the first law of thermodynamics, in the case that the appended magnetic field B is held fixed. In doing so, we essentially use the first law to derive expressions for the angular velocity Ω and the electrostatic potential Φ. [1] We then extend the discussion, treating B also as a thermodynamic variable by introducing an extra contribution $-\mu dB$ in the first law, where μ has an interpretation as an induced magnetic moment. The explicit expressions for Ω, Φ and μ may be calculated exactly, but their forms are rather complicated. However they simplify considerably if one works to low orders in B or q. We show also that the thermodynamic variables in the extended system obey a Smarr-type relation.

In Sec. VI, we attempt to compare our thermodynamic formalism with some work of Wald [10,11]. We minimize the total energy E with respect to the total charge Q, at fixed values of the energy, angular momentum and magnetic field of the black hole. Our result resembles that of Wald in general form, but differs in detail.

In Sec. VII we examine some further properties of the magnetized black holes, including the Meissner effect whereby as one approaches extremality, the magnetic flux penetrating the horizon vanishes. In other words, flux is expelled [12]. We find, that if the total charge on the hole Q vanishes, then the magnetic field on the horizon does indeed vanish as one approaches extremality, consistent with earlier work.

In Sec. VIII we extend our discussion of the conserved angular momentum to the case of the STU supergravity model, which comprises four-dimensional $\mathcal{N}=2$ supergravity coupled to three vector multiplets. We apply our results to the case of the magnetization of certain four-charge static black holes that have been investigated recently in [13]. The paper ends with conclusions in Sec. IX.

II. CONSERVED CHARGES IN EINSTEIN-MAXWELL THEORY

Here we present a discussion of some aspects of the Wald procedure Wald2 for calculating conserved charges, applied to the case of the four-dimensional Einstein-Maxwell theory. Our motivation for doing so will be as part of an investigation of the thermodynamics of the magnetized Kerr-Newman black holes that were recently studied in [7]. We shall find that some subtleties arise in this context that make it necessary to pay close attention to some of the details of the Wald procedure.

Starting from the Einstein-Maxwell Lagrangian

$$\mathcal{L}_4 = \frac{1}{16\pi}(R * \mathbb{1} - 2 * F\wedge F), \tag{2.1}$$

and following a calculation developed by Wald [10,11], one can use the Noether procedure to derive a current $\mathcal{J}$, given by

$$\mathcal{J} = -d * d\xi - 4 * F\wedge d(\xi^\mu A_\mu), \tag{2.2}$$

where $\xi = \xi_\mu dx^\mu$ and $\xi^\mu \partial_\mu$ is a Killing vector. [2] Since $d\mathcal{J}=0$, we can write $\mathcal{J} = -d\mathcal{P}$ and hence derive the conserved charge

$$\mathcal{Q}[\xi] = \frac{1}{16\pi G}\int_{S^2} \mathcal{P}. \tag{2.3}$$

From this point on we shall work in units where $G=1$.

One way to obtain a local expression for $\mathcal{P}$ is to note that the Maxwell equation $d * F = 0$ allows us to extract an exterior derivative from the second term in (2.2) and write

$$\mathcal{P} = *d\xi + 4 * F(\xi^\mu A_\mu). \tag{2.4}$$

This is the the form in which the conserved charge was obtained in [10].

An objection one may raise to the expression (2.4) is that it is not invariant under gauge transformations of A. Specifically, if we send $A \to A + d\lambda$, then we shall have

$$\mathcal{P} \to \mathcal{P} + 4 * F(\xi^\mu \partial_\mu \lambda). \tag{2.5}$$

Since the Killing vector ξ^μ generates a symmetry of the solution, it follows that the Lie derivative of F will vanish, $\mathfrak{L}_\xi F = 0$. We may assume that a gauge choice for A is made so that $\mathfrak{L}_\xi A = 0$ also. However, there can still remain a residual gauge freedom that preserves this choice, namely when the gauge parameter λ satisfies

$$\xi^\mu \partial_\mu \lambda = c, \tag{2.6}$$

where c is a constant. This can be seen from the fact that gauge transformations preserving $\mathfrak{L}_\xi A = 0$ must satisfy $\mathfrak{L}_\xi d\lambda = (i_\xi d + d i_\xi) d\lambda = d i_\xi d\lambda = d(\xi^\mu \partial_\mu \lambda) = 0$. [3] The conserved charge in (2.3) will then undergo a gauge transformation of the form

[1] To be precise, Ω and Φ represent the *differences* $\Omega = \Omega_H - \Omega_\infty$ and $\Phi = \Phi_H - \Phi_\infty$ between the values on the horizon and the values at infinity. Ω_H and Φ_H are easily computed directly, but the asymptotics of the magnetized black hole solutions make it difficult to define Ω_∞ and Φ_∞ directly.

[2] We shall present a detailed derivation in Sec. VIII of the analogous result in the more complicated context of the STU supergravity model.

[3] Here i_ξ denotes the interior product of $\xi = \xi^\mu \partial_\mu$ with a p-form $\omega = (1/p!)\omega_{\mu_1\ldots\mu_p} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$. Its action is defined by $i_\xi \omega = (1/(p-1)!)\xi^{\mu_1}\omega_{\mu_1\ldots\mu_p} dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p}$. Note that if ω and ν are a p-form and a q-form, then $i_\xi(\omega\wedge\nu) = (i_\xi\omega)\wedge\nu + (-1)^p \omega\wedge(i_\xi \nu)$. The Lie derivative of any p-form is given by $\mathfrak{L}_\xi \omega = (d i_\xi + i_\xi d)\omega$.

$$\mathcal{Q}[\xi] \to \mathcal{Q}[\xi] + cQ, \tag{2.7}$$

where $Q = 1/(4\pi)\int *F$ is the electric charge.

An alternative way of extracting an exterior derivative from the expression (2.2) for $\mathcal{J}$ is to introduce a dual gauge potential $\tilde{A}$ such that $*F \equiv \tilde{F} = d\tilde{A}$, and then write $\mathcal{J} = -d\tilde{\mathcal{P}}$, where

$$\tilde{\mathcal{P}} = *d\xi + 4\tilde{A}\wedge d(\xi^\mu A_\mu). \tag{2.8}$$

It is evident that the corresponding conserved charge $\tilde{\mathcal{Q}}[\xi]$ obtained by substituting $\tilde{\mathcal{P}}$ into (2.3) will be invariant under the residual gauge transformations of A, which satisfy (2.6).

Our principle interest in this paper will be to apply the Wald construction to the calculation of the angular momentum. The two alternative expressions (2.4) and (2.8) for a 2-form whose exterior derivative gives $\mathcal{J}$ then essentially correspond to the standard Wald expressions that one obtains by using either the original Lagrangian (2.1) [leading to (2.4)] or else the dual Lagrangian [4]

$$\tilde{\mathcal{L}}_4 = \frac{1}{16\pi G}(R * \mathbb{1} - 2 * \tilde{F}\wedge\tilde{F}), \tag{2.10}$$

where $\tilde{F} = *F = d\tilde{A}$. To see this, consider the analogue of the Wald expression (2.4) that one would derive from the dual Lagrangian (2.10):

$$\mathcal{P}_{\text{dual}} = *d\xi + 4 * \tilde{F}(\xi^\mu \tilde{A}_\mu). \tag{2.11}$$

The difference between this and $\tilde{\mathcal{P}}$ [defined in (2.8)] is therefore

$$\mathcal{P}_{\text{dual}} - \tilde{\mathcal{P}} = 4 * \tilde{F} i_\xi \tilde{A} - 4\tilde{A}\wedge d i_\xi A. \tag{2.12}$$

Assuming that ξ^μ is a Killing vector, so that the Lie derivative of the field strength F in a solution vanishes, $\mathfrak{L}_\xi F = (d i_\xi + i_\xi d)F = 0$, we may choose gauges where $\mathfrak{L}_\xi A = 0$ and $\mathfrak{L}_\xi \tilde{A} = 0$. In particular, this means that $d i_\xi A = -i_\xi dA = -i_\xi F$. Using also that $*\tilde{F} = -F$, we see that

$$\begin{aligned}\mathcal{P}_{\text{dual}} - \tilde{\mathcal{P}} &= -4 i_\xi \tilde{A} F + 4\tilde{A}\wedge i_\xi F,\\ &= -4 i_\xi(\tilde{A}\wedge F).\end{aligned} \tag{2.13}$$

[4]The dual Lagrangian can be obtained by adding a Lagrange multiplier term to (2.1) and writing

$$\mathcal{L} = \frac{1}{16\pi G}(R * \mathbb{1} - 2 * F\wedge F + 4 d\tilde{A}\wedge F), \tag{2.9}$$

where now F and $\tilde{A}$ are viewed as fundamental fields. The equation of motion for $\tilde{A}$ implies the usual Bianchi identity for F. If instead we eliminate F via its (algebraic) equation of motion, we obtain (2.10). The original and the dual Lagrangian differ on shell by the total derivative term $-4(d\tilde{A}\wedge dA)/(16\pi G)$.

In particular, if we consider the case when $\xi = \partial/\partial\phi$ is the Killing vector that generates azimuthal rotations, then it follows from the final line of (2.13) that $(\mathcal{P}_{\text{dual}} - \tilde{\mathcal{P}})$ has no pullback onto the 2-sphere over which we integrate to obtain a conserved charge. This means that $\mathcal{P}_{\text{dual}}$ and $\tilde{\mathcal{P}}$ would give identical expressions for the angular momentum.

In the following section, we shall discuss the dimensional reduction of the theory, and its solutions, on the azimuthal Killing vector $\partial/\partial\phi$. This will provide us with a formalism that is particularly well adapted to computing the angular momentum for the solutions we are interested in.

III. CONSERVED CHARGE, ANGULAR MOMENTUM AND MASS VIA DIMENSIONAL REDUCTION

A convenient way of calculating the conserved charges is to perform a Kaluza-Klein dimensional reduction on the ϕ coordinate. Thus we write [5]

$$\begin{aligned}ds_4^2 &= e^{2\varphi} d\bar{s}_3^2 + e^{-2\varphi}(d\phi + 2\bar{\mathcal{A}})^2,\\ A &= \bar{A} + \chi(d\phi + 2\bar{\mathcal{A}}),\end{aligned} \tag{3.1}$$

where, whenever there is an ambiguity, we place a "bar" on three-dimensional quantities to distinguish them from the unbarred four-dimensional ones. Note that $F = \bar{F} + d\chi\wedge(d\phi + 2\bar{\mathcal{A}})$. The equations of motion for the three-dimensional fields then follow from the dimensionally reduced Lagrangian

$$\begin{aligned}\mathcal{L}_3 = \frac{\Delta\phi}{16\pi}\sqrt{-\bar{g}}[\bar{R} - 2(\partial\varphi)^2 - 2e^{2\varphi}(\partial\chi)^2\\ - e^{-4\varphi}\bar{\mathcal{F}}^2 - e^{-2\varphi}\bar{F}^2],\end{aligned} \tag{3.2}$$

$$\bar{\mathcal{F}} = d\bar{\mathcal{A}}, \qquad \bar{F} = d\bar{A} + 2\chi d\bar{\mathcal{A}}, \tag{3.3}$$

where $\Delta\varphi$ is the period of the azimuthal coordinate φ. The equations of motion for $\bar{A}$ and $\bar{\mathcal{A}}$ imply that we can write

$$e^{-2\varphi}\bar{*}\bar{F} = d\psi, \qquad e^{-4\varphi}\bar{*}\bar{\mathcal{F}} = d\sigma - 2\chi d\psi. \tag{3.4}$$

Here, ψ and σ are the axionic scalar duals of the 1-form potentials $\bar{A}$ and $\bar{\mathcal{A}}$.

A. Conserved electric charge

Since $*F = e^{-2\varphi}\bar{*}\bar{F}\wedge(d\phi + 2\bar{\mathcal{A}}) + e^{2\varphi}\bar{*}d\chi$, the conserved electric charge is given by

[5]Note that the reduction ansatz for A is compatible with the partial gauge condition $\mathfrak{L}_\xi A = 0$ that we discussed previously, since $i_\xi A = \chi$ and $i_\xi dA = -d\chi$, so $(d i_\xi + i_\xi d)A = 0$.

$$Q = \frac{1}{4\pi}\int_{S^2} *F = \frac{\Delta\phi}{4\pi}\int e^{-2\phi}\bar{*}\bar{F},$$
$$= \frac{\Delta\phi}{4\pi}\int d\psi = \frac{\Delta\phi}{4\pi}[\psi]_{\theta=0}^{\theta=\pi}. \tag{3.5}$$

Note that here, and henceforth, we are allowing for the possibility that the period $\Delta\varphi$ of the azimuthal coordinate ϕ might be different from 2π. In particular, this happens in the case of the magnetized black hole solutions that we shall be considering in this paper.

B. Conserved angular momentum

We first calculate the angular momentum $J = \mathcal{Q}[\xi]$ using (2.3) with $\mathcal{P}$ given by (2.4), and with $\xi = \partial/\partial\tilde{\phi}$ where $\tilde{\phi}$ is the canonically normalized azimuthal angular coordinate with period 2π. It will, in general, be related to φ by $\varphi = \alpha\tilde{\varphi}$, with $\alpha = \Delta\phi/(2\pi)$. As a 1-form, ξ will be given, in terms of the three-dimensional quantities, by

$$\xi = \alpha e^{-2\varphi}(d\phi + 2\bar{\mathcal{A}}), \tag{3.6}$$

and furthermore $\xi^\mu A_\mu = \alpha\chi$, so from (2.4)

$$\mathcal{P} = *[2\alpha e^{-2\varphi}\bar{\mathcal{F}} + 4\alpha\chi\bar{F}$$
$$- 2\alpha(e^{-2\varphi}d\varphi - 2\chi d\chi)\wedge(d\phi + 2\bar{\mathcal{A}})]. \tag{3.7}$$

Thus using (3.4) we have

$$\mathcal{P} = 2\alpha(e^{-4\phi}\bar{*}\bar{\mathcal{F}} + 2\chi e^{-2\phi}\bar{*}\bar{F})\wedge(d\phi + 2\bar{\mathcal{A}})$$
$$- 2\alpha(\bar{*}d\phi - 2\chi e^{2\phi}\bar{*}d\chi),$$
$$= 2\alpha d\sigma\wedge(d\phi + 2\bar{\mathcal{A}}) - 2\alpha(\bar{*}d\phi - 2\chi e^{2\phi}\bar{*}d\chi). \tag{3.8}$$

Only the first term has a nonzero pullback onto the 2-sphere, and so this gives a conserved angular momentum

$$J = \frac{1}{16\pi}\int_{S^2}\mathcal{P} = \frac{\alpha\Delta\phi}{8\pi}\int d\sigma = \frac{(\Delta\phi)^2}{16\pi^2}[\sigma]_{\theta=0}^{\theta=\pi}. \tag{3.9}$$

As we discussed in Sec. II, a different choice for the definition of the angular momentum is to perform a dualization of the four-dimensional field strength F, and work instead with $\tilde{F} = *F$ as the fundamental electromagnetic field strength. As discussed in [7], in the three-dimensional language this dualization amounts to interchanging the three-dimensional fields χ and ψ. At the same time, the field σ must be redefined, so that in the dual formulation we shall have tilded fields given in terms of the original ones by [7]

$$\tilde{\chi} = \psi, \qquad \tilde{\psi} = \chi, \qquad \tilde{\sigma} = \sigma - 2\chi\psi. \tag{3.10}$$

It follows that in this dualized formalism, the angular momentum defined in (3.9) would be replaced by

$$\tilde{J} = \frac{(\Delta\phi)^2}{16\pi^2}[\tilde{\sigma}]_{\theta=0}^{\theta=\pi}. \tag{3.11}$$

It is instructive to look at the behavior of the two expressions under gauge transformations. The quantity $\mathcal{P}$ defined in (2.3) which we used in order to calculate the angular momentum (3.9) is in general gauge dependent, since the potential A_μ appears explicitly in its construction. This can be seen in the three-dimensional language as follows. If we perform a gauge transformation $A \to A' = A + d\lambda$ on the four-dimensional gauge potential, then this will be compatible with the Kaluza-Klein reduction ansatz (3.1) for A provided that λ is restricted to have the form

$$\lambda = \bar{\lambda} + c\phi, \tag{3.12}$$

where $\bar{\lambda}$ depends only on the three-dimensional coordinates and c is a constant. Specifically, comparing with the reduction ansatz

$$A' = \bar{A}' + \chi'(d\phi + 2\bar{\mathcal{A}}'), \tag{3.13}$$

we see that the three-dimensional fields will transform as

$$\chi' = \chi + c, \qquad \bar{A}' = \bar{A} - 2c\bar{\mathcal{A}} + d\bar{\lambda}, \qquad \bar{\mathcal{A}}' = \bar{\mathcal{A}}. \tag{3.14}$$

Since $\bar{F} = d\bar{A} + 2\chi d\bar{\mathcal{A}}$, it follows that

$$\bar{F}' = d\bar{A}' + 2\chi' d\bar{\mathcal{A}} = d\bar{A} + 2\chi d\bar{\mathcal{A}} = \bar{F}, \tag{3.15}$$

and therefore from (3.4) we see that

$$\psi' = \psi. \tag{3.16}$$

Since we also have $\bar{\mathcal{F}}' = \bar{\mathcal{F}}$ it also follows from (3.4) that

$$d\sigma' - 2\chi' d\psi' = d\sigma - 2\chi d\psi, \tag{3.17}$$

and so using $\chi' = \chi + c$ and $\psi' = \psi$ we see that

$$\sigma' = \sigma + 2c\psi. \tag{3.18}$$

It follows from (3.18) that if we perform the gauge transformation in (3.12) that is parametrized by the constant c, then the angular momentum given by (3.9) will transform to

$$J' = J + cQ\frac{\Delta\phi}{2\pi}, \tag{3.19}$$

where Q is the conserved electric charge given by (3.5).

If, on the other hand, we consider the angular momentum $\tilde{J}$ defined by (3.11), then we see that under the gauge transformations parametrized by c we shall have

$$\tilde{\sigma}' = \sigma' - 2\chi'\psi' = \sigma + 2c\psi - 2\chi\psi - 2c\psi$$
$$= \sigma - 2\chi\psi = \tilde{\sigma}, \tag{3.20}$$

and so $\tilde{J}$ is gauge invariant. To be more precise, the expression (3.11) for $\tilde{J}$ is invariant under gauge transformations of the original potential A. Conversely, the expression (3.9) for J is invariant under gauge transformations of the *dual* potential $\tilde{A}$. Correspondingly, J does depend on gauge transformations of A, whilst $\tilde{J}$ depends on gauge transformations of $\tilde{A}$.

In the reduced three-dimensional description, the residual gauge transformations of the dual potential $\tilde{A}$, preserving $\mathfrak{L}_\xi \tilde{A} = 0$, correspond to sending

$$\psi \to \psi + b, \qquad \chi \to \chi, \qquad \sigma \to \sigma, \tag{3.21}$$

where b is a constant parameter. This implies that the $\tilde{\sigma} \equiv \sigma - 2\chi\psi$, which is invariant under the original residual gauge transformations, will transform as

$$\tilde{\sigma} \to \tilde{\sigma} - 2b\chi \tag{3.22}$$

under the dual residual gauge transformations. However, if there is no magnetic charge, and thus $[\chi]_{\theta=0}^{\theta=\pi} = 0$, then the angular momentum $\tilde{J}$ calculated using (3.11) will be invariant also under (3.21).

It should be noted also that if we are able to make a gauge transformation of the form (3.14) that sets χ to zero on the z axis, then the gauge-invariant expression (3.11) for the angular momentum of an electrically charged solution will coincide with the expression, in general gauge dependent, following from (3.9).

IV. MASS OF THE KERR-NEWMAN-MELVIN BLACK HOLES

In this section, we shall be describing an approach to calculating the mass of the magnetized black holes by means of a dimensional reduction to three dimensions. In order to avoid a profusion of annotations on the three-dimensional equations we shall, *in this section only*, adopt the convention that four-dimensional quantities are denoted with hats, while three-dimensional ones are unadorned.

A. Hamiltonian formalism

The original four-dimensional theory is given by

$$\hat{I} = \frac{1}{16\pi G_4}\int_{\hat{M}} (\hat{R} - \hat{F}^2)\sqrt{-\hat{g}}d^4x + \frac{1}{8\pi G_4}\oint_{\partial\hat{M}} \hat{K}\,\sqrt{|\hat{\gamma}|}\,d^3x, \tag{4.1}$$

where $\hat{K}$ is the extrinsic curvature of the three-dimensional boundary $\partial\hat{M}$, which has the induced metric $\hat{\gamma}_{\mu\nu}$. Upon dimensional reduction on a circle using the standard Kaluza-Klein ansatz

$$d\hat{s}_4^2 = e^{2\varphi}ds_3^2 + e^{-2\varphi}(d\phi + 2\mathcal{A})^2,$$
$$\hat{A} = A + \chi(d\phi + 2\mathcal{A}), \tag{4.2}$$

we obtain the three dimensional theory

$$I = \frac{1}{16\pi G_3}\int_M (R - 2\Box\varphi - 2(\partial\varphi)^2 - 2e^{2\varphi}(\partial\chi)^2 - e^{-4\varphi}\mathcal{F}^2 - e^{-2\varphi}F^2)\sqrt{-g}d^3x + \frac{1}{8\pi G_3}\oint_{\partial M}(K + n^\mu\partial_\mu\varphi)\sqrt{|\gamma|}d^2x,$$
$$G_4 = (\Delta\phi)G_3, \tag{4.3}$$

where $\Delta\phi$ is the period of the reduction coordinate φ. In the following, we set $G_4 = 1$, and therefore $G_3 = 1/(\Delta\phi)$. After integration by parts,

$$I = \frac{\Delta\phi}{16\pi}\int_M (R - 2(\partial\varphi)^2 - 2e^{2\varphi}(\partial\chi)^2 - e^{-4\varphi}\mathcal{F}^2 - e^{-2\varphi}F^2) \times \sqrt{-g}d^3x + \frac{\Delta\phi}{8\pi}\oint_{\partial M} K\sqrt{|\gamma|}d^2x. \tag{4.4}$$

Adding Lagrange multipliers $4d\psi\wedge(F - 2\chi\mathcal{F}) + 4d\sigma\wedge\mathcal{F}$ and eliminating F and $\mathcal{F}$, we arrive at the dualized Lagrangian describing three-dimensional gravity coupled to a sigma model

$$I = \frac{\Delta\phi}{16\pi}\int_M (R - 2\Sigma_{AB}(\phi)\partial\phi^A\partial\phi^B)\sqrt{-g}d^3x + \frac{\Delta\phi}{8\pi}\oint_{\partial M} K\sqrt{|\gamma|}d^2x, \tag{4.5}$$

where φ^A represents all the scalars. The sigma-model metric is

$$d\Sigma^2 = d\varphi^2 + e^{2\varphi}(d\chi^2 + d\psi^2) + e^{4\varphi}(d\sigma - 2\chi d\psi)^2. \tag{4.6}$$

As stated before, the dualized action differs from the original one by a total derivative term, and this will modify the definition of energy. [6] The original Lagrangian cannot easily be used to calculate the energy because the corresponding Hamiltonian contains terms such as $\oint_{S_\infty} A d\psi$ and $\oint_{S_\infty} \mathcal{A} d\sigma$, whose evaluation is unclear. However, these term are absent in the dualized Lagrangian, rendering the calculation more well defined. We shall therefore carry out our calculations, and give a thermodynamic interpretation, using the dualized form of the Lagrangian. This can be viewed as a choice of regularization scheme for giving a definition of mass that is applicable in the rather unusual asymptotic geometry of the magnetized black hole solution.

[6] It is easy to see this in the Wald procedure, where adding a total derivative term $d\nu$ to the Lagrangian will shift the canonical charge associated with Killing vector ξ by $i_\xi\nu$.

In the ADM decomposition, the three-dimensional metric is recast into the form

$$ds_3^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (4.7)$$

It follows that the Hamiltonian defined on the constant t surface takes the form [14]

$$\begin{aligned} H = &\int_{\Sigma^t} d^2x (N\mathcal{H} + N^i \mathcal{H}_i) \\ &- \oint_{S^t_\infty} dx \sqrt{\sigma} \left[\frac{\Delta\phi}{8\pi} Nk + \frac{2}{\sqrt{h}} N^i P_{ij} n^j \right], \end{aligned} \quad (4.8)$$

where $\mathcal{H}$ and $\mathcal{H}_i$ are the total Hamiltonian constraint and the momentum constraint. Using the extrinsic curvature K_{ij} of Σ^t, the momentum P^{ij}, conjugate to h_{ij}, can be expressed as

$$\begin{aligned} P_{ij} &= \frac{\Delta\phi}{16\pi} \sqrt{h} (K_{ij} - K h_{ij}), \\ K_{ij} &= \frac{1}{2N} (\dot{h}_{ij} - 2\nabla_{(i} N_{j)}), \end{aligned} \quad (4.9)$$

where ∇_i is defined with respect to h_{ij}. S^t_∞, defined at $t = \text{const}$ and $r = \infty$, is a hypersurface inside Σ^t with outward unit normal vector n^i. The quantity $k \equiv h^{ij} \nabla_i n_j$ is the trace of the extrinsic curvature of S^t_∞. In general, the above expression for the Hamiltonian diverges. To obtain a meaningful result, we must regularize the Hamiltonian by making a subtraction in the surface term:

$$\begin{aligned} H = &\int_{\Sigma^t} d^2x (N\mathcal{H} + N^i \mathcal{H}_i) \\ &- \oint_{S^t_\infty} dx \sqrt{\sigma} \left[\frac{\Delta\phi}{8\pi} N(k - k_0) + \frac{2}{\sqrt{h}} N^i p_{ij} n^j \right], \end{aligned} \quad (4.10)$$

where k_0 is the extrinsic curvature of S^t_∞ embedded in a certain two-dimensional reference background.

B. Mass of the Kerr-Newman black hole

Before computing the mass of the Kerr-Newman-Melvin black hole, we first show how the three-dimensional Hamiltonian we have derived reproduces the standard mass for the Kerr-Newman black hole.

On shell, we have $\mathcal{H} = \mathcal{H}^i = 0$, and the Hamiltonian receives contributions only from the boundary terms. According to the reduction ansatz (4.2), the three-dimensional metric induced from the four-dimensional Kerr-Newman black hole is given by

$$\begin{aligned} ds^2_{3\,\mathrm{KN}} &= -\Delta \sin^2\theta dt^2 + \Sigma \sin^2\theta \left(\frac{dr^2}{\Delta} + d\theta^2 \right), \\ \rho^2 &= r^2 + a^2 \cos^2\theta, \\ \Delta &= r^2 - 2mr + a^2 + q^2, \\ \Sigma &= (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta. \end{aligned} \quad (4.11)$$

This 3-metric is static, and so $p^{ij} = 0$. The extrinsic curvature of $S^t_{r=r_0}$ in Σ^t can be computed, giving

$$k = \frac{1}{\sqrt{\sigma}} \frac{\partial \sqrt{\sigma}}{\partial n} = \frac{1}{2 \sin\theta} \sqrt{\frac{\Delta}{\Sigma}} \frac{\partial_r \Sigma}{\Sigma} \bigg|_{r=r_0}, \quad (4.12)$$

where $\sigma = g_{\theta\theta}$ is the determinant of the one-dimensional boundary metric and $\partial/\partial n$ is the derivative with respect to the unit normal in the radial direction at $r = r_0$.

To compute k_0, we recall that the reference metric for the four-dimensional Kerr-Newman black hole is the four-dimensional Minkowski metric, which upon dimensional reduction gives rise to the three-dimensional reference metric

$$ds^2 = -R^2 \sin^2\theta dt^2 + R^4 \sin^2\theta \left(\frac{dR^2}{R^2} + d\theta^2 \right). \quad (4.13)$$

The calculation of k_0 requires us to embed $S^t_{r=r_0}$ into the above background in such a way that the metric on $S^t_{r=r_0}$ induced from the reference metric should be isometric to the metric on $S^t_{r=r_0}$ induced from Σ^t. Thus the $t = \text{constant}$ boundary at $R = R_0$ in the reference metric should be matched to the $t = \text{constant}$ boundary at $r = r_0$ in the reduction of the Kerr-Newman metric, implying

$$R_0^4 = \Sigma|_{r=r_0}. \quad (4.14)$$

This gives

$$k_0 = \frac{1}{\sqrt{\sigma_0}} \frac{\partial \sqrt{\sigma_0}}{\partial n} = \frac{2}{\sqrt{\Sigma} \sin\theta} \bigg|_{r=r_0}, \quad (4.15)$$

where $\sigma_0 = g_{\theta\theta}$ is the determinant of the one-dimensional boundary metric in the reference metric (4.13). Bearing in mind that the azimuthal coordinate φ has period $\Delta\varphi = 2\pi$ in the Kerr-Newman metric, we therefore find from (4.10) that

$$E_{\mathrm{KN}} = -\frac{1}{4} \oint_{S^t_\infty} dx \sqrt{\sigma} N(k - k_0) = m, \quad (4.16)$$

which reproduces the mass for the Kerr-Newman black hole.

C. Mass of the Kerr-Newman-Melvin black hole

We now turn to the calculation of the the mass of the Kerr-Newman-Melvin black hole. The calculation closely resembles the previous case, since the dimensionally reduced 3-metric of the Kerr-Newman-Melvin black hole is identical to that for the Kerr-Newman case, given in (4.11). The three-dimensional evaluation of the mass differs in only one respect, namely that the period $\Delta\varphi$ of the azimuthal coordinate is no longer 2π, and so the mass is now given by

$$E_{\rm KNM} = \frac{\Delta\phi}{2\pi} m. \tag{4.17}$$

D. Euclidean action

The Lorentzian action is related to the Hamiltonian by

$$I = \int d^4x (P^{ij}\dot{h}_{ij} + \pi_A \dot{\phi}^A) - \int dt H. \tag{4.18}$$

Therefore, for stationary solutions, the Euclidean action is given by

$$I_E = \beta H. \tag{4.19}$$

The Hamiltonian will include a contribution from the horizon. The total on-shell Hamiltonian then takes the form [14]

$$\begin{aligned} H = &-\oint_{S^t_\infty} dx\sqrt{\sigma}\left[\frac{\Delta\phi}{8\pi} N(k-k_0) + \frac{2}{\sqrt{h}} N^i p_{ij} n^j\right] \\ &-\oint_{S^t_H} dx\sqrt{\sigma}\left[\frac{\Delta\phi}{8\pi} n^i\partial_i N - \frac{2}{\sqrt{h}} N^i p_{ij} n^j\right]. \end{aligned} \tag{4.20}$$

The first is equal to the energy E, and the second term gives rise to $-TS$. Thus the Euclidean action of the dualized action is equal to the Helmholtz free energy $F = E - TS$, suggesting that the dualized action provides a canonical ensemble description for black-hole thermodynamics.

V. CONSERVED CHARGES FOR KERR-NEWMAN-MELVIN BLACK HOLES

A. Kerr-Newman black holes

Before turning to the magnetized black hole metric, let us first illustrate the three-dimensional procedure for calculating the conserved charges by considering the original four-dimensional Kerr-Newman solution itself, for which the reduction to three dimensions gives [7]

$$\begin{aligned} d\bar{s}_3^2 &= \sin^2\theta\Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) - \Delta\sin^2\theta dt^2, \\ \chi &= \frac{aqr\sin^2\theta}{\rho^2}, \qquad \psi = -\frac{q(r^2+a^2)\cos\theta}{\rho^2}, \\ \sigma &= -\frac{2am\cos\theta[r^2(3-\cos^2\theta) + a^2(1+\cos^2\theta)]}{\rho^2} \\ &\quad - \frac{2a^3q^2r\cos\theta\sin^4\theta}{\rho^4}, \end{aligned} \tag{5.1}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2mr + a^2 + q^2, \\ \Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta. \tag{5.2}$$

For the mass, we saw that (4.16) gave the expected result

$$E = m. \tag{5.3}$$

The angular velocity and the electrostatic potential on the horizon are given, in the three-dimensional calculation, by

$$\Omega = -2i_k\bar{\mathcal{A}}|_{r=r_+} = \frac{a}{r_+^2 + a^2}, \\ \Phi_H = -i_k\bar{A}|_{r=r_+} = \frac{qr_+}{r_+^2 + a^2}. \tag{5.4}$$

The electric charge and angular momentum, given by (3.5) and (3.9), are

$$Q = \frac{1}{2}[\psi]^{\theta=\pi}_{\theta=0} = q, \qquad J = \frac{1}{4}[\sigma]^{\theta=\pi}_{\theta=0} = am, \\ \tilde{J} = \frac{1}{4}[\tilde{\sigma}]^{\theta=\pi}_{\theta=0} = am, \tag{5.5}$$

since in this Kerr-Newman example the period $\Delta\varphi$ of the azimuthal coordinate φ is 2π.

As we discussed in Sec. III B, the expression for J given in (3.9) is not invariant under the residual gauge transformations

$$\chi \to \chi + c, \qquad \sigma \to \sigma + 2c\psi, \qquad \psi \to \psi, \tag{5.6}$$

and in fact, from (3.19), we will have

$$J \to J + cQ \tag{5.7}$$

in this case. Thus the fact that J in (5.5) has turned out to give the correct result for the angular momentum is a consequence of happy choice of gauge; as can be seen from (5.1), it is the one in which χ goes to zero on the axis at $\theta = 0$ and $\theta = \pi$, thus implying that it is regular there. Furthermore, it goes to zero at infinity.

By contrast, as we discussed in Sec. III B, the expression (3.11) for $\tilde{J}$ *is* invariant under the residual gauge transformations (5.6), and so it is not subject to the same ambiguities.

Finally, we remark that a simple way to calculate the angular momentum of a dyonically charged Kerr-Newman

[7] Expressions for the three-dimensional fields ψ, χ and σ can be found from those in [7], by specializing to the case where the external magnetic field is set to zero. Note that some sign conventions in [7], associated with the definition of the orientation of the 2-spheres, differ from ours.

black hole is to perform first a duality transformation to the case with purely electric charge, then and use the gauge-invariant expression (3.11).

B. Charges and thermodynamics for the Kerr-Newman-Melvin metrics

Here, we use the Kaluza-Klein formalism of Sec. III to calculate the electric charge and angular momentum for the Kerr-Newman-Melvin solutions, which take the form

$$ds^2 = H\left[-f dt^2 + R^2\left(\frac{dr^2}{\Delta} + \theta^2\right)\right] + \frac{\Sigma \sin^2\theta}{HR^2}(d\phi - \omega dt)^2,$$
$$A = \Phi_0 dt + \Phi_3(d\phi - \omega dt). \tag{5.8}$$

The various functions appearing in the metric and gauge potential are rather complicated, and we refer the reader to [7], where they are presented. The charge and angular momentum are again given by (3.5) and (3.9) or (3.11), where expressions for the scalar fields σ, χ and ψ can be found in Appendices A and B of [7]. The period of φ, determined by the requirement of there being no conical singularity on the axis at $\theta = 0$ and $\theta = \pi$, is now given by [7]

$$\Delta\phi = 2\pi\left[1 + \frac{3}{2}q^2B^2 + 2aqmB^3 + \left(a^2m^2 + \frac{1}{16}q^4\right)B^4\right]. \tag{5.9}$$

It is straightforward to see that the expression for χ given in [7] is nonvanishing on the axis at $\theta = 0$ and $\theta = \pi$: we have

$$\chi|_{\theta=0} = \chi|_{\theta=\pi} = \gamma \equiv \frac{\pi B}{4\Delta\phi}[12q^2 + 24amqB + (q^4 + 16a^2m^2)B^2]. \tag{5.10}$$

It is therefore natural, in the light of the previous calculation for the Kerr-Newman black hole, to make a gauge transformation of the form (5.6) with $c = -\gamma$ before evaluating the gauge-dependent expression (3.9) for the angular momentum. Assuming that we do this, we then find

$$Q = q + 2amB - \frac{1}{4}q^3B^2,$$
$$J = \tilde{J} = am - q^3B - \frac{3}{2}amq^2B^2 - \frac{1}{4}qB^3(8a^2m^2 + q^4) - \frac{1}{16}amB^4(16a^2m^2 + 3q^4). \tag{5.11}$$

(The calculation of Q is discussed in [7].) By having chosen the gauge where χ vanishes on the axis, we obtain the same expression J for the angular momentum as we get from the gauge-invariant expression $\tilde{J}$ given by (3.11).

For the mass, we see from (4.17) that result for the Kerr-Newman-Melvin metric will be just the usual factor m, however now scaled by a factor of $(\Delta\varphi)/(2\pi)$, where $\Delta\varphi$ is the period of the azimuthal angle φ, given in (5.9). Thus we find that the mass is given by

$$E = m\left[1 + \frac{3}{3}q^2B^2 + 2aqmB^3 + \left(a^2m^2 + \frac{1}{16}q^4\right)B^4\right]. \tag{5.12}$$

The area A_H of the outer horizon and the surface gravity κ can be straightforwardly calculated from the Kerr-Newman-Melvin metrics given in [7], leading to

$$A_H = 4\pi\left(1 + a^2m^2B^4 + 2amB^3q + \frac{3B^2q^2}{2} + \frac{B^4q^4}{16}\right)(a^2 + (m + \sqrt{m^2 - a^2 - q^2})^2),$$
$$\frac{\kappa}{8\pi} = \frac{\sqrt{m^2 - a^2 - q^2}}{8\pi(a^2 + (m + \sqrt{m^2 - a^2 - q^2})^2)}. \tag{5.13}$$

Assuming for now that we hold the external magnetic field B fixed, we can expect that the first law should take the form

$$dE = \frac{\kappa}{8\pi}dA_H + \Omega dJ + \Phi dQ, \tag{5.14}$$

where $\Omega = \Omega_H - \Omega_\infty$ is the difference between the angular velocity of the horizon and the angular velocity at infinity, and $\Phi = \Phi_H - \Phi_\infty$ is the potential difference between the horizon and infinity. Because of subtleties associated with the asymptotic structure of the Kerr-Newman-Melvin metrics at infinity it is not obvious how to calculate Ω_∞ and Φ_∞. We can, however, proceed by using our results above for the other thermodynamic quantities, and then seeking solutions for Φ and Ω such that the first law (5.14) holds. We find that solutions do indeed exist. This is in fact nontrivial, since with three independent parameters being varied in (5.14) we have three equations for the two unknowns Ω and Φ.[8] The solutions for Ω and Φ are rather

[8]The fact that a solution exists for Ω and Φ also provides nontrivial support for the validity of our expression (5.12) for the mass of the Kerr-Newman-Melvin solution.

complicated, and we shall not present them in detail here. Later, we shall present leading-order terms in Ω and Φ in useful approximations.

Firstly, however, we remark that we can also allow B to become an additional thermodynamic variable in the first law, which will now be generalized to

$$dE = \frac{\kappa}{8\pi} dA_H + \Omega dJ + \Phi dQ - \mu dB, \tag{5.15}$$

where μ has the interpretation of being the magnetic moment of the system. (Analogous expressions have been obtained by for the case of Einstein-Dilaton-Maxwell theory in the Kaluza-Klein case by Yazadjiev [15].) Again, it is nontrivial that a solution for μ exists. Having obtained μ, which is also rather complicated in general, it is straightforward to verify that the various thermodynamic quantities satisfy the Smarr-like relation

$$E = \frac{\kappa}{4\pi} A_H + 2\Omega J + \Phi Q + \mu B. \tag{5.16}$$

As we mentioned above, the solutions for Ω, Φ and μ are rather complicated in general. It is instructive to look at the leading-order forms of these quantities. Up to linear order in q, we find

$$\mu = aq(1 + a^2 m^2 B^4) + \mathcal{O}(q^2). \tag{5.17}$$

To linear order in B, we find

$$\Omega = \frac{a}{r_+^2 + a^2} - \frac{2qBr_+}{r_+^2 + a^2} + \mathcal{O}(B^2),$$
$$\Phi = \frac{qr_+}{r_+^2 + a^2} + \frac{3aq^2 B}{(r_+^2 + a^2)} + \mathcal{O}(B^2). \tag{5.18}$$

Note that from (5.17) we have, to lowest order, that $\mu = Jq/m$, reproducing the gyromagnetic ratio $g = 2$ as found by Carter [16]. We also see that the second term in the expression for Ω in (5.18) agrees with the standard formula for the Larmor precession frequency $\Omega_L = \mu B/J$, in the limit that we may approximate r_+ by $2m$.

C. The case $q = -amB$

It was shown in [7] that in general the magnetized Kerr-Newman black holes have an ergoregion that extends out to infinity close to the axis of rotation. A special case arises if the charge parameter q of the original Kerr-Newman solution is chosen to satisfy [7]

$$q = -amB, \tag{5.19}$$

where m and a are the mass and rotation parameters of the Kerr-Newman metric. Under these circumstances we find that the conserved charge and angular momentum, given in general by (5.11), simplify considerably, and become

$$Q = amB\sqrt{\frac{\Delta\phi}{2\pi}} = -q\sqrt{\frac{\Delta\phi}{2\pi}} = -Q_0\sqrt{\frac{\Delta\phi}{2\pi}},$$
$$J = am\frac{\Delta\phi}{2\pi} = J_0\frac{\Delta\phi}{2\pi}, \tag{5.20}$$

where $Q_0 = q$ and $J_0 = am$ are the conserved charge and angular momentum of the original Kerr-Newman solution. The period of the azimuthal coordinate is now

$$\frac{\Delta\phi}{2\pi} = \left(1 + \frac{1}{4}a^2 m^2 B^2\right)^2. \tag{5.21}$$

The area of the event horizon, given in general by (5.13), can now be written as

$$A_H = A_H^0 \frac{\Delta\phi}{2\pi}, \tag{5.22}$$

where A_H^0 is the area of the event horizon of the Kerr-Newman black hole. Of course we still also have, from (4.17), that the mass is given by

$$E = E_0 \frac{\Delta\phi}{2\pi}, \tag{5.23}$$

where $E_0 = m$ is the mass of the Kerr-Newman black hole, while the surface gravity κ is, as always, just equal to its value in the Kerr-Newman solution [see Eq. (5.13)].

VI. ENERGY MINIMIZATION

Defining $E_0 = m$ and $J_0 = am$ as the mass and the angular momentum of the Kerr-Newman black hole (i.e. the $B = 0$ specialization), we may eliminate q between the expressions for Q and E in (5.11) and (5.12), thereby obtaining an equation that determines E in terms of Q, J_0 and B:

$$E^3 - E^2(17 + 3B^4 J_0^2)E_0 + \frac{1}{2}E(160 - 192B^4 J_0^2 + 6B^8 J_0^4 + 136B^3 J_0 Q - 11B^2 Q^2)E_0^2$$
$$- \bigg(64 + 48B^4 J_0^2 + 12B^8 J_0^4 + B^{12} J_0^6 - 128 J_0 Q - 32B^7 J_0^3 Q + 68B^2 Q^2 + 17B^6 J_0^2 Q^2$$
$$- 2B^5 J_0 Q^3 + \frac{1}{16}B^4 Q^4\bigg)E_0^3 = 0. \tag{6.1}$$

Extremizing E with respect to Q, while holding E_0, J_0 and B_0 fixed then implies

$$E = \frac{(512BJ_0 + 128B^5J_0^3 - 544Q - 136B^4J_0^2Q + 24B^3J_0Q^2 - B^2Q^3)E_0}{4(11Q - 68BJ_0)}. \tag{6.2}$$

From these equations we can now obtain expressions for $\bar{E}$ and $\bar{Q}$, the values of E and Q at the extremum, as function of E_0, J_0 and B. For Q, we find that $\bar{Q}$ is given by the roots of the factorized polynomial $P(Q) = P_1(Q)P_2^2(Q)$, where

$$\begin{aligned} P_1 &= B^2Q^3 - 12B^3J_0Q^2 + 16(4 + B^4J_0^2)(3Q - 4BJ_0), \\ P_2 &= B^2Q^3 - 30B^3J_0Q^2 - 4(392 - 75B^4J_0^2)Q + 8BJ_0(588 - 125B^4J_0^2). \end{aligned} \tag{6.3}$$

Expanding around $B = 0$ we find just one real root for $P_1(Q) = 0$, giving

$$\begin{aligned} \bar{Q} &= \frac{4}{3}BJ_0 + \frac{8}{81}B^5J_0^3 + \cdots, \\ \bar{E} &= E_0 + \frac{1}{3}E_0B^4J_0^2 + \cdots. \end{aligned} \tag{6.4}$$

For $P_2(Q) = 0$ we find three real roots, with

$$\begin{aligned} \bar{Q} &= 3BJ_0 + \cdots, \\ \bar{E} &= 8E_0 - \frac{3}{4}E_0B^4J_0^2 + \cdots, \end{aligned} \tag{6.5}$$

or

$$\begin{aligned} \bar{Q} &= \pm\frac{28\sqrt{2}}{B} + \frac{27}{2}BJ_0 \mp \frac{21}{32\sqrt{2}}B^3J_0^2 + \cdots, \\ \bar{E} &= -48E_0 \mp 7\sqrt{2}E_0B^2J_0 + \frac{15}{8}E_0B^4J_0^2 + \cdots. \end{aligned} \tag{6.6}$$

VII. FURTHER PROPERTIES OF UNCHARGED BLACK HOLES

In this section, we explore various properties of magnetized Kerr-Newman black holes in the special case where the physical charge Q vanishes.

A. Angular momentum of uncharged magnetized black holes

Using the expressions (5.11) for the physical charge Q and the angular momentum J of a magnetized Kerr-Newman black hole, we may express J in terms of $J_0 = am$ and B in the case that Q is required to be zero. Note that here J_0 is the angular momentum of the unmagnetized Kerr-Newman seed solution. We find that J, J_0 and B are then related by

$$\begin{aligned} &B^4J^3 + B^4J_0(79 + 3B^4J_0^2)J^2 \\ &\quad - (256 - 944B^4J_0^2 + 248B^8J_0^4 - 3B^{12}J_0^6)J \\ &\quad + J_0(4 + B^4J_0^2)^4 = 0. \end{aligned} \tag{7.1}$$

Expanding in powers of B, the branch that reduces to $J = J_0$ in the case that B vanishes gives

$$J = J_0 + 5B^4J_0^3 + 21B^8J_0^5 + 94B^{12}J_0^7 + 454B^{16}J_0^9 + \cdots. \tag{7.2}$$

In order that J remain real the product B^2J_0 should not exceed a maximum value, given by

$$B^2J_0|_{\max} = \frac{2}{3\sqrt{3}}. \tag{7.3}$$

This corresponds to

$$J|_{\max} = \frac{128}{27}J_0|_{\max} = \frac{256}{81\sqrt{3}B^2}. \tag{7.4}$$

B. Meissner effect for extremal black holes

The electromagnetic field in the magnetized Kerr-Newman solution takes the form $A = \bar{A} + \chi(d\phi + 2\bar{\mathcal{A}}dt)$, as in (3.1), where the various quantities may be found in Appendix B of [7]. The magnetic flux threading the upper hemisphere S_+^2 of the horizon is given by

$$\mathcal{F}_H = \frac{1}{4}\pi\int_{S_+^2} F = \frac{\Delta\phi}{4\pi}[\chi]_{\theta=\frac{1}{2}\pi}^{\theta=\pi}. \tag{7.5}$$

Consider the case where the physical charge Q on the black hole vanishes. From (5.11), this is achieved if the magnetic field is given by $B = B_\pm$ where

$$B_\pm = \frac{2}{q^3}[2am \pm \sqrt{4a^2m^2 + q^4}]. \tag{7.6}$$

Suppose furthermore that the black hole is extremal, which means that the inner and outer horizons at $r = r_\pm$ coincide at

$$r_\pm = m, \qquad m = \sqrt{q^2 + a^2}. \tag{7.7}$$

Inserting the zero-charge condition (7.6) and the extremality condition (7.7) into the expression for χ given in [7], we find that χ is constant on the horizon, and it is given by

$$\chi|_H = \pm \frac{q^3}{2(q^2+2a^2)}. \quad (7.8)$$

Evidently therefore, from (7.5), it follows that the magnetic flux threading the upper hemisphere of the horizon is zero. This is consistent with much previous work (see [12] for a review).

VIII. ANGULAR MOMENTUM IN STU SUPERGRAVITY

In this section, we extend our earlier discussion of the conserved angular momentum in Einstein-Maxwell theory to the case of the four-dimensional STU model, which is $\mathcal{N}=2$ supergravity coupled to three vector multiplets. For our purposes, it suffices to focus just on the bosonic sector of the theory. The bosonic Lagrangian, in the notation of [17], is

$$\begin{aligned}\mathcal{L}_4 = {}& R*\mathbb{1} - \frac{1}{2}*d\varphi_i\wedge d\varphi_i - \frac{1}{2}e^{2\varphi_i}*d\chi_i\wedge d\chi_i - \frac{1}{2}e^{-\varphi_1}(e^{\varphi_2-\varphi_3}*F_{(2)1}\wedge F_{(2)1}\\ &+ e^{\varphi_2+\varphi_3}*F_{(2)2}\wedge F_{(2)2} + e^{-\varphi_2+\varphi_3}*\mathcal{F}^1_{(2)}\wedge\mathcal{F}^1_{(2)} + e^{-\varphi_2-\varphi_3}*\mathcal{F}^2_{(2)}\wedge\mathcal{F}^2_{(2)})\\ &+\chi_1(F_{(2)1}\wedge\mathcal{F}^1_{(2)} + F_{(2)2}\wedge\mathcal{F}^2_{(2)}),\end{aligned} \quad (8.1)$$

where the index i labeling the dilatons ϕ_i and axions χ_i ranges over $1\le i\le 3$. The four field strengths can be written in terms of potentials as

$$\begin{aligned}F_{(2)1} &= dA_{(1)1} - \chi_2 d\mathcal{A}^2_{(1)},\\ F_{(2)2} &= dA_{(1)2} + \chi_2 d\mathcal{A}^1_{(1)} - \chi_3 dA_{(1)1} + \chi_2\chi_3 d\mathcal{A}^2_{(1)},\\ \mathcal{F}^1_{(2)} &= d\mathcal{A}^1_{(1)} + \chi_3 d\mathcal{A}^2_{(1)},\\ \mathcal{F}^2_{(2)} &= d\mathcal{A}^2_{(1)}.\end{aligned} \quad (8.2)$$

A. Derivation of the conserved angular momentum

The conserved charge associated with a diffeomorphism ξ can be calculated using the standard Wald procedure. Thus, we first calculate $\delta\mathcal{L}(\Phi) = d\Theta + \text{e.o.m.}$, where all the fields Φ are varied using the Lie derivatives $\delta\Phi = \mathfrak{L}_\xi\Phi$. For example, for the metric we have $\delta g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$, and for gauge potentials $\delta A_\mu = \xi^\nu\nabla_\nu A_\mu + A_\nu\nabla_\mu\xi^\nu$. In the standard way, we then define

$$\mathcal{J} = \Theta - i_\xi\mathcal{L}, \quad (8.3)$$

where i_ξ denotes the contraction of the vector ξ with the Lagrangian 4-form $\mathcal{L}$, as defined in footnote [3]. It follows that $d\mathcal{J}=0$ and hence we can write

$$\mathcal{J} = -d\mathcal{P}. \quad (8.4)$$

After considerable algebra, we find that for the Lagrangian (8.1) we shall have

$$\mathcal{P} = \mathcal{P}_{\text{Ein}} + \mathcal{P}_{\text{Kin}} + \mathcal{P}_{\text{CS}}, \quad (8.5)$$

where

$$\begin{aligned}\mathcal{P}_{\text{Ein}} &= *d\xi,\\ \mathcal{P}_{\text{Kin}} &= e^{-\varphi_1+\varphi_2-\varphi_3}*F_{(2)1}\xi^\mu(A_{\mu 1} - \chi_2\mathcal{A}^2_\mu)\\ &\quad + e^{-\varphi_1+\varphi_2+\varphi_3}*F_{(2)2}\xi^\mu(A_{\mu 2} + \chi_2\mathcal{A}^1_\mu - \chi_3 A_{\mu 1} + \chi_2\chi_3\mathcal{A}^2_\mu)\\ &\quad + e^{-\varphi_1-\varphi_2+\varphi_3}*\mathcal{F}^1_{(2)}\xi^\mu(\mathcal{A}^1_\mu + \chi_3\mathcal{A}^2_\mu) + e^{-\varphi_1-\varphi_2-\varphi_3}*\mathcal{F}^2_{(2)}\xi^\mu\mathcal{A}^2_\mu,\\ \mathcal{P}_{\text{CS}} &= -\chi_1[(\xi^\mu A_{\mu 1})d\mathcal{A}^1_{(1)} + (\xi^\mu\mathcal{A}^1_\mu)dA_{(1)1} + (\xi^\mu A_{\mu 2})d\mathcal{A}^2_{(1)} + (\xi^\mu\mathcal{A}^2_\mu)dA_{(1)2}].\end{aligned} \quad (8.6)$$

Here $\mathcal{P}_{\text{Ein}}$ is the contribution from the Einstein-Hilbert term in (8.1), $\mathcal{P}_{\text{Kin}}$ is the contribution from the four kinetic terms for the gauge field strengths, and $\mathcal{P}_{\text{CS}}$ is the contribution from the Chern-Simons terms.

We now make the spacelike dimensional reduction

$$ds_4^2 = e^{-\varphi_4}d\bar{s}_3^2 + e^{\varphi_4}(d\phi + \bar{\mathcal{B}}_{(1)})^2, \quad (8.7)$$

and

$$\begin{aligned}A_{(1)1} &= \bar{A}_{(1)1} + \sigma_1(d\phi + \bar{\mathcal{B}}_{(1)}), \qquad A_{(1)2} = \bar{A}_{(1)2} + \sigma_2(d\phi + \bar{\mathcal{B}}_{(1)}),\\ \mathcal{A}^1_{(1)} &= \bar{\mathcal{A}}^1_{(1)} + \sigma_3(d\phi + \bar{\mathcal{B}}_{(1)}), \qquad \mathcal{A}^2_{(1)} = \bar{\mathcal{A}}^2_{(1)} + \sigma_4(d\phi + \bar{\mathcal{B}}_{(1)}),\end{aligned} \quad (8.8)$$

otherwise following the notation of the timelike reduction described in [17]. In particular, in three dimensions the four reduced field strengths and the Kaluza-Klein field strength $\bar{\mathcal{G}}_{(2)} = d\bar{\mathcal{B}}_{(1)}$ are reexpressed in terms of scalar fields, by means of dualizations [17]:

$$\begin{aligned}
-e^{-\varphi_1+\varphi_2-\varphi_3+\varphi_4} \bar{*}\bar{F}_{(2)1} &= d\psi_1 + \chi_3 d\psi_2 - \chi_1 d\sigma_3 - \chi_1\chi_3 d\sigma_4, \\
-e^{-\varphi_1+\varphi_2+\varphi_3+\varphi_4} \bar{*}\bar{F}_{(2)2} &= d\psi_2 - \chi_1 d\sigma_4, \\
-e^{-\varphi_1-\varphi_2+\varphi_3+\varphi_4} \bar{*}\bar{\mathcal{F}}^1_{(2)} &= d\psi_3 - \chi_2 d\psi_2 - \chi_1 d\sigma_1 + \chi_1\chi_2 d\sigma_4, \\
-e^{-\varphi_1-\varphi_2-\varphi_3+\varphi_4} \bar{*}\bar{\mathcal{F}}^2_{(2)} &= d\psi_4 + \chi_2 d\psi_1 - \chi_3 d\psi_3 - \chi_1 d\sigma_2 + \chi_2\chi_3 d\psi_2 \\
&\quad - \chi_1\chi_2 d\sigma_3 + \chi_1\chi_3 d\sigma_1 - \chi_1\chi_2\chi_3 d\sigma_4, \\
e^{2\varphi_4} \bar{*}\bar{\mathcal{G}}_{(2)} &= d\chi_4 + \sigma_1 d\psi_1 + \sigma_2 d\psi_2 + \sigma_3 d\psi_3 + \sigma_4 d\psi_4.
\end{aligned} \tag{8.9}$$

We then find after some algebra that with $\xi = \partial/\partial\varphi$ we shall have

$$\begin{aligned}
\mathcal{P}_{\rm Ein} &= (d\chi_4 + \sigma_1 d\psi_1 + \sigma_2 d\psi_2 + \sigma_3 d\psi_3 + \sigma_4 d\psi_4)\wedge d\phi + \cdots, \\
\mathcal{P}_{\rm Kin} &= [-\sigma_1 d\psi_1 - \sigma_2 d\psi_2 - \sigma_3 d\psi_3 - \sigma_4 d\psi_4 + \chi_1 d(\sigma_1\sigma_3 + \sigma_2\sigma_4)]\wedge d\phi + \cdots, \\
\mathcal{P}_{\rm CS} &= -\chi_1 d(\sigma_1\sigma_3 + \sigma_2\sigma_4)\wedge d\phi + \cdots,
\end{aligned} \tag{8.10}$$

where the ellipses denote terms that have vanishing pullback to the 2-sphere. Thus, from (8.5) we conclude that $\mathcal{P} = d\chi_4\wedge d\phi + \cdots$, and so the conserved charge associated with the Killing vector $\xi = (\Delta\phi/(2\pi))\partial/\partial\phi$ is given by

$$J = \frac{1}{16\pi}\int_{S^2} \mathcal{P} = \frac{(\Delta\phi)^2}{32\pi^2}\int d\chi_4 = \frac{(\Delta\phi)^2}{32\pi^2}[\chi_4]^{\theta=\pi}_{\theta=0}. \tag{8.11}$$

[The $(\Delta\phi/(2\pi))$ factor in the choice of the Killing vector takes account of the fact that angular momentum should be defined with respect to a canonically normalized azimuthal angle having period 2π.]

This result is the analogue of the expression we derived in (3.9) for the angular momentum for the Einstein-Maxwell black holes. It also has the same feature as in that case, of not being invariant under gauge transformations of the electromagnetic potentials. Specifically, we have four Abelian $U(1)$ gauge symmetries in the STU model, under which the potentials transform as

$$A^{[i]}_{(1)} \to A^{[i]}_{(1)}{}' = A^{[i]}_{(1)} + d\lambda_i, \tag{8.12}$$

where $A^{[i]}_{(1)}$ for $i = 1$, 2, 3 and 4 denotes $(A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)})$ respectively. The subset of gauge transformations where

$$\lambda_i = \bar{\lambda}_i + c_i\phi, \tag{8.13}$$

with $\bar{\lambda}_i$ being independent of ϕ and c_i being constants, preserve the form of the Kaluza-Klein reductions (8.8). For these gauge transformations, the three-dimensional gauge potentials and the σ_i fields therefore transform as

$$\bar{A}^{[i]}_1{}' = \bar{A}^{[i]}_1 - c_i\bar{\mathcal{B}}_{(1)} + d\bar{\lambda}_i, \qquad \sigma_i' = \sigma_i + c_i. \tag{8.14}$$

From these, it follows that the quantities $d\bar{A}^{i]}_{(1)} + \sigma_i d\bar{\mathcal{B}}_{(1)}$ from which three-dimensional field strengths $\bar{F}^{[i]}_{(2)}$ are constructed, and hence the three-dimensional field strengths themselves, are inert under the gauge transformations. This in turn implies that the scalar fields ψ_i are inert,

$$\psi_i{}' = \psi_i. \tag{8.15}$$

Finally, since $\bar{\mathcal{F}}_{(2)}$ is inert, it follows from (8.14) and (8.15) that χ_4 transforms as

$$\chi_4' = \chi_4 - \sum_i c_i\psi_i. \tag{8.16}$$

Thus we see that under the gauge transformations (8.12), the angular momentum J defined in (8.11) transforms as

$$J' = J - \frac{\Delta\phi}{8\pi}\sum_i c_i Q_i, \tag{8.17}$$

where

$$Q_i = \frac{\Delta\phi}{4\pi}[\psi_i]^{\theta=\pi}_{\theta=0} \tag{8.18}$$

are the electric charges carried by the four field strengths.

As in the Einstein-Maxwell case discussed earlier, we can derive a different expression for the angular momentum, which *is* gauge invariant, by performing dualizations of all the four gauge fields. This is easily done in the three-dimensional description, where it amounts to sending σ_i, ψ_i and χ_4 to tilded quantities, defined by

$$\tilde{\sigma}_i = \psi_i, \qquad \tilde{\psi}_i = \sigma_i, \qquad \tilde{\chi}_4 = \chi_4 + \sum_i \sigma_i \psi_i. \quad (8.19)$$

Repeating the calculation of the angular momentum for the dualized theory will give

$$\tilde{J} = \frac{(\Delta\phi)^2}{32\pi^2} [\tilde{\chi}_4]_{\theta=0}^{\theta=\pi}. \quad (8.20)$$

Using our results above for the gauge transformations of σ_i, ψ_i and χ_4, it is easily seen that $\tilde{\chi}_4$, and hence $\tilde{J}$, is gauge invariant. We can then argue, in a manner analogous to our argument in the Einstein-Maxwell case, that (8.20) would be the appropriate expression to use if all four of the charges carried by the gauge fields were electric.

B. Angular momentum for the magnetized static STU-model black holes

In the case of the four-charge black holes in the STU model discussed in [17], and with external fields in [13], the field strengths numbered 1 and 3 carry magnetic charges, whilst those numbered 2 and 4 carry electric charges. It follows, therefore, that rather than using (8.20) directly in order to calculate the angular momentum, we should first "undualize" the contributions associated with fields 1 and 3, meaning that $\tilde{\chi}_4$ in (8.20), which was defined in (8.19), should be replaced by $\tilde{\chi}_4 - \sigma_1\psi_1 - \sigma_3\psi_3$. Thus the proposal for the angular momentum in this case is now

$$J_e = \frac{(\Delta\phi)^2}{32\pi^2} [\chi_4 + \sigma_2\psi_2 + \sigma_4\psi_4]_{\theta=0}^{\theta=\pi}. \quad (8.21)$$

Substituting the expressions for the scalar fields obtained in [13], we obtain the result

$$J_e = \frac{1}{4}\Pi_q \sum_{i=1}^{4} \frac{B_i}{q_i} + \frac{1}{16}\Pi_B \Pi_q \sum_{i=1}^{4} \frac{q_i}{B_i}, \quad (8.22)$$

for the angular momentum of the magnetized four-charge black holes immersed in the background of the four external fields B_i, where $\Pi_B = \prod_i B_i$ and $\Pi_q = \prod_i q_i$. Here q_i are the four charges of the original static black hole solutions, prior to the magnetization. This expression reduces, as it should, to $\tilde{J}$ given in (5.11) if we set all four charge parameters q_i equal and set $a = 0$ in (5.11).

An alternative way to calculate the angular momentum is to use the four-charge analogue of the expression (3.9) that we considered in the Einstein-Maxwell case. In the present context, this amounts to starting from the expression (8.11), and then dualizing the contributions from fields 1 and 3 to take account of the fact that they actually carry magnetic charges. This gives

$$J_m = \frac{(\Delta\phi)^2}{32\pi^2} [\chi_4 + \sigma_1\psi_1 + \sigma_3\psi_3]_{\theta=0}^{\theta=\pi}. \quad (8.23)$$

Since (8.11) is gauge dependent, it is then necessary to perform gauge transformations to ensure that the four functions $(\sigma_1, \psi_2, \sigma_3, \psi_4)$ vanish on the axis at $\theta = 0$ and $\theta = \pi$. After doing this, we obtain a result that agrees with the gauge-invariant one given in (8.22).

IX. CONCLUSIONS

In this paper we have obtained expressions for the energy and angular momentum of magnetized Kerr-Newman black holes. We showed how these quantities can be conveniently calculated by making a Kaluza-Klein reduction of the four-dimensional Einstein-Maxwell theory, and the black hole solutions, on the azimuthal coordinate φ. Using these expressions, we have verified the first law of thermodynamics and the associated Smarr formulas for rotating black holes immersed in an external magnetic field. We also extended the the first law to include variations of the magnetic field, and hence we obtained the induced magnetic moment. In an attempt to make contact with some early work of Wald in which the magnetic field was treated at the test level, ignoring backreaction, we have calculated the electric charge that minimizes the energy, holding the initial energy and angular momentum fixed and at constant magnetic field. Our results resemble qualitatively those of Wald but differ quantitatively. Finally we extended our calculation of the angular momentum to the case of the STU model of four-dimensional $\mathcal{N} = 2$ supergravity coupled to three vector multiplets, in preparation for a future paper [13] on that subject.

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