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# Modified Friedmann equation from nonminimally coupled theories of gravity

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In this work we study how nonminimally coupled theories of gravity modify the usual Friedmann equation, and develop two methods to treat these. The ambiguity in the form of the Lagrangian density of a perfect fluid is emphasized, and the impact of different dominant matter species is assessed. The cosmological constant problem is also discussed.

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## I. INTRODUCTION

Despite its great experimental success (see e.g. Refs. [1,2]), it is well known that general relativity (GR) is not the most encompassing way to couple matter with curvature. Indeed, these can be coupled, for instance, in a nonminimal way [3] (see also Refs. [4–6] for early proposals in cosmology), a fact that can have a bearing on the dark matter [7,8] and dark energy [9,10] problems, as well as inflation [11,12] and structure formation [13]. This putative nonminimal coupling modifies the well-known energy conditions [14] and can give rise to several implications, from Solar System [15] and stellar dynamics [16–19] to close timelike curves [20] and wormholes [21].

Another interesting issue that arises in the context of gravity theories with a nonminimal coupling between curvature and matter is the fact that the Lagrangian degeneracy in the description of a perfect fluid, encountered in GR [22,23] is lifted [24]: indeed, since the Lagrangian density explicitly appears in the modified equations of motion, two Lagrangian densities leading to the same energy-momentum tensor have different dynamical implications, whereas in GR they are physically indistinguishable.

Thus, one considers that the Einstein-Hilbert action is extended by the action functional [3],

$$S = \int [\kappa f_1(R) + f_2(R)\mathcal{L}]\sqrt{-g}d^4x, \qquad (1)$$

where $f_i(R)$ ($i = 1, 2$) are arbitrary functions of the scalar curvature, $R$, $g$ is the determinant of the metric and $\kappa = c^4/16\pi G$. The above encompasses the well-known $f(R)$ theories, which are widely used to study the effect of modifications of gravity in a plethora of scenarios, e.g. the Starobinsky inflationary model $f(R) = R + \alpha R^2$ [25], the accelerated expansion of the Universe [26], and Solar System tests [27], amongst others.

Following the argument that $f(R)$ theories should be derived from a more complete theory as low-energy phenomenological models, one also finds strong fundamental motivation for the presence of a nonminimal coupling, as it arises from one-loop vacuum-polarization effects in the formulation of quantum electrodynamics in a curved spacetime [28], as well as in the context of multi-scalar-tensor theories, when considering matter scalar fields [29] (as explicitly shown in Ref. [30]). Furthermore, a nonminimal coupling was put forward in an earlier proposal [31], developed in the context of Riemann-Cartan geometry.

A related approach is found in the so-called Horndeski's scalar-tensor theories [32], where the most general action that includes a scalar field $\phi$ and leads to second-order equations of motion is developed: in that context, one finds that a linear coupling between the scalar curvature and the scalar field is allowed, although appropriate "counterterms" in derivatives of $\phi$ must be added to cancel higher-order terms in the field equations.

In contrast, $f(R)$ theories (and, by extension, the model here considered) yield fourth-order equations of motion, but can be cast as a well-defined Cauchy problem exhibiting second-order equations in both the metric and the scalar curvature. This approach is reminiscent of Ostrogradski's Hamiltonian formulation of $f(R)$ theories, where the curvature is promoted to a canonical variable [33]. Horndeski's scalar-tensor theories are not considered here since one does not aim at including a scalar field as an additional matter species, but instead seeks to find how a nonminimal coupling of the scalar curvature with normal matter (described as a perfect fluid) can leave an imprint on the Friedmann equation.

In what follows, two methods to relate modifications of the Friedmann expansion rate equation with the functions of the scalar curvature appearing in the action functional

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Eq. (1) are developed—considering both a nonlinear curvature term and a nonminimal coupling (NMC) between the latter and matter. The impact of the form of the Lagrangian density in this identification is assessed, with no particular choice for this quantity or the dominant matter species (aside from the assumption of a perfect fluid). This work aims at complementing a previous study of the impact of a NMC in a cosmological context [9].

This work is structured as follows: in Sec. II, we describe the nonminimally coupled models of interest, and consider their impact in cosmology in Sec. III; in Secs. IV and V, we discuss two different regimes of the NMC models; Secs. VI–X concern specific explicit solutions; in Sec. XI, we address the issue of generating a cosmological constant; finally, in Sec. XII we present our conclusions.

## II. THE MODEL

Variation of Eq. (1) with respect to the metric yields the modified field equations,

$$\left(F_1+\frac{F_2\mathcal{L}}{\kappa}\right)G_{\mu\nu}=\frac{1}{2\kappa}f_2T_{\mu\nu}+\Delta_{\mu\nu}\left(F_1+\frac{F_2\mathcal{L}}{\kappa}\right)+\frac{1}{2}g_{\mu\nu}\left(f_1-F_1R-\frac{F_2R\mathcal{L}}{\kappa}\right), \quad (2)$$

with $F_i\equiv df_i(R)/dR$ and $\Delta_{\mu\nu}\equiv\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box$. As expected, GR is recovered by setting $f_1(R)=R$ and $f_2(R)=1$.

The trace of Eq. (2) reads

$$(\kappa F_1+F_2\mathcal{L})R=\frac{1}{2}f_2T-3\Box(\kappa F_1+F_2\mathcal{L})+2\kappa f_1. \quad (3)$$

Resorting to the Bianchi identities, one concludes that the energy-momentum tensor of matter may not be (covariantly) conserved, since

$$\nabla_\mu T^{\mu\nu}=\frac{F_2}{f_2}(g^{\mu\nu}\mathcal{L}-T^{\mu\nu})\nabla_\mu R \quad (4)$$

can be nonvanishing. Given the analogy between the nonminimally coupled $f(R)$ theories here considered and a two-scalar-tensor theory [30], this can be interpreted as due to an energy exchange between the perfect fluid and the latter.

## III. COSMOLOGY

In order to study the effect of the modified dynamics arising from a NMC, one assumes a spatially flat, homogeneous and isotropic Universe, thus considering the Friedmann-Robertson-Walker (FRW) line element,

$$ds^2=-dt^2+a^2(t)dV, \quad (5)$$

where $a(t)$ is the scale factor, and it is assumed that matter is a perfect fluid with an energy-momentum tensor,

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu+pg_{\mu\nu}\to T=3p-\rho, \quad (6)$$

with $u^\mu u_\nu=-1$ and $u^\mu u_{\mu;\nu}=0$; since one uses comoving coordinates, $u^\mu=\delta^\mu_0$ and

$$T_{00}=\rho, \qquad T_{rr}=\frac{T_{\theta\theta}}{r^2}=a^2p. \quad (7)$$

The $tt$ component of the field Eqs. (2) yields the modified Friedmann equation,

$$H^2=\frac{h(R,\mathcal{L},\rho)}{6\kappa},$$
$$h(R,\mathcal{L},\rho)\equiv\frac{\kappa}{F_1+\frac{F_2\mathcal{L}}{\kappa}}\left[\frac{f_2\rho}{\kappa}-6H\partial_t\left(F_1+\frac{F_2\mathcal{L}}{\kappa}\right)+\left(F_1+\frac{F_2\mathcal{L}}{\kappa}\right)R-f_1\right], \quad (8)$$

where $H\equiv\dot{a}/a$; clearly, for $f_1(R)=R$ and $f_2(R)=1$, the usual result $h(\rho)=\rho$ ensues.

The purpose of this work is to ascertain the forms for $f_1(R)$ and $f_2(R)$ compatible with a specific form $h(\rho)$ for the Friedmann-like equation; this requires that both the Lagrangian density $\mathcal{L}$ and the scalar curvature $R$ are expressed in terms of the energy density $\rho$. The former is simply attained by writing

$$\mathcal{L}=-\alpha\rho, \qquad \alpha=\begin{cases}1, & \mathcal{L}=-\rho\\ -\omega, & \mathcal{L}=p\end{cases}, \quad (9)$$

where $\omega=p/\rho$ is the equation of state (EOS) parameter. This captures the two possible Lagrangian formulations of a perfect fluid (although it was argued in Ref. [24] that $\mathcal{L}=-\rho$ is the most adequate choice), including the possibility of effectively describing a scalar field where $\mathcal{L}=p(\varphi,\dot{\varphi})$.

Replacing Eq. (9) into Eq. (8), one gets

$$h(R,\rho)\equiv\frac{\kappa}{F_1-\alpha F_2\frac{\rho}{\kappa}}\left[\frac{f_2\rho}{\kappa}-f_1+6H\partial_t\left(F_1-\alpha F_2\frac{\rho}{\kappa}\right)+\left(F_1-\alpha F_2\frac{\rho}{\kappa}\right)R\right], \quad (10)$$

while the trace Eq. (3) yields

$$\left(F_1-\alpha\frac{F_2\rho}{\kappa}\right)R=-\frac{1-3\omega}{2}\frac{f_2\rho}{\kappa}-3\Box\left(F_1-\alpha\frac{F_2\rho}{\kappa}\right)+2f_1, \quad (11)$$

and the nontrivial $\nu=t$ component of the energy conservation Eq. (4) becomes

$$\dot{\rho} + 3H(1+\omega)\rho = \frac{F_2}{f_2}(\alpha - 1)\rho\dot{R}. \tag{12}$$

In the following sections one derives the aforementioned relation between the functions $f_1(R)$, $f_2(R)$ and $h(\rho)$, using two different methods. First, one considers only a perturbative regime, so that the scalar curvature is approximately given by its usual GR expression, $R(\rho) = R_0(\rho) \equiv -T/(2\kappa)$. Second, one attempts to obtain a correspondence valid even in a nonperturbative scenario, by imposing instead that the solution to the above system of equations obeys the condition $\kappa F_1 + F_2\mathcal{L} = \text{constant}$.

## IV. PERTURBATIVE REGIME

In this section, one establishes a relation between the nontrivial forms for $f_1(R)$, $f_2(R)$ and the modified Friedmann equation. Following Ref. [34], one writes

$$f_1(R) = R + \varphi_1(R), \qquad f_2(R) = 1 + \varphi_2(R), \tag{13}$$

where $\varphi_i(R)$ are assumed to be perturbative, $\varphi_1(R) \ll R$ and $\varphi_2(R) \ll 1$. By the same token, the scalar curvature is assumed to be approximately given by

$$R(\rho) \approx R_0(\rho) = (1-3\omega)\frac{\rho}{2\kappa}, \tag{14}$$

so that

$$\frac{d\varphi_i}{dR} = \frac{2\kappa}{1-3\omega}\varphi_i{}'(\rho), \tag{15}$$

where the prime denotes differentiation with respect to the energy density $\rho$. This perturbative approach cannot be applied for radiation or relativistic matter, as both have an EOS parameter $\omega = 1/3$, so that the ensuing curvature vanishes according to GR, $R_0 = 0$.

It should be noted that this classical result can be evaded, since corrections to the value of the EOS parameter $\omega = 1/3$ (of the order $10^{-2}$) appear due to the effect of a trace anomaly arising from quantum effects in a photon gas in a $SU(N)$ gauge theory [35].

Using the covariant conservation of the energy-momentum tensor, Eq. (12), and writing $h(\rho) = \rho + \delta(\rho)$, Eq. (10) becomes, to first order in $\varphi_i$,

$$\delta(\rho) = -\kappa\varphi_1 + \left(\varphi_2 - \kappa\frac{1+3\omega}{1-3\omega}\varphi_1'\right)\rho + \frac{6(1+\omega)\kappa\varphi_1'' - \alpha(5+3\omega)\varphi_2'}{1-3\omega}\rho^2 - \frac{6(1+\omega)\alpha\varphi_2''}{1-3\omega}\rho^3. \tag{16}$$

This linear nonhomogeneous differential equation enables a direct translation between the form of the modified Friedmann equation and the nontrivial forms for $f_1(R)$, $f_2(R)$ giving origin to it. Setting $\varphi_i(R) = 0$ trivially yields $\delta(\rho) = 0$, while the inclusion of a cosmological constant (CC) term in the action, $\varphi_1(R) = -2\Lambda$ and $\varphi_2(R) = 0$, leads to $H^2 = (\rho/6\kappa) + \Lambda/3$, as expected. Notice, however, that a dominant CC, which occurs since $z \sim 0.4$ [36], cannot be accommodated by the mechanism outlined above, as it is nonperturbative, i.e. $R \sim 4\Lambda \neq R_0$ (since $\Omega_\Lambda \sim 0.7 > \Omega_m \sim 0.3$).

### A. "Neutral" solutions

One notices that since Eq. (16) is linear, its general solution for a given modification $\delta(\rho)$ of the Friedmann equation includes the solution of the corresponding homogeneous differential equation, i.e. the one for which $\delta(\rho) = 0$. Physically, this simply states that any perturbation $\varphi_{1H}(R)$ in the action has a counterpart $\varphi_{2H}(R)$ that cancels out its dynamical effect. These functions, here dubbed as "neutral," are related by

$$0 = -\kappa\varphi_{1H} + \left(\varphi_{2H} - \kappa\frac{1+3\omega}{1-3\omega}\varphi_{1H}'\right)\rho + \frac{6(1+\omega)\kappa\varphi_{1H}'' - \alpha(5+3\omega)\varphi_{2H}'}{1-3\omega}\rho^2 - \frac{6(1+\omega)\alpha\varphi_{2H}''}{1-3\omega}\rho^3. \tag{17}$$

By the same token, if one sets $\varphi_{2H} = 0$ and solves the above differential equation for $\varphi_{1H}$ (and vice versa), one finds that functions of the form $\varphi_{1H}(R) \sim R^{n_1}$ and $\varphi_{2H}(R) \sim R^{n_2}$, with

$$n_1 = \frac{7 + 9\omega \pm \sqrt{73 + 78\omega + 9\omega^2}}{12(1+\omega)},$$
$$n_2 = \frac{1 + 3\omega \pm \sqrt{(1+3\omega)^2 + \frac{24}{\alpha}(1+\omega)(1-3\omega)}}{12(1+\omega)}, \tag{18}$$

do not modify the Friedmann equation at first order, since their contribution to Eq. (16) vanish.

For completeness, one lists the values of the above exponents for the relevant matter species:

(i) Nonrelativistic dust ($\omega = 0$, $\alpha = 1$):

$$n_1 = \frac{7 \pm \sqrt{73}}{12}, \qquad n_2 = \frac{1 \pm 5}{12}. \tag{19}$$

(ii) Ultrastiff matter ($\omega = 1$, $\alpha = 1$):

$$n_1 = \frac{4 \pm \sqrt{10}}{6}, \qquad n_2 = \frac{1 \pm i\sqrt{5}}{6}. \tag{20}$$

(iii) Scalar field ($\omega = 1$, $\alpha = -\omega = -1$):

$$n_1 = \frac{4 \pm \sqrt{10}}{6}, \qquad n_2 = \frac{1 \pm \sqrt{7}}{6}. \tag{21}$$

Notice the unphysical, complex value of $n_2$ for ultrastiff matter. The conditions under which the method outlined here can be applied to a power-law modification of the Friedmann equation will be discussed in a subsequent section.

## V. NONPERTURBATIVE, RELAXED REGIME

In this section one obtains a correspondence between the functions $f_1(R)$, $f_2(R)$ and the right-hand side of the Friedmann Eq. (8); instead of assuming a perturbative regime, the possibility that the scalar curvature may strongly deviate from its GR expression, $R(\rho) \neq R_0(\rho)$, is addressed.

To tackle this situation, the hypothesis that the combination $F_1 + F_2\mathcal{L}/\kappa$ is constant is posited. This could stem, for instance, from a fixed point in the dynamical system constituted by Eq. (2): indeed, the presence of the kinetic term $\Delta_{\mu\nu}(F_1 + F_2\mathcal{L}/\kappa)$ in these is suggestive that some fixed points could obey the constraint

$$F_1 + \frac{F_2\mathcal{L}}{\kappa} = F_1 - \alpha F_2 \frac{\rho}{\kappa} = A \neq 0. \tag{22}$$

Notice that GR yields $A = 1$, so that one expects this quantity to be either always equal to unit, or to depend on additional parameters related to the functions $f_1(R)$, $f_2(R)$ in such a way that $A \to 1$ renders GR.

Inserting Eq. (22) into Eq. (10) yields

$$Ah = f_2\rho + A\kappa R - \kappa f_1, \tag{23}$$

while the trace Eq. (11) becomes

$$AR = -\frac{1-3\omega}{2\kappa} f_2\rho + 2f_1. \tag{24}$$

Solving for $f_1(R)$ and inserting back into Eq. (23), one gets

$$2h = \kappa R + \frac{3}{2A}(1+\omega) f_2\rho. \tag{25}$$

The above is further simplified by writing the scalar curvature explicitly,

$$R = 6(\dot{H} + 2H^2) = 6\left(\frac{(\dot{H^2})}{2H} + 2H^2\right) = \\ = \frac{1}{\kappa}\left(\frac{\dot{h}}{2H} + 2h\right) = \frac{1}{\kappa}\left(\frac{h'(\rho)}{2}\frac{\dot{\rho}}{H} + 2h\right). \tag{26}$$

One may resort to Eq. (12) to eliminate $\dot{\rho}$. For this, one assumes that the scalar curvature can be written as a function of the energy density alone, $R = R(\rho)$ (as will be shown below), so that

$$f_2(R) = f_2(R(\rho)) \to F_2\dot{R} = f_2'(\rho)\dot{\rho}; \tag{27}$$

i.e. one differentiates with respect to the energy density $\rho$, instead of the scalar curvature. In the remainder of this work, the notation $F_i = df_i(R)/dR$ and $f_i' = df(R(\rho))/d\rho$ is adopted, for brevity. Thus, Eq. (12) implies that

$$\dot{\rho} = -\frac{3H(1+\omega)f_2\rho}{f_2 + (1-\alpha)f_2'\rho}. \tag{28}$$

Replacing this into Eq. (26) yields

$$R(\rho) = \frac{1}{\kappa}\left(2h - \frac{3}{2}\frac{(1+\omega)f_2 h'\rho}{f_2 + (1-\alpha)f_2'\rho}\right), \tag{29}$$

thus showing that the scalar curvature can be expressed solely as a function of the energy density, as claimed above.

Inserting this into Eq. (25) leads to

$$A\frac{(1+\omega)f_2 h'\rho}{f_2 + (1-\alpha)f_2'\rho} = (1+\omega)f_2\rho. \tag{30}$$

Assuming that $f_2 + (1-\alpha)f_2'\rho \neq 0$ and $(1+\omega)f_2\rho \neq 0$ (so that a $\omega \neq -1$ scalar field in slow roll is excluded), one finally obtains

$$f_2 + (1-\alpha)f_2'\rho = Ah'. \tag{31}$$

This relationship provides the connection between the modified form of the Friedmann equation and the NMC; it is worth remarking that if $\mathcal{L} = -\rho \to \alpha = 1$, then the NMC is directly read, $f_2(\rho) = Ah'$. Inserting this into Eq. (29), one gets

$$R(\rho) = \frac{1}{\kappa}\left[2h - \frac{3}{2A}(1+\omega)f_2\rho\right]. \tag{32}$$

Finally, Eq. (11) allows one to read

$$f_1 = \frac{1}{\kappa}\left(Ah - \frac{1+3\omega}{2}f_2\rho\right). \tag{33}$$

In order to identify the functions $f_1(R)$ and $f_2(R)$ that enable a particular form for $h(\rho)$, one must first solve Eq. (31) for $f_2(\rho)$. One can then compute $R(\rho)$ from Eq. (32) and invert it to obtain $\rho = \rho(R)$; replacing this back into $f_2(\rho)$ yields the aimed NMC. A similar procedure using Eq. (33) yields the curvature term $f_1(R)$. Condition Eq. (22) can then be explicitly checked (it can also be shown to be valid for a general $h(\rho)$, although that is not shown here).

In the following sections some specific forms for the right-hand side of the Friedmann equation, i.e. the function $h(\rho)$, are explored, with the aim of obtaining the functions $f_i(R)$ that give rise to the latter using the two methods proposed above.

## VI. APPLICATION: COSMOLOGICAL CONSTANT

One remarks that Eq. (31) is linear, so that its solution is the sum of a particular solution plus the general solution of the corresponding homogeneous equation. The latter amounts to $h'(\rho) = 0$, yielding a constant Hubble parameter $H$ (i.e. a de Sitter phase)—and can thus be related to the CC.

Assuming that the matter contribution is negligible, one thus has $h(\rho) \approx 6\kappa H_0^2$: for this reason, one cannot rely on the perturbative approach, as the scalar curvature is approximately constant $R = 2h/\kappa = 12H_0^2$ and cannot be considered to be a small deviation from the form $R = R_0 = (1 - 3\omega)\rho/2\kappa$, as argued above. Consequently one must rely solely on the method developed for the relaxed regime; in particular, Eq. (31) becomes homogeneous,

$$f_2 + (1 - \alpha)f_2'\rho = 0 \to f_2(\rho) \propto \rho^{1/(\alpha-1)}. \tag{34}$$

Inserting into Eq. (29), one gets a constant, as expected; as a result, one cannot invert it to write $\rho = \rho(R)$, and the obtained NMC cannot be expressed as a function of the scalar curvature.

Thus, one concludes instead that, although the energy density may be subdominant with respect to the contribution of the CC, it must also be accounted for. This translates into the choice

$$h(\rho) = \rho + 2\kappa\Lambda, \tag{35}$$

so that Eq. (31) reads

$$f_2 + (1 - \alpha)f_2'\rho = A = 1 \to f_2(\rho) = 1 + C\left(\frac{\rho}{2\kappa\Lambda}\right)^{1/(\alpha-1)}, \tag{36}$$

where $C$ is an integration constant and $A = 1$ was set to enforce $f(0) = 1$. If $\mathcal{L} = -\rho \to \alpha = 1$, the exponent on the right-hand side is singular, so one has to separately consider both options $\alpha = 1$ or $\alpha = -\omega$.

### A. $\mathcal{L} = -\rho$ case

If $\alpha = 1$, then Eq. (31) straightforwardly reads $f_2(R) = 1$, so that a minimal coupling between curvature and matter is recovered. Naturally, Eq. (32) collapses into the GR result

$$R(\rho) = 4\Lambda + \frac{(1 - 3\omega)\rho}{2\kappa}, \tag{37}$$

and Eq. (33) reads

$$f_1 = 2\Lambda + \frac{(1 - 3\omega)\rho}{2\kappa} = R - 2\Lambda, \tag{38}$$

thus yielding the trivial result that, in the absence of a NMC, a CC can be produced by simply inserting it into $f_1(R)$.

### B. $\mathcal{L} = -p$ case

One is thus left with the possibility $\mathcal{L} = p \to \alpha = -\omega$, which is read directly from Eq. (36),

$$f_2(\rho) = 1 + C\left(\frac{2\kappa\Lambda}{\rho}\right)^{1/(1+\omega)}. \tag{39}$$

Inserting this into Eq. (32) yields

$$R(\rho) = 4\Lambda + \left[1 - 3\omega - 3(1 + \omega)C\left(\frac{2\kappa\Lambda}{\rho}\right)^{1/(1+\omega)}\right]\frac{\rho}{2\kappa}. \tag{40}$$

Although the above cannot be easily inverted in order to obtain $\rho = \rho(R)$, Eq. (11) allows one to read

$$f_1(R) = 2\Lambda + \left[1 - 3\omega - (1 + 3\omega)C\left(\frac{2\kappa\Lambda}{\rho}\right)^{1/(1+\omega)}\right]\frac{\rho}{2\kappa} = R - 2\Lambda, \tag{41}$$

so that the usual expression for GR with a CC is recovered.

One also obtains

$$R'(\rho) = \frac{1 - 3\omega}{2\kappa} - \frac{3\omega C}{2\kappa}\left(\frac{2\kappa\Lambda}{\rho}\right)^{1/(1+\omega)}. \tag{42}$$

Up to now, the integration constant $C$ has not been specified. One is only left with the relaxation condition Eq. (22) to look for its value; it reads

$$F_1 + \omega\frac{F_2\rho}{\kappa} = 1 + \omega\frac{f_2'(\rho)\rho}{\kappa R'(\rho)} = 1 - 2\frac{\omega}{1 + \omega}C\frac{\left(\frac{2\kappa\Lambda}{\rho}\right)^{1/(1+\omega)}}{1 - 3\omega - 3\omega C\left(\frac{2\kappa\Lambda}{\rho}\right)^{1/(1+\omega)}}, \tag{43}$$

which is only constant (and equal to $A = 1$) if $C = 0$. This, however, collapses Eqs. (39) and (40) to their GR form—and so one concludes that the only way of obtaining a CC obeying the relaxation condition Eq. (22) is by trivially setting $f_1(R) = R - 2\Lambda$ and $f_2(R) = 1$.

This result complements a similar argument, discussed in Ref. [9], which was valid only for a pure de Sitter expansion, i.e. disregarding the contribution of the energy density of matter, so that $h(\rho) \sim 2\kappa\Lambda$. If the scalar curvature $R$ does not evolve with time, neither do $F_1$ and $F_2$; since the energy density $\rho$ varies [albeit in a modified fashion, as given by Eq. (28)], then it is impossible to enforce the relaxation condition $F_1 - \alpha F_2 \rho/\kappa$ (except for the trivial case obtained). Here, it is shown that this result does not change even if the matter contribution is considered. This does not preclude the possibility of obtaining a CC from a NMC, although the two methods developed above cannot be applied. Such an issue is deferred to Sec. XI.

Given the above, one drops the homogeneous solution $\sim\rho^{-1/(1+\omega)}$ from the scenarios studied in the subsequent paragraphs, thus considering that the Friedmann equation [as given by $h(\rho)$] cannot include a constant term.

## VII. APPLICATION: POWER-LAW FORM

In this section one addresses the possibility that the modified Friedmann equation is written as

$$h(\rho) = \rho\left[1 + \epsilon\left(\frac{\rho}{\rho_c}\right)^{n-1}\right], \quad n \neq 1, \tag{44}$$

where $\epsilon = \pm 1$ marks the sign of the power-law addition. Positive quadratic corrections, $\epsilon = 1$ and $n = 2$, arise in braneworld scenarios [37–39], while loop quantum gravity [40] gives rise to a negative quadratic term, $\epsilon = -1$. The so-called Cardassian models postulate deviations of the form above [41], with $\epsilon = 1$ and $n < 2/3$. By following the two methods here devised, one may ascertain the related form of the functions $f_1(R)$ and $f_2(R)$.

### A. Perturbative regime

The perturbation $\delta(\rho) \equiv h(\rho) - \rho = \epsilon\rho_c(\rho/\rho_c)^n$ is now considered. Inspection of Eq. (16) shows that the functions $\varphi_i(\rho)$ should also be power laws, writing

$$\varphi_1(\rho) = K_1 \frac{\rho_c}{\kappa}\left(\frac{\rho}{\rho_c}\right)^{n_1}, \qquad \varphi_2(\rho) = K_2\left(\frac{\rho}{\rho_c}\right)^{n_2}, \tag{45}$$

where $K_1$ and $K_2$ are dimensionless constants, and thus

$$\epsilon = -\left(1 + \frac{7 + 9\omega - 6(1+\omega)n_1}{1 - 3\omega} n_1\right) K_1 \left(\frac{\rho}{\rho_c}\right)^{n_1 - n} + \left(1 + \frac{1 + 3\omega - 6(1+\omega)n_2}{1 - 3\omega}\alpha n_2\right) K_2 \left(\frac{\rho}{\rho_c}\right)^{n_2 - n + 1}, \tag{46}$$

so that one must have $n = n_1 = n_2 + 1$. Replacing into the above, one obtains the following relation between the constants $K_1$ and $K_2$:

$$\epsilon = -\left(1 + \frac{7 + 9\omega - 6(1+\omega)n}{1 - 3\omega} n\right) K_1 + \left(1 + \frac{7 + 9\omega - 6(1+\omega)n}{1 - 3\omega}\alpha(n-1)\right) K_2. \tag{47}$$

The result above plainly shows that at a perturbative level, a $\delta(\rho) \sim \rho^n$ power-law modification of the Friedmann equation can be achieved by either a nonlinear curvature term,

$$f_1(R) = R\left[1 + \frac{2K_1}{1 - 3\omega}\left(\frac{2\kappa R}{\rho_c(1 - 3\omega)}\right)^{n-1}\right], \tag{48}$$

a power-law NMC,

$$f_2(R) = 1 + K_2\left(\frac{2\kappa R}{\rho_c(1 - 3\omega)}\right)^{n-1}, \tag{49}$$

or a combination of both.

Furthermore, due to the linearity of Eq. (16), one concludes that if the perturbative modification of the Friedmann equation can be expanded in powers of $\rho$,

$$\delta(\rho) = -\sum_{n=1}^{\infty} a_n \left(\frac{\rho}{\rho_c}\right)^n, \tag{50}$$

it is related to the functions

$$f_1(R) = R\left[1 + \sum_{n=1}^{\infty}\frac{2a_n K_{1n}}{1 - 3\omega}\left(\frac{2\kappa R}{\rho_c(1 - 3\omega)}\right)^{n-1}\right],$$
$$f_2(R) = 1 + \sum_{n=1}^{\infty} a_n K_{2n}\left(\frac{2\kappa R}{\rho_c(1 - 3\omega)}\right)^{n-1}, \tag{51}$$

where the $K_{in}$ $(i = 1, 2)$ coefficients obey Eq. (47) for each $n$.

Finally, one recalls that, from the previous discussion about homogeneous solutions of Eq. (16), power-law perturbations $\varphi_i(R)$ with exponents given by Eq. (18) do not have an impact on the Friedmann equation (at first perturbative order).

This means that any power-law perturbation $\delta(\rho) \sim \rho^n$ to the Friedmann equation can be considered. As an example, suppose one is aiming at implementing a $h(\rho) \sim \rho^{3/2}$ perturbation for a dust-dominated universe: although a NMC $\varphi_2(R) \sim R^{1/4}$ has no impact on Eq. (46), the function $\varphi_1 \sim R^{3/2}$ gives rise to the desired power-law modification. Conversely, $h(\rho) \sim \rho^{\frac{7+\sqrt{73}}{12}}$ requires a NMC $\varphi_2(R) \sim \rho^{\frac{-5+\sqrt{73}}{12}}$, since $\varphi_1(R) \sim R^{\frac{7+\sqrt{73}}{12}}$ would have no dynamical impact in a universe dominated by a nonrelativistic perfect fluid.

Furthermore, one notices that the listed neutral exponents are always noninteger, so they do not contradict the assumed Taylor expansion, as discussed above.

### B. Relaxed regime

One now uses Eq. (31) to read the NMC,

$$f_2 + (1-\alpha)f_2'\rho = A\left[1+\epsilon n\left(\frac{\rho}{\rho_c}\right)^{n-1}\right] \rightarrow$$
$$f_2(R) = 1 + \frac{\epsilon n}{n+\alpha(1-n)}\left(\frac{\rho}{\rho_c}\right)^{n-1}, \quad (52)$$

where $A=1$ was fixed so that $f_2(\rho)=1$ when $\rho \ll \rho_c$ and GR is recovered.

One may now replace the obtained expression into Eq. (32), leading to

$$R(\rho) = \frac{\rho}{2\kappa}\left[1-3\omega+\epsilon\frac{4\alpha(1-n)+n(1-3\omega)}{n+\alpha(1-n)}\left(\frac{\rho}{\rho_c}\right)^{n-1}\right], \quad (53)$$

and, from Eq. (33),

$$f_1 = \frac{\rho}{2\kappa}\left[1-3\omega+\epsilon\frac{2\alpha(1-n)+n(1-3\omega)}{n+\alpha(1-n)}\left(\frac{\rho}{\rho_c}\right)^{n-1}\right]. \quad (54)$$

In order to write $f_1$ and $f_2$ in terms of the scalar curvature, one must invert Eq. (53), which requires solving a $n$-order algebraic equation. For a generic value of the exponent $n$ and the EOS parameter $\omega$, one can proceed numerically; for completion, the examples considered in the previous section are worked out below:

(i) Nonrelativistic dust ($\omega=0$, $\alpha=1$):

$$R(\rho) = \frac{\rho}{2\kappa}\left[1+\epsilon(4-3n)\left(\frac{\rho}{\rho_c}\right)^{n-1}\right],$$
$$f_1(R) = \frac{\rho}{2\kappa}\left[1+\epsilon(2-n)\left(\frac{\rho}{\rho_c}\right)^{n-1}\right],$$
$$f_2(R) = 1+\epsilon n\left(\frac{\rho}{\rho_c}\right)^{n-1}. \quad (55)$$

(ii) Radiation ($\omega=1/3$, $\alpha=-\omega=-1/3$ and $n\neq 1/4$):

$$R(\rho) = 2\epsilon\frac{\rho_c}{\kappa}\frac{n-1}{4n-1}\left(\frac{\rho}{\rho_c}\right)^{n},$$
$$f_1(R) = \epsilon\frac{\rho_c}{\kappa}\frac{n-1}{4n-1}\left(\frac{\rho}{\rho_c}\right)^{n} = \frac{R}{2},$$
$$f_2(R) = 1+\epsilon\frac{3n}{4n-1}\left(\frac{\rho}{\rho_c}\right)^{n-1}$$
$$= 1+\frac{3n}{\sqrt{\epsilon(4n-1)}}\left(\frac{\kappa R}{2(n-1)\rho_c}\right)^{(n-1)/n}. \quad (56)$$

(iii) Relativistic matter ($\omega=1/3$, $\alpha=1$):

$$R(\rho) = 2\epsilon(1-n)\frac{\rho_c}{\kappa}\left(\frac{\rho}{\rho_c}\right)^{n},$$
$$f_1(R) = \epsilon(1-n)\frac{\rho_c}{\kappa}\left(\frac{\rho}{\rho_c}\right)^{n} = \frac{R}{2},$$
$$f_2(R) = 1+\epsilon n\left(\frac{\rho}{\rho_c}\right)^{n-1}$$
$$= 1+\epsilon^{1/n}n\left(\frac{\kappa R}{2(1-n)\rho_c}\right)^{(n-1)/n}. \quad (57)$$

(iv) Ultrastiff matter ($\omega=1$, $\alpha=1$):

$$R(\rho) = -\frac{\rho}{\kappa}\left[1+\epsilon(3n-2)\left(\frac{\rho}{\rho_c}\right)^{n-1}\right],$$
$$f_1(\rho) = -\frac{\rho}{\kappa}\left[1+\epsilon(2n-1)\left(\frac{\rho}{\rho_c}\right)^{n-1}\right],$$
$$f_2(\rho) = 1+\epsilon n\left(\frac{\rho}{\rho_c}\right)^{n-1}. \quad (58)$$

(v) Scalar field ($\omega=1$, $\alpha=-\omega=-1$):

$$R(\rho) = -\frac{\rho}{\kappa}\left[1+\epsilon\frac{n-2}{2n-1}\left(\frac{\rho}{\rho_c}\right)^{n-1}\right],$$
$$f_1(\rho) = -\frac{\rho}{\kappa}\left[1+\frac{\epsilon}{2n-1}\left(\frac{\rho}{\rho_c}\right)^{n-1}\right],$$
$$f_2(\rho) = 1+\epsilon\frac{n}{2n-1}\left(\frac{\rho}{\rho_c}\right)^{n-1}. \quad (59)$$

One concludes that radiation or relativistic matter are not suited for implementing a power-law modification of the Friedmann equation, as they require that $f_1(R)=R/2$.

## VIII. APPLICATION: QUADRATIC FORM

The specific case

$$h(\rho) = \rho\left(1+\epsilon\frac{\rho}{\rho_c}\right), \quad (60)$$

is of particular interest, as it arises from braneworld scenarios [37–39], loop quantum gravity (where it is found to be valid even for very high densities, $\rho \lesssim \rho_c$) [40] and, phenomenologically, it can be viewed as a leading-order correction to the linear form of the Friedmann equation. One now replaces $n=2$ into the previous results, for illustration.

### A. Perturbative regime

Using Eqs. (48) and (49) one finds that, at perturbative level, the modified Friedmann Eq. [40]) stems from both a quadratic curvature term

$$f_1(R) = R + \frac{4\kappa K_1}{(1-3\omega)^2}\frac{R^2}{\rho_c}, \tag{61}$$

and a linear NMC

$$f_2(R) = 1 + \frac{2\kappa K_2}{(1-3\omega)}\frac{R}{\rho_c}, \tag{62}$$

with the constraint

$$\epsilon = 9\frac{1+\omega}{1-3\omega}K_1 - \left(\frac{5+3\omega}{1-3\omega}\alpha - 1\right)K_2. \tag{63}$$

The above agrees with the particular case previously studied in Ref. [34], with a minimally coupled scalar field ($\omega = -\alpha = 1$) and a negative quadratic term, $\epsilon = -1$,

$$f_1(R) = R + \frac{\kappa R^2}{9\rho_c}, \qquad f_2(R) = 1. \tag{64}$$

For comparison, one finds that a scalar field can also give rise to the same quadratic modification resorting only to a linear NMC,

$$f_1(R) = R, \qquad f_2(R) = 1 + \frac{R}{3\rho_c}. \tag{65}$$

Notice that although the NMC has a positive slope, the negative curvature $R \approx -\rho/\kappa < 0$ found in Eq. (59) leads to a subtractive quadratic term $h(\rho) = -\rho^2/\rho_c$.

### B. Relaxed regime

From Eqs. (52)–(54), one gathers that

$$R(\rho) = \frac{\rho}{\kappa}\left(\frac{1-3\omega}{2} + \epsilon\frac{1-3\omega-2\alpha}{2-\alpha}\frac{\rho}{\rho_c}\right) \to \rho(R) = \epsilon\frac{(2-\alpha)(1-3\omega)}{4(1-2\alpha-3\omega)}\rho_c\left(\sqrt{1+\epsilon\frac{16(1-3\omega-2\alpha)}{(2-\alpha)(1-3\omega)^2}\frac{\kappa R}{\rho_c}} - 1\right),$$

$$f_1(R) = \frac{\rho}{\kappa}\left(\frac{1-3\omega}{2} + \epsilon\frac{1-3\omega-\alpha}{2-\alpha}\frac{\rho}{\rho_c}\right) = \frac{1-3\omega-\alpha}{1-3\omega-2\alpha}R - \epsilon\frac{\rho_c}{8\kappa}(2-\alpha)\alpha\left(\frac{1-3\omega}{1-3\omega-2\alpha}\right)^2\left(\sqrt{1+\epsilon\frac{16(1-2\alpha-3\omega)}{(2-\alpha)(1-3\omega)^2}\frac{\kappa R}{\rho_c}} - 1\right),$$

$$f_2(R) = 1 + \frac{2\epsilon}{2-\alpha}\frac{\rho}{\rho_c} = 1 + \frac{1-3\omega}{2(1-2\alpha-3\omega)}\left(\sqrt{1+\epsilon\frac{16(1-2\alpha-3\omega)}{(2-\alpha)(1-3\omega)^2}\frac{\kappa R}{\rho_c}} - 1\right), \tag{66}$$

where the positive branch of the square root was chosen when obtaining $\rho = \rho(R)$ so that $\rho \sim 2\kappa R/(1-3\omega)$ if $\rho \ll \rho_c$. All expressions collapse to their GR counterparts $f_1(R) \approx R$ and $f_2(R) \approx 1$ when $\rho \ll \rho_c$, as expected.

One may compute the obtained expressions for specific matter contents, namely:

(i) Nonrelativistic dust ($\omega = 0$, $\alpha = 1$):

$$R(\rho) = \frac{\rho}{2\kappa}\left(1 - 2\epsilon\frac{\rho}{\rho_c}\right),$$

$$f_1(R) = \epsilon\frac{\rho_c}{8\kappa}\left(1 - \sqrt{1 - 16\epsilon\frac{\kappa R}{\rho_c}}\right),$$

$$f_2(R) = \frac{1}{2}\left(3 - \sqrt{1 - 16\epsilon\frac{\kappa R}{\rho_c}}\right). \tag{67}$$

(ii) Radiation ($\omega = 1/3$, $\alpha = -\omega = -1/3$):

$$R(\rho) = \frac{2\epsilon}{7}\frac{\rho^2}{\kappa\rho_c},$$

$$f_1(R) = \frac{R}{2},$$

$$f_2(R) = 1 + 3\sqrt{\frac{2\epsilon}{7}\frac{\kappa R}{\rho_c}}. \tag{68}$$

(iii) Relativistic matter ($\omega = 1/3$, $\alpha = 1$):

$$R(\rho) = -2\epsilon\frac{\rho^2}{\kappa\rho_c},$$

$$f_1(R) = \frac{R}{2},$$

$$f_2(R) = 1 - \sqrt{-2\epsilon\frac{\kappa R}{\rho_c}}. \tag{69}$$

(iv) Ultrastiff matter ($\omega = 1$, $\alpha = 1$):

$$R(\rho) = -\frac{\rho}{\kappa}\left(1 + 4\epsilon\frac{\rho}{\rho_c}\right),$$

$$f_1(R) = \frac{3}{4}R + \epsilon\frac{\rho_c}{32\kappa}\left(1 - \sqrt{1 - 16\epsilon\frac{\kappa R}{\rho_c}}\right),$$

$$f_2(R) = \frac{1}{4}\left(3 + \sqrt{1 - 16\epsilon\frac{\kappa R}{\rho_c}}\right). \tag{70}$$

(v) Scalar field ($\omega = 1$, $\alpha = -\omega = -1$):

$$R(\rho) = -\frac{\rho}{\kappa},$$
$$f_1(R) = R + \frac{\epsilon}{3}\frac{\kappa R^2}{\rho_c},$$
$$f_2(R) = 1 - \frac{2\epsilon}{3}\frac{\kappa R}{\rho_c}. \tag{71}$$

The different forms for $f_i(R)$ obtained for radiation vs relativistic matter and for ultrastiff matter vs scalar field show the relevance of the choice of the form for $\mathcal{L}$. Although these pairs of matter types are characterized by the same EOS parameter ($\omega = 1/3$ and $\omega = 1$, respectively), their differing Lagrangian densities (relativistic and ultrastiff matter are assumed to be described by $\mathcal{L} = -\rho$ (as discussed in Ref. [24]), while radiation and a scalar field have $\mathcal{L} = p$) account for the distinct results.

Finally, one notices that a NMC scalar field can lead to a quadratic Friedmann equation, with both a linear NMC and a quadratic curvature term.

## IX. APPLICATION: NONINTEGER POWER-LAW FORM

The previous section considered a power-law addition to the usual linear term found in the Friedmann equation; another modification of the latter can assume instead that the behaviour of the Friedmann equation is almost linear, so that

$$h(\rho) = \rho\left(\frac{\rho}{\rho_c}\right)^\beta, \quad \beta \sim 0. \tag{72}$$

While the previously considered power-law addition may be viewed as arising from a Taylor expansion of a more general modification of the Friedmann equation, the above presents the opposite case of a nonanalytical form for $h(\rho)$ (for noninteger $\beta$). Regardless of this, both methods developed here still apply, as is shown in the following paragraphs.

### A. Perturbative regime

Since one considers an exponent $\beta \sim 0$, the quantity

$$\delta(\rho) \equiv h(\rho) - \rho = \rho\left[\left(\frac{\rho}{\rho_c}\right)^\beta - 1\right], \tag{73}$$

is a small perturbation, and one may resort to Eq. (16) to translate $h(\rho)$ into the corresponding functions $f_1(R)$ and $f_2(R)$. Since the former is linear, its solution (disregarding nondynamical contributions obeying Eq. (17), as discussed before) is given by the linear combination of Eqs. (48) and (49) with $(n = \beta + 1, \epsilon = 1)$ and $(n = 1, \epsilon = -1)$,

$$\varphi_1(\rho) = A_1\frac{\rho}{\kappa} + K_1\frac{\rho}{\kappa}\left(\frac{\rho}{\rho_c}\right)^\beta,$$
$$\varphi_2(\rho) = A_2 + K_2\left(\frac{\rho}{\rho_c}\right)^\beta. \tag{74}$$

Replacing the above into Eq. (16), one finds that the pairs $(K_1, K_2)$ and $(A_1, A_2)$ still obey Eq. (47) (with $n = \beta + 1$, $\epsilon = 1$ and $n = 1$, $\epsilon = -1$, respectively),

$$1 = -\left[1 + \frac{1+3\omega-6(1+\omega)\beta}{1-3\omega}(\beta+1)\right]K_1 + \left[1 + \frac{1+3\omega-6(1+\omega)\beta}{1-3\omega}\alpha\beta\right]K_2,$$
$$1 = \frac{2}{1-3\omega}A_1 - A_2, \tag{75}$$

as can be verified explicitly.

Setting $A_2 = K_2 = 0$, one finds that Eq. (72) admits a minimal coupling,

$$A_1 = \frac{1-3\omega}{2},$$
$$K_1 = -\left[1 + \frac{1+3\omega-6(1+\omega)\beta}{1-3\omega}(\beta+1)\right]^{-1} \approx -\frac{1-3\omega}{2}, \tag{76}$$

so that

$$f_1(R) \approx R\left(2 - \left[\frac{2\kappa R}{(1-3\omega)\rho_c}\right]^\beta\right) \sim R, \tag{77}$$

and $f_2(R) = 1$.

Conversely, one may resort solely to a NMC, so that $f_1(R) = R$ and

$$A_2 = -1,$$
$$K_2 = \left[1 + \frac{1+3\omega-6(1+\omega)\beta}{1-3\omega}\alpha\beta\right]^{-1}, \tag{78}$$

thus yielding

$$f_2(R) \approx \left[\frac{2\kappa R}{(1-3\omega)\rho_c}\right]^\beta. \tag{79}$$

Both expressions clearly show the validity of the perturbative regime, since they approach their GR counterpart, $f_2 \sim 1$ when $\beta$ vanishes.

### B. Relaxed regime

Instead of applying the procedure outlined above, one may simply resort to Eqs. (52)–(54) found in the preceding section for a positive addition, $\epsilon = 1$. Replacing the exponent $n \to 1 + \beta$ and disregarding the usual term arising from GR, i.e. considering only the $\rho_c$ terms, one obtains

$$R(\rho) = \frac{(1+\beta)(1-3\omega) - 4\alpha\beta}{1 + (1-\alpha)\beta} \frac{\rho}{2\kappa} \left(\frac{\rho}{\rho_c}\right)^\beta, \tag{80}$$

$$f_1(R) = \left(1 + \frac{2\alpha\beta}{(1+\beta)(1-3\omega) - 4\alpha\beta}\right) R, \tag{81}$$

$$f_2(R) = \frac{1+\beta}{1+(1-\alpha)\beta} \times \left(\frac{1+(1-\alpha)\beta}{(1+\beta)(1-3\omega) - 4\alpha\beta} \frac{2\kappa R}{\rho_c}\right)^{\frac{\beta}{1+\beta}}. \tag{82}$$

Clearly, $\beta = 0$ yields the usual GR results. One sees that the deviation from linearity in the Friedmann equation arises from a small correction to the strength of gravity [via the correction to $f_1(R)$] and the new dynamics imprinted by the NMC. Again, both radiation and relativistic matter yield unphysical results, as replacing $\omega = 1/3$ into the above yields $f_1(R) = R/2$.

## X. OTHER MODIFICATIONS OF THE FRIEDMANN EQUATION

Another class of modifications of the Friedmann equation relies on the introduction of additional terms on $H$, while keeping the linear energy density term. Indeed, in Ref. [42] the alternative formulation,

$$H^2 - \frac{H^\beta}{r_c^{2-\beta}} = \frac{\rho}{6\kappa}, \quad \beta < 2, \tag{83}$$

is considered. For a given value of $\beta$, inverting the above (even if this can only be attained numerically) yields the form here considered, $H^2 = h(\rho)/6\kappa$. One may then apply either of the two methods presented here.

However, the complexity of the ensuing computations (even for a simple linear modification, $\beta = 1$, where $H^2 = H^2(\rho)$ is easily obtained) implies that an analytic solution is very cumbersome, and perhaps best approached numerically. Since this is not the purpose of this study, the above modification is addressed only in the perturbative regime, as one can advantageously take a further step and insert $H^2 \approx \rho/6\kappa$ into Eq. (81), obtaining

$$H^2 = \frac{\rho}{6\kappa} + \frac{1}{r_c^2}(r_c H)^\beta \approx \frac{\rho}{6\kappa} + \frac{1}{r_c^2}\left(\frac{r_c^2 \rho}{6\kappa}\right)^{\beta/2}, \tag{84}$$

showing that the problem collapses into the previously addressed power-law modification, Eq. (44), with $\epsilon = 1$, $n = \beta/2$ and $\rho_c = 6\kappa/r_c^2$.

## XI. COSMOLOGICAL CONSTANT FROM A NONMINIMAL COUPLING

In this section, one considers the putative relation between a CC and the nonminimally coupled $f(R)$ theory posited by Eq. (1); given the inability of the two methods discussed in Sec. VI to tackle this issue, one approaches it by assuming a weaker condition of exponential expansion of the Universe as a solution of Eq. (8). Inserting

$$a(t) = a_0 e^{H_0 t} \to R = 12 H_0^2 = 4\Lambda \tag{85}$$

into the latter, one finds, for the 0–0 component of Eq. (2),

$$\kappa f_{01} - f_{02}\rho = 2[\kappa F_{01} - (4 + 3\omega)\alpha F_{02}\rho]\Lambda, \tag{86}$$

while the $r - r$ component reads

$$\kappa f_{01} + f_{02}\omega\rho = 2[\kappa F_{01} + (4 + 3\omega)\omega\alpha F_{02}\rho]\Lambda, \tag{87}$$

where one defines $f_{0i} \equiv f_i(4\Lambda) = \text{constant}$ and $F_{0i} \equiv F_i(4\Lambda) = \text{constant}$ and uses the covariant conservation of energy-momentum, $\dot{\rho} = -3H(1+\omega)\rho$, which stems from Eq. (12), since the scalar curvature is constant. Clearly, the constraint Eq. (22) cannot be enforced here, since $f_{01}$ and $f_{02}$ are constants, while $\rho$ varies (as argued previously in Ref. [9]).

Inspection shows that Eqs. (86) and (87) are composed of both constant terms and those linear in the energy density. Equating these one obtains, for $\omega \neq -1$,

$$f_{01} = 2F_{01}\Lambda, \tag{88}$$

$$f_{02} = 2(4 + 3\omega)\alpha F_{02}\Lambda, \tag{89}$$

which, again, are not differential equations for $f_1(R)$ and $f_2(R)$, but algebraic ones relating the value of the relevant quantities evaluated at the exponentially expanding phase with a constant scalar curvature.

If one assumes a minimal coupling $f_{02} = f_2(R) = 1$, then Eq. (89) is ill defined, since this phase is only obtained if the energy density contribution is negligible. This amounts to considering only Eq. (88): a trivial solution is, as expected, given by $f_1(R) = R - 2\Lambda \to f_{10} = 2\Lambda$ and $F_{01} = 1$—although other forms are allowed, such as $f_1(R) = 2\Lambda e^{(R-4\Lambda)/2\Lambda} \approx R - 2\Lambda$.

The two expressions for $f_1(R)$ presented above are in fact solutions of the differential equation

$$f_1(R) = 2f_1'(R)\Lambda, \tag{90}$$

and as such hold for the particular scenario of a de Sitter phase, $R = 4\Lambda =$ constant.

However, one is not restricted to these general solutions: any form for $f_1(R)$ is admissible, even if it does not obey Eq. (90). If so, Eq. (88) sets the scale of $\Lambda$–as can be seen from the example below, where a power-law form is considered,

$$f_1(R) = R_1\left(1 - \frac{R}{4\Lambda_1}\right)^n \to F_1(R) = \frac{nf_1(R)}{R - 4\Lambda_1}. \quad (91)$$

Replacing into Eq. (88), one gets

$$\Lambda = \frac{\Lambda_1}{1 - \frac{n}{2}}. \quad (92)$$

This, of course, does not shed any light on the CC problem (see Refs. [43,44]) or why the CC has its observed value, as it merely shifts the question to the value of the parameter $\Lambda_1$.

Also, notice that the strength $R_1$ is not constrained: demanding that $f_1(4\Lambda) = 2\Lambda$, so that the value of the curvature term is identical to its GR value in a de Sitter phase, then

$$R_1 = 2\Lambda\left(\frac{n-2}{n}\right)^n, \quad (93)$$

and one may rewrite Eq. (91) as

$$f_1(R) = 2\Lambda\left(1 + \frac{R - 4\Lambda}{2n\Lambda}\right)^n. \quad (94)$$

If one also considers a NMC, then Eq. (89) has to be considered. Since the curvature term cannot be set to its GR form $f_1(R) = R$ [as this violates Eq. (88)], this merely adds a layer of complexity to the problem.

However, this can be circumvented if one assumes that the density terms in Eqs. (86) and (87) are much larger than the constant contributions, so that Eq. (88) can be safely neglected; setting $f_1(R) = R$, this yields $\alpha F_{02}\rho \gg 1$ (as a remark, a power-law expansion $a(t) \sim t^\beta$ required the inverse inequality [9]).

A straightforward solution for Eq. (89) is given by

$$f_2(R) = 1 + \frac{2}{(4+3\omega)\alpha - 2}\frac{R}{4\Lambda}, \quad (95)$$

where the integration constant was chosen so that $f_2(0) = 1$; notice that the denominator $(4+3\omega)\alpha - 2$ is always positive or negative, respectively, for either $\alpha = 1$ or $\alpha = -\omega$ and a positive EOS parameter $\omega > 0$.

As discussed above, a more complex NMC such as

$$f_2(R) = \exp\left[\frac{R}{2(4+3\omega)\alpha\Lambda}\right], \quad (96)$$

and a power-law form,

$$f_2(R) = \left[1 + \frac{R - 4\Lambda}{2\Lambda(4+3\omega)n\alpha}\right]^n, \quad (97)$$

are also suitable.

## XII. THE COSMOLOGICAL CONSTANT PROBLEM

The preceding section shows that a CC may be obtained from a suitable NMC, if the contribution of the time evolving energy density dominates the modified dynamics, embodied by Eq. (8). This stemmed from the decomposition of the terms in the latter into constant or evolving with $\rho$, and precluded the possibility that the energy density is also constant.

In this section, one addresses the possibility that a matter species with EOS $\rho = -p$ dominates the dynamics, so that one has $\omega = -1$ and $\alpha = 1$, regardless of the choice of Lagrangian density (since $\alpha = 1$ or $\alpha = -\omega$). Inspection of Eq. (12) shows that the energy density is then constant, $\dot{\rho} = 0$, regardless of the form for the NMC.

One thus inserts $-p = \rho \equiv \rho_\Lambda$ into Eqs. (86) and (87), which yield

$$2\Lambda = \frac{\kappa f_{01} - f_{02}\rho_\Lambda}{\kappa F_{01} - F_{02}\rho_\Lambda} = \frac{f_{01} - 2f_{02}\Lambda_0}{F_{01} - 2F_{02}\Lambda_0}, \quad (98)$$

where one defines $\Lambda_0 = \rho_\Lambda/2\kappa$. Naturally, setting $f_{01} = 4\Lambda$ and $f_{02} = 1$ yields $\Lambda = \Lambda_0$.

The cosmological constant problem lies in the fact that there are approximately 120 orders of magnitude between the observed and expected value for the CC, $\Lambda \sim \Lambda_0 \times 10^{-120}$, assuming that $\rho_\Lambda$ expresses the energy density of the quantum vacuum [43].

Clearly, one can set appropriate forms for $f_1(R)$ and $f_2(R)$ so that Eq. (98) is satisfied. The general implications of considering either a nonlinear curvature term or a NMC are outlined below. One begins by assuming a minimal coupling $f_2(R) = 1$, so that the above collapses into

$$2\Lambda = \frac{f_{01} - 2\Lambda_0}{F_{01}} \to F_{01} = \frac{f_{01} - 2\Lambda_0}{2\Lambda}. \quad (99)$$

If one further requires that the nonlinear curvature is perturbative, $f_{01} \sim R = 4\Lambda$ but $F_{01} \neq 1$, this becomes

$$F_{01} \approx 2 - \frac{\Lambda_0}{\Lambda} \approx -\frac{\Lambda_0}{\Lambda} \sim -10^{120}. \quad (100)$$

Conversely, if one only assumes a perturbative NMC, $f_1(R) = R$ and $f_{02} \sim 1$, but $F_{02} \neq 0$, Eq. (98) becomes

$$F_{02} = \frac{f_{02}}{2\Lambda} - \frac{1}{2\Lambda_0} \approx \frac{1}{2\Lambda}. \qquad (101)$$

This condition is satisfied e.g. by the NMC in Eq. (95), as can be checked by substituting $\alpha = -\omega = 1$.

One finds that a putative solution of the CC problem using a nonlinear curvature term $f_1(R) \neq R$ requires a new dimensionless scale $F_{01} \sim -\Lambda_0/\Lambda$ to reconcile the 120 order of magnitude difference between $\Lambda$ and $\Lambda_0$, as shown in Eq. (100). On the contrary, the observed value $\Lambda$ arises naturally if only a NMC is considered with a characteristic scale $F_{02} \approx 1/(2\Lambda)$.

## XIII. DISCUSSIONS AND OUTLOOK

In this work, we have shown that phenomenological modifications of the Friedmann expansion rate equation are related to the fundamental form of the action functional, i.e. the curvature term $f_1(R)$ and NMC $f_2(R)$.

We have proposed two methods to relate these functions with the modifications of the Friedmann equation: the first method assumes that the latter are perturbative, and expands upon the previous work reported in Ref. [34], while the second relies instead on the condition $F_1 - \alpha F_2/\kappa =$ constant, which can only be implemented in NMC models.

We have shown that both methods successfully translate a number of specific modifications of the Friedmann equation found in the literature, using both nontrivial functions $f_1(R)$ and $f_2(R)$, or just a nontrivial $f_1(R) \neq R$ (i.e. $f(R)$ theories) or a nonminimal $f_2(R) \neq 1$.

We have also addressed the possibility of replicating a phase of accelerated expansion of the Universe, by considering the impact of a constant scalar curvature on the modified field equations. We find that the latter is compatible with a nonminimally coupled perfect fluid with any EOS parameter $\omega \neq -1$, provided that Eqs. (88) and (89) are satisfied.

Finally, we consider the cosmological constant problem, i.e. how to reconcile the 120 order of magnitude difference between the observed value $\Lambda$ for the cosmological constant and its expected value $\Lambda_0$, obtained by considering a perfect fluid with EOS $p_\Lambda = -\rho_\Lambda =$ constant. We find that this difference can be accounted for by either a nontrivial curvature term $f_1(R) \neq R$ or a NMC $f_2(R) \neq 1$ (or a combination of both).

In the first case, this requires the introduction of a dimensionless quantity $F_{01} \approx -\Lambda_0/\Lambda \sim -10^{120}$, which merely reframes the cosmological constant problem, shifting it to the question of what is the origin of such a large number.

The use of only a NMC yields a rather interesting result, namely that by introducing the characteristic scale $F_{02} \approx 1/2\Lambda$, the "bare" cosmological constant $\Lambda_0$ is driven towards its observed value $\Lambda$, regardless of the former: although this mechanism does not account for the value of $\Lambda$, the 120 orders of magnitude difference from $\Lambda_0$ is effectively removed from the problem.

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