Novel method for the physical scale setting on the lattice and its application to $N_f = 4$ simulations

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This paper reports on a new procedure for the lattice spacing setting that takes advantage of the very precise determination of the strong coupling in the Taylor scheme. Although it can be applied for the physical scale setting with the experimental value of $\Lambda_{\overline{\mathrm{MS}}}$ as an input, the procedure is particularly appropriate for relative "calibrations." The method is here applied for simulations with four degenerate light quarks in the sea and leads to prove that their physical scale is compatible with the same one for simulations with two light and two heavy flavors.

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I. INTRODUCTION

The field theory of the strong interactions, QCD, is essentially nonperturbative in its low energy domain. There, its asymptotic states differ from the noninteracting elementary fields and it should properly account for the main features of the strong phenomenology: chiral symmetry breaking and confinement. One of the most fruitful nonperturbative approaches to the QCD low-energy phenomenology is the lattice field theory [1] which, more and more in the past few years, is providing with accurate numerical results to account for the rich phenomenology of QCD.[1] To this goal, a key role is played by the physical scale setting or lattice "calibration": the adjustment of the lattice spacing to reproduce properly a low-energy experimental value: masses, decay constants, etc.

The purpose of this paper is to present a novel technique to perform this scale setting, which is based on the direct computation of the strong coupling constant from the gauge and ghost propagators in Landau gauge. This computation results from applying the so-called Taylor renormalization scheme to the ghost-gluon vertex, which prescribes an incoming vanishing ghost momentum for the kinematics configuration at the renormalization point [3,4]. In the past, gluonic quantities, as the string tension for the linear static interquark potential [5–8], has been used to perform a relative calibration: to fix the lattice spacing for one simulation from that known from another different simulation. The method presented here avails for a relative calibration from gluonic quantities but, the strong coupling being directly accessible from experiments, also for an absolute lattice calibration with Λ_{QCD} as an input. At this point, it is worthwhile to recall that the running with momentum for the Taylor coupling lattice data have been successfully described by applying continuum computations, based on perturbation theory at the four-loop level and operator product expansion (OPE) nonperturbative corrections roughly above 2 GeV [9,10] and on Dyson-Schwinger equations (DSE) solutions for the deep IR limit [11–13].

The method we propose is particularly useful for simulations with many degenerate light flavors, as those to compute renormalization constants in the flavor massless limit [14] or motivated by the expected similarities of many-light-flavors QCD with Walking models for technicolor [15] as Refs. [16–20]. In those cases, there is no physical quantity to compare with for the scale setting, but Λ_{QCD} can be well defined by assuming the strong coupling running not to depend on the quark masses, at least far away from the flavor thresholds. Furthermore, for more than two light degenerate flavors and three Goldstone

[1]See the lattice review of [2] or the contributions from plenary presentations of 30th (2012) Lattice conference that can be found here: http://pos.sissa.it/cgi-bin/reader/conf.cgi?confid=164.

bosons, the standard chiral behavior cannot be reliably applied to guide the chiral fits of masses or decay constants. On the other hand, $\Lambda_{\rm QCD}$ being the fundamental scale of QCD, to which many different experiments refer, to use it for the scale setting could be taken as a theoretical "ace." Last but not least, the strong coupling running being obtained from data for different simulations, the results can be compared to each other and directly confronted to continuum QCD predictions. This provides with a very valuable cross-check for the scale setting reliability and ensures the best accuracy.

II. THE MATCHING BY THE TAYLOR COUPLING

The strategy is to get the ratios of lattice spacings from different simulations by the intercomparison of a renormalization-group invariant quantity, as the one defining a coupling, computed with the lattice gauge field configurations obtained from the simulations.

A. The matching strategy

In the following, we will first describe the matching scale setting, without loss of generality, for any appropriate lattice quantity. Let us call Q this quantity that could be computed from lattice QCD such that one would have

$$Q_{\rm phys}(p) = Q_{\rm Latt}(p_L) + \mathcal{O}(a), \tag{1}$$

where the physical and the lattice momenta are related such that $p_L = a(\beta,\mu)p$, where $a(\beta,\mu)$ stands for the lattice spacing. We consider a particular simulation in a $N^3 \times N_t$ lattice with β, for the bare lattice coupling, and μ, standing for any other relevant setup parameter (in our next application, the twisted mass of the light[2] degenerated quarks [21,22]). After the appropriate Fourier transform of data in configuration space from the simulations, one would be left with

$$p_L^2 = \left(\frac{2\pi}{N}\right)^2 \left(n_x^2 + n_y^2 + n_z^2 + \frac{N^2}{N_t^2} n_t^2\right), \tag{2}$$

defined by the four integers n_x, n_y, n_z and n_t. In the right-hand side of Eq. (1), we included terms of the order a to account for the lattice artifacts that should tend to disappear when approaching the continuum limit. $Q_{\rm phys}$ will be now supposed not to depend on the lattice setup parameters at sufficiently high energy where the matching is possible, such that, for two different simulations with parameters (β_1,μ_1) and (β_2,μ_2), after neglecting (or properly correcting) the lattice artifacts, we can write

$$Q_{\rm Latt}^{(\beta_1,\mu_1)}(p_L) \equiv Q_{\rm phys}(p) \equiv Q_{\rm Latt}^{(\beta_2,\mu_2)}(p'_L), \tag{3}$$

[2]The dependence in the heavy masses will be dealt with at length in this paper.

where $p'_L/a(\beta_2,\mu_2) = p_L/a(\beta_1,\mu_1) = p$. Then, the ratio of lattice spacings, $a(\beta_2,\mu_2)/a(\beta_1,\mu_1)$ is to be obtained by computing Q from the two different simulations and impose the results to match as Eq. (3) requires. The latter implies that, where the matching is required, any dependence of Q on β and μ has been supposed to be captured by the lattice spacing through the scale setting. This will be confirmed, in our procedure, by the comparison of the running of Q with the momentum for the different simulations, after the scale setting.

B. The Taylor coupling

In order to apply now the matching procedure above described, we need to make an appropriate choice for Q. In particular, we will use the running coupling defined in the so-called Taylor scheme [3,4],

$$Q_{\rm Latt}(p_L) \equiv \alpha_T^{\rm Latt}(p_L) = \frac{g_0^2(a)}{4\pi}\tilde{Z}_3^2(p_L,a)Z_3(p_L,a), \tag{4}$$

where $\tilde{Z}_3$ and Z_3 are the ghost and gluon propagator renormalization constants in MOM scheme and in Landau gauge. The latter is a main advantage for the Taylor coupling to be used as, only involving two-point Green functions to be computed, it can be very accurately estimated from lattice simulations. Furthermore, its running with momenta obtained from the lattice has been exhaustively studied and proven to be very well described by continuum predictions for $N_f = 0$ [4], 2 [23] and $2+1+1$ [9,10,24,25] flavor numbers, after properly dealing with the UV cutoff effects.

After learning from these studies the appropriate lessons, we will compute the Taylor coupling from different lattice simulations as Eq. (4) reads and then apply the so-called $H(4)$-extrapolation procedure [26–28], that exploits the remaining symmetry which is restricted to the $H(4)$ isometry group for the elimination of $O(4)$-breaking lattice artifacts,

$$\alpha_T^{\rm Latt}(p_L) \Rightarrow \alpha_T^{H(4)}(p_L). \tag{5}$$

We will next correct for the $O(4)$-invariant lattice artifacts as shown in Refs. [9,24],

$$\alpha_T^{H(4)}(p_L) = \alpha_T^{\rm phys}(a^{-1}p_L) + c_{a^2p^2}p_L^2. \tag{6}$$

A brief description of the way this procedure works to cure from the UV lattice artifacts can be also found in Appendix A of Ref. [10]. Thus, we will finally be left with the "continuum" Taylor coupling that has been shown to be, in practice, very well described by [9,10,24]

$$\alpha_T^{\rm phys}(p) = \alpha_T(p^2) + \frac{d_x}{p^x}, \tag{7}$$

with

$$\alpha_T(p^2) = \alpha_T^{\rm pert}(p^2)\left(1+\frac{9}{p^2}R(\alpha_T^{\rm pert}(p^2),\alpha_T^{\rm pert}(q_0^2))\right.$$
$$\left.\times\left(\frac{\alpha_T^{\rm pert}(p^2)}{\alpha_T^{\rm pert}(q_0^2)}\right)^{1-\gamma_0^{A^2}/\beta_0}\frac{g_T^2(q_0^2)\langle A^2\rangle_{R,q_0^2}}{4(N_C^2-1)}\right), \quad (8)$$

derived from the OPE description of ghost and gluon dressing functions in terms of the dimension-two gluon condensate.[3] The OPE Wilson coefficient also accounts for higher-order corrections beyond the leading logarithm which appears included in $R(\alpha,\alpha_0)$ for Eq. (8), while the purely perturbative running is given by $\alpha_T^{\rm pert}$ up to four-loops through the integration of the β function [2],

$$\alpha_T^{\rm pert}(\mu^2) = \frac{4\pi}{\beta_0 t}\left(1-\frac{\beta_1}{\beta_0^2}\frac{\log(t)}{t}+\frac{\beta_1^2}{\beta_0^4}\frac{1}{t^2}\left(\left(\log(t)-\frac{1}{2}\right)^2\right.\right.$$
$$\left.+\frac{\beta_2\beta_0}{\beta_1^2}-\frac{5}{4}\right)+\frac{1}{(\beta_0 t)^3}\left(\frac{\beta_3}{2\beta_0}+\frac{1}{2}\left(\frac{\beta_1}{\beta_0}\right)^3\right.$$
$$\times\left(-2\log^3(t)+5\log^2(t)\right.$$
$$\left.\left.\left.+\left(4-6\frac{\beta_2\beta_0}{\beta_1^2}\right)\log(t)-1\right)\right)\right) \quad (9)$$

in terms of $t=\ln(p/\Lambda_T)$ where $\Lambda_T/\Lambda_{\overline{\rm MS}}=0.5608$ for $N_f=4$. In Ref. [9], Eqs. (6)–(8) have been successfully applied to fit the running of the Taylor coupling obtained from unquenched lattice simulations with two light degenerate quark flavors and two heavier nondegenerate ones ($N_f=2+1+1$). The results, recently updated in Ref. [10], for the best-fit parameters are $\Lambda_{\overline{\rm MS}}\bar{a}(1.90,0)=0.1413(32)$, $g^2\langle A^2\rangle\bar{a}^2(1.90,0)=0.76(11)$, $x=5.73(27)$ and $d_x\bar{a}^x(1.90,0)=-0.157(10)$; expressed in units of $\bar{a}(1.90,0)$ (we used here $\bar{a}$ for simulations with $N_f=2+1+1$ and keep a for $N_f=4$ degenerate flavors).

C. The procedure and its validity

Then, for any simulation with setup parameters (β,μ), according to Eqs. (3)–(6), one can write

$$\alpha^{H(4)}_{T,(\beta,\mu)}(p_L) = \alpha_T^{\rm phys}\left(\frac{p'_L}{\bar{a}(1.90,0)}\right)+c_{a^2p^2}p_L^2, \quad (10)$$

where $p'_L=p_L\bar{a}(1.90,0)/a(\beta,\mu)$, p_L being the lattice momentum for the simulation, Eq. (2), and the running of $\alpha_T^{\rm phys}$ given by Eqs. (7) and (8) and expressed in units of $\bar{a}(1.90,0)$,

$$\alpha_T^{\rm phys}\left(\frac{p'_L}{\bar{a}(1.90,0)}\right) = \alpha_T\left(\frac{p'^2_L}{\bar{a}^2(1.90,0)}\right)+\frac{d_x\bar{a}^x(1.90,0)}{p'^x_L}, \quad (11)$$

with $x=5.73$ and the central value for the parameters $\Lambda_{\overline{\rm MS}}\bar{a}(1.90,0)$, $g^2\langle A^2\rangle\bar{a}^2(1.90,0)$ and $d_x\bar{a}^x(1.90,0)$ above presented. The latter is a consequence of our main assumption: $\Lambda_{\overline{\rm MS}}$ and the nonperturbative corrections, coded by $g^2\langle A^2\rangle$ and d_x, are supposed to depend only on the number of active quarks and, far above the quark mass thresholds, their masses should not matter so much. As the matching of coupling data for simulations with μ and μ' setup parameters naturally implies[4]

$$\frac{\Lambda^{\mu'}_{\overline{\rm MS}}}{\Lambda^{\mu}_{\overline{\rm MS}}}\simeq\left(\frac{g^2\langle A^2\rangle^{\mu'}}{g^2\langle A^2\rangle^{\mu}}\right)^{1/2}\simeq\left(\frac{d_x^{\mu'}}{d_x^{\mu}}\right)^{1/x}\simeq 1, \quad (12)$$

this main assumption will appear supported *a posteriori*: as can be seen in Fig. 2, a perfect matching for the Taylor coupling data is obtained from three simulations with different β's at $N_f=4$ and one with $\beta=2.10$ at $N_f=2+1+1$. Furthermore, the data from this last simulation have been strikingly shown to behave as continuum computations predicts for above 1.7 GeV [9,10], where our fits will be performed. Thus, taking the ratios in Eq. (12) to be exactly 1, the ratio of lattice spacings, $\bar{a}(1.90,0)/a(\beta,\mu)$, and the coefficient $c_{a^2p^2}$ are the two only free parameters to be determined by the best fit of Eqs. (10) and (11) to the Taylor coupling lattice data above 1.7 GeV.

One more question is very in order here for discussion: the Taylor coupling is a Landau-gauge quantity, obtained from gauge-dependent ghost and gluon propagators, which suffers from the problem of the Gribov ambiguity [33]. Thus, the standard Landau-gauge fixing procedure by the minimization of a functional of the gauge field, A^a_μ, verifying $\partial_\mu A^a_\mu=0$ with the Fadeev-Popov operator being positive, leads to many local minima of the gauge orbit, usually called "Gribov copies." Such an ambiguity on the gauge fixing may introduce disrupting deviations for the confrontation of continuum and Landau-gauge lattice quantities. On the lattice, this ambiguity has been scrutinized by comparing the results from a "best copy," selected as the minimum of the functional for a sample of random copies, with the ones from the "first copy" resulting from the minimization [34,35]. The selection of the best copy has been also improved in recent investigations by the application of the so-called simulated annealing gauge-fixing algorithm [36–38]. The main conclusion from these investigations is that Gribov-copy effects are found not to have any impact on SU(2) gluon and ghost propagators above a

[3]The OPE power corrections of Landau-gauge gluon and ghost propagators have been found to be dominated by a nonvanishing gauge-dependent dimension-two gluon condensate [29–32].

[4]Deviations from Eq. (12) should be included in the procedure's systematic uncertainties.

given momentum, $p_{\min}$. This momentum $p_{\min}$ is also found to decrease with the lattice size in physical units, L, for a hypercubic lattice. The authors of Ref. [38] studied results from simulations with L roughly ranging from 1 to 8 GeV and generally concluded that Gribov-copy effects were relevant for $p < 1$ GeV. In particular, their Fig. 5 shows that $p_{\min} \simeq 0.7$ GeV for $L \simeq 5$ fm.

In our case, we studied SU(3) Taylor coupling results from nonhypercubic lattice simulations where the spatial size roughly ranges from 2 to 3 fm (while the time-direction one is twice the spatial). Therefore, performing fits for the matching above $p \simeq 1.7$ GeV, the results are expected to be free of Gribov-copies ambiguities. However, as far as the matching works for a fitting window reaching the UV domain, whatsoever the lower bound might be, the ratios of lattice spacings will be the same in the UV region, free of ambiguities, as for any IR momenta inside the window. Then, the scale setting can be proven *a posteriori* to be safe from the Gribov-ambiguity problem through, again, the quality of the matching for the Taylor coupling results obtained from the different involved simulations. As will be seen in Fig. 2, this is indeed the case for the results here presented.

III. THE RESULTS

In the following, the above-described procedure will be applied to estimate the lattice spacing for simulations with $N_f = 4$ degenerate twisted-mass flavors [14] (Table I gathers their setup parameters), produced by ETM collaboration (ETMC) to apply the massless renormalization. To our knowledge, no other method allows for such a reliable scale setting in this case, as the Taylor coupling can be properly taken not to depend very much[5] on the setup parameters for $N_f = 4$ and $N_f = 2 + 1 + 1$ simulations.

A. Relative calibration

We will take the lattice spacing to depend on the bare gauge coupling, β, and on the dynamical degenerate-flavor mass only through the bare polar mass, M_0 (see Table I and Ref. [14]). Then, we compute the Taylor coupling, α_T^{Latt}, given by Eq. (4) for each lattice ensemble. Next, we average for the two ensembles with roughly the same m_{PCAC} but opposite sign, as explained in Ref. [14], in order to achieve approximatively the $O(a)$ improvement though working out of the maximal twist. We apply the $H4$ extrapolation procedure to remove the hypercubic artifacts and, finally, the cured results for the coupling is fitted with Eqs. (10) and (11), as explained in the previous section. Thus, we obtain the ratios of lattice spacings, $\bar{a}(1.90, 0)/a(\beta, aM_0)$, and $c_{a^2p^2}$ as the best-fit parameters

[5]This is the case, as the matching we reach shows, at least for quark masses varying not too much, as happens for our simulations.

TABLE I. Setup parameters, m_{PCAC} and the bare polar mass for the ensembles here exploited (borrowed from Ref. [14]).

β	$a\mu$	am_{PCAC}	aM_0	Configurations
1.90	0.0080	−0.0390(01)	0.0285(01)	130
1.90	0.0080	0.0398(01)	0.0290(01)	130
1.90	0.0080	−0.0358(02)	0.0263(01)	200
1.90	0.0080	0.0356(01)	0.0262(01)	200
1.90	0.0080	−0.0318(01)	0.0237(01)	200
1.90	0.0080	0.0310(02)	0.0231(01)	200
1.90	0.0080	−0.0273(02)	0.0207(01)	130
1.90	0.0080	0.0275(04)	0.0209(01)	130
1.95	0.0085	−0.0413(02)	0.0329(01)	130
1.95	0.0085	0.0425(02)	0.0338(01)	130
1.95	0.0085	−0.0353(01)	0.0285(01)	130
1.95	0.0085	0.0361(01)	0.0285(01)	130
1.95	0.0020	−0.0363(01)	0.0280(01)	120
1.95	0.0020	0.0363(01)	0.0274(01)	120
1.95	0.0180	−0.0160(02)	0.0218(01)	130
1.95	0.0180	0.0163(02)	0.0219(01)	130
1.95	0.0085	−0.0209(02)	0.0182(01)	130
1.95	0.0085	0.0191(02)	0.0170(01)	130
1.95	0.0085	−0.0146(02)	0.0141(01)	130
1.95	0.0085	0.0151(02)	0.0144(01)	130
2.10	0.0078	−0.00821(11)	0.0102(01)	180
2.10	0.0078	0.00823(08)	0.0102(01)	180
2.10	0.0064	−0.000682(13)	0.0084(01)	180
2.10	0.0064	0.00685(12)	0.0084(01)	180
2.10	0.0046	−0.00585(08)	0.0066(01)	120
2.10	0.0046	0.00559(14)	0.0064(01)	120
2.10	0.0030	−0.00403(14)	0.0044(01)	240
2.10	0.0030	0.00421(13)	0.0045(01)	240

TABLE II. Ratios of lattice spacings obtained as explained in the text. The values obtained by performing a chiral extrapolation down to a zero light quark mass are also shown. The quality of the fits is characterized by the χ^2/d.o.f. All the errors have been derived by applying the jackknife method.

β	aM_0	$\bar{a}(1.90, 0)/a(\beta, aM_0)$	$c_{a^2p^2}$	χ^2/d.o.f.
1.90	0.0288	0.932(18)	−0.0074(14)	5.6/44
1.90	0.0263	0.969(17)	−0.0067(5)	3.3/45
1.90	0.0234	0.969(11)	−0.0080(10)	7.6/45
1.90	0.0208	1.004(25)	−0.0056(12)	2.4/46
1.90	0	1.049(46)		
1.95	0.0334	1.024(12)	−0.0079(8)	4.4/46
1.95	0.0285	1.059(12)	−0.0088(9)	10.2/49
1.95	0.0277	1.019(20)	−0.0099(9)	5.6/46
1.95	0.0219	1.086(20)	−0.0069(9)	3.4/47
1.95	0.0176	1.105(11)	−0.0066(11)	19.8/48
1.95	0.0143	1.115(18)	−0.0054(7)	9.9/50
1.95	0	1.134(18)		
2.10	0.0102	1.530(15)	−0.0053(4)	142/93
2.10	0.0084	1.518(15)	−0.0048(5)	90.3/93
2.10	0.0065	1.533(19)	−0.0049(5)	161/93
2.10	0.0045	1.578(42)	−0.0055(10)	240/94
2.10	0	1.533(35)		

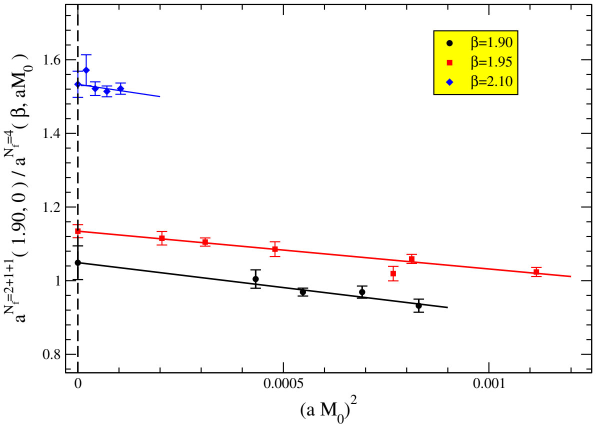

FIG. 1 (color online). Ratios of lattice spacings (see Table II) obtained by the matching procedure of the Taylor coupling and the corresponding chiral extrapolation in the solid line.

gathered in Table II. The results appear also plotted in Fig. 1, where a linear extrapolation on M_0^2, as the use of a $\mathcal{O}(a)$-improved lattice action suggests, down to the chiral limit is also shown. It should be noticed that the fitted parameters for the coefficient correcting the $O(4)$-invariant lattice artifacts, $c_{a^2p^2}$, shows no important dependence on the light quark mass, as expected, and fairly well agree with the same parameter obtained for our previous analysis with simulations for $N_f = 2+1+1$ [9,10].

In Table III, the ratios of $N_f = 4$ lattice spacings over that at $\beta = 1.90$ for $N_f = 2+1+1$ from Table II, after the chiral extrapolation, are shown in comparison with ratios of the same lattice spacings for $N_f = 2+1+1$, borrowed from Refs. [10,39]. They all agree within the errors, although the lattice spacings for $N_f = 2+1+1$ appear to be systematically larger (~5%) than those for $N_f = 4$.

B. Absolute calibration from $\Lambda_{\overline{\rm MS}}$

In the previous section, the matching of the Taylor coupling led to a relative scale setting for the analyzed simulations, i.e. in terms of a given lattice spacing for another simulation ($\beta = 1.90$ and $N_f = 2+1+1$, with chiral light flavors). Then, the "absolute" calibration of the former, in physical units, requires from the latter's knowledge. On the other hand, the Taylor coupling from lattice data confronted to Eqs. (6) and (8) provided with an estimate for $\Lambda_{\overline{\rm MS}}$ in terms of the lattice spacing. Such an estimate was used in Refs. [9,10,24] to compute, after the scale setting from ETMC, $\Lambda_{\overline{\rm MS}}$ in physical units and hence $\alpha_{\overline{\rm MS}}(m_Z^2)$. Alternatively, one can also take the experimental value for $\Lambda_{\overline{\rm MS}}$ and use it to estimate the lattice spacing. We have $\bar{a}(1.90,0)\Lambda_{\overline{\rm MS}} = 0.1413(32)$, from lattice data with $N_f = 2+1+1$ unquenched flavors, as mentioned above, and $\Lambda_{\overline{\rm MS}}^{N_f=4} = 296(10)$ MeV from PDG [2]. Then, for $N_f = 2+1+1$, one would have $\bar{a}(1.90,0) = 0.0940(38)$fm, which compares fairly well to the very recent ETMC result: $\bar{a}(1.90,0) = 0.0885(36)$ fm [39]. It should be furthermore noticed that the determination of $\bar{a}(1.90,0)\Lambda_{\overline{\rm MS}}$ in Ref. [10] takes into account systematic uncertainties we do not include in the present calibration. These uncertainties could be drastically reduced by performing a simulation at as larger a β parameter as possible, to reach larger physical momenta but keeping the higher-order hypercubic artifacts under control.

TABLE III. Comparison of the ratios of lattice spacings for $N_f = 4$ (noted as a) obtained here and those for $N_f = 2+1+1$ simulations (noted as $\bar{a}$) from Refs. [10,39].

β	$\bar{a}(1.90,0)/a(\beta,0)$	$a(1.90,0)/a(\beta,0)$	$\bar{a}(1.90,0)/\bar{a}(\beta,0)$
1.90	1.049(46)	1	1
1.95	1.134(18)	1.081(50)	[39]: 1.085(59)
2.10	1.533(35)	1.461(72)	[10]: 1.477(28)
			[39]: 1.429(71)

Thus, we can take our estimate for the lattice spacing and set the physical scale for all the lattices in Table I, with the help of the ratios from Table II. Then, we verify that the running of $\alpha_T^{\rm phys}$, defined by Eq. (6), is the same for all of them and the same as for $N_f = 2+1+1$, as can be seen in Fig. 2. In particular, in the chiral limit, we obtain

$$\frac{a(\beta,0)}{1\ \text{fm}} = \begin{cases} 0.0896(53) & \beta = 1.90 \\ 0.0829(36) & \beta = 1.95 \\ 0.0613(29) & \beta = 2.10, \end{cases} \tag{13}$$

for the $N_f = 4$ simulations. The results from Eq. (13) compare pretty well with those obtained for the same ETMC simulations by setting the scale through chiral fits of the pseudoscalar meson masses in terms of the renormalized light quark mass [39]: 0.0885(36) at $\beta = 1.90$, 0.0815(30) at $\beta = 1.95$ and 0.0619(18) at $\beta = 2.10$; that have been

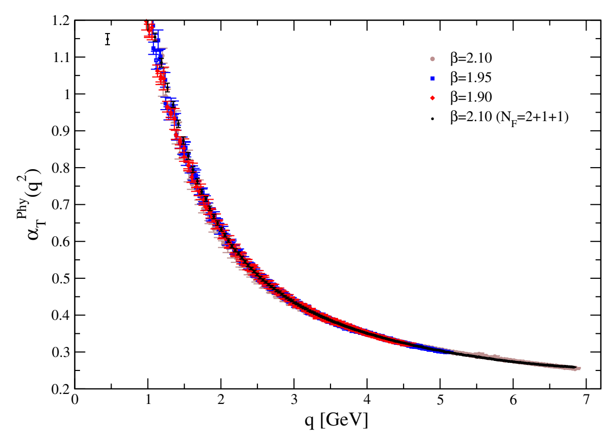

FIG. 2 (color online). The physical running of the Taylor coupling, defined by Eq. (6), for all the properly calibrated lattices from Table I. $N_f = 2+1+1$ data from [10] are included for comparison.

used for the computation of the renormalization constants with $N_f = 4$ simulations, within the massless quark renormalization scheme approach. It might be worthwhile to recall that our scale-setting procedure does not invoke any hadronic quantity in order to provide with the results of Eq. (13).

IV. CONCLUSIONS

We have proposed a novel method for the scale setting on lattice simulations that only needs the evaluation of gauge and ghost propagators to determine the strong coupling running and requires for it, after the appropriate removal of lattice artifacts, to be the same for different simulations, when the scale is properly fixed. The method allows for a relative calibration of lattices, the lattice spacing for them being expressed in terms of the one in another given simulation, but also for an absolute calibration with $\Lambda_{\overline{\rm MS}}$ as an input. A major advantage of this procedure comes from the necessity of reproducing with the lattice data a running with momenta which is well known from continuum calculations, the validity of the results relying on the appropriate fulfillment of this requirement.

The method has been successfully applied to perform the scale setting for unquenched simulations including four degenerate light flavors. We have found that, within our statistical uncertainties, the lattice spacings for $N_f = 4$ and $N_f = 2 + 1 + 1$ simulations appear to be compatible. Thus, we should conclude that, with our procedure, heavy-quark mass effects on the scale setting for our $N_f = 2 + 1 + 1$ simulations are of the same order as the statistical errors.

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