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Quark masses, mixings, and *CP* violation from spontaneous breaking of flavor $SU(3)^3$

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A $\mathcal{G}_\mathcal{F} = SU(3)_Q \times SU(3)_u \times SU(3)_d$ invariant scalar potential breaking spontaneously the quark flavor symmetry can explain the standard model flavor puzzle. The approximate alignment in flavor space of the vacuum expectation values of the up and down "Yukawa fields" is explained as a dynamical effect, and the observed quark mixing angles, the weak *CP* violating phase, and hierarchical quark masses can be reproduced without introducing hierarchical parameters.

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I. INTRODUCTION

Although the standard model (SM) is an extremely successful theory in describing the fundamental building blocks of our Universe, it contains some unpleasant features, among which a puzzling flavor structure, which is characterized by large hierarchies between Yukawa couplings and by a quark mixing matrix that, without any apparent reason, is approximately proportional to the identity. In the absence of Yukawa interactions, the SM has a large global symmetry $G = U(3)_Q \times U(3)_u \times U(3)_d \times U(3)_\ell \times U(3)_e \times U(1)_H$ [1], where the subscripts refer respectively to the quark $SU(2)$ doublets Q and up and down singlets u, d, lepton doublets ℓ and singlets e, and to the Higgs field H. When Yukawa interactions are turned on, the surviving symmetry reduces to $U(1)_Y \times U(1)_B \times \Pi_\alpha U(1)_{L_\alpha}$ corresponding to hypercharge, baryon, and lepton flavor numbers, which are linear combinations of $U(1)_H$ and of the Abelian factors contained in the five $U(3) = SU(3) \otimes U(1)$. Of course, the lepton flavor symmetries are broken by the mixing between massive neutrinos. However, describing this breaking requires some hypothesis about the specific extension of the SM model responsible for neutrino masses, which is not unique. Therefore, in this paper we will concentrate on the quark sector, for which the flavor symmetry breaking pattern is fully contained within the SM, and experimentally known with good accuracy. Following a general approach [2–12], we assume that at some large energy scale the quark flavor symmetry $\mathcal{G}_\mathcal{F} = SU(3)_Q \times SU(3)_u \times SU(3)_d \subset G$ is an exact (global or local) symmetry of nature.[1] We are interested in the spontaneous breaking of $\mathcal{G}_\mathcal{F}$ which occurs when the scalar "Yukawa fields" coupled to quark bilinears via dimension-five operators, and transforming nontrivially under the symmetry, acquire vacuum expectation values (vevs) which give rise to the Yukawa couplings we observe.

The effective dimension-five Yukawa operators read

$$-\mathcal{L}_Y = \sum_{q=u,d} \left[\frac{\tilde{\kappa}_q}{\Lambda} \bar{Q} Y_q q H_q + \text{H.c.} \right], \tag{1}$$

where $H_d = H$ is the Higgs field and $H_u = i\sigma_2 H$, $\tilde{\kappa}_{u,d}$ are dimensionless (complex) couplings, $Y_{u,d}$ are the up- and down-type Yukawa fields, and Λ is the high scale where the effective operators arise.[2] At lower energy the flavor symmetry gets broken, and the SM Yukawa couplings $\mathcal{Y}_q = \tilde{\kappa}_q \langle Y_q \rangle / \Lambda$ are eventually generated.

Of course, the theoretical challenge is to find a $\mathcal{G}_\mathcal{F}$-invariant scalar potential $V(Y_q, Z)$ (where Z denotes generically additional scalars coupled to Y_q in a symmetry invariant way) which can spontaneously break $\mathcal{G}_\mathcal{F}$ yielding a set of vevs $\langle Y_q \rangle$ with the observed structure of the SM Yukawa couplings.

In Ref. [11] it was found that the most general renormalizable $\mathcal{G}_\mathcal{F}$-invariant potential with only Y_u and Y_d admits the tree-level vacuum configuration $\langle Y_q \rangle \sim \text{diag}(0, 0, v_q)$. This appeared as a promising starting point to account for the hierarchies $m_t \gg m_{c,u}$ and $m_b \gg m_{s,d}$. However, in Ref. [15] it was proven that the vanishing entries in $\langle Y_q \rangle$ cannot be lifted to nonvanishing values by any type of perturbative effects (loop corrections or higher dimensional operators involving $Y_{u,d}$ only).[3] Nevertheless, in Ref. [15]

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[1]We will not be interested here in the Abelian factors contained in G. Some possible roles of these factors are analyzed for example in [11] for the quarks and in [13] for the leptons.

[2]We neglect effective operators of dimension higher than five. This is justified for example if $\Lambda \gtrsim 10^9$ GeV (as is needed in case the flavor symmetry is global, to suppress sufficiently the flavor changing neutral current (FCNC) couplings of the Nambu-Goldstone bosons with quarks) which corresponds to a top-quark coupling $y_t \sim \langle Y_{u,33} \rangle / \Lambda \lesssim 0.6$ [14].

[3]Stated in another way, this implies that no type of perturbative effect can further break the little groups left unbroken by the minima of the tree-level potential. We will then assume that nonrenormalizable operators remain subdominant with respect to the renormalizable ones, in which case neglecting them leaves our results qualitatively unchanged.

it was also shown that by including additional scalar multiplets transforming in fundamental representations of the $SU(3)$ factors of $\mathcal{G}_\mathcal{F}$, a scalar potential which admits a hierarchical ground state $\langle Y_q\rangle \sim v_q \mathrm{diag}(\epsilon', \epsilon, 1)$ with $\epsilon' \ll \epsilon \ll 1$ can be constructed.

In this work, we show that realistic quark masses and Cabibbo-Kobayashi-Maskawa (CKM) mixing [16,17] as well as the weak CP violating phase can indeed be obtained from the spontaneous breaking of $\mathcal{G}_\mathcal{F}$. In Sec. II we introduce a general classification of the different field monomials that can appear in $V(Y_q, Z)$. In Sec. III we rederive the hierarchical solution $\langle Y_q\rangle \sim v_q \mathrm{diag}(\epsilon', \epsilon, 1)$ obtained in Ref. [15]. In Sec. IV we discuss which is the minimal field content needed to obtain nontrivial quark mixings. In Sec. V we write down the most general $\mathcal{G}_\mathcal{F}$-invariant potential for this minimal field content, and we show that it automatically implies that one weak CP violating phase is generated at the potential minimum. In Sec. VI we discuss briefly a numerical example which produces realistic quark masses, mixing angles, and the weak CP violating phase. The counting of physical complex parameters in $V(Y_q, Z)$ is carried out in the Appendix.

II. GENERALITIES AND NOTATIONS

Constructing a $\mathcal{G}_\mathcal{F}$-invariant potential which can yield at its minimum a symmetry breaking pattern with Yukawa vevs $\langle Y_q\rangle$ in agreement with observations requires, besides the Yukawa fields $Y_{u,d}$, the inclusion of additional scalars that we generically denote with Z. The resulting potential $V(Y_q, Z)$ contains various operators describing interactions and self-interactions between the different fields which, by themselves, tend to break $\mathcal{G}_\mathcal{F}$ in some specific way. It is then useful to introduce a classification of these operators based on their dynamical properties with respect to minimization.

Let us start, as a first example, with the $SU(3)_Q \times SU(3)_q$ invariant potential for a single Yukawa field $Y_q \sim (\mathbf{3}, \bar{\mathbf{3}})$. We have three invariants:

$$\begin{aligned} T_q &= \mathrm{Tr}(Y_q Y_q^\dagger), \\ A_q &= \frac{1}{2}[T_q^2 - \mathrm{Tr}(Y_q Y_q^\dagger Y_q Y_q^\dagger)], \\ \mathcal{D}_q &= \det Y_q, \end{aligned} \tag{2}$$

and the scalar potential $V(Y_q)$ is [11]

$$V(Y_q) = V_\mathcal{I} + V_{\mathcal{A}\mathcal{R}} + V_\mathcal{A}, \tag{3}$$

$$V_\mathcal{I} = \lambda[T_q - v_q^2]^2, \tag{4}$$

$$V_{\mathcal{A}\mathcal{R}} = \lambda_A A_q, \tag{5}$$

$$V_\mathcal{A} = \tilde{\mu}_q \mathcal{D}_q + \mathrm{H.c.} = 2\mu_q D_q \cos\delta_q, \tag{6}$$

where in the last line we have introduced $\mu_q = |\tilde{\mu}_q|$, $D_q = |\mathcal{D}_q|$, and $\delta_q = \mathrm{Arg}\mathcal{D}_q$ (the phase of μ_q can always be reabsorbed by redefining δ_q, see below).[4]

As discussed in [11] this potential allows for two patterns of breaking the $SU(3)_Q \times SU(3)_q$ symmetry: $\langle Y_q\rangle^s \sim v_q \mathrm{diag}(1,1,1)$ which yields the maximal little group $SU(3)_{Q+q}$, and $\langle Y_q\rangle^h \sim v_q \mathrm{diag}(0,0,1)$ which yields the maximal little group $SU(2)_Q \times SU(2)_q \times U(1)$. We will define as *attractive* ($\mathcal{A}$) those terms in the potential that tend to break the symmetry to the largest maximal little group [in this case $SU(3)_{Q+q}$ with eight generators], and *repulsive* ($\mathcal{R}$) those terms that tend to break the symmetry to the smallest maximal little group [in this case $SU(2)_Q \times SU(2)_q \times U(1)$ with seven generators], and as *flavor irrelevant* ($\mathcal{I}$) those operators that are blind to particular configurations of the flavor symmetry breaking minimum.

T_q in $V_\mathcal{I}$ [Eq. (4)] is an example of a flavor irrelevant operator. This is because T_q is invariant under the accidental symmetry $SO(18)$ which is much larger than the flavor symmetry, and that gets broken to $SO(17)$ by the vev $\langle T_q\rangle \neq 0$. This operator is then flavor irrelevant because the value of its vev does not depend on any particular flavor configuration, given that $SO(17)$ transformations can rotate e.g., $\langle Y_q\rangle^h$ into $\langle Y_q\rangle^s$. Then the role of $V_\mathcal{I}$ [Eq. (4)] is just that of determining the "length" of the vev of Y_q (defined as $\sqrt{T_q}$), while it does not contribute to the determination of any specific flavor direction.

$V_{\mathcal{A}\mathcal{R}}$ in Eq. (5) contains the operator A_q which corresponds to a Hermitian monomial. A_q contributes to determine the flavor structure, and it can be both attractive or repulsive: it is easily seen that for $\lambda_A < 0$ the action of A_q is attractive, since the minimum of the potential is lowered for the largest possible value of $\langle A_q\rangle$, which is obtained for $\langle Y_q\rangle = \langle Y_q\rangle^s$. If $\lambda_A > 0$, then the action of A_q is repulsive, since its minimum value $\langle A_q\rangle = 0$ is obtained for $\langle Y_q\rangle = \langle Y_q\rangle^h$.

Operators which correspond to non-Hermitian monomials are included in $V_\mathcal{A}$. Non-Hermitian monomials are always attractive, as is the case for $\mathcal{D}_q$ in Eq. (6). For example, when $\langle \mathcal{D}_q\rangle$ is nonvanishing, minimization drives its phase $\delta_q \to \pi$ ($\cos\delta_q \to -1$). Then the potential minimum gets lowered for the largest possible value of D_q, which is obtained for $\langle Y_q\rangle = \langle Y_q\rangle^s$. Besides being relevant for determining the flavor structure, in those cases in which $V_\mathcal{A}$ contains physical complex phases (see Sec. V), it will also have the important role of determining the value of the SM weak CP violating phase.

For interactions between different fields the jargon *attractive*, *repulsive*, *flavor irrelevant*, acquires a more

[4]As long as $\langle H^\dagger H\rangle/\Lambda^2 \ll 1$ the coupling with the Higgs, $H^\dagger H T$, can be omitted from Eq. (3). Regarding the effects of such coupling on the Higgs potential, electroweak symmetry breaking at the correct scale would require a certain degree of fine-tuning in the term $H^\dagger H(\langle T\rangle - \mu_H^2)$.

intuitive meaning. Let us introduce for example two additional fields Z_Q and Z_q transforming under $SU(3)_Q \times SU(3)_q$ respectively as $(\mathbf{3},\mathbf{1})$ and $(\mathbf{1},\mathbf{3})$. Also in this case we have terms that are invariant under extended accidental symmetries, and that are flavor irrelevant:[5]

$$|Z_Q|^2, \quad |Z_q|^2 \qquad SO(6), \tag{7}$$

$$|Z_Q|^2 \cdot |Z_q|^2 \qquad SO(6) \times SO(6), \tag{8}$$

$$T_q|Z_Q|^2, \quad T_q|Z_q|^2 \qquad SO(18) \times SO(6). \tag{9}$$

We assign this type of terms to $V_{\mathcal{I}}$. Hermitian monomials like

$$\alpha_q |Y_q Z_q|^2, \qquad \alpha_q^Q |Y_q^\dagger Z_Q|^2 \tag{10}$$

can be attractive or repulsive depending if their real couplings are negative or positive, and are assigned to $V_{\mathcal{A}R}$. Assuming for example that Y_q acquires the vev $\langle Y_q \rangle^h \sim v_q \mathrm{diag}(0,0,1)$, we see that $\alpha_q^Q < 0$ favors the aligned configuration $\langle Z_Q^T \rangle \sim v_{Z_Q}(0,0,1)$ because it maximizes the vev $\langle |Y_q^\dagger Z_Q|^2 \rangle$, while $\alpha_q^Q > 0$ would favor the orthogonal configuration $\langle Z_Q^T \rangle \sim v_{Z_Q}(c,s,0)$ (with $c^2 + s^2 = 1$) which yields $\langle |Y_q^\dagger Z_Q|^2 \rangle = 0$. Operators corresponding to non-Hermitian monomials are included in $V_{\mathcal{A}}$. An example is

$$\tilde{\nu}_q Z_Q^\dagger Y_q Z_q + \mathrm{H.c.} = 2\nu_q |Z_Q^\dagger Y_q Z_q| \cos\phi, \tag{11}$$

where $\nu_q = |\tilde{\nu}_q|$ while ϕ denotes the overall phase of the term. These operators are always attractive, since at the minimum $\phi \to \pi$ and thus they always give a negative contribution to the potential.

It should be clear by now that attractive terms (non-Hermitian monomials and Hermitian monomials with negative couplings) tend to align in flavor space the vevs of different multiplets, while repulsive terms (Hermitian monomials with positive couplings) favor "orthogonal" or more generically "maximally misaligned" vevs configurations.[6]

III. HIERARCHICAL YUKAWA COUPLINGS

We start by describing, following Ref. [15], which type of scalar potential is needed to generate a vev $\langle Y_q \rangle \sim v_q \mathrm{diag}(\epsilon', \epsilon, 1)$ with hierarchical entries $\epsilon' \ll \epsilon \ll 1$. Focusing on just one type of quark ($q = u$ or d) the flavor symmetry is $SU(3)_Q \times SU(3)_q$ under which Y_q transforms as a bifundamental representation $Y_q \sim (\mathbf{3},\bar{\mathbf{3}})$. Additional fields are needed to generate solutions different from $\langle Y_q \rangle^{h,s}$ [15] and thus we add two scalar multiplets transforming respectively in the fundamental of each one of the two $SU(3)$ factor $Z_Q \sim (\mathbf{3},\mathbf{1})$ and $Z_q \sim (\mathbf{1},\mathbf{3})$.[7] With this field content, the most general renormalizable scalar potential invariant under $SU(3)_Q \times SU(3)_q$ is

$$V(Y_q, Z_q, Z_Q) = V_{\mathcal{I}} + V_{\mathcal{A}R} + V_{\mathcal{A}}, \tag{12}$$

where

$$\begin{aligned} V_{\mathcal{I}} = {} & \lambda_q (T_q - v_q^2)^2 + \lambda_Q (|Z_Q|^2 - v_{Z_Q}^2)^2 + \lambda_{Z_q} (|Z_q|^2 - v_{Z_q}^2)^2 \\ & + [g_q (T_q - v_q^2) + g_Q (|Z_Q|^2 - v_{Z_Q}^2) + g_{Z_q} (|Z_q|^2 - v_{Z_q}^2)]^2, \end{aligned} \tag{13}$$

$$V_{\mathcal{A}R} = \lambda_{A_q} A_q + \alpha_q |Y_q Z_q|^2 + \alpha_q^Q |Y_q^\dagger Z_Q|^2, \tag{14}$$

$$V_{\mathcal{A}} = \tilde{\mu}_q \mathcal{D}_q + \tilde{\nu}_q Z_Q^\dagger Y_q Z_q + \mathrm{H.c.} \tag{15}$$

The way $V_{\mathcal{I}}$ is written makes it clear that it is flavor irrelevant, and only determines the "lengths" of T_q, Z_Q, and Z_q at the minimum. In the following we adopt the convention of denoting complex quantities with a tilde, $\tilde{x}$, while the modulus and the phase of $\tilde{x}$ will be denoted respectively with x and ϕ_x. The two phases ϕ_{μ_q} and ϕ_{ν_q} in Eq. (15) can be removed by redefining the fields,

$$Y_q \to e^{-i\phi_{\mu_q}/3} Y_q, \qquad Z_q \to e^{i(\phi_{\mu_q}/3 - \phi_{\nu_q})} Z_q, \tag{16}$$

so that the potential is manifestly *CP* invariant. In order to study the ground state, we take all the fields to be background classical fields (i.e., spacetime independent). However, to avoid overcluttering the notations, we keep using the same symbols $Y = Y^c$, $Z = Z^c$ as for the spacetime dependent fields $Y = Y(x)$, $Z = Z(x)$, since the difference should be clear from the context. We can make use of the $SU(3)_Q \times SU(3)_q$ symmetry to choose, without loss of generality, a convenient basis by removing $3+3$ moduli and $5+5$ phases. The generic 3×3 matrix Y_q has nine moduli and nine phases. By rotating away six moduli and six phases it can be brought to diagonal form, which we will denote by $\hat{Y}_q$. Two additional transformations generated by λ_3^{Q-q} and λ_8^{Q-q} [where $\lambda_{3,8}$ are the

[5]Here and in the rest the paper, we use the modulus square notation $|A|^2 = A^\dagger A$ (where A is generically a column vector that can also result from the product of vectors and square matrices) to put in evidence the Hermiticity of the corresponding monomials.

[6]According to the standard usage, "orthogonal" describes the situation in which the product of two vevs vanishes exactly. With "maximally misaligned" we will instead refer to the situation in which, at fixed lengths, the product of two or more vevs is made as small as possible in absolute value.

[7]Through operators of dimension six or higher, the vevs of $Z_{Q,q}$ can give rise to new FCNC effective operators. They can however be forbidden by a suitable choice of the representations of the messenger fermions that generate the effective operators [Eq. (1)].

diagonal $SU(3)$ Gell-Mann matrices] allow one to remove two phases from two diagonal entries. However, since λ_3^{Q+q} and λ_8^{Q+q} both leave $\hat{Y}_q$ invariant, the third phase cannot be removed. We can however make use of these two $U(1)$ symmetries to remove one phase from Z_Q and another one from Z_q. For example, a valid choice of basis is $\hat{Y}_q = \mathrm{diag}(y_q^1, y_q^2, \tilde{y}_q^3)$, $Z_Q^T = (\tilde{z}_Q^1, \tilde{z}_Q^2, z_Q^3)$ and $Z_q^T = (z_q^1, \tilde{z}_q^2, \tilde{z}_q^3)$ where the tilde denotes complex components while the others are real and positive.

Let us first study whether CP violation can occur spontaneously, that is if at the minimum, the vevs of some of the remaining five phases are forced to acquire a value $\neq 0, \pi$. $V_{\mathcal{A}}$ in Eq. (15) is the sum of four pairs of complex conjugate terms which depend on different combinations of the five phases. Then their optimal minimization (i.e., all $\phi \to \pi$) can all be satisfied by fixing the value of three phases and of one phase difference. For example, for the choice of basis given above we obtain $\phi_{y_q^3} = \phi_{z_Q^1} = \phi_{z_q^2} - \phi_{z_Q^2} = \pi$ and $\phi_{z_q^3} = 0$. Thus, the CP conserving potential $V(Y_q, Z_q, Z_Q)$ also yields a CP conserving minimum.

To study the possible flavor configurations of the minima, we can now take the potential of classical background fields to be a function of real parameters and of real classical fields, after accounting for a minus sign in front of the four pairs of complex conjugate terms in Eq. (15) which then read

$$V_{\mathcal{A}}^{\min} = -2\mu_q \Pi_i y_q^i - 2\nu_q \sum_i z_Q^i z_q^i y_q^i. \tag{17}$$

As already said, $V_{\mathcal{I}}$ in Eq. (13) fixes the length $\sqrt{T_q} = v_q$. We chose $\lambda_{A_q} > 0$ in Eq. (14) so that the first term is repulsive and favors the configuration $\hat{Y}_q^h \simeq v_q \mathrm{diag}(0,0,1)$ [11], which is the starting point for generating a hierarchical solution. Choosing $\alpha_q > 0$ and $\alpha_q^Q > 0$ implies that the last two terms in $V_{\mathcal{AR}}$ are also repulsive, and tend to generate a maximal misalignment between Z_Q, Z_q and $\hat{Y}_q$, which corresponds to $Z_Q^T = v_{Z_Q}(c_Q, s_Q, 0)$ and $Z_q^T = v_{Z_q}(c_q, s_q, 0)$ with $c_Q^2 + s_Q^2 = c_q^2 + s_q^2 = 1$. On the other hand, the terms proportional to ν_q in Eq. (17) which are always attractive prefer to align Z_Q and Z_q with $\hat{Y}_q$ in order to get a nonvanishing (and possibly large) negative contribution.

The crucial point is that, while the positive definite terms with couplings α_q, α_q^Q are proportional to $(y_q^i)^2$, that is to the square of the entries in $\hat{Y}_q$, the negative terms with coupling ν_q are linearly proportional to y_q^i. Then one vanishing diagonal entry in $\hat{Y}_q$ gets lifted to a nonzero value proportional to ν_q. This entry gets aligned with the nonvanishing entries in $Z_{q,Q}$. Written explicitly, we have $\hat{Y}_q \sim \mathrm{diag}(0, \nu_q, v_q)$ together with $Z_q^T \simeq v_{Z_q}(0,1,0)$ and $Z_Q^T \simeq v_{Z_Q}(0,1,0)$.[8] However, there is one more attractive operator, that is the determinant in the first term in Eq. (17), which favors a nonvanishing third entry in $\hat{Y}_q$. A nonzero value proportional to the product $\mu_q \cdot \nu_q$ is thus induced, yielding $\hat{Y}_q \propto \mathrm{diag}(\mu_q \nu_q / v_q, \nu_q, v_q)$. In the end, if ν_q and μ_q are adequately small (that is $\nu_q, \mu_q \ll v_q$) a hierarchical solution is obtained. With $\alpha_q^Q = \alpha_q = \lambda_{A_q} > 0$ and the simplified choice $v_{Z_Q} = v_{Z_q} = v_q$ we obtain, at the minimum, the configuration $\hat{Y}_q \simeq v_q \mathrm{diag}(\epsilon_q{}', \epsilon_q, 1)$ with [15][9]

$$\begin{aligned} \epsilon_q &= \frac{\lambda_{A_q} \nu_q / v_q}{3\lambda_{A_q}^2 - \mu_q^2 / v_q^2}, \\ \epsilon_q{}' &= \frac{\mu_q / v_q}{\lambda_{A_q}} \epsilon_q, \\ V_q^{\min} &= -\nu_q \epsilon_q v_q^3. \end{aligned} \tag{18}$$

Equation (18) seems to suggest that to obtain e.g., $\epsilon_q{}' \sim 10^{-4}$ and $\epsilon_q \sim 10^{-2}$, in rough agreement with the hierarchy in the up-quark Yukawa sector, a mild parametric hierarchy like $\mu_u / v_u \sim \nu_u / v_u \sim 10^{-2}$ is needed (the milder hierarchy in the down-quark sector can be obtained with $\mu_d / v_d \sim \nu_d / v_d \sim 10^{-1}$). In fact, as we argue in the next section, after coupling the up and down Yukawa sectors, a dynamical (as opposite to parametric) suppression of ϵ_q' and ϵ_q arises. The only requirement to seed this suppression is μ_q, $\nu_q \lesssim v_q$, which is, however, a natural requirement. In fact, in the limit μ_q, $\nu_q \to 0$ (that is $V_{\mathcal{A}} \to 0$), only Hermitian monomials survive, and the potential acquires three $U(1)$ symmetries corresponding trivially to rephasing of the field multiplets Z_Q, Z_q and Y_q. This ensures that higher order corrections to μ_q and ν_q will be proportional to these same parameters, while in contrast v_q can receive larger corrections proportional for example to the square of the cutoff scale Λ in Eq. (1). In conclusion, the emergence of the large hierarchies observed in the SM Yukawa sector is triggered by μ_q, $\nu_q \lesssim v_q$, which in turn is what should be expected from simple naturalness considerations.

IV. QUARK MIXINGS

In this section we extend the previous construction by including both the u and d quark sectors. The flavor symmetry corresponds to the full flavor group $\mathcal{G}_{\mathcal{F}}$.

[8] Because of the attractive nature of the ν_q term, Z_Q and Z_q get aligned with the second largest diagonal component of $\hat{Y}_q$, which implies c_Q, $c_q \to 0$ and s_Q, $s_q \to 1$.

[9] The analytical derivation of the result [Eq. (18)] is more easily carried out by assuming that the lengths of the vevs remains fixed at the values determined by $V_{\mathcal{I}}$ alone (respectively v_q, v_{Z_q} and v_{Z_Q}). In fact, $V_{\mathcal{AR}}$ and $V_{\mathcal{A}}$ can induce shifts of order $\nu_q \epsilon_q / \lambda_q$ in the lengths. However, this gives only negligible corrections to the hierarchical solution.

As can be guessed from the previous section, in order to obtain hierarchical solutions for both the Yukawa fields $Y_u \sim (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$ and $Y_d \sim (\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$, we need to introduce three scalar multiplets transforming in fundamental representations of the three group factors: $Z_{Q_1} \sim (\mathbf{3}, \mathbf{1}, \mathbf{1})$, $Z_u \sim (\mathbf{1}, \mathbf{3}, \mathbf{1})$ and $Z_d \sim (\mathbf{1}, \mathbf{1}, \mathbf{3})$. However, as we will argue in the following, to get three nonvanishing quark mixings we will need in fact to introduce one additional multiplet $Z_{Q_2} \sim (\mathbf{3}, \mathbf{1}, \mathbf{1})$.

Let us write the matrices of background Yukawa fields as

$$Y_u = (\vec{y}_{u1}, \vec{y}_{u2}, \vec{y}_{u3}), Y_d = (\vec{y}_{d1}, \vec{y}_{d2}, \vec{y}_{d3}), \tag{19}$$

where $\vec{y}_{ui}$ and $\vec{y}_{di}$ are column vectors with three components, which we arrange according to $|\vec{y}_{u1}| < |\vec{y}_{u2}| < |\vec{y}_{u3}|$ and $|\vec{y}_{d1}| < |\vec{y}_{d2}| < |\vec{y}_{d3}|$. Quark mixing can be described as a misalignment in flavor space between the background Yukawa matrices Y_u and Y_u. This is better understood if we choose a special basis as follows: with $SU(3)_Q \times SU(3)_q$ rotations we can bring one Y_q in diagonal form (with one complex phase). We choose to rotate Y_u and we denote its diagonal matrix as $\hat{Y}_u$, that is $\vec{y}_{ui} \propto (\delta_{1i}, \delta_{2i}, \delta_{3i})^T$. Transformations generated by the generators λ_3^{Q+u}, λ_8^{Q+u} and by the eight generators of $SU(3)_d$ leave invariant $\hat{Y}_u$, and we can use this remaining freedom to remove three moduli and seven phases from Y_d, which can thus be written as $Y_d = K\hat{Y}_d$ with K a special unitary matrix (with one complex phase) and $\hat{Y}_d$ diagonal (with another complex phase). Clearly the matrix K describes the misalignment between Y_u and Y_d, and corresponds to the CKM matrix.

Since we will defer to the next section the study of *CP* violation, for the time being we take all the parameters of the scalar potential as well as all the background field components to be real. A simple graphical illustration can help to understand how mixings can be induced. Let us first draw the vectors $\vec{y}_{ui}$ and $\vec{y}_{di}$ as in Fig. 1, where the three axes x_1, x_2 and x_3 define a three-dimensional flavor space, on which Y_u and Y_d get projected respectively with components $\vec{y}_{ui}$ and $\vec{y}_{di}$. Let us first consider the Y_u-Y_d interaction corresponding to the term

$$\lambda_{ud} \mathrm{Tr}|\hat{Y}_u^\dagger K \hat{Y}_d|^2. \tag{20}$$

This term, by itself, will align or maximally misalign Y_u and Y_d. If $\lambda_{ud} < 0$ the interaction is attractive, and the largest possible values for the products between the components of Y_u and Y_d are favored, which means that $\vec{y}_{u3}$ aligns with $\vec{y}_{d3}$, $\vec{y}_{u2}$ with $\vec{y}_{d2}$, and $\vec{y}_{u1}$ with $\vec{y}_{d1}$. This configuration clearly yields $K = I_{3\times3}$ that is no mixing, as is shown in Fig. 1(a). If $\lambda_{ud} > 0$ the interaction is repulsive, and Y_u and Y_d will get maximally misaligned, that is the vector $\vec{y}_{u3}$ aligns with $\vec{y}_{d1}$ and $\vec{y}_{u1}$ with $\vec{y}_{d3}$ suppressing the two largest entries ($\vec{y}_{u2}$ remains aligned with $\vec{y}_{d2}$). Also in this configuration all the mixings vanish, in the sense that K becomes antidiagonal with unit entries. In other words, in the basis in which the entries in $\hat{Y}_u$ and $\hat{Y}_d$ are ordered in opposite ways, again we have $K = I_{3\times3}$. Clearly, this means that the heaviest quarks get coupled to the lightest ones, i.e., $V_{td} = V_{ub} = 1$, which conflicts with observations. We thus learn that $\lambda_{ud} < 0$ is the correct choice to get $K = I_{3\times3}$ as a first approximation and, as was first noted in Ref. [2], that with just two Yukawa fields Y_u and Y_d, it is not possible to generate any mixing: attractive interactions yield exact alignment, and repulsive interactions result in maximal misalignment.

With $\lambda_{ud} < 0$, the term in Eq. (20) besides aligning $\hat{Y}_u$ and $\hat{Y}_d$, has also the important effect of enhancing the hierarchies among their diagonal entries. We have first detected this effect in our numerical study, but there is a simple way to understand the way it works. Consider for the matrices $\hat{Y}_q$ ($q = u, d$) the following two configurations:

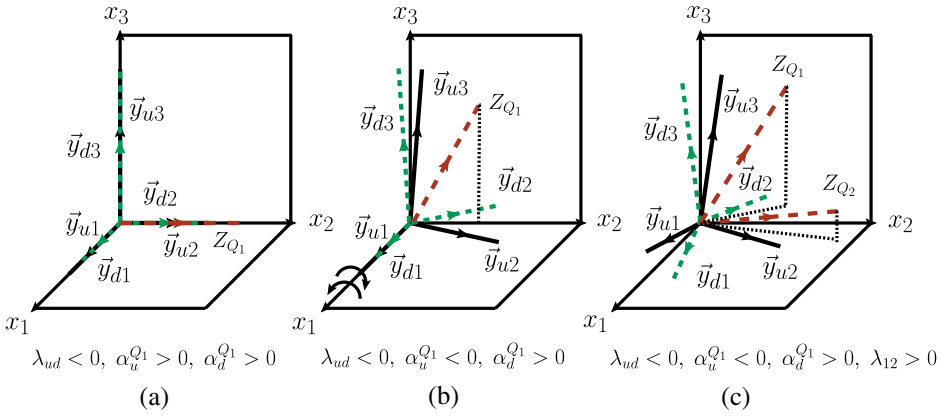

FIG. 1 (color online). In (a), we illustrate the configuration with no mixing. In (b), we illustrate the configuration with one nonvanishing mixing angle θ_{23}. In (c), we illustrate the configuration for nonvanishing θ_{12}, θ_{23} and θ_{13}.

$$\frac{1}{v}\hat{Y}_1 = \mathrm{diag}(\delta', \delta, 1), \tag{21}$$

$$\frac{1}{v}\hat{Y}_2 = \mathrm{diag}\left(\epsilon', \epsilon, 1 + \frac{1}{2}(\delta'^2 + \delta^2)\right). \tag{22}$$

With $\epsilon', \epsilon \ll \delta'$, $\delta \ll 1$, $\hat{Y}_1$ is mildly hierarchical, while $\hat{Y}_2$ is strongly hierarchical. Up to $\mathcal{O}(\epsilon'^2, \epsilon^2)$ and $\mathcal{O}(\delta'^4, \delta^4)$, $\hat{Y}_1$ and $\hat{Y}_2$ are of equal length $\frac{1}{v^2}\mathrm{Tr}(\hat{Y}_1^\dagger \hat{Y}_1) = \frac{1}{v^2}\mathrm{Tr}(\hat{Y}_2^\dagger \hat{Y}_2) = 1 + \delta'^2 + \delta^2$. Consider now the contributions to the potential at the minimum from the λ_{ud} and the determinant terms. Setting for simplicity $\mu_u = \mu_d = \mu$ and neglecting higher powers of the small parameters we have

$$\frac{1}{v^4}\delta V(Y_1) \sim -\lambda_{ud} - \frac{4\mu}{v}\delta'\delta + \mathcal{O}(\delta^4), \tag{23}$$

$$\frac{1}{v^4}\delta V(Y_2) \sim -\lambda_{ud} - 2\lambda_{ud}(\delta^2 + \delta'^2) + \mathcal{O}(\epsilon'\epsilon). \tag{24}$$

Clearly if $\frac{2\mu}{v} \lesssim \lambda_{ud}$ the strongly hierarchical configuration $\frac{1}{v}\hat{Y}_2 \sim \mathrm{diag}(\epsilon', \epsilon, 1)$ gives the lowest minimum. By carrying out a more detailed analysis including also higher powers of ϵ' and ϵ, one obtains that $\epsilon' \ll \epsilon$ also lowers the minimum. The conclusion of this simple discussion is that when the up and down sectors are coupled together, the naive hierarchical pattern $y_1 \sim \frac{\nu}{v}\mu$, $y_2 \sim \mu$ and $y_3 \sim v$ derived at the end of the previous section gets amplified by the effect of the λ_{ud} term. Numerically, we find (see Sec. VI) that $\frac{\mu}{v} \sim 10^{-1}\lambda_{ud}$ is sufficient to generate the strong hierarchies of the quark mass matrices. It is quite remarkable that strong hierarchies can arise dynamically from a potential in which the fundamental parameters are nonhierarchical, and are only required to satisfy some generic naturalness condition.

Coming back to the issue of mixing angles, let us now include just the minimal number of fields needed to generate the hierarchies, which is Z_{Q_1}, Z_u and Z_d. We thus have in total three "vectors" Z_{Q_1}, Y_u and Y_d transforming under the L-handed factor $SU(3)_Q$. Looking at the terms which couple the Z's and the Y's, the following $\mathcal{G}_\mathcal{F}$-invariant (attractive/repulsive) terms are relevant to generate mixings:

$$\alpha_u^{Q_1}|Z_{Q_1}^\dagger \hat{Y}_u|^2, \qquad \alpha_d^{Q_1}|Z_{Q_1}^\dagger K\hat{Y}_d|^2, \tag{25}$$

$$\alpha_u|Z_u^\dagger \hat{Y}_u^\dagger|^2, \qquad \alpha_d|Z_d^\dagger \hat{Y}_d^\dagger|^2. \tag{26}$$

In the second term in Eq. (25) the matrix K is sandwiched between Z_{Q_1} and $\hat{Y}_d$, and this implies that Z_{Q_1} plays a direct role in determining the mixing pattern. The roles of Z_u and Z_d are instead indirect since they either do not couple directly to K as in Eq. (26), or they couple to field combinations in which K is sandwiched between the other two fields, as for example in $(Z_{Q_1}^\dagger K\hat{Y}_d)Z_d$ or in $Z_u^\dagger(\hat{Y}_u^\dagger K\hat{Y}_d)Z_d$. These last two monomials are non-Hermitian and thus always attractive, so that they favor aligned configurations, for example between $(Z_{Q_1}^\dagger K\hat{Y}_d)$ and Z_d. In short, the relevant parameters which determine the mixing are $\alpha_u^{Q_1}$ which tends to align or maximally misalign Z_{Q_1} and $\hat{Y}_u$, and $\alpha_d^{Q_1}$ which affects the structure of the matrix K. If both $\alpha_u^{Q_1}$ and $\alpha_d^{Q_1}$ are positive, Z_{Q_1} will repulse from both Y_u and Y_d, a situation which is optimally realized e.g., by the configuration $Z_{Q_1}^T = v_{Z_{Q_1}}(0, 1, 0)$. Clearly in this case there is no mixing [see Fig. 1(a)].[10] Now if we switch the sign of one of the $\alpha_q^{Q_1}$ couplings, something interesting occurs. Choosing for example $\alpha_u^{Q_1} < 0$ and $\alpha_d^{Q_1} > 0$, there is attraction between Z_{Q_1} and Y_u from the first term [Eq. (25)], while there is repulsion between Z_{Q_1} and Y_d from the second term. As a result, we obtain $Z_{Q_1}^T = v_{Z_{Q_1}}(0, c, s)$ $(c^2 + s^2 = 1)$ as is depicted in Fig. 1(b). Notice that Z_{Q_1} will always lie in the x_2-x_3 plane because the attraction (repulsion) with the component of Y_u (Y_d) along the $x_{2,3}$ axis dominates over the effects of the components along x_1. Due to the attraction/repulsion, Y_u and Y_d will then rotate about the x_1 axis, a nonzero θ_{23} is induced [see Fig. 1(b)], but the other two mixing angles remain vanishing.

From this analysis, one can guess that to get three nonvanishing mixing angles, at least four interacting "vector" transforming under the L-handed $SU(3)_Q$ flavor symmetry are need. To generate the other two mixing angles θ_{12} and θ_{13} let us then introduce another scalar field Z_{Q_2} transforming in the same way as Z_{Q_1} under $SU(3)_Q$. Its interactions with $Y_{u,d}$ can be obtained directly from Eq. (25) by replacing $Z_{Q_1} \to Z_{Q_2}$ and $\alpha_{u,d}^{Q_1} \to \alpha_{u,d}^{Q_2}$. If both $\alpha_{u,d}^{Q_2}$ are positive, then the corresponding terms will favor also in this case $Z_{Q_2}^T \propto (0, 1, 0)$. However, we also have the following interactions involving the invariant $Z_{Q_1}^\dagger Z_{Q_2}$:

$$\lambda_{12}|Z_{Q_1}^\dagger Z_{Q_2}|^2 + [\tilde{\eta}_{12}(Z_{Q_1}^\dagger Z_{Q_2})^2 + \mathcal{M}_{12}^2 Z_{Q_1}^\dagger Z_{Q_2} + \mathrm{H.c.}]. \tag{27}$$

In the above expression $\mathcal{M}_{12}^2$ is a field dependent quantity that reads

$$\mathcal{M}_{12}^2 = \tilde{\mu}_{12}^2 + \tilde{\eta}_u T_u + \tilde{\eta}_d T_d, \tag{28}$$

where $\tilde{\mu}_{12}^2, \tilde{\eta}_u, \tilde{\eta}_d$ are complex parameters. It should be clear that small variations of $\mathcal{M}_{12}^2$ are uninfluential to determine

[10] Naively we would expect $Z_{Q_1}^T = v_{Z_{Q_1}}(c, s, 0)$ (with $c^2 + s^2 = 1$). However, the attractive terms $Z_{Q_1}^\dagger Y_u Z_u$ and $Z_{Q_1}^\dagger Y_d Z_d$ favor alignments with the second largest diagonal entries of $\hat{Y}_u$ and $\hat{Y}_d$.

the flavor structure, so that in the minimization problem it is a consistent simplification approximating $\mathcal{M}_{12}^2$ with its background value,

$$\tilde{m}_{12}^2 \equiv \langle \mathcal{M}_{12}^2 \rangle \simeq \tilde{\mu}_{12}^2 + \tilde{\eta}_u v_u^2 + \tilde{\eta}_d v_d^2, \tag{29}$$

with $\tilde{m}_{12}^2$ a complex quantity. The two non-Hermitian monomials within square brackets in Eq. (27) produce as usual attractive interactions. If in addition λ_{12} in the first term is negative, Z_{Q_2} will align with Z_{Q_1}. In this case they will both remain on the x_2-x_3 plane and we do not get any new mixing. However, if $\lambda_{12} > 0$ this term becomes repulsive and if, as we will assume, its effect is the dominant one [which can be ensured by taking $\lambda_{12} > |\tilde{m}_{12}^2|/(v_u^2 + v_d^2)$], then Z_{Q_1} and Z_{Q_2} can get sufficiently misaligned only if they both leave the x_2-x_3 plane, as is illustrated in Fig. 1(c). Their couplings to Y_u and Y_d will then induce rotations of the Yukawa matrices around the x_2 and x_3 axes, with the result that nonvanishing values for θ_{13} and θ_{12} are generated [see Fig. 1(c)].

In conclusion, while to obtain hierarchical solutions for both $\hat{Y}_u$ and $\hat{Y}_d$ the set of auxiliary scalar fields $Z_{Q_1} \sim (\mathbf{3},\mathbf{1},\mathbf{1})$, $Z_u \sim (\mathbf{1},\mathbf{3},\mathbf{1})$ and $Z_d \sim (\mathbf{1},\mathbf{1},\mathbf{3})$ is sufficient, this field content can only generate one nontrivial mixing angle θ_{23}. In order to generate the other two mixings θ_{12} and θ_{13} at least one additional multiplet $Z_{Q_2} \sim (\mathbf{3},\mathbf{1},\mathbf{1})$ is needed. As is well known, in the SM three nonvanishing mixing angles are a necessary condition to allow for *CP* violation [17,18]. In the next section we will argue that the same set of scalar multiplets is also sufficient to ensure that the ground state of the scalar potential violates *CP* and induces a nonvanishing complex phase in the quark mixing matrix K.

V. *CP* VIOLATION

We have seen that a scalar sector containing the two Yukawa fields Y_u and Y_d and the four auxiliary multiplets $Z_{Q_{1,2}} \sim (\mathbf{3},\mathbf{1},\mathbf{1})$, $Z_u \sim (\mathbf{1},\mathbf{3},\mathbf{1})$ and $Z_d \sim (\mathbf{1},\mathbf{1},\mathbf{3})$ can break the flavor group in such a way that all the diagonal entries in the Yukawa matrices are nonvanishing and naturally hierarchical, and moreover can misalign Y_u and Y_d along all the three flavor directions inducing three nonvanishing CKM mixing angles. In deriving these results we have taken for simplicity all quantities to be real. In this section we will complete our study by addressing the issue of *CP* violation and, in order to do this, we will allow for complex parameters and complex values of the background fields. With the given field content the most general renormalizable $\mathcal{G}_\mathcal{F}$-invariant potential is

$$V(Y_q, Z_q, Z_{Q_i}) = V_\mathcal{I} + V_{\mathcal{A}R} + V_\mathcal{A}, \tag{30}$$

where

$$V_\mathcal{I} = \sum_{q=u,d} \lambda_q (T_q - v_q^2)^2 + \sum_{q=u,d} \lambda_{Z_q} (|Z_q|^2 - v_{Z_q}^2)^2 + \sum_{i=1,2} \lambda_{Q_i} (|Z_{Q_i}|^2 - v_{Z_{Q_i}}^2)^2 + \Big\{ \sum\nolimits_{q=u,d} [g_q (T_q - v_q^2) + g_{Z_q} (|Z_q|^2 - v_{Z_q}^2)] + \sum\nolimits_{i=1,2} g_{Q_i} (|Z_{Q_i}|^2 - v_{Z_{Q_i}}^2) \Big\}^2, \tag{31}$$

$$V_{\mathcal{A}R} = \sum_{q=u,d} \left(\lambda_{A_q} A_q + \alpha_q |Z_q^\dagger Y_q^\dagger|^2 + \sum_{i=1,2} \alpha_q^{Q_i} |Z_{Q_i}^\dagger Y_q|^2 \right) + \lambda_{ud} \mathrm{Tr}|Y_u^\dagger Y_d|^2 + \lambda_{12} |Z_{Q_1}^\dagger Z_{Q_2}|^2, \tag{32}$$

$$V_\mathcal{A} = \sum_{q=u,d} \left[\tilde{\mu}_q \mathcal{D}_q + \sum_{i=1,2} \tilde{\nu}_{iq} Z_{Q_i}^\dagger Y_q Z_q \right] + \tilde{\gamma}_{ud} Z_u^\dagger Y_u^\dagger Y_d Z_d + \tilde{\eta}_{12} (Z_{Q_1}^\dagger Z_{Q_2})^2 + \tilde{m}_{12}^2 Z_{Q_1}^\dagger Z_{Q_2} + \mathrm{H.c.} \tag{33}$$

$V_\mathcal{A}$ contains nine complex couplings.[11] As shown in the Appendix, field redefinitions allow one to remove five phases, leaving just four physical phases that can be chosen as the two phase differences $\Delta_q (q = u, d)$ between $\tilde{\nu}_{1q}$ and $\tilde{\nu}_{2q}$, the phase φ_γ of the $(Y_u Z_u)$-$(Y_d Z_d)$ coupling term, and the phase difference Δ_Q between the two complex parameters $\tilde{\eta}_{12}$ and $\tilde{m}_{12}^2$ of the Z_{Q_1}-Z_{Q_2} sector. All the remaining parameters in $V_\mathcal{A}$ can be taken, without loss of generality, real and positive. In particular, the determinant terms can be written as $\sum_q \mu_q \mathcal{D}_q + \mathrm{H.c.}$

We address now the issue of weak *CP* violation. As is discussed in Sec. IV, without loss of generality we can parametrize the vevs of the matrices of Yukawa fields as

$$Y_u = \hat{Y}_u, \tag{34}$$

$$Y_d = K \hat{Y}_d, \tag{35}$$

with K a special unitary matrix depending on one phase δ_K, and $\hat{Y}_{u,d}$ both diagonal depending respectively on the two phases $\delta_{u,d}$. These latter two phases are fixed by the minimization conditions for the determinants $\delta_{u,d} \to \pi$ so that at the minimum both $\hat{Y}_{u,d}$ are real. As regards the mixing matrix K, given that $\det K = +1$, the value of its phase δ_K is left undetermined by the minimization of the determinants, but will be fixed after minimizing the remaining terms in $V_\mathcal{A}$. These terms can be rewritten as [see Eq. ((A3)) in the Appendix]:

[11] Dropping the simplification $\mathcal{M}_{12}^2 \to \tilde{m}_{12}^2 \equiv \langle \mathcal{M}_{12}^2 \rangle$ we would have in fact two additional complex parameters $\tilde{\eta}_u$ and $\tilde{\eta}_d$, see Eq. (28). However, the only role of the phases of these couplings is that of contributing to the overall phase of $\tilde{m}_{12}^2$ at the minimum. Adopting the simplified form of $V_\mathcal{A}$ [Eq. (33)] is thus justified.

$$
\begin{aligned}
V_{\mathcal{A}} \supset\, & 2\sum_{i=1,2}\nu_{iu}\sum_j z^j_{Q_i} y^j_u z^j_u \cos(\mp\Delta_u - \phi^j_{Q_i} + \phi^j_u) \\
& + 2\sum_{i=1,2}\nu_{id}\sum_{jk} z^j_{Q_i} K_{jk} y^k_d z^k_d \cos(\mp\Delta_d - \phi^j_{Q_i} + \xi^{jk}_{\delta_K}) \\
& + 2\gamma_{ud}\sum_{jk} z^j_u y^j_u K_{jk} y^k_d z^k_d \cos(\phi_\gamma - \phi^j_u + \xi^{jk}_{\delta_K}) \\
& + 2\eta_{12}\sum_{jk} z^j_{Q_1} z^j_{Q_2} z^k_{Q_1} z^k_{Q_2} \cos(\Delta_Q - \phi^j_{Q_1} + \phi^j_{Q_2} - \phi^k_{Q_1} + \phi^k_{Q_2}) \\
& + 2m^2_{12}\sum_j z^j_{Q_1} z^j_{Q_2} \cos\left(-\frac{1}{2}\Delta_Q - \phi^j_{Q_1} + \phi^j_{Q_2}\right), \qquad (36)
\end{aligned}
$$

where we denote the field components as $(\hat{Y}_q)^{jj} = y^j_q$, $(Z_{Q_i})^j = z^j_{Q_i} e^{i\phi^j_{Q_i}}$, $(Z_q)^j = z^j_q e^{i\phi^j_q}$ and in the first two lines the minus sign in front of $\Delta_{u,d}$ holds for $i = 1$ and the plus sign for $i = 2$. In Eq. (36) the quantities $\phi^{jk}(\delta_K) \equiv \arg(K_{jk})$ can be regarded as functions of the phase δ_K of the mixing matrix. They always appear in the combinations $\xi^{jk}_{\delta_K} = \phi^{jk}(\delta_K) + \phi^k_d$. A few remarks are in order:

(1) Although the functions $\phi^{jk} = \phi^{jk}(\delta_K)$ of δ_K always come together with the phase of a Z_d component, these phase combinations satisfy $\xi^{jk}_{\delta_K} + \xi^{lm}_{\delta_K} - \xi^{jm}_{\delta_K} - \xi^{lk}_{\delta_K} = \phi^{jk} + \phi^{lm} - \phi^{jm} - \phi^{lk}$, and thus the Jarlskog invariant [18] $J = \mathrm{Im}(K_{jk} K_{lm} K^*_{jm} K^*_{lk})$ (no sum over repeated indices) is proportional to $\sin(\phi^{jk} + \phi^{lm} - \phi^{jm} - \phi^{lk})$ and can be evaluated (numerically) without any ambiguity.

(2) There is a special choice of the phases that besides rendering $\tilde{\nu}_{iq}$, $\tilde{\gamma}_{ud}$, $\tilde{\eta}_{12}$ and $\tilde{m}^2_{12}$ all real, also implies that at the minimum $J = 0$ and there is no CP violation. At fixed values of the moduli of the parameters, this choice of phases corresponds to the lowest possible minimum. This can be shown in a simple analytical way. By means of the redefinitions

$$
Z_q \to e^{i\Delta_q} Z_q, \qquad Z_{Q_2} \to e^{\frac{i}{2}\Delta_Q} Z_{Q_2}, \qquad (37)
$$

$V_{\mathcal{A}}$ can be rewritten in the basis in which m^2_{12} and ν_{1q} are real [cf. Eq. (A3)], while the four complex parameters become

$$
\tilde{\nu}_{2q} = \nu_{2q} e^{i(2\Delta_q - \frac{1}{2}\Delta_Q)}, \qquad (38)
$$

$$
\tilde{\gamma}_{ud} = \gamma_{ud} e^{i(\varphi_\gamma - \Delta_u + \Delta_d)}, \qquad (39)
$$

$$
\tilde{\eta}_{12} = \eta_{12} e^{2i\Delta_Q}. \qquad (40)
$$

By choosing the four phase combinations above all equal to π, it is easy to check that the minimum of $V_{\mathcal{A}}$ is obtained for

$$
\phi^j_{Q_2} - \phi^j_{Q_1} = \pi, \qquad \xi^{jk}_{\delta_K} = \phi^j_u = \phi^j_{Q_2}. \qquad (41)
$$

Since in this case all the cos $\to -1$, this corresponds to the best possible minimum. Finally, given that $\phi^{jk}(\delta_K)$ can be written as $\phi^{jk} = \phi^j_u - \phi^k_d$, $J = 0$ follows straightforwardly.

(3) Instead, we have not been able to prove analytically that for generic values of Δ_q, Δ_Q and φ_γ, $J \neq 0$ is generally obtained. (Expressing J as a function of the fundamental phases would obviously be an even more awkward task.) However, some arguments can be put forth to suggest that this is indeed what should be expected. Imagine for example to fix the value of δ_K to 0 or π. There are several terms in $V_{\mathcal{A}}$ and, for generic values of the phases, not enough field variables to drive all the cos $\to -1$. This is obvious for example for the last two lines in Eq. (36), as well as in the first and in the second line once the difference $\phi^j_{Q_2} - \phi^j_{Q_1}$ gets fixed in terms of Δ_Q. Minimization at fixed δ_K would then yield some value $V^{\min}_{\mathcal{A}}$ higher than the best minimum of the example above. Leaving now δ_K free, it is reasonable to expect that some cosine term could be made smaller when the value of δ_K departs from 0, π, and a lower minimum could then be reached. In any case, the results of our numerical analysis confirm that such an expectation is correct, and that this is precisely what happens.

VI. A NUMERICAL EXAMPLE

In our study, the final verdict if spontaneous breaking of the flavor symmetry is able to account for the entire set of observables in the quark sector, has been settled only by means of numerical minimization of the full $\mathcal{G}_{\mathcal{F}}$-invariant scalar potential. In particular, we have not attempted to carry out multidimensional global fits to the SM observables, which would have required a prohibitive amount of CPU time, but we have just assumed a simple set of values for most of the fundamental parameters and then, by varying the remaining (crucial) ones, we have attempted to approximate the experimental values of the observables. Of course, in carrying out this procedure we have been guided by a good understanding of the role of each term in the potential, which we have gained by inspecting several partial analytical results.[12] An example of the type of results that can be obtained is given below. We work in the basis in which ν_{1q} ($q = u, d$) and m^2_{12} are real and positive, and the complex parameters are $\tilde{\nu}_{2q}$, $\tilde{\gamma}_{ud}$ and $\tilde{\eta}_{12}$.

[12]Numerical minimizations have been carried out with the built-in minimization routines of the MATHEMATICA package. To seek for the global minimum, we have used a *random search method*: we start with a random generation of initial search points in field space and proceed with the minimization routine. The minima of the potential resulting from different initial search points are compared, and the lowest one is selected. The set of initial search points is then augmented until there is no change in the final result.

In $V_{\mathcal{I}}$ we fix all the vevs to be equal $v_q = v_{Z_q} = v_{Z_{Q_i}} = v$ and all the couplings to be equal to 1: $\lambda_q = \lambda_{Z_q} = \lambda_{Q_i} = g_q = g_{Z_q} = g_{Q_i} = 1$. In $V_{\mathcal{AR}}$ we also fix $\lambda_{A_q} = \alpha_q = \alpha_q^{Q_2} = \alpha_d^{Q_1} = 1$ while

$$\alpha_u^{Q_1} = -1 \quad \text{and} \quad \lambda_{ud} = -1.3. \tag{42}$$

In $V_{\mathcal{A}}$ the dimensional parameters, the dimensionless couplings and the phases are fixed as ($q = u, d$)

$$\begin{aligned} &\mu_q = \nu_{1q} = \nu_{2q} = v/10, \qquad m_{12}^2 = 0.15v^2, \\ &\gamma_{ud} = 0.81, \qquad \eta_{12} = 0.1, \qquad \lambda_{12} = 1.27, \\ &\phi_{\gamma_{ud}} = 0.98\pi, \qquad \phi_{\eta_{12}} = 0.92\pi, \qquad \phi_{\nu_{2q}} = 0.95\pi. \end{aligned} \tag{43}$$

With these inputs, the resulting parameters of the SM quark sector are

$$\begin{aligned} |\hat{Y}_u| &= v\,\mathrm{diag}(0.0003, 0.009, 1.4), \\ |\hat{Y}_d| &= v\,\mathrm{diag}(0.0007, 0.02, 1.2), \\ K &= \begin{pmatrix} 0.974 & 0.223 & 0.027 \\ 0.224 & 0.974 & 0.042 \\ 0.017 & 0.046 & 0.999 \end{pmatrix}, \\ J &= 2.9 \times 10^{-5}. \end{aligned} \tag{44}$$

Let us note that having the largest entries in $\hat{Y}_{u,d}$ of similar size, which follows from $v_d = v_u$, does not constitute any problem. The value of the b-quark mass can be easily suppressed by means of a $U(1)$ spurion vev, along the lines described for example in [11], or by extending the Higgs sector to a two doublets model with $\langle H_d \rangle \ll \langle H_u \rangle$. Also, the fact that the phases in the last line of Eq. (43) are all close to π, which implies somewhat small imaginary parts for the complex parameters, is a simple consequence of our attempt to reproduce the observed value of J with an angle θ_{13} a bit too large. We include below for completeness also the resulting values of the moduli of the auxiliary fields:

$$\begin{aligned} |Z_{Q_1}^T| &= v(0.05, 0.10, 0.98), \\ |Z_{Q_2}^T| &= v(0.04, 0.79, 0.05), \\ |Z_u^T| &= v(0.0001, 0.70, 0.37), \\ |Z_d^T| &= v(0.0007, 0.66, 0.43). \end{aligned} \tag{45}$$

VII. CONCLUSIONS

In this paper we have shown that eight observables of the SM quark sector (four mass ratios, three mixing angles and δ_K) can be reproduced by starting from the simple idea that the complete breaking of the quark flavor symmetry results as the dynamical effect of driving a suitable scalar potential towards its minimum. We have identified the minimum set of multiplets in simple (fundamental and bifundamental) representations of the group needed to break $\mathcal{G}_{\mathcal{F}} \to 0$, and we have shown that this same set of fields is also sufficient to generate one weak *CP* violating phase. Besides the quantitative results, through this study we have gained important qualitative understandings of various mechanisms that might underlie some of the most puzzling features of the SM quark sector. We list them in what we think is their order of importance.

(1) $K = V_{\mathrm{CKM}} \approx I_{3\times3}$.—The interaction between the two Yukawa fields Y_u and Y_d tends to generate an exact alignment of their vevs in flavor space, resulting in $V_{\mathrm{CKM}} = I_{3\times3}$ [2]. If the interaction is repulsive ($\lambda_{ud} > 0$) the alignment occurs when the eigenvalues of the two matrices are ordered by size in an opposite way. When the interaction is attractive ($\lambda_{ud} < 0$) the alignment occurs when the ordering is the same. This second possibility is the one observed in nature. To generate three nonvanishing mixing angles, that is to (slightly) misalign Y_u and Y_d in all flavor directions, at least two other multiplets transforming under the L-handed factor $SU(3)_Q$ are needed. Their presence will induce perturbation in the exact alignment, but if the Y_u-Y_d interaction is sufficiently strong, $V_{\mathrm{CKM}} \approx I_{3\times3}$ will be maintained.

(2) *Yukawa hierarchies.*—Hierarchies between the different entries in Y_u and Y_d are seeded by taking for a subset of the dimensional parameters values somewhat smaller than the overall scale of the vevs: μ_q, $\nu_{1q}, \nu_{2q} \sim v/10$. This can be justified by the fact that when these parameters are set to zero, the scalar potential gains some additional $U(1)$ invariances. The initial (mild) suppression of some entries in $Y_{u,d}$ can get enhanced down to the observed values of the quark mass ratios by dynamical effects. Hierarchical Yukawa couplings can then be generated without strong hierarchies in the fundamental parameters.

(3) *Weak CP violation.*—Once the flavor symmetry is completely broken, generating the CKM *CP* violating phase does not require complicating further the model. The set of scalar multiplets needed to obtain $\mathcal{G}_{\mathcal{F}} \to 0$ ensures that several complex phases cannot be removed regardless of field redefinitions, and this ensures that the scalar potential contains *CP* violating terms. For generic values of these phases, a *CP* violating ground state for $Y_{u,d}$ is obtained.

Indeed, one could object that in our construction there are many more fundamental parameters than there are observables. This of course affects its predictivity, and in some respects also its elegance. We cannot object to such a criticism, but it is worth stressing that the proliferation of parameters is a result of the usual quantum field theory prescription for building renormalizable Lagrangians: we have identified the minimum number of multiplets needed to break completely $\mathcal{G}_{\mathcal{F}}$, and next we have simply written

down the complete set of renormalizable operators allowed by the symmetry. After all, as it has been argued e.g., in [19], the apparent lack of simple relations between the observables of the quark sector might well be due to the fact that, as in our case, they are determined by a very large number of fundamental parameters.

Direct evidences of the scenario we have been studying might arise from the fact that if the flavor symmetry is global, then spontaneous symmetry breaking implies the presence of Nambu-Goldstone bosons that could show up in yet unseen hadron decays or in rare flavor violating processes. If the flavor symmetry is instead gauged, then to ensure the absence of gauge anomalies additional fermions must be introduced [9], and their detection could then represent a smoking gun for this type of models. All this remains, however, a bit speculative, especially because the theory provides no hint of the scale at which the flavor symmetry gets broken, and very large scales would suppress most, if not all, types of signatures.

Note added—.In Ref. [20] we put forth the idea that the spontaneous breaking of the quark-flavor symmetry could automatically solve the strong CP problem. The mechanism underlying this idea was that, after rotating $\theta_{\rm QCD}$ and all other potentially dangerous phases into the scalar potential, at the minimum the vevs of the Yukawa field matrices satisfy $\mathrm{Arg}[\det(\langle Y_u\rangle\langle Y_d\rangle)] = 0(\mathrm{mod}\,2\pi)$. While this is true, when the Yukawa field vevs are reinserted into the effective operators [Eq. (1)], unremovable phases (and in particular $\theta_{\rm QCD}$) reappear in the Yukawa matrices $\mathcal{Y}_u$ and $\mathcal{Y}_d$ [defined after Eq. (1)]. Therefore, the claim made in [20] is incorrect. The issue whether spontaneous breaking of the quark-flavor symmetry can provide some alternative mechanism to solve the strong CP problem is presently under investigation.

APPENDIX: PHYSICAL PHASES OF THE SCALAR POTENTIAL

The complex parameters of the scalar potential [Eq. (30)] all appear in $V_{\mathcal{A}}$ [Eq. (33)] which can be rewritten as

$$\begin{aligned} V_{\mathcal{A}} = & \sum_{q=u,d}\left[e^{i\phi_{\mu_q}}\mu_q\mathcal{D}_q + \sum_{i=1,2} e^{i\phi_{\nu_{iq}}}\nu_{iq}Z^\dagger_{Q_i}Y_qZ_q\right] \\ & + e^{i\phi_{\gamma_{ud}}}\gamma_{ud}Z^\dagger_u Y^\dagger_u Y_d Z_d + e^{i\phi_{\eta_{12}}}\eta_{12}(Z^\dagger_{Q_1}Z_{Q_2})^2 \\ & + e^{i\phi_{m^2_{12}}}m^2_{12}Z^\dagger_{Q_1}Z_{Q_2} + \mathrm{H.c.}, \end{aligned} \tag{A1}$$

where in the last term we have absorbed the vacuum expectation value of $\sum_q\tilde{\eta}_qT_q \approx \sum_q\tilde{\eta}_qv^2_q$ $(q = u, d)$. Let us redefine the fields as follows:

$$\begin{aligned} & Y_q \to e^{-\frac{i}{3}\phi_{\mu_q}}Y_q, Z_q \to e^{-\frac{i}{2}(\phi_{\nu_{1q}}+\phi_{\nu_{2q}}-\frac{2}{3}\phi_{\mu_q})}Z_q, Z_{Qi} \\ & \to e^{\pm\frac{i}{8}(\phi_{\eta_{12}}+2\phi_{m^2_{12}})}Z_{Qi}, \end{aligned} \tag{A2}$$

where in the last line the plus sign is for Z_{Q_1} and the minus sign for Z_{Q_2}. After these redefinitions Eq. (A1)) becomes

$$\begin{aligned} V_{\mathcal{A}} = & \sum_{q=u,d}[\mu_q\mathcal{D}_q + (e^{-i\Delta_q}\nu_{1q}Z^\dagger_{Q_1} + e^{i\Delta_q}\nu_{2q}Z^\dagger_{Q_2})Y_qZ_q] \\ & + e^{i\varphi_\gamma}\gamma_{ud}Z^\dagger_u Y^\dagger_u Y_d Z_d + e^{i\Delta_Q}\eta_{12}(Z^\dagger_{Q_1}Z_{Q_2})^2 \\ & + e^{-\frac{i}{2}\Delta_Q}m^2_{12}Z^\dagger_{Q_1}Z_{Q_2} + \mathrm{H.c.}, \end{aligned} \tag{A3}$$

where in terms of the initial phases in Eq. (A1), we have $\Delta_q = \frac{1}{2}(\phi_{\nu_{2q}} - \phi_{\nu_{1q}}) + \frac{1}{8}(\phi_{\eta_{12}} + 2\phi_{m^2_{12}})$ (for $q = u,\ d$), $\Delta_Q = \frac{1}{2}\phi_{\eta_{12}} - \phi_{m^2_{12}}$ and $\varphi_\gamma = \phi_{\gamma_{ud}} + \frac{1}{2}(\phi_{\nu_{1u}} - \phi_{\nu_{1d}} + \phi_{\nu_{2u}} - \phi_{\nu_{2d}})$.

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