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Half-flat quantum hair

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By wrapping $D3$-branes over 3-cycles on a half-flat manifold, we construct an effective supersymmetric black hole in the $\mathcal{N} = 2$ low-energy theory in four dimensions. Specifically, we find that the torsion cycles present in a half-flat compactification, corresponding to the mirror symmetric image of electric Neveu-Schwarz flux on a Calabi–Yau manifold, manifest in the half-flat black hole as *quantum hair*. We compute the electric and magnetic charges related to the quantum hair and also the mass contribution to the effective black hole. We find that by wrapping a number of $D3$-branes equal to the order of the discrete group associated to the torsional part of the half-flat homology, the effective charge and mass terms vanish. We compute the variation of entropy and the corresponding temperature associated with the loss of quantum hair. We also comment on the equivalence between canceling Freed–Witten anomaly and the assumption of self-duality for the 5-form field strength. Finally from a K-theoretical perspective, we compute the presence of discrete Ramond–Ramond charge of D-branes wrapping torsional cycles in a half-flat manifold.

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I. INTRODUCTION

The inclusion of flux backgrounds in the search of string compactification scenarios leading to fully symmetric four-dimensional spaces with some or all moduli stabilized has been studied in a huge effort and detail in the last decade (see, for example, Ref. [1] and references therein). Within this context, there have been enormous advances in the construction of string models in which (supersymmetric) Standard-Model-like scenarios are immersed ([2–4]).

On the other hand, supersymmetric black holes (SBHs) have been constructed in the context of type II string theory fluxless compactifications on a Calabi–Yau (CY) manifold by wrapping Dp-branes on internal p cycles [5–16] and in the corresponding low-energy limit given by supergravity $\mathcal{N} = 2$ [17–22]. Integer electric and magnetic charges are computed from the corresponding massless Ramond–Ramond (RR) potential fields. However, as it is well known, string compactifications on Calabi–Yau manifolds are far from being realistic for various reasons, among which are the moduli stabilization problem and the *a priori* selection of vibrational modes of the quantum string.

Hence, it would be desirable to construct black holes on the grounds of a more general theory where RR and (Neveu-Schwarz–Neveu-Schwarz) NS-NS fluxes are turned on. The main problem arises from the fact that in the most general background, since the fluxes are translated into the mirror symmetric manifold as torsion contributions to the Levi-Cività connection, most of the forms [as the holomorphic (3,0) form Ω and the Kähler form J] do not close under the standard differential operator d, and the relation among harmonic fields and standard cohomology is lost. Without this identification, effective theories in four dimensions cannot being constructed as easily as in the CY case.

Huge efforts have been made in the past years to construct $\mathcal{N} = 2$ gauged supergravities from compactifications of type II strings on generalized manifolds, as half-flat manifolds, such as internal spaces with $SU(3)$ or $SU(3) \times SU(3)$ structures, etc. Construction of black hole solutions within the context of gauged supergravity has also been studied in the last years [23–26]. On the other hand, nonperturbative corrections to the prepotential in type IIA string theory compactifications on Calabi–Yau manifolds and for self-mirror manifolds have been considered recently to find analytical black hole solutions involving nonextremal solutions as well as an interesting solution describing a supersymmetric black hole [27,28].

It is important to remark that the inclusion of NS–NS fluxes in the construction of SBHs could made some of the involved branes unstable to decay into closed strings, as shown in Ref. [29] due to the presence of a Freed–Witten (FW) anomaly [30–32]. If the FW anomaly is cancelled, a SBH constructed in such scenarios seems to be stable [33]. Nevertheless, the amount of fluxes considered must be small; otherwise, we cannot assure a small backreaction due to fluxes in the supergravity approach. We are interested precisely on this point and study the effects on the black hole by wrapping $D3$-branes on an internal manifold in which the backreaction has been considered.

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Therefore, one can foresee two options for constructing a SBH: by wrapping $D3$-branes on a CY manifold threaded with a slight amount of NS–NS flux or by wrapping $D3$-branes on a generalized manifold [29,33–36]. The main goal of this work is to establish the first steps to study the physics of a black hole constructed by wrapping D-branes on a generalized manifold.

Concretely, we focus our study on the construction of SBHs by *wrapping D-branes on torsional cycles of a half-flat manifold* without considering quantum corrections to the superpotential or prepotential. In this sense, we are studying at tree level how to construct SBHs on manifolds which already have backreacted to the presence of NS–NS fluxes. The backreaction is then manifested by the appearance of torsional components on which wrapped D-branes contribute to extra degrees of freedom to the low-energy theory identified with (a supersymmetric version of) *quantum hair* studied in Refs. [37–43] and measure it by the presence of a four-dimensional string.

Our work is organized as follows. In Sec. II, we briefly review the standard construction of a SBH in a type IIB string CY compactification by wrapping $D3$-branes on supersymmetric 3-cycles. In Sec. III, we compute and review some of the most important properties of a half-flat manifold including the derivation of cohomology groups and the expansion of some fields in terms of torsional forms, followed by the calculation of the electric and magnetic discrete charges and the torsional mass contribution. With this, we can compute the entropy variation of the state by increasing the number of torsional $D3$-branes and the associated temperature once the black hole loses its hair as the number of torsional branes reaches the order of the discrete group. Finally, in Sec. IV we use K theory in order to compute the discrete charge associated to D-branes in torsional cycles with the purpose of elucidating the nature of discrete charge without using an extra object as a four-dimensional string manifested as the Aharanov–Bohm effect. Our final comments are given in Sec. V followed by a couple of Appendices. In Appendix A, we refer to the usual notation for the symplectic cohomology basis, while in Appendix B, we explicitly show a review on the construction of the low-energy theory corresponding to a type IIB compactification on a half-flat manifold on which the 5-form field strength is expanded in terms of torsional forms.

II. SUPERSYMMETRIC BLACK HOLES FROM WRAPPED $D3$-BRANES

It is well known from the past years that a SBH in four dimensions can be constructed by wrapping D-branes in internal nontrivial cycles. The physics of the effective Bogomolnyi-Prasad-Sommerfield (BPS) object can be derived from different approaches [5,6,8,10,11,15,35]. According to our purposes, we would like to review the construction of a SBH in the type IIB scenario in which $D3$-branes wrap internal 3-cycles of a CY manifold X_3, closely following Ref. [6].

A massive SBH is obtained by wrapping a large number of $D3$-branes on the corresponding cycles (otherwise, they simply describe elementary massive particles). The gauge field $\mathcal{A}_1$ related to the electric and magnetic charges in the effective four-dimensional $\mathcal{N}=2$ theory is constructed from the self-dual RR field strength $\mathcal{F}_5$, given by $\mathcal{F}_5=\mathcal{F}_2\wedge F_3$, with $d\mathcal{A}_1=\mathcal{F}_2$, and through the decomposition driven by compactifying the extra six dimensions on X_3.

To see that, consider N $D3$-branes wrapping an internal 3-cycle $\mathcal{C}_3\subset X_3$ given by a linear combination of the symplectic basis of 3-cycles (A^I,B_I) with $\mathcal{PD}_6(A^I)=\beta^I$, $\mathcal{PD}_6(B_I)=\alpha_I$, and $I=0,1,\ldots,h^{(2,1)}(X_3)$. The basis of 3-forms (α_I,β^I) is chosen to satisfy, as usual,

$$\int_{A^J}\alpha_I=-\int_{B_I}\beta^J=\delta_I^J. \tag{2.1}$$

Defining the RR potential related to these $D3$-branes by

$$C_4=A_1\wedge\sum_I(e^I\alpha_I-m_I\beta^I), \tag{2.2}$$

the (non-self-dual) electric part of $\mathcal{F}_5$ can be written as

$$F_5=F_2\wedge F_3=F_2\wedge\sum_I(e^I\alpha_I-m_I\beta^I). \tag{2.3}$$

The electric charge Q_e is computed by integrating $*_{10}F_5$ over a 5-cycle Γ_5 identified as the boundary of $\Gamma_6=\mathbf{B}^3\times\Gamma_3$. Actually, since Γ_3 belongs to $\mathrm{H}_3(X_3;\mathbf{Z})$, only the four-dimensional component of this cycle has boundary. Both electric and magnetic charges are then given by

$$\begin{aligned}Q_e&=\int_{\mathbf{S}^2\times\Gamma_3}\star F_2\wedge\mathcal{PD}_6(\mathcal{C}_3)=-qN,\\ Q_m&=\int_{\mathbf{S}^2\times\mathcal{C}_3}F_2\wedge\mathcal{PD}_6(\Gamma_3)=pN,\end{aligned} \tag{2.4}$$

where $\mathcal{PD}_6(\mathcal{C}_3)=*F_3$, $\mathcal{PD}_6(\Gamma_3)=F_3$ with the intersection number $\Gamma_3\cap\mathcal{C}_3=N$. From this, it follows that

$$\mathcal{C}_3=-p_IA^I+q^IB_I, \tag{2.5}$$

with

$$\begin{aligned}q^I&=e^JA_J^I-m_JC^{IJ},\\ p_I&=-e^JB_{IJ}-m_JA_I^J,\end{aligned} \tag{2.6}$$

with the matrices A, B, and C defined through integration of the wedge product between (α_I,β^I) and their duals as depicted in relations (A2). The total charge of the system can also be computed by integrating the self-dual 5-form $\mathcal{F}_5 = F^I \wedge \alpha_I - G_I \wedge \beta^I$ over the cycle $\mathcal{C}_3 \cup \Gamma_3$ as

$$Q_T = \int_{\mathbf{S}^2 \times (\mathcal{C}_3 \cup \Gamma_3)} \mathcal{F}_5 = N(p - q), \tag{2.7}$$

where

$$F^I = e^I F_2 + q^I \star F_2, \quad \text{and} \quad G_I = m_I F_2 + p_I \star F_2. \tag{2.8}$$

On the other hand, since $D3$-branes are BPS states of the theory, it is expected that the pointlike object in the effective theory should be a BPS object as well. This means that it represents a massive state in the short multiplet of the $\mathcal{N} = 2$ supersymmetric theory with a metric given by [17]

$$ds^2 = -e^{2U(\tau)} dt^2 + \frac{e^{-2U(\tau)}}{\tau^4} d\tau^2 + \frac{e^{-2U(\tau)}}{\tau^2} d\Omega^2, \tag{2.9}$$

where $U(\tau)$ vanishes as $\tau \to 0$ and diverges at the horizon. Hence, the RR 5-form is self-dual in this metric provided

$$F_2 = \sin\theta d\theta \wedge d\phi \quad \text{and} \quad \star F_2 = e^{2U} dt \wedge d\tau. \tag{2.10}$$

The effective scalar potential $\mathcal{V}(r)$, computed by dimensionally reducing the 10-dimensional term $F_5 \wedge *_{10} F_5$ (with the corresponding self-duality being imposed afterward) reads

$$\mathcal{V}(r) = \tau^4 \mathcal{V}_{\rm BH}, \tag{2.11}$$

where

$$\begin{aligned} \mathcal{V}_{\rm BH} &= \int_{X_6} F_3 \wedge * F_3 = e^{\mathcal{K}} (D_I \mathcal{W} D_{\bar{J}} \bar{\mathcal{W}} K^{I\bar{J}} + 3|\mathcal{W}|^2), \\ &= -e^I p_I + m_I q^I = N, \end{aligned} \tag{2.12}$$

with the superpotential $\mathcal{W}$ given by

$$\mathcal{W} = \int_{X_6} F_3 \wedge \Omega_3. \tag{2.13}$$

Being a BPS state, a SBH is extremal by construction. The sum of the squared charges $Q^2 = Q_e^2 + Q_m^2$ equals the mass of the expected supersymmetric BPS object in the four-dimensional $\mathcal{N} = 2$ supergravity theory on which one obtains that $F_3 = \mathrm{Re}(C\Omega_3)$ for an arbitrary complex constant C with Ω_3 being the unique holomorphic (3, 0) form in $\mathrm{H}^3(X_3\mathbf{Z})$. Although the SBH's mass is formally computed through the use of the special symplectic geometry [17,44,45], we shall reconstruct it following the prescription given in Refs. [11,46], which fits our purposes better.

The mass can be directly computed from the Dirac–Born–Infeld action of the $D3$-branes wrapping $\mathcal{C}_3$,

$$S_{D3} = \int_{\gamma \times \mathcal{C}_3} \sqrt{-G} = -M_{\rm BPS} \int_\gamma ds, \tag{2.14}$$

where $G_{\rm MN}$ is the world-volume metric of the $D3$-branes, decomposing as $G_{\rm MN} = \mathbf{1} \otimes g_{\rm mn}$. After assuming preservation of supersymmetry in four dimensions (which implies that $\mathcal{C}_3$ is a special Lagrangian cycle), it is possible to show that

$$M_{\rm BPS}^2 = e^{\mathcal{K}} |\mathcal{W}|^2 = \frac{1}{2\mathrm{Im}(\bar{\tau}_{IJ} X^I \bar{X}^J)} |e_I X^I - m^I F_I|^2, \tag{2.15}$$

where $\mathcal{K} = -\ln i \int \Omega_3 \wedge \bar{\Omega}_3 = -\ln i(\bar{X}^I F_I - X^I \bar{F}_I)$ and $F_I = \tau_{IJ} X^J$. As noticed in Ref. [6], the total charge and the mass are equal, as corresponding to a BPS object, by considering only the graviphoton mass. At the end of the day, we have a BPS pointlike object with a horizon, within the extremal condition on which its charge equals its mass. It is interesting to notice that the black hole charge in four dimensions can be understood as a linking number among the three-dimensional ball with S^2 as its boundary and a pointlike object. From the internal space point of view, the quantity $\int_{\mathcal{C}_3} F_3$ also represents a linking number[1] among the internal components of the RR 3-form and the cycle on which the integration is performed. One could say after such observation that electric and magnetic charges of four-dimensional objects constructed from extended branes in higher-dimensional spaces correspond to an arrangement of those branes such that there is a linking number in four dimensions and in the internal space. However, the fact the mass equals its charge is not so evident from this perspective since the mass is computed through an integral which does not represent a linking number. Mass and charge are equal due to the fact that the cycles over which they are computed are supersymmetric.

Finally, the field content in the background theory with $\mathcal{N} = 2$ in four dimensions is constructed from an expansion of the 10-dimensional massless RR fields on a basis of cohomological forms in the internal space in order to describe massless states in four dimensions. The existence of scalar fields in a background dominated by the black hole is not in contradiction with the famous no-hair theorem involving a classical black hole. The no-hair theorems applied to four-dimensional black holes can be followed from the fact that all degrees of freedom related to the SBH are computed from surface integrals of massless sates in four dimensions. Values of nonzero scalar fields are fixed at the horizon through the so-called attractor

[1]This is also reflected in the definition of Poincaré duals between (A^I, B_I) and (α_I, β^I).

mechanism [47]. Finally notice that, since there are not extra fluxes, especially NS–NS fluxes, all $D3$-branes are free from the FW anomaly. This becomes an important restriction in the construction of black holes in a background threaded with NS–NS fluxes.

III. BLACK HOLES FROM HALF-FLAT MANIFOLDS

So far, we have reviewed the standard construction of supersymmetric black holes by wrapping $D3$-branes on homological cycles of a Calabi–Yau manifold. In this section, we shall concentrate our analysis on constructing black holes by wrapping $D3$-branes on a half-flat manifold. As we shall see, this implies wrapping $D3$-branes on torsional cycles and leads to the existence of extra degrees of freedom associated to the torsional group [42].

A. Why select half-flat manifolds?

As studied in the last few years, generalized CY manifolds can be characterized by torsional components of the Levi-Cività connection as representations of the internal structure group $SU(3)$ [1]. Under this perspective, the Kähler and the holomorphic (3, 0) forms satisfy

$$dJ = \frac{3}{2}\mathrm{Im}(\bar{W}_1\Omega_3) + W_4\wedge J + W_3,$$
$$d\Omega_3 = W_1 J^2 + W_2\wedge J + \bar{W}_5\wedge\Omega_3, \tag{3.1}$$

where W_i's are the representations on $SU(3)$ of the intrinsic torsional components of the connection ∇. The intrinsic torsion $\mathcal{T}$ is defined as the antisymmetrization of the contorsion κ, which in turn is given as follows. Consider differentiation of a generic p-cochain $d\sigma_p = (\nabla\sigma)_{\mu_1\cdots\mu_p} dx^{\mu_1}\cdots dx^{\mu_p}$. If σ_p is not closed under d, then $d\sigma_p = \kappa\sigma_p$, where κ defines the contorsion. Then we can define a differential operator $d^{(T)}$ with torsion such that $d^{(T)}\sigma_p = 0$ with

$$d^{(T)} = d - \kappa. \tag{3.2}$$

To wrap D3-branes on cycles of the internal manifold Y_3, we concentrate on those manifolds where the nonvanishing terms of dJ and $d\Omega_3$ have the following two properties: 1) torsional components of the connection are represented by torsional components of the (co)homology such that it is still possible to wrap $D3$-branes in a geometrical way, and 2) we need to relate the $SU(3)$ representations of the cohomology groups of Y_3 with the $SU(3)$ representations of the intrinsic torsion. This forces us to consider the case in which $W_4 = W_5 = 0$ since the corresponding cohomology groups vanish in a CY manifold.

There is a variety of manifolds for which $W_4 = W_5 = 0$ such as Calabi–Yau, almost Kähler, nearly Kähler, special Hermitian, and half-flat [1]. We are going to focus on the simplest and more studied case of the half-flat manifold, in which torsional cohomology components are easy to compute, and therefore a detailed study of how $D3$-branes wrap such components can be carried out straightforwardly. Nevertheless, it is important to mention that by selecting a half-flat manifold as a background to built black holes, some extra effects (with respect to the standard supersymmetric black hole in a Calabi–Yau) would come precisely from torsional branes, by which we mean $D3$-branes wrapping torsional cycles.

B. Half-flat manifolds

Let us start by reviewing the construction of the cohomology groups associated to a half-flat manifold Y_3. Under mirror symmetry, compactification of type IIA string theory on a CY manifold X_3 threaded with electric Neveu–Schwarz flux is mapped into a mirror manifold Y_3 referred to as a half-flat manifold [48,49] on which type IIB is compactified. Mirror symmetry is guaranteed once we have that on Y_3 $d\mathrm{Im}\,\Omega_3 = 0$ and $d\times\mathrm{Re}\Omega_3 = e_i\tilde{\omega}_i$, where e_i comes from turning on the electric part of the NS–NS field strength in type IIA compactification, while $\tilde{\omega}_i$ are the 4-forms in $\mathrm{H}^4(Y_3;\mathbf{Z})$. Here we want to stress that this fact leads to the existence of torsional components in the (co) homology of Y_3 as shown in Refs. [50,51].

Take the zero components of a symplectic 3-form basis (α_I, β^I) with $I = 0, \ldots.., h^{(2,1)}(Y_3)$ satisfying

$$d\alpha_0 = e_i\tilde{\omega}^i, \tag{3.3}$$

$$d\omega_i = e_i\beta^0, \tag{3.4}$$

where $i, j = 1, \ldots.., h^{(2,1)}(Y_3)$.

Writing the right-hand side of Eq. (3.3) as $e_i\tilde{\omega}^i = k(n_i\tilde{\omega}^i)$, where $k = \gcd(e_1, \cdots, e_{h^{1,1}})$ for some integers n_i, a basis for $\mathrm{H}^{(2,2)}(Y_3;\mathbf{Z})$ is then given by

$$(n_1\tilde{\omega}^1 + n_a\tilde{\omega}^a, \tilde{\omega}^a), \tag{3.5}$$

with $a = 2, \ldots, h^{(1,1)}$. It is clear that $n_1\tilde{\omega}^1 + n_a\tilde{\omega}^a$ is torsional since $k(n_1\tilde{\omega}^1 + n_a\tilde{\omega}^a) = d\alpha_0$, but $\tilde{\omega}^a$ is not. Hence,

$$\mathrm{H}^{(2,2)}(Y_3;\mathbf{Z}) = \mathbf{Z}^{h^{(1,1)}-1}\oplus\mathbf{Z}_k. \tag{3.6}$$

Following the notation used in Ref. [42], we shall denote by $\hat{\Omega}^p(Y_3)$ all those nonclosed p forms such that $d\sigma_p = k\lambda_{p+1}$, implying in turn that $\lambda_{p+1} \in \mathrm{Tor}\,\mathrm{H}^p(Y_3;\,Z)$. Therefore, from Eq. (3.4), we observe that 2-forms ω_i are nonclosed under differentiation and that $d(n^i\omega_i) = k(n^i n_i\beta^0)$, implying that $[n^i n_i\beta^0] \equiv \beta^{0,\mathrm{tor}} \in \mathrm{Tor}\,\mathrm{H}^3(Y_3)$ and $n_i\omega_i \in \hat{\Omega}^2(Y_3)$ (we have taken $n^i n_i = 1$). Hence, there is a single 3-form which is torsional ($\beta^{0,\mathrm{tor}}$) and another which is nonclosed ($\alpha_0 \equiv \hat{\alpha}_0$). From this, it is concluded that

$$H^3(Y_3;\ Z) = \mathbf{Z}^{2h^{(1,2)}} \oplus \mathbf{Z}_k. \tag{3.7}$$

With respect to the 2-forms, we can construct a basis of $H^2(Y_3;\mathbf{Z})$ given by

$$\left(\omega_1, \eta_a = \omega_a - \frac{e_a}{e_1}\omega_1\right). \tag{3.8}$$

The forms η^a are all closed, but ω_1 is not. Notice also that none of them (including ω^1) is torsional. Hence,

$$H^{(1,1)}(Y_3;\ Z) = \mathbf{Z}^{h^{(1,1)}-1}. \tag{3.9}$$

The results are summarized in Table I.

The existence of torsional forms in a given manifold and their closure under the action of the Laplacian [50] leads to the fact that it is possible to expand RR potentials in terms of forms belonging to Tor $H^p(Y_3, \mathbf{Z}) \oplus \hat{\Omega}(Y_3)$. Notice that under differentiation,

$$d\colon \hat{\Omega}^p(Y_3) \to \mathrm{Ker}[\mathrm{Tor}\ H^{p+1}(Y_3;\mathbf{Z})]. \tag{3.10}$$

$$\tag{3.11}$$

Therefore, following the notation used in Ref. [42], it is possible to define a basis of 3-forms in the half-flat manifold as $(\hat{\alpha}_0, \beta^{0,\mathrm{tor}})$ supported in the pair $(\Sigma_3^{\mathrm{tor}}, \hat{\Pi}_3)$ conformed by a 3-cycle and a 3-chain with

$$k\Sigma_3^{\mathrm{tor}} = \partial\hat{\Pi}_4, \quad \partial\hat{\Pi}_3 = k\Sigma_2^{\mathrm{tor}}, \tag{3.12}$$

where $\Sigma_3^{\mathrm{tor}} \in \mathrm{Tor}\ H_3(Y_3;\ Z)$ and $\hat{\Pi}_3 \in \hat{\Omega}_3(Y_3)$. This establishes an isomorphism between the spaces Tor $H^3(Y_3) \oplus \hat{\Omega}^3$ and $\hat{\Omega}_3 \oplus$Tor $H_3(Y_3)$, meaning that the trivial element in the field is given by integration of a torsional (nonclosed) form over a nonclosed (torsional) cycle. This in turn defines an extra isomorphism between Tor $H^3(Y_3)$ and Tor $H^4(Y_3)$ as expected by Poincaré duality and the universal coefficient theorem [42,52]. Specifically, we have that

TABLE I. Cohomology groups for Y_3.

	$H^n(Y_3;\mathbf{Z})$	Tor $H^n(Y_3)$	Exact mod k	Nonclosed
$n=0$	$\mathbf{Z}$	...	...	...
$n=1$	...	...	...	...
$n=2$	$\mathbf{Z}^{h^{(1,1)}-1}$	...	...	$n^i\omega_i \equiv \hat{\omega}_2$
$n=3$	$\mathbf{Z}^{2h^{(2,1)}}$	$\mathbf{Z}_k$	$n^i n_i \beta^0 \equiv \beta^{0,\mathrm{tor}}$	$\hat{\alpha}_0$
$n=4$	$\mathbf{Z}^{h^{(1,1)}-1}$	$\mathbf{Z}_k$	$n_i\tilde{\omega}^i \equiv \omega_4^{\mathrm{tor}}$	...
$n=5$	...	...	...	...
$n=6$	$\mathbf{Z}$	...	...	...

$$\int_{\Sigma_3^{\mathrm{tor}}} \hat{\alpha}_0 = -\int_{\hat{\Pi}_3} \beta^{0,\mathrm{tor}} = \int_{Y_3} \hat{\alpha}_0 \wedge \beta^{0,\mathrm{tor}} = 1,$$
$$\int_{\hat{\Pi}_4} \omega_4^{\mathrm{tor}} = -\int_{\Sigma_2^{\mathrm{tor}}} \hat{\omega}_2 = \int_{Y_3} \omega_4^{\mathrm{tor}} \wedge \hat{\omega}_2 = 1, \tag{3.13}$$

in accordance with the basis chosen in Refs. [48,49,53,54] and where we made use of

$$\int_{\Sigma_3^{\mathrm{tor}}} \hat{\alpha}_0 = \frac{1}{k}\int_{\partial\hat{\Pi}_4} \hat{\alpha}_0 = \frac{1}{k}\int_{\hat{\Pi}_4} d\hat{\alpha}_0 = \int_{\hat{\Pi}_4} \omega_4^{\mathrm{tor}} = 1. \tag{3.14}$$

From these relations, we can also obtain that

$$\mathcal{P}D_6(\hat{\alpha}_0) = \hat{\Pi}_3,$$
$$\mathcal{P}D_6(\beta^{0,\mathrm{tor}}) = \Sigma_3^{\mathrm{tor}}, \tag{3.15}$$

with $\mathcal{P}D_6\colon$ Tor $H^3(Y_3;\mathbf{Z}) \oplus \hat{\Omega}^3(Y_3) \longleftrightarrow$ Tor$H_3(Y_3;\mathbf{Z}) \oplus \hat{\Omega}_3(Y_3)$. Notice that the above integrals define the linking number between Σ_3^{tor} and $\hat{\Pi}_3$.

Using this structure, it is possible to write down expressions for the Kähler form and the holomorphic 3-form depending on the noncohomological forms [48,49]. Consider the Kähler 2-form $J = v^i\omega_i$, which can be written in terms of $\hat{\omega}_2$ as

$$J = v^i n_i \hat{\omega}_2, \tag{3.16}$$

from which it follows that

$$dJ = v^i n_i d\hat{\omega}_2 = v^i n_i k \beta^{\mathrm{tor},0}, \qquad n_i \in \mathbb{Z}. \tag{3.17}$$

Similarly, the holomorphic (3, 0)-form Ω_3 satisfies [48,49]

$$d\Omega = d\hat{\alpha}_0 = k\omega_4^{\mathrm{tor}}, \tag{3.18}$$

for which it is straightforward to set the most general expression for Ω_3,

$$\Omega_3 = \Omega_3^0 + \tilde{\Omega}_3 = X^i\alpha_i - F_i\beta^i + \hat{\alpha}_0 - F_0\beta^{0,\mathrm{tor}}, \tag{3.19}$$

with $\tilde{\Omega}_3$ corresponding to the components of Ω_3 expanded in the basis $(\hat{\alpha}_0, \beta^{0,\mathrm{tor}})$ and where the periods are given by the integrals

$$F_I = (F_0, F_i) = \left(\int_{\hat{\Pi}_3} \tilde{\Omega}_3, \int_{B_i} \Omega_3^0\right),$$
$$X^I = (X^0, X^i) = \left(\int_{\Sigma_3^{\mathrm{tor}}} \tilde{\Omega}_3, \int_{A^i} \Omega_3^0\right). \tag{3.20}$$

Notice that for the half-flat manifold, the nonclosed parts of J and Ω parametrize how different a half-flat manifold is compared with a CY manifold. Particularly, for the half-flat, the torsional components of the geometrical connection are identified with torsional components of cohomology. This is a key ingredient in our method to construct black hole by wrapping D-branes on internal cycles, since the extra information we have in relation with a CY manifold is now encoded in torsional homology cycles, which one can use to wrap D-branes. Notice that this is just the half-flat version of the well-known example in which NS–NS flux is transformed into torsional cohomology at the level of the tori compactification [50].

C. Discrete electric gauge charge from half-flat manifolds

In contrast with the supersymmetric compactification on a CY manifold, a half-flat manifold has torsional cycles. Following the prescription reviewed in Sec. II, one wonders what are the consequences of wrapping $D3$-branes around some of these spaces on the black hole physics. It is then the purpose of this section to study the physical implications of wrapping branes on torsional 3-cycles. For that, as we have seen, we must in principle also consider chains in $\hat{\Omega}_3(Y_3)$.

Let us start by wrapping N $D3$-branes on a general chain $\tilde{\mathcal{C}}_3 \in \mathrm{Tor}\ \mathrm{H}_3(Y_3;\mathbf{Z})\oplus\hat{\Omega}_3(Y_3)$ given by

$$\tilde{\mathcal{C}}_3 = p^0\hat{\Pi}_3 - q_0\Sigma_3^{\mathrm{tor}}, \tag{3.21}$$

with a world volume of the $D3$-branes given by $\mathcal{W}_4 = \gamma\times\tilde{\mathcal{C}}_3$. It follows that the electric charge is computed by

$$Q_3 = \int_{\Gamma_6} \mathcal{PD}(\mathcal{W}_4) = \int_{\Gamma_6} \mathcal{PD}_4(\gamma)\wedge\mathcal{PD}_6(\tilde{\mathcal{C}}_3). \tag{3.22}$$

Contrary to the SBH in which $\Gamma_6 = \mathbf{B}^3\times\Gamma_3$ with $\Gamma_3\in\mathrm{H}_3(Y_3;\mathbf{Z})$, in this case we can capture a discrete charge value by integrating the current $\mathcal{PD}_6(\tilde{\mathcal{C}}_3)$ over the chain

$$\Gamma_6 \equiv \mathbf{B}^3\times\tilde{\Gamma}_3 = \mathbf{B}^3 \times\left(\frac{e^0}{k}\hat{\Pi}_3 - m_0\Sigma_3^{\mathrm{tor}}\right), \tag{3.23}$$

which is nothing else than the world volume of a $D3$-brane wrapping the torsional 2-cycle

$$\partial^2\Gamma_6 = \mathbf{S}^2\times e^0\Sigma_2^{\mathrm{tor}}, \tag{3.24}$$

precisely corresponding to the fractional charge computed by the Aharanov–Bohm effect through the holonomy of a four-dimensional string around the pointlike BH constructed by wrapping $D3$-branes on $\tilde{\mathcal{C}}_3$ [38,39,41,42,55,56]. Therefore, it follows that

$$\begin{aligned} *F_3 &= \mathcal{PD}_6(\tilde{\mathcal{C}}_3) = p^0\hat{\alpha}_0 - q_0\beta^{0,\mathrm{tor}},\\ F_3 &= \mathcal{PD}_6(\tilde{\Gamma}_3) = \frac{e^0}{k}\hat{\alpha}_0 - m_0\beta^{0,\mathrm{tor}}, \end{aligned} \tag{3.25}$$

with

$$\begin{aligned} p^0 &= \frac{e^0}{k}A_0^0 - m_0C^{00},\\ -q_0 &= m_0A_0^0 + \frac{e^0}{k}B_{00}, \end{aligned} \tag{3.26}$$

and the real matrix elements given by

$$\begin{aligned} A_0^0 &= -\int\hat{\alpha}_0\wedge *\beta^{0,\mathrm{tor}},\\ B_{00} &= \int\hat{\alpha}_0\wedge *\hat{\alpha}_0,\\ C^{00} &= -\int\beta^{0,\mathrm{tor}}\wedge *\beta^{0,\mathrm{tor}}. \end{aligned} \tag{3.27}$$

Thus, the RR field strength associated to those $D3$-branes is then given by

$$F_5 = F_2\wedge\left[\frac{e^0}{k}\hat{\alpha}_0 - m_0\beta^{0,\mathrm{tor}}\right], \tag{3.28}$$

from which it is straightforward to compute the effective charges. However, before computing the corresponding electric and magnetic charges, it is worth mentioning that, as shown in Refs. [48,54] for the half-flat compactification, it is also necessary to consider the presence of a nontrivial NS–NS flux[2] given by $H_3 = e_0\beta^{0,\mathrm{tor}}$. Nevertheless, the existence of this flux is potentially dangerous for $D3$-branes wrapping regions on which the NS–NS flux is supported, since it renders the branes anomalous [30–32]. To cancel this Freed–Witten anomaly, it is necessary that

$$\int_{\tilde{\mathcal{C}}_3} e_0\beta^{0,\mathrm{tor}} = 0, \tag{3.29}$$

implying that $p^0 = 0$. Therefore, the Freed–Witten anomaly cancelation leads us to a relation between the winding numbers e^0 and m_0 by

$$m_0 = \frac{e^0}{k}\frac{A}{C}, \tag{3.30}$$

where we have adopted the notation of A, B, and C to refer to the corresponding matrix elements in Eq. (3.27). Let us emphasize two important remarks:

[2]Notice that, although we have a NS–NS flux, the low-energy limit preserves a $\mathcal{N}=2$ supersymmetry since we are not considering an extra RR flux F_3 and therefore the tadpole contribution to the $D3$-brane charge vanishes [57].

(1) For $e^0 = k$ the world volume of the four-dimensional string becomes trivial, and no measurement of fractional charge is obtained. Therefore, the value of the quotient e^0/k vanishes if it equals an integer; i.e., we must refer to it as $e^0/k \bmod 1$.

(2) By canceling the Freed–Witten anomaly, the internal 3-form F_3 reduces its degrees of freedom from 2 to 1. By writing F_5 as

$$F_5 = e^0 F_2 \wedge \hat{\alpha}_0 - m_0 F_2 \wedge \beta^{0,\mathrm{tor}} = F^0 \wedge \hat{\alpha}_0 - G_0 \wedge \beta^{0,\mathrm{tor}}, \tag{3.31}$$

it is possible to eliminate G_0, since it does not carry degrees of freedom. This actually was shown in Refs. [48,54] by compactifying type IIB string theory on a half-flat manifold and by demanding self-duality on the 5-form field strength. Therefore, it seems that self-duality on F_5 is in agreement with the cancelation of the Freed–Witten anomaly on $D3$-branes wrapping $\hat{\Pi}_3$. Notice that this implies that the chain $\tilde{\mathcal{C}}_3$ reduces to a torsional cycle; i.e., $D3$-branes are only wrapping torsional components in the homology of Y_3. After making use of the FW anomaly cancelation, $\tilde{\mathcal{C}}_3$ and $\tilde{\Gamma}_3$ reduce to

$$\tilde{\mathcal{C}}_3 = -\left(\frac{e^0}{k} \bmod 1\right) \frac{1}{C} \Sigma_3^{\mathrm{tor}}, \quad \tilde{\Gamma}_3 = \left(\frac{e^0}{k} \bmod 1\right) \left[\hat{\Pi}_3 - \frac{A}{C} \Sigma_3^{\mathrm{tor}}\right] \tag{3.32}$$

and

$$F_5 = \left(\frac{e^0}{k} \bmod 1\right) F_2 \wedge \left(\hat{\alpha}_0 - \frac{A}{C} \beta^{0,\mathrm{tor}}\right). \tag{3.33}$$

We have now all the necessary ingredients to compute the black hole electric and magnetic charges, which read

$$Q_e = Q \int_{\tilde{\Gamma}_3} *F_3 = \frac{Q}{C} \left(\frac{e^0}{k} \bmod 1\right)^2, \quad Q_m = P \int_{\tilde{\mathcal{C}}_3} F_3 = -\frac{P}{C} \left(\frac{e^0}{k} \bmod 1\right)^2, \tag{3.34}$$

where we have used that the effective charges are

$$Q = \int_{\mathbf{S}^2} *F_2, \quad \text{and} \quad P = \int_{\mathbf{S}^2} F_2. \tag{3.35}$$

The total charge can also be computed by integrating the self-dual 5-form $\mathcal{F}_5 = F_5 + *_{10} F_5$ over the cycle $\mathcal{W}_5$ given by $\mathcal{W}_5 = \mathbf{S}^2 \times (\tilde{\Gamma}_3 \cup \tilde{\mathcal{C}}_3)$, and it is given by

$$Q_{\mathrm{TOT}} = \int_{\Gamma_5} \mathcal{F}_5 = \frac{1}{C} (Q - P) \left(\frac{e^0}{k} \bmod 1\right)^2, \tag{3.36}$$

where we have used the Freed–Witten anomaly cancellation condition (3.30). Notice that once we have wrapped k of $D3$-branes on $\tilde{\mathcal{C}}_3$, their world volume becomes trivial in homology, and in consequence Q_{TOT} vanishes. This is exactly the mirror symmetric picture of the disappearance of D-branes in a background threaded with NS–NS flux with support on the homology cycles on which the D-branes are wrapped [29,33].

Under this perspective, measuring a discrete charge by an Aharanov–Bohm mechanism through the presence of a four-dimensional string indicates the existence of extra degrees of freedom associated to the black hole as pointed out in Refs. [39,42]. Therefore, by considering a small number k and a huge number of $D3$-branes wrapping torsional cycles in Y_3, the effects of torsional $D3$-branes are manifested at the quantum level. Besides this, it is also possible to show that at the low-energy level, there are massive scalars which are charged under the graviphoton, with a discrete charge as well (see Appendix B). In conclusion, all together, these features point out to the presence of the so called quantum hair of a black hole, which in our case is supersymmetric.

Before computing some extra consequences in the mass of the black hole, let us make one last comment concerning the electric charge we have computed from torsional branes:

(1) We know that under mirror symmetry, the electric component of a NS–NS flux is mapped into the geometry of a manifold called *half-flat*. Now, by considering the construction of SBHs in such backgrounds, we can safely say that a SBH constructed in a CY manifold threaded with electric NS–NS flux is mapped into the mirror symmetric picture in which a SBH has quantum hair.

(2) These discrete charges are associated to massive gauge bosons, which obtain their masses by the breakdown of a continuous symmetry $U(1)$ into a discrete $\mathbf{Z}_k$ (for details, see Refs. [39,42]). In consequence, these massive gauge fields are relevant at the scale just below the breaking of symmetry.[3]

D. Corrections of black hole mass

Up to now, we have computed the discrete charge of a SBH related to $D3$-branes wrapping a torsional 3-cycle $\tilde{\mathcal{C}}_3$. It is therefore of importance to compute the contribution to the mass of the black hole given by those D-branes. As it has been remarked, after using k $D3$-branes, the 3-cycle $\tilde{\mathcal{C}}_3$ becomes trivial in homology and collapses into a point,

[3]For a string scenario in which discrete symmetries arise by the rupture of a continuous symmetry, see Ref. [58].

rendering all the involved $D3$-branes to become unstable, canceling out all their RR charge and transforming into closed strings [31,32,50]. It is therefore expected to associate a discrete value of the mass. An important point to remark is that while accumulating $D3$-branes, the system is stable and behaves as a BPS object[4] in the four-dimensional extended space.

Hence, with the purpose of computing the mass from a base point of view, let us closely follow the black hole's mass computation given in Ref. [11]. Let us start by considering the Dirac–Born–Infeld (DBI) action of a bunch of $D3$-branes wrapping $\tilde{\mathcal{C}}_3$ given by[5]

$$S_{\mathrm{DBI}} = -\int_{\mathcal{W}_4} \sqrt{-h} * \mathbf{1}, \tag{3.37}$$

where h is the determinant of the pullback of the 10-dimensional metric to the world volume $\mathcal{W}_4 = \gamma \times \tilde{\mathcal{C}}_3$ of the $D3$-branes considered stable unless we have a number of k $D3$-branes. Therefore, let us take only a number $e^0 < k$ of $D3$-branes wrapping $\tilde{\mathcal{C}}_3$.

Under this assumption, all our branes are stable and therefore are wrapping a chain which minimizes their energy. In such geometric regions, it is possible to show [46,49] that two conditions hold: i) $J^* = 0$ on $\tilde{\mathcal{C}}_3$, and ii) the superpotential has a constant phase. These two properties lead to the possibility to write the DBI action as

$$S_{\mathrm{DBI}} = -\int_{\gamma} \mathcal{V}_{D3} * \mathbf{1}, \tag{3.38}$$

where $\mathcal{V}_{D3}$ is the volume of $D3$-branes playing the role of the four-dimensional mass M_{BH}, which in terms of the holomorphic 3-cochain Ω_3 reads

$$\begin{aligned} M_{\mathrm{BH}} &= e^{\mathcal{K}/2} \left| \int_{\mathcal{C}_3 \cup \tilde{\mathcal{C}}_3} \Omega_3 \right| \\ &= e^{\mathcal{K}/2} \left| \int_{\mathcal{C}_3} \Omega_3^0 + \int_{\tilde{\mathcal{C}}_3} \tilde{\Omega}_3 \right|. \end{aligned} \tag{3.39}$$

Therefore, using the expression (3.19) for the torsional component of Ω_3, the mass term given by the $D3$-branes wrapping $\tilde{\mathcal{C}}_3$, which actually is the mass contribution to the SBH by adding torsional $D3$-branes, reads

$$\Delta M = e^{\mathcal{K}/2} \int_{\tilde{\mathcal{C}}_3} \tilde{\Omega}_3 = -e^{\mathcal{K}/2} \left(\frac{e^0}{k} \mathrm{mod} 1 \right) \frac{1}{C}. \tag{3.40}$$

Hence, the total mass of the black hole conformed by $D3$-branes wrapping a 3-cycle $\mathcal{C}_3$ in $\mathrm{H}_3(Y_3; \mathbf{Z})$ and by $D3$-branes wrapping the torsional cycle $\tilde{\mathcal{C}}_3$ is given by

$$M_{\mathrm{BH}}^2 = (M_{\mathrm{BPS}} + \Delta M)^2 = e^{\mathcal{K}} \left| m_{\mathrm{BPS}} - \left(\frac{e^0}{k} \mathrm{mod}\, 1 \right) \frac{1}{C} \right|^2, \tag{3.41}$$

with m_{BPS} given by Eq. (2.13) as

$$m_{\mathrm{BPS}} = -\mathcal{W} = -(e_i X^i - m^i F_i). \tag{3.42}$$

Notice that here, for the half-flat case, the index i in the superpotential runs from 1 to $h^{(1,1)}(Y_3)$, contrary to the superpotential in a CY manifold where the index also takes the zero value. Therefore, in analogy, we can write the black hole total mass in terms of a new superpotential given by

$$\begin{aligned} \mathcal{W}_{\mathrm{TOTAL}} &= \mathcal{W} + \int_{Y_3} * F_3 \wedge \tilde{\Omega}_3, \\ &= \mathcal{W} + \lambda \mathcal{W}_{\mathrm{HF}}, \end{aligned} \tag{3.43}$$

where $\lambda = i(\frac{e^0}{k} \mathrm{mod}\, 1) \frac{1}{C} \frac{1}{k n_i v^i}$, and

$$\mathcal{W}_{\mathrm{HF}} = \int_{Y_3} i dJ \wedge \tilde{\Omega}_3, \tag{3.44}$$

which under the flux conditions in our setup (no RR fluxes and cancellation of the Freed–Witten anomaly derived from the presence of a nontrivial NS–NS flux) is precisely the superpotential related to a half-flat manifold as shown in Refs. [48,54]. Two comments are given in order. First, notice that demanding that the black hole mass satisfies the relation $M_{\mathrm{BPS}}^2 = e^{\mathcal{K}} |\mathcal{W}|^2$, where $\mathcal{W}$ is a superpotential, provides an alternative way to derive the superpotential of the half-flat manifold. Second, we see that the contribution to the mass by torsional branes is also proportional to $(\frac{e^0}{k} \mathrm{mod}\, 1)$, indicating that after wrapping k $D3$ in $\tilde{\mathcal{C}}_3$, the extra mass term vanishes.

E. Loss of quantum hair

As shown below, quantum degrees of freedom, or quantum hair can be associated to the black hole by wrapping $D3$-branes on torsional cycles. Besides this electric discrete charge, we have also computed the mass contribution and seen that it also has a discrete value, meaning that upon completion of k $D3$-branes wrapping $\tilde{\mathcal{C}}_3$, the mass of the black hole will collapse to the original value M_{BPS} (i.e., without considering torsional branes), and it will lose all its quantum hair.

This is quite interesting since it implies that a stable and extremal black hole with an associated vanishing temperature would emit some radiation (consisting of closed strings) once the number of torsional branes reaches k. Once the SBH loses all its quantum hair, it would return to another stable state with a lower mass. Therefore, there must be an emission of closed strings localized in time, and

[4]Notice that this is valid since in this case, the charge computed through dJ over a chain in $\hat{\Omega}_3(Y_3)$ does not vanish.

[5]We are taking all numerical coefficients equal to 1.

in consequence we expect a variation in the entropy by the loss of all torsional degrees of freedom. A previous mirror symmetric picture of this mechanism was partially studied in Ref. [29].

Hence, let us compute the change in entropy and the associated temperature for the radiation the SBH would emit once completing k-torsional $D3$-branes, and let us start by reviewing the way in which entropy is computed in a CY manifold. The associated entropy for a SBH constructed on a CY manifold is computed by extremizing the action [15,59]

$$S = -\frac{\pi}{4}[e^{-\mathcal{K}(X,\bar{X})} + 2i\mathcal{W}(X) - 2i\bar{\mathcal{W}}(\bar{X})] \tag{3.45}$$

and evaluating the extreme at the attractor point on which the involved superpotential vanishes, rendering the system supersymmetric. The Kähler potential is

$$\mathcal{K} = \mathcal{K}_0 + \tilde{\mathcal{K}} = i \times \log\left(\int \Omega^0 \wedge \bar{\Omega}^0 + \int \tilde{\Omega}_3 \wedge \bar{\tilde{\Omega}}_3\right), \tag{3.46}$$

with $\tilde{K} = -\frac{1}{2}\mathrm{Im}X_0 F^0$. Notice that, although at this point we are not considering the presence of torsional $D3$-branes, the Kähler potential contains some information coming from torsional cohomology since it is related to the geometry of the internal space independently of the presence of $D3$-branes. Since the supersymmetric black hole constructed by wrapping $D3$-branes on 3-cycles satisfies the BPS bound, i.e., it is an extremal black hole, it does not radiate since it is in a state of minimal energy. Therefore, the associated temperature is zero (see Ref. [21] and references therein).

Now, let us think on a system consisting of just $D3$-branes wrapping supersymmetric 3-cycles on which we start adding torsional $D3$-branes, taking care that the number of these branes does not overpass k. Since we are adding extra degrees of freedom (parametrized by e^0), it is expected that entropy will grow with respect to the entropy associated to the SBH. Its variation must come precisely from the extra components in the superpotential denoted by $\lambda\mathcal{W}_{\mathrm{HF}}$; i.e., the entropy variation can be computed by extremizing the action

$$\tilde{S} = \frac{\pi}{4}(e^{-\tilde{K}} - 4\mathrm{Im}\lambda\mathcal{W}_{\mathrm{HF}}). \tag{3.47}$$

A direct calculation shows that $\tilde{S}$ has an extreme at

$$X^0_{\mathrm{min}} = \frac{i}{C^{00}}\left(\frac{e^0}{k}\mathrm{mod}\,1\right)\frac{1}{(\mathrm{Im}\tau)_{00}}, \tag{3.48}$$

at which, upon substitution, gives the entropy associated to torsional branes [and by taking $\mathcal{W}_{\mathrm{HF}}(X^0_{\mathrm{min}}) = 0$]:

$$\Delta\mathcal{S} = \frac{\pi}{2}\frac{1}{C^{00}}\frac{1}{\mathrm{Im}\tau_{00}}\left(\frac{e^0}{k}\mathrm{mod}\,1\right)^2 = \frac{\pi}{2}e^{-\mathcal{K}/2}\left(\frac{e^0}{k}\mathrm{mod}\,1\right)\Delta M. \tag{3.49}$$

Therefore, the entropy related to quantum hair goes like $(\frac{e^0}{k}\mathrm{mod}\,1)^2$, meaning that by increasing the number of torsional branes, the entropy of the system also becomes larger. Once we add k $D3$-branes, the system becomes unstable to decay into the original supersymmetric setup, and all torsional branes radiate into closed strings. Although a precise description of this transition is beyond the scope of this work, we can mention some interesting features.

First of all, by reaching the number k of $D3$-branes, the extra mass and entropy vanish. The transition consists of a black hole which suddenly loses part of its mass and goes from a stable state with a zero temperature to another stable state with a smaller mass. During this transition, the system is not represented by a BPS state since $\tilde{\mathcal{C}}_3$ is a trivial cycle and collapses into a point. It is then natural to associate a temperature related to the emission of the energy contained in the system of $D3$-branes wrapping a trivial cycle, which, being a nonsupersymmetric and unstable state, can be estimated from $dS/dM = 1/\Delta T$. Therefore,

$$\frac{1}{\Delta T} \sim \frac{\pi}{2}e^{-\mathcal{K}_0/2}. \tag{3.50}$$

From this, we can also notice the following: consider two black holes with the same total mass, but one has a larger amount of mass coming from supersymmetric $D3$-branes. Therefore, the mass contribution from torsional branes is smaller in the first black hole than in the second one. In that sense, the black hole with more discrete charge is also the one with more entropy. Notice that if these black hole would be nonextremal, we would say that a black hole with a more discrete charge would be also cooler. These features are pretty similar to the properties one expects (in a supersymmetric point of view) from a black hole with quantum hair with an associated temperature, as predicted in Ref. [38].

Finally, from this supersymmetric construction, still there is a question we can address and that was already pointed out in Ref. [39]. It would be desirable to compute the discrete charge of a black hole without considering the presence of a four-dimensional string. We consider that this can be accomplished by the use of K theory.

IV. QUANTUM HAIR AND K THEORY

The discrete electric charge computed in the previous sections relies on the presence of an extra object. Therefore, quantum hair seems to be detectable only if we could take into account a four-dimensional string and perform a holonomy around the black hole. This of course triggers a question about how to compute such discrete charge without using an extra extended object. In the context of string theory, computation of classical properties of a SBH requires the use of (co)homology, while the quantum regime of the black hole should be described in a appropriate way related to the computation of D-brane RR charges. For some years, we have known that such a mathematical structure is encoded in K theory, and for that reason, we expect that discrete charges must be derived from some version of K theory. In this section, we shall use the Atiyah–Hirzebruch spectral sequence (AHSS), connecting cohomology to K theory in order to derive how the discrete charge appears by computing the corresponding K-theoretical charge of the D-branes used for the construction of the black hole.

With the purpose of presenting a clearer argument and in order to show that SBH can also be constructed by compactifying type IIA string theory on a half-flat manifold, we shall present our analysis in this background, i.e., the construction of black holes by wrapping $D2$- and $D4$-branes in type IIA theory compactified on a half-flat manifold.

A. Atiyah–Hirzebruch spectral sequence

Let us start by briefly reviewing the AHSS in the context of string theory. Essentially, the AHSS is an algorithm which connects integral cohomology to K theory [60–62]. The main goal of this approach is to compute the K-theory group $K(X)$ related to the RR charge of D-brane supported on the submanifold X with dimension d. For that, the AHSS makes use of a sequence of successive approximations starting from integral cohomology and gradually considering successive orders of approximation, which involves the cohomology of differential maps d^n, where $d^n\colon H^p(X;\mathbf{Z})\to H^{p+n}(X;\mathbf{Z})$. In each step, the n-cohomology group E_p^n for a given n is computed by the quotient $K_p(X)/K_{p+1}(X)$ where $K_p(X)$ is a subgroup of $K(X)$ which classifies all stable $D(d-p)$-branes supported on a $(d-p)$-dimensional submanifold of X but trivial in $(d-p-1)$ submanifolds via the RR field strengths (p-forms). Computing $K_p(X)$ involves solving the following exact short sequence:

$$0\to K_{p+1}\to K_p\to K_p/K_{p+1}\to 0. \tag{4.1}$$

If all extensions are trivial for all p, the K-theory group $K(X)$ is computed just by adding the subgroups E_p^n, i.e., by $K(X)=\oplus_p E_p^n$. Notice, therefore, that in a fluxless CY compactification, D-brane charges or equivalently the RR charge is simply computed through the cohomology groups. By turning on an extra NS–NS flux, the AHSS requires a second step of approximation involving the groups E_p^3 ([32,61,63]). Hence, in the absence of extra fluxes (notice that $H_3=e_0\beta^{0,\mathrm{tor}}$ does not have an influence in the sequence), $K(X)=H^p(X)$ up to solving the extension problem (4.1).

B. Discrete charge from K theory

A SBH in the context of type IIA string theory is constructed by wrapping $D2$-and $D4$-branes on two- and four-dimensional chains in Y_3. Therefore, the electric and magnetic charges are computed by integrating the RR field strength (a 4-and 6-form, respectively) over some suitable submanifold of the 10-dimensional space-time; this is

$$Q_e^{IIA}=\int_{\mathbf{S}^2\times\Gamma_4}*F_2\wedge\mathcal{P}D_6(\mathcal{C}_2),\quad \text{and}$$
$$Q_m^{IIA}=\int_{\mathbf{S}^2\times\Gamma_2}F_2\wedge\mathcal{P}D_6(\mathcal{C}_4), \tag{4.2}$$

with both currents $\mathcal{P}D_6(\mathcal{C}_2)$ and $\mathcal{P}D_6(\mathcal{C}_4)$ in $\mathrm{H}^4(Y_3;\mathbf{Z})$ and $\mathrm{H}^2(Y_3;\mathbf{Z})$, respectively. After extending the integration to the whole internal space, the charge of a black hole in four dimensions is determined by a 6-form flux in $\mathrm{H}^6(Y_3;\mathbf{Z})$, proportional to $\mathcal{P}D_6(\mathcal{C}_2)\wedge J$ for a $D2$-brane on $\mathcal{C}_2$ and $\mathcal{P}D_6(\mathcal{C}_4)\wedge J^2$ for a $D4$-brane wrapping $\mathcal{P}D_6(\mathcal{C}_4)$. Since it is not obvious from the above expressions that the corresponding charges are fractional without considering the presence of a four-dimensional string, our goal here is to elucidate its nature from the K-theory perspective.

Within the context of the AHSS, the relevant short sequence involves the following filtrations (where $h^{(1,1)}$ is the Hodge number of Y_3):

(1) $K_5=K_6=\mathbf{Z}$, which measures the charge carried by the 6-form $\mathcal{P}D_6(\mathcal{C}_2)\wedge J_2$ by computing the cohomology group $\mathrm{H}^6(Y_3;\mathbf{Z})=\mathbf{Z}$.

(2) $K_4=\mathbf{Z}\oplus\mathbf{Z}^{h^{(1,1)}}$, which measures the K-theoretical charge related to the 6-form $\mathcal{P}D_6(\mathcal{C}_2)\wedge J_2$ and stable $D2$-branes wrapping $h^{(1,1)}$ 2-cycles in Y_3.

(3) $K_4/K_5=H^4(Y_3;\mathbf{Z})=\mathbf{Z}^{h^{(1,1)}}\oplus\mathbf{Z}_k$ concerning the group of 4-forms, related only to $D2$-branes wrapping 2-cycles of Y_3. Notice the presence of torsional components for the half-flat manifold.

Therefore, the relevant extension problem is given by

$$0\overset{\times k}{\to}H^6(Y_3:\mathbf{Z})=\mathbf{Z}\to\mathbf{Z}\oplus\mathbf{Z}^{h(1,1)}\to H^4(Y_3;\mathbf{Z})$$
$$=\mathbf{Z}^{h(1,1)}\oplus\mathbf{Z}_k\to 0, \tag{4.3}$$

where we have also assumed that there is no difference between cohomology and K theory in the sequential steps. By demanding the sequence to be exact, we notice that there must be a shift of fractional charge i/k for each element in $\mathbf{Z}_k$, with $i=1,\ldots,k$.

Therefore, for each $D2$-brane wrapping a torsional cycle in $\mathbf{Z}_k$, the corresponding torsional element induces a fractional charge in the generator $G_6\in\mathrm{H}^6(Y_3;\mathbf{Z})$, which, as said, contributes to the four-dimensional charge of the black hole. Notice that a similar situation holds for the magnetic part, i.e., by wrapping $D4$-branes on 4-cycles and the fractional charge induction is independent of the presence of one or another. This in fact confirms that it is possible to associate a fractional K-theoretical charge of branes wrapping torsional cycles.

V. FINAL COMMENTS

In this work, we have constructed a supersymmetric black hole in the effective low-energy theory by wrapping $D3$-branes on 3-cycles of a half-flat manifold. As it is well known, type II string compactification on half-flat manifold is the mirror symmetric image of a compactification on a Calabi–Yau manifold threaded with electric NS flux. Therefore, we are wrapping $D3$-branes on a manifold which has backreacted under the presence of the electric NS flux, and, in consequence, it is expected that the black hole constructed in such a scenario contains some characteristics inherited from the electric NS flux.

Those effects manifest in the black hole's physics, primarily by the presence of torsional components in the (co)homology of the half-flat manifold with an associated discrete group denoted by $\mathbf{Z}_k$. A number of N $D3$-branes wrapping torsional cycles correspond in the low-energy level to stable and supersymmetric pointlike objects with a discrete value $N/k\,\mathrm{mod}1$ for the mass and for the electric and magnetic charges. Expansion of the corresponding RR potential on these torsional components of cohomology leads to the existence of effective massive gauge bosons and massive scalars with discrete gaugings.

Since k is finite, by wrapping a large number of $D3$ on nontorsional cycles, a massive SBH is constructed with $M_{\mathrm{BH}} \gg M_{\mathrm{pl}}$. This is a very good approximation of a classical SBH. However, if the number of D-branes wrapping the nontorsional cycles are of order k, the massive states related to the torsional part must become relevant. These degrees of freedom must correspond to some hair on the SBH which manifests in a quantum regime as quantum hair studied in Refs. [39,41].

Thinking on a black hole which increases its mass by adding torsional $D3$-branes, it is possible to compute its variation on mass and entropy up to the nonstatical stage in which the bunch of torsional branes complete the number k and annihilate each other, departing from the stable BPS state. As for the electric and magnetic charges, the variation of mass and entropy resulting from increasing the number of torsional branes goes like $N/k \times \mathrm{mod}\ 1$. In consequence, once the number of torsional branes reaches the number k, the black hole transits from a stable state with a mass and charge larger than a black hole conformed only by $D3$-branes wrapping homological cycles to another stable state corresponding to the supersymmetric black hole. Both states are stable, but during the transition, the system conformed by the torsional branes becomes an unstable set of branes wrapping a trivial cycle. Therefore, all the entropy gained during the addition of torsional branes is emitted in the form of closed strings, and we can estimate a temperature related to this process. We observe that for two black holes with the same total mass, the one with more discrete charge has a bigger entropy than the second one. This resembles some properties expected from quantum hair in a supersymmetric version.

Keeping the number of torsional branes less than k, we notice some other important features: since in a half-flat manifold there is a nontrivial NS–NS flux, it is important to cancel the Freed–Witten anomaly on all those branes wrapping submanifolds on which the flux is supported. This implies vanishing half of the degrees of freedom, associated with the 5-form field strength F_5. This is compatible with the same loss of degrees of freedom by restricting the 5-form field strength to be self-dual as shown in Ref. [54]. Therefore, we conclude that in half-flat compactification where F_5 is taken to be self-dual, all $D3$-branes are free from Freed–Witten anomalies.

Nevertheless, in this setup, computation of discrete quantum hair requires the presence of a four-dimensional string. With the purpose to compute the discrete charge of the black hole without requiring the presence of an extra object, we use the Atiyah–Hirzebruch spectral sequence to compute the $D3$-branes charge from a K-theoretical perspective. We find that, as in the presence of orientifolds, torsional components of cohomology induce a lift on the generators of the D-brane charge in fraction, rendering the total charge discrete.

However, there are still many features to study, among which we can mention some. First, although it seems possible that magnetic quantum hair appears by constructing black holes on mirror symmetric manifolds to those on which magnetic components of a NS flux have been taken into account, still it is not clear how to wrap D-branes on those backgrounds. Constructing black holes in general manifolds would be an important task to perform as well as the relation between those black holes and the solutions in the gauged supergravity side. Second, it would be interesting to elucidate some string mechanism which leads us to the construction of non-Abelian quantum hair. Black holes in a background mirror symmetric to a compactification on a CY threaded with magnetic NS fluxes could be the string construction of the magnetic quantum hair as described in Ref. [39]. It would be interesting to explore such an issue. Finally, it is necessary to study how stable a black hole in a half-flat manifold is due to the presence of an effective scalar field V_g originated by the nonzero curvature of the half-flat manifold. We leave this feature for a future work.

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APPENDIX A: NOTATION

For a fluxless compactification on a Calabi–Yau, we have that

$$\begin{aligned} \int_X \alpha_I \wedge * \alpha_J &= \int_X \alpha_J \wedge * \alpha_I = B_{IJ}, \\ \int_X \alpha_I \wedge * \beta^J &= \int_X \beta^J \wedge * \alpha_I = -A^I_J, \\ \int_X \beta^I \wedge * \beta^J &= \int_X \beta^I \wedge \beta^J = -C^{IJ}, \end{aligned} \tag{A1}$$

which in turn defines the complex matrix $\mathcal{M}_{IJ}$ through

$$\begin{aligned} A &= (\mathrm{Re}\mathcal{M})(\mathrm{Im}\mathcal{M})^{-1}, \\ B &= -(\mathrm{Im}\mathcal{M}) - (\mathrm{Re}\mathcal{M})(\mathrm{Im}\mathcal{M})^{-1}(\mathrm{Re}\mathcal{M}), \\ C &= (\mathrm{Im}\mathcal{M})^{-1}. \end{aligned} \tag{A2}$$

APPENDIX B: LOW-ENERGY THEORY

In this section, we review the low-energy limit of type IIB string theory compactified on a half-flat manifold closely following Refs. [48,53,54].

First of all, it is important to notice that as shown in Ref. [42], all fields expanded in terms of $\hat{\alpha}_0$ and $\beta^{0,\mathrm{tor}}$ are massive in the four-dimensional effective theory, and since $[\nabla^2, d] = 0$, it is possible to show that

$$\nabla^2 \hat{\omega}_2 = -n_i M^j_i \omega_j = \hat{\omega}_2, \tag{B1}$$

where $-M^i_j = \delta^i_j$ is the corresponding mass matrix. Similarly, we have that

$$\begin{aligned} \nabla^2 \hat{\alpha}_0 &= \hat{\alpha}_0, \\ \nabla^2 \beta^{0,\mathrm{tor}} &= \beta^{0,\mathrm{tor}}, \\ \nabla^2 \omega_4^{\mathrm{tor}} &= \omega_4^{\mathrm{tor}}. \end{aligned} \tag{B2}$$

The squared masses are all of order of the Planck mass (we have taken $M_{\mathrm{pl}} = 1$), and therefore the Laplace operator acting on these fields gives terms of order $(\mathrm{flux})^2$, implying that it is possible to ignore massive Kaluza-Klein (KK) states since the order of the fluxes is smaller than the order of the compactification scale rendering the supergravity approach valid.

The massive scalar fields and massive gauge vector arising from compactification on a half-flat manifold can be shown by directly computing the effective low-energy scale as in Ref. [54]. Let us review this computation for our specific case in which we are turning on an internal field related only with the presence of wrapped $D3$-branes; i.e., we are not considering extra NS–NS or RR fluxes.

Consider the NS–NS and RR potentials given by

$$\begin{aligned} B_2 &= b_2 + b^a \omega_a + (b^i n_i)\hat{\omega}_2 \quad \text{and} \\ C_2 &= c_2 + c^a \omega_a + (c^i n_i)\hat{\omega}_2, \end{aligned} \tag{B3}$$

where b^i and c^i are constant real scalar fields and b_2 and c_2 are 2-forms supported in the four-dimensional extended space-time (not necessarily constants) and ω_a is the 2-form basis in $\mathrm{H}^2(Y_3; \mathbf{Z})$ with a constant b^a and c^a. The corresponding field strengths for these potentials read

$$\begin{aligned} H_3 &= dB_2 = db_2 + (k(b^i n_i) + e_0)\beta^{0,\mathrm{tor}}, \\ F_3 &= dC_2 - C_0 dB_2 = dc_2 + k(c^i n_i)\beta^{0,\mathrm{tor}} - C_0 H_3. \end{aligned} \tag{B4}$$

By expanding the RR potential C_4 as

$$C_4 = A_1 \wedge \left[e^i \alpha_i - m_i \beta^i + \left(\frac{e^0}{k} \mathrm{mod}\, 1 \right) \hat{\alpha}_0 - m_0 \beta^{0,\mathrm{tor}} \right], \tag{B5}$$

the 5-form field strength reads

$$\begin{aligned} F_5 = {} & F_2 \wedge \left[e^i \alpha_i - m_i \beta^i + \left(\frac{e^0}{k} \mathrm{mod}\, 1 \right) \hat{\alpha}_0 - m_0 \beta^{0,\mathrm{tor}} \right] \\ & - A_1 e^0 \bar{\omega}_4^{\mathrm{tor}}. \end{aligned} \tag{B6}$$

Comparing this expression for F_5 with that given in Eq. (3.28), we see that the last term in the right-hand side does not contribute to the charge, and that is why it was not considered in the calculation of the black hole charge, although it plays an important role in the low-energy effective theory we are computing.

Therefore, using the above expressions for C_4 and F_5, together with the complex moduli Z^i and v^i from Eqs. (3.19) and (3.16), it is possible to construct the multiplets for the effective theory $\mathcal{N} = 2$ in four dimensions where the gravity multiplet consists of the graviton $g_{\mu\nu}$ and the vector field (graviphoton) $(\frac{e^0}{k} \mathrm{mod} 1) A_1$. The vector multiplet is given by $(e^a A_1, Z^a)$ with $a = 1, \ldots, h^{(2,1)}(Y_3)$ and the scalars conforming the hypermultiplets $(\phi, C_0, \star b_2, \star c_2, b^i, c^i, v^i)$ where $\star$ is the Hodge dual in four dimensions.

Not being a Ricci-flat manifold, compactification on a half-flat manifold [54] leads to an effective potential induced by the internal curvature and is given by

$$V_g^{\mathrm{HF}} = -\frac{\kappa_0}{16\mathcal{K}} e^{2\phi} k^2 n_i n_j g^{ij}, \tag{B7}$$

where g^{ij} is the metric of the scalar moduli space. After incorporating self-duality on F_5, it was shown that [48] some of the fields carry nonphysical degrees of freedom.

In particular, it is possible to show that $m_I F_2$ can be eliminated. Notice that this is compatible with cancelation of the Freed–Witten anomaly once $D3$-branes are considered as in our case. Therefore, the low-energy action reads

$$S_{IIB} = \int -\frac{1}{2}R * \mathbf{1} + \frac{1}{2}\mathrm{Im}\mathcal{M}_{IJ}F^I \wedge * F^J + \frac{1}{2}\mathrm{Re}\mathcal{M}_{IJ}F^I \wedge F^J - h_{\mu\nu}Dq^\mu \wedge Dq^\nu - V_{\mathrm{eff}} * \mathbf{1}, \tag{B8}$$

where, following the notation in Ref. [48], $q = (\phi, a, \xi^I, \tilde{\xi}_I)$, with the scalar fields given by

$$\begin{aligned} a &= 2\star b_2 + C_0 \star c_2, \\ \xi^I &= (C_0, C_0 b^i - c^i), \\ \tilde{\xi}_I &= \left(-\star C_2 - \frac{C_0}{6}\mathcal{K}_{ijk}b^i b^j b^k + \frac{1}{2}\mathcal{K}_{ijk}b^i b^j b^k, \frac{C_0}{2}\mathcal{K}_{ijk}b^j b^k - \mathcal{K}_{ijk}b^j c^k\right), \end{aligned} \tag{B9}$$

and the covariant derivatives read

$$\begin{aligned} D\tilde{\xi}_I &= d\tilde{\xi}_I - k\left(\frac{e^0}{k}\mathrm{mod}\ 1\right)n_I A_1, \\ Da &= da + k\left(\frac{e^0}{k}\mathrm{mod}\ 1\right)A_1 n_I \xi^I, \end{aligned} \tag{B10}$$

with $d\tilde{\xi}_I = (-d\star C_2, \frac{dC_0}{2}\mathcal{K}_{ijk}b^j b^k)$. Then it follows that

$$h_{\mu\nu}Dq^\mu \wedge * Dq^\nu = d\phi \wedge * d\phi + g_{ab}dZ^a \wedge d\bar{Z}^b + \mathcal{L}_{\mathrm{mscalars}} + \mathcal{L}_{\mathrm{mgauge}}, \tag{B11}$$

with

$$\begin{aligned} \mathcal{L}_{\mathrm{mscalars}} &= \frac{e^{4\phi}}{4}\left[Da - \xi^I\left(d\tilde{\xi}_I - kn_I\left(\frac{e^0}{k}\ mod1\right)A_1\right)\right] \wedge * \left[Da - \xi^I\left(d\tilde{\xi}_I - kn_I\left(\frac{e^0}{k}\mathrm{mod}1\right)A_1\right)\right], \\ \mathcal{L}_{\mathrm{mgauge}} &= -\frac{e^{2\phi}}{2}C^{IJ}\left(d\tilde{\xi}_I - kn_I\left(\frac{e^0}{k}\mathrm{mod}1\right)A_1\right) \wedge * \left(d\tilde{\xi}_J - kn_J\left(\frac{e^0}{k}\mathrm{mod}1\right)A_1\right). \end{aligned} \tag{B12}$$

Notice then that the gaugings from $\mathcal{L}_{\mathrm{mscalars}}$ show that the scalar a is charged under the graviphoton and also that the scalars b^i and c^i become massive through the terms ξ_I. Concerning the Lagrangian term $\mathcal{L}_{\mathrm{mgauge}}$, this is exactly a Stückelberg Lagrangian, showing that the involved photons $(\frac{e^0}{k}\mathrm{mod}1)A_1$ are massive [42].

Finally, let us comment on the effective scalar potential given by

$$V_{\mathrm{eff}} = V_g^{\mathrm{HF}} - \frac{\kappa_0}{2}e^{4\phi}(kn_I\xi^I)^2 + \frac{V_{\mathrm{BH}}}{r^4}, \tag{B13}$$

where V_{BH} is given by expression (2.12). The stability of such a system depends essentially on parameters n_I (coming from the mirror symmetric image of electric NS fluxes in type IIA on a CY manifold X_3), the curvature of the internal manifold, and the contribution of the supersymmetric part of the black hole through the term V_{BH}. The stability of a SBH in a background threaded with fluxes was studied in Ref. [33]. For the present case treated in this paper, we leave the study of this important fact for a future work.

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