# Neutrino mixing matrix and masses from a generalized Friedberg-Lee model

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The overall characteristics of the solar and atmospheric neutrino oscillation are approximately consistent with a tribimaximal form of the mixing matrix $U$ of the lepton sector. Exact tribimaximal mixing leads to $\theta_{13} = 0$. However, recent results from the Daya Bay and RENO experiments have established a nonzero value for $\theta_{13}$. Keeping the leading behavior of $U$ as tribimaximal, we use a generalized Friedberg-Lee neutrino mass model along with a complementary ansatz to incorporate a nonzero $\theta_{13}$ along with $CP$ violation. We generalize this model in two stages: In the first stage, we assume $\mu - \tau$ symmetry and add imaginary components which leads to nonzero phases. In the second stage, we add a perturbation with real components which breaks the $\mu - \tau$ symmetry, and this leads to a nonzero value for $\theta_{13}$. The combination of these two generalizations leads to $CP$ violation. Using only two sets of the experimental data, we can fix all of the parameters of our model and predict not only values for the other experimental data, which agree well with the available data, but also the masses of neutrinos and the $CP$-violating phases and parameters. These predictions include the following: $\langle m_{\nu_e} \rangle \approx (0.033 - 0.037)$ eV, $\langle m_{\nu_\mu} \rangle \approx (0.043 - 0.048)$ eV, $\langle m_{\nu_\tau} \rangle \approx (0.046 - 0.051)$ eV, and $59.21° \lesssim \delta \lesssim 59.34°$.

## I. INTRODUCTION

The remarkable experimental achievements resulting in experimental data for solar, atmospheric, accelerator, and reactor neutrinos [1] give us information about neutrino masses and mixing, which can be summarized as follows in Table I[2,3].

The lepton mixing matrix in the standard parametrization is given by [4]

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\rho} & 0 \\ 0 & 0 & e^{i\sigma} \end{pmatrix}, \quad (1)$$

where $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$ [for $i, j = (1, 2), (1,3)$, and $(2,3)$]. The phase $\delta$ is called the Dirac phase, analogous to the CKM phase, and the phases $\rho$ and $\sigma$ are called the Majorana phases and are relevant for Majorana neutrinos.

A relatively successful phenomenological ansatz for the neutrino mixing matrix was proposed by Harrison *et al.* in 2002 [5] and is known as the tribimaximal mixing matrix (TBM). It has the following form:

$$U_{\text{TBM}} = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (2)$$

In this matrix, the third mixing angle, i.e., $\theta_{13}$, is exactly zero. However, the results of the Daya Bay and RENO collaborations have shown that $\theta_{13} = 0$ is now rejected at a significance level higher than $8\sigma$. A combined analysis of the data coming from the T2K, MINOS, Double Chooz, and Daya Bay experiments shows that the best-fit value of $\theta_{13}$ is given by $\sin^2 \theta_{13} = 0.026(0.027)^{+0.003}_{-0.004}$ for normal (or inverted) mass hierarchy. From these significant data, it has to be concluded that the simple picture of tribimaximal mixing matrix fails. However, the smallness of $\theta_{13}$ as compared to the other two mixing angles encourages us to examine whether a small perturbation on the basic tribimaximal structure could lead to a nonzero value for $\theta_{13}$ and a more realistic neutrino mixing matrix.

A successful phenomenological neutrino mass model with flavor symmetry that is suitable for both the Dirac and the Majorana neutrinos was proposed by Friedberg and Lee (FL). This model is interesting because when $\mu - \tau$

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TABLE I. The experimental data for the neutrinos' mixing parameters. When multiple sets of allowed ranges are stated, the upper row corresponds to normal hierarchy and the lower row to inverted hierarchy.

| Parameter | Experimental data | Best fit ($\pm 1\sigma$) |
|---|---|---|
| $\Delta m_{21}^2 [10^{-5} \text{ eV}^2]$ | 7.12–8.20 | 7.43–7.81 |
| $|\Delta m_{31}^2| [10^{-3} \text{ eV}^2]$ | 2.13–2.74 | 2.46–2.61 |
| | 2.21–2.64 | 2.37–2.50 |
| $\sin^2 \theta_{12}$ | 0.27–0.37 | 0.303–0.336 |
| $\sin^2 \theta_{13}$ | 0.017–0.033 | 0.0218–0.0275 |
| | | 0.0223–0.0276 |
| $\sin^2 \theta_{23}$ | 0.36–0.68 | 0.400–0.461 and 0.573–0.635 |
| | 0.37–0.67 | 0.569–0.626 |
| $\delta$ | 0–$2\pi$ | 0–$2\pi$ |

symmetry is assumed, the resulting neutrino mixing matrix reduces to the $U_{\text{TBM}}$. Usually, the ansatz for the neutrino mass matrices contains more parameters than can be measured in realistic experiments. Thus, the presence of certain conditions or simplifications for the neutrino mass matrix is useful. What might first come to mind is the presence of zeros in the mass matrix [6]. However, conditions on basis-independent quantities, namely, the trace and the determinant, are more suitable to consider. The simplest conditions are to set these quantities to zero. The condition $\det(M_\nu) = 0$ [7] leads to zero mass for one of the neutrinos. A zero determinant can be motivated on various grounds [8]. The second simplest basis-independent condition is a vanishing trace, i.e., $\text{Tr}(M_\nu) = 0$, which is called the complementary ansatz. Its consequences have first been investigated in Ref. [9] applying a three neutrino framework that simultaneously explains the anomalies of solar and atmospheric neutrino oscillation experiments as well as the LSND experiment. In Ref. [10], a *CP* conserving traceless $M_\nu$ has been investigated for the simpler case of explaining only the atmospheric and solar neutrino deficits. Motivations for traceless mass matrices can be provided by models in which $M_\nu$ is constructed through a commutator of two matrices, as it happens in models of radiative mass generation [11]. More interestingly, an approximately traceless $M_\nu$ can be the consequence of exact $b - \tau$ unification at high energy scales within type II seesaw models [12], which in this framework is also the reason for maximal atmospheric neutrino mixing [13,14]. The type II seesaw mechanism was the original motivation of the traceless $M_\nu$ condition as investigated in Ref. [9]. Also, in Ref. [15], the condition $\text{Tr}(M_\nu) = 0$ has been used to determine the values of neutrino masses and the *CP* phases when *CP* violation is considered. In the present paper, we will use the FL model as a starting point and assume the complementary ansatz.

In the FL model, the mass eigenstates of three charged leptons are identified with their flavor eigenstates.

Therefore, the neutrino mixing matrix can be simply described by a $3 \times 3$ unitary matrix $U$, which transforms the neutrino mass eigenstates to the flavor eigenstates, $(\nu_e, \nu_\mu, \nu_\tau)$. The Dirac neutrino mass operator in the FL model can be written as

$$\mathcal{M}_{\text{FL}} = a(\bar{\nu}_\tau - \bar{\nu}_\mu)(\nu_\tau - \nu_\mu) + b(\bar{\nu}_\mu - \bar{\nu}_e)(\nu_\mu - \nu_e)$$
$$+ c(\bar{\nu}_e - \bar{\nu}_\tau)(\nu_e - \nu_\tau) + m_0(\bar{\nu}_e \nu_e + \bar{\nu}_\mu \nu_\mu + \bar{\nu}_\tau \nu_\tau). \tag{3}$$

All the parameters in this model ($a$, $b$, $c$, and $m_0$) are assumed to be real. In the original FL model, also known as the pure FL model, $m_0 = 0$, and in this case, $M_{\text{FL}}$ has the following symmetry: $\nu_e \rightarrow \nu_e + z$, $\nu_\mu \rightarrow \nu_\mu + z$, and $\nu_\tau \rightarrow \nu_\tau + z$, where $z$ is an element of the Grassmann algebra. For constant $z$, this symmetry is called the FL symmetry [16], in which case, the kinetic term is also invariant. However, the other terms of the electroweak Lagrangian do not have such a symmetry. The $m_0$ term breaks this symmetry explicitly. However, we may add that the FL symmetry leads to a magic matrix, and this property is not spoiled by the $m_0$ term. The magic property has many manifestations, which we shall discus in detail. Also, it has been reasoned that the FL symmetry is the residual symmetry of the neutrino mass matrix after the $SO(3) \times U(1)$ flavor symmetry breaking [17]. The mass matrix can be displayed as

$$M_{\text{FL}} = \begin{pmatrix} b + c + m_0 & -b & -c \\ -b & a + b + m_0 & -a \\ -c & -a & a + c + m_0 \end{pmatrix}, \tag{4}$$

where $a \propto (Y_{\mu\tau} + Y_{\tau\mu})$, $b \propto (Y_{e\tau} + Y_{\tau e})$, $c \propto (Y_{\tau e} + Y_{e\tau})$, and $Y_{\alpha\beta}$ denote the Yukawa coupling constants. The proportionality constant is the expectation value of the Higgs field. It is apparent that $M_{\text{FL}}$ possesses exact $\mu - \tau$ symmetry only when $b = c$. Setting $b = c$ and using the Hermiticity of $M_{\text{FL}}$, a straightforward diagonalization procedure yields $U^T M_{\text{FL}} U = \text{diag}\{m_1^{(0)}, m_2^{(0)}, m_3^{(0)}\}$, where

$$m_1^{(0)} = 3b + m_0$$
$$m_2^{(0)} = m_0, \quad \text{and} \quad U = U_{\text{TBM}},$$
$$m_3^{(0)} = 2a + b + m_0. \tag{5}$$

Note that in the pure FL model, one of the neutrino masses is exactly zero. For a general exact TBM neutrino mixing, regardless of the model, the mixing angles are given by $\sin^2 \theta_{12} = \frac{1}{3}$ ($\theta_{12} \approx 35.3°$), $\sin^2 \theta_{23} = \frac{1}{2}$ ($\theta_{23} = 45°$), $\theta_{13} = 0$, and the *CP*-violating phases can be chosen to be zero. In order to have *CP* violation in the standard parametrization given in Eq. (1), the necessary condition is $\delta \neq 0$ and $\theta_{13} \neq 0$. In this model, these conditions necessarily mandate that $\mu - \tau$ symmetry should

be broken. Another interesting question is whether $\theta_{23} = 45°$ holds after the $\mu - \tau$ symmetry breaking.

There are four independent $CP$-even quadratic invariants, which can conveniently be chosen as $U_{11}^* U_{11}$, $U_{13}^* U_{13}$, $U_{21}^* U_{21}$, and $U_{23}^* U_{23}$ and three independent $CP$-odd quartic invariants [18],

$$
\begin{aligned}
J &= \mathcal{I}m(U_{11} U_{12}^* U_{21}^* U_{22}), \\
I_1 &= \mathcal{I}m[(U_{11}^* U_{12})^2], \\
I_2 &= \mathcal{I}m[(U_{11}^* U_{13})^2].
\end{aligned}
\tag{6}
$$

The Jarlskog rephasing-invariant parameter $J$ [19] is relevant for $CP$ violation in lepton number conserving processes like neutrino oscillations. $I_1$ and $I_2$ are relevant for $CP$ violation in lepton number violating processes like neutrinoless double beta decay. Oscillation experiments cannot distinguish between the Dirac and Majorana neutrinos. The detection of neutrinoless double beta decay would provide direct evidence of lepton number nonconservation and the Majorana nature of neutrinos. Many theoretical and phenomenological works have discussed massive neutrino models that break $\mu - \tau$ symmetry as a prelude to $CP$ violation [20].

In this paper, we start with the simple FL model with exact $\mu - \tau$ symmetry, which leads to the $U_{\text{TBM}}$ mixing matrix. We then generalize this model by adding complex parameters to the elements of $M_{\text{FL}}$, which can ultimately be linked to complex Yukawa coupling constants. This generalization can be broken down to two clearly distinguishable pieces. In this paper, we first study the results of each piece separately and then investigate the results when both pieces are applied simultaneously. Separately, both pieces can be solved exactly but the combination cannot. First, we add a real term proportional to the matrix representation of the permutation group element $S^{132}$, merely to break the $\mu - \tau$ symmetry. As we shall show, it suffices to treat this piece as a perturbation. We find that this generates a nonzero $\theta_{13}$ whose smallness justifies the use of perturbation theory. Second, we

add arbitrary imaginary coefficients while preserving $\mu - \tau$ symmetry. This results in the generation of nonzero phases, including $\delta$, which turn out to be large. For combining the two steps, we first do the complexification step exactly and finally add the $\mu - \tau$ symmetry breaking perturbation. We then find that nonzero values are generated for both $\theta_{13}$ and $\delta$, and this leads to $CP$ violation.

This paper is organized as follows. In Sec. II, we introduce our model and show the results of the two aforementioned generalizations, separately. We then combine the two generalizations and show the results. We use some self-consistency arguments along with the complementary ansatz to reduce the number of our free parameters. In Sec. III, we map two sets of the experimental data onto the allowed region of our parameter space. We then check the consistency of all other experimental data with the two overlap regions, which selects only one region. This region is extremely small and almost pinpoints all of our parameters. The results indicate that the complexification had to be treated nonperturbatively, since the phases turn out to be large. However, the coefficient for $\mu - \tau$ symmetry breaking justifies the use of perturbation theory for that part, since $\theta_{13}$ turns out to be small. We then not only check the consistency of all of our results with the available experimental data but also present our predictions for the actual masses and $CP$-violation parameters. In Sec. IV, we state our conclusions.

## II. THE MODEL

In this section, we study the effects of generalizations of the FL model that break the $\mu - \tau$ and $CP$ symmetries. In the first stage, we perturb the mass matrix so as to break the $\mu - \tau$ symmetry. We choose the mass matrix to be of the form $M_\nu = M_\nu^0 + M_\nu' = M_\nu^0 + \lambda S^i$, where $M_\nu^0 = M_{\text{FL}}$ with $b = c$, $\lambda$ is a perturbation parameter with dimension of mass, and $S^i$ is an element of the permutation matrices which breaks the $\mu - \tau$ symmetry. For example, for $S^i = S^{(132)}$, we have

$$
M_\nu = M_\nu^0 + M_\nu' = M_\nu^0 + \lambda S^{(132)} = \begin{pmatrix} 2b + m_0 & -b & -b \\ -b & a + b + m_0 & -a \\ -b & -a & a + b + m_0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\tag{7}
$$

Notice that $M_\nu$ and $M_{\text{FL}}$ are both magic and symmetric matrices since they both commute with the magic $S$ matrix defined by

$$
S = \begin{pmatrix} F & T & T \\ T & F & T \\ T & T & F \end{pmatrix}.
\tag{8}
$$

Therefore, one of the eigenstates must be $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The elements of the basis in which $M_\nu^0$ is diagonal are simply the columns of $U_{\text{TBM}}$, as shown in Eq. (2) and are as follows:

$$\nu_1^{(0)} = \begin{pmatrix} -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \nu_2^{(0)} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \quad \nu_3^{(0)} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$
(9)

Here, $M_\nu^0$ and $M_\nu'$ are real matrices. Needless to say, this may generate a nonzero $\theta_{13}$ but will not lead to $CP$ violation since this computation necessarily yields $\delta = 0$. As stated in the Introduction, although this problem is exactly solvable, it not only suffices to treat this part as a perturbation since $\theta_{13}$ turns out to be small, but also it is advantageous to do so for comparison purposes with the part when all generalizations are combined. The perturbation expansion of the mass eigenstates of $M_\nu$ to first order is

$$\nu_i = \nu_i^{(0)} + \sum_{i \neq j} \frac{\langle \nu_j^{(0)} | M_\nu' | \nu_i^{(0)} \rangle}{m_i^0 - m_j^0} \nu_j^{(0)}.$$
(10)

The $\nu_i$'s are simply the columns of the mixing matrix $U$ and are as follows:

$$\nu_1 = \begin{pmatrix} -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} - \frac{\sqrt{\frac{3}{32}}\lambda}{(a-b)} \\ \frac{1}{\sqrt{6}} + \frac{\sqrt{\frac{3}{32}}\lambda}{(a-b)} \end{pmatrix}, \quad \nu_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix},$$

$$\nu_3 = \begin{pmatrix} -\frac{\lambda}{2\sqrt{2}(a-b)} \\ -\frac{1}{\sqrt{2}} + \frac{\lambda}{4\sqrt{2}(a-b)} \\ \frac{1}{\sqrt{2}} + \frac{\lambda}{4\sqrt{2}(a-b)} \end{pmatrix}.$$
(11)

Comparing $\nu_3$ given in Eq. (11) with the third column of $U_{\text{PMNS}}$ as given by Eq. (1), we immediately obtain all of the mixing angles $(\theta_{13}, \theta_{12}, \theta_{23})$ and the $CP$-violating phase $\delta$

in terms of the parameters of our model, i.e., $a$, $b$, and $\lambda$, as follows:

$$\sin^2\theta_{13} = \frac{2\lambda^2}{16(a-b)^2 + 3\lambda^2},$$

$$\sin^2\theta_{12} = \frac{1}{3\cos^2\theta_{13}} = \frac{16(a-b)^2 + 3\lambda^2}{48(a-b)^2 + 3\lambda^2},$$

$$\sin^2\theta_{23} = \frac{1}{2} - \frac{4\lambda(a-b)}{16(a-b)^2 + \lambda^2},$$

$$\tan\delta = 0.$$
(12)

Notice that we managed to obtain a nonzero $\theta_{13}$.

The first-order corrections to the neutrino masses are obtained from $\langle \nu_i^{(0)} | M_\nu' | \nu_j^{(0)} \rangle = m_i^{(1)} \delta_{ij}$. Therefore, the masses up to first-order correction are as follows:

$$m_1^{(\lambda)} = 3b + m_0 - \frac{\lambda}{2}, \qquad m_2^{(\lambda)} = m_0 + \lambda,$$

$$m_3^{(\lambda)} = 2a + b + m_0 - \frac{\lambda}{2}.$$
(13)

From the complementary ansatz we obtain $a = \frac{-(4b+3m_0)}{2}$. In the limit $\lambda \to 0$, the mixing matrix $U$ must reduce to $U_{\text{TBM}}$. Therefore, we find $(a-b) < 0$, which has also been obtained in Ref. [21]. Using these conditions along with the usual convention of the FL model in which $m_0 > 0$, one can easily show that $-\frac{1}{2}m_0 \lesssim b \lesssim 0$, $-\frac{3}{2}m_0 \lesssim a \lesssim -\frac{1}{2}m_0$, $m_3 < 0$, $m_3 < m_1$, and $\Delta m_{31}^2 = m_3^2 - m_1^2 > 0$. Therefore, we have normal hierarchy at this stage. As we shall see, this property remains true in all stages of our model.

In the second stage, we let all of the coefficients $a$, $b$, $c$ of the original $M_{\text{FL}}$ be complex with the restriction $b = c$, which preserves the exact $\mu - \tau$ symmetry of the original $M_\nu^0$. This mass matrix can be displayed as

$$M_\nu^{c0} = \begin{pmatrix} 2(b+iB) + m_0 & -(b+iB) & -(b+iB) \\ -(b+iB) & (a+iA) + (b+iB) + m_0 & -(a+iA) \\ -(b+iB) & -(a+iA) & (a+iA) + (b+iB) + m_0 \end{pmatrix},$$
(14)

where the superscript "c" stands for "complex." The $M_\nu^{c0}$ is a non-Hermitian matrix, so, in general, we need two distinct unitary matrices $U$ and $V$ to diagonalize it, i.e., $M_{\text{diag}}^{c0} = U^\dagger M_\nu^{C0} V$. These matrices can be easily obtained by diagonalizing $M_\nu^{c0} M_\nu^{c0\dagger}$ and $M_\nu^{c0\dagger} M_\nu^{c0}$, separately. The matrices $U$ and $V$ are the conventional transformation matrices for the left-handed and right-handed neutrinos, respectively. An interesting point is that for $M_\nu^{c0}$, one can easily show that $U = V = U_{\text{TBM}}$ due to the $\mu - \tau$ symmetry. Therefore, the basis in which $M_\nu^{c0}$ is diagonal is the same as that of $M_\nu^0$, i.e., $\{\nu_i^{(0)}, i = 1, 2, 3\}$ given by Eq. (9).

The eigenvalues of $M_\nu^{c0}$ are as follows:

$$m_1^{c0} = (3b + m_0) + i(3B), \qquad m_2^{c0} = m_0,$$

$$m_3^{c0} = (2a + b + m_0) + i(2A + B).$$
(15)

From the complementary ansatz, i.e., $\text{Tr}(M_\nu^{c0}) = 0$, we obtain

$$a = \frac{-(4b + 3m_0)}{2}, \quad \text{and} \quad A = -2B.$$
(16)

It is worth mentioning that in this case because of the imaginary terms in the mass matrix $M_\nu^{c0}$, we have two phases that appear in the mass eigenvalues shown in Eq. (15), i.e., $\exp[i \arctan(\frac{3B}{3b+m_0})]$ and $\exp[i \arctan(\frac{2A+B}{2a+b+m_0})]$. For the Dirac neutrinos, these phases can be removed, and for the Majorana neutrinos, these phases remain as Majorana phases and contribute to $CP$ violation.

Notice that for the complex FL model with $\mu - \tau$ symmetry, $\theta_{13} = 0$. Moreover, the "zero sum" condition, i.e., $m_1^{c0} + m_2^{c0} + m_3^{c0} = 0$, is equivalent to the traceless condition, since $\mathrm{Tr}(M_\nu^{c0}) = \mathrm{Tr}(U^\dagger V M_{\mathrm{diag}}^{c0}) = \mathrm{Tr}(M_{\mathrm{diag}}^{c0}) = 0$. However, notice that, in general, $\mathrm{Tr}(M) \neq \mathrm{Tr}(M_{\mathrm{diag}})$ [10].

The experimental data have now definitely confirmed that $\Delta m_{21}^2 = m_2^2 - m_1^2 > 0$. Therefore, using Eq. (15), we can define the following new variable:

$$B' := |y|\sqrt{-b'\left(b' + \frac{2}{3}\right)}, \qquad (17)$$

where $-1 < y < 1$, and we have defined dimensionless parameters $b' \equiv \frac{b}{m_0}$, $B' \equiv \frac{B}{m_0}$, and $a' \equiv \frac{a}{m_0}$. The limit $|y| \to 1$ leads to $|m_2^{c0}| \to |m_1^{c0}|$. The conditions stated for $a$ and $b$ below Eq. (13), i.e., $-\frac{1}{2} \lesssim b' \lesssim 0$ and $-\frac{3}{2} \lesssim a' \lesssim -\frac{1}{2}$, are consistent with the definition of $B'$.

In the final stage, we combine two basic generalizations to the $M_{\mathrm{FL}}$, namely, the addition of imaginary components while preserving $\mu - \tau$ symmetry as a finite distortion, and the addition of a permutation matrix with a real coefficient to break the $\mu - \tau$ symmetry as a perturbation. Then we obtain $M_\nu^c$ as follows:

$$
\begin{aligned}
M_\nu^c &= M_\nu^{c0} + M_\nu' \\[4pt]
&= \begin{pmatrix} 2(b+iB)+m_0 & -(b+iB) & -(b+iB) \\ -(b+iB) & (a+iA)+(b+iB)+m_0 & -(a+iA) \\ -(b+iB) & -(a+iA) & (a+iA)+(b+iB)+m_0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\end{aligned} \qquad (18)
$$

where $M_\nu^{c0}$ is a complex symmetric non-Hermitian matrix given in Eq. (14) and $M_\nu' = \lambda S^{(132)}$, as given by Eq. (7), is a real matrix. The columns of the mixing matrix $U$ in Eq. (1) are eigenstates of $M_\nu^{c\dagger} M_\nu^c = M_\nu^{c0\dagger} M_\nu^{c0} + M_\nu^{c0\dagger} M_\nu' + M_\nu'^\dagger M_\nu^{c0}$, where we have dropped a term, which is $\mathcal{O}(\lambda^2)$. To proceed, we recall that $M_\nu^{c0\dagger} M_\nu^{c0}$ is Hermitian, and its eigenstates are the same as the columns that produce $U_{\mathrm{TBM}}$, and its eigenvalues are $|m_1^{(c0)}|^2$, $|m_2^{(c0)}|^2$, and $|m_3^{(c0)}|^2$. The basis in which $M_\nu^c$ is diagonal is given by

$$\nu_i^c = \nu_i^{(0)} + \sum_{i \neq j} \frac{\langle \nu_j^{(0)} | M_\nu^{c0\dagger} M_\nu' + M_\nu'^\dagger M_\nu^{c0} | \nu_i^{(0)} \rangle}{(m_i^{c0})^2 - (m_j^{c0})^2} \nu_j^{(0)}. \qquad (19)$$

A straightforward calculation yields

$$
\nu_1^c = \frac{2\sqrt{3}}{\sqrt{12 + \frac{(36B^2 + (3-2m_0)^2)\lambda^2}{(1+2b)^2(3-2m_0)^2}}} \begin{pmatrix} -\sqrt{\frac{2}{3}} \\[6pt] \frac{1}{\sqrt{6}} - \frac{(3-2m_0 - i6B)\lambda}{2\sqrt{6}(1+2b)(-3+2m_0)} \\[6pt] \frac{1}{\sqrt{6}} + \frac{(3-2m_0 - i6B)\lambda}{2\sqrt{6}(1+2b)(-3+2m_0)} \end{pmatrix}, \qquad \nu_2^c = \begin{pmatrix} \frac{1}{\sqrt{3}} \\[6pt] \frac{1}{\sqrt{3}} \\[6pt] \frac{1}{\sqrt{3}} \end{pmatrix},
$$

$$
\nu_3^c = \begin{pmatrix} \dfrac{-\sqrt{\frac{2}{3}}(3-2m_0 + i6B)\lambda}{(1+2b)(-3+2m_0)\sqrt{12 + \frac{(36B^2 + (3-2m_0)^2)\lambda^2}{(1+2b)^2(3-2m_0)^2}}} \\[14pt] \dfrac{-\frac{1}{\sqrt{2}} + \frac{(3-2m_0 + i6B)\lambda}{6\sqrt{2}(1+2b)(-3+2m_0)}}{\frac{1}{2}\sqrt{4 + \frac{(36B^2 + (3-2m_0)^2)\lambda^2}{(1+2b)^2(3-2m_0)^2}}} \\[14pt] \dfrac{\frac{1}{\sqrt{2}} + \frac{(3-2m_0 + i6B)\lambda}{6\sqrt{2}(1+2b)(-3+2m_0)}}{\frac{1}{2}\sqrt{4 + \frac{(36B^2 + (3-2m_0)^2)\lambda^2}{(1+2b)^2(3-2m_0)^2}}} \end{pmatrix}. \qquad (20)
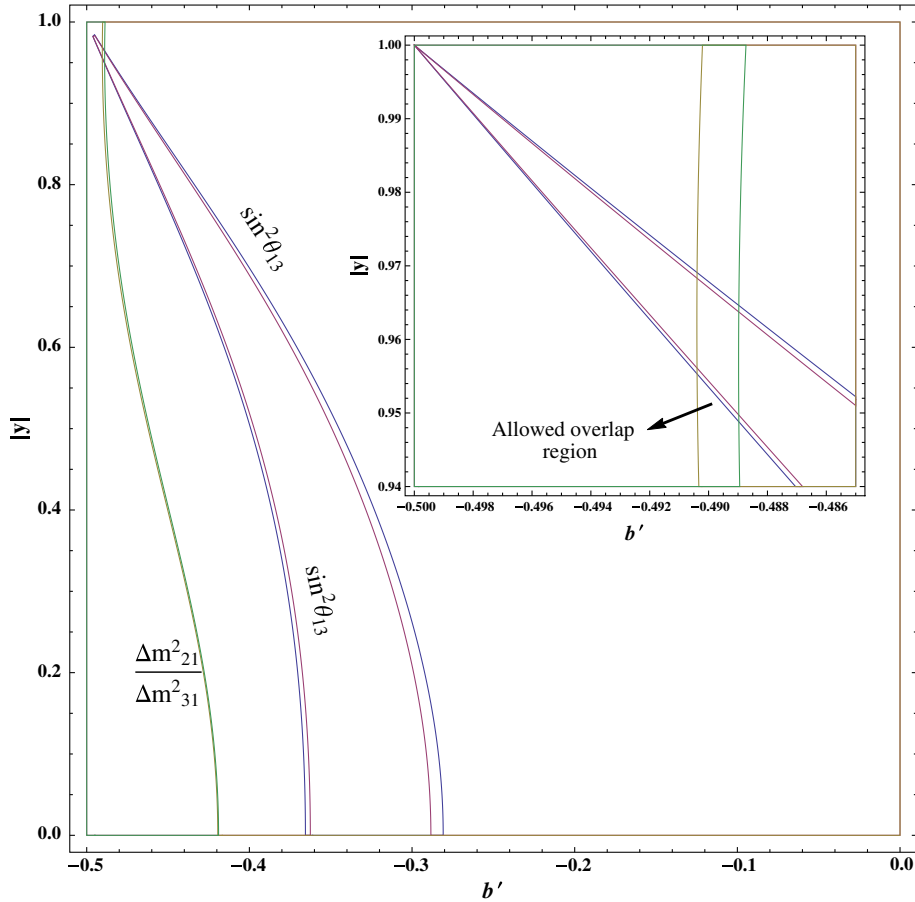$$

FIG. 1 (color online). In this figure, the whole region of the $|y|$-$b'$ plane which is allowed by our model is shown. The overlap region of the experimental values for $\frac{\Delta m_{21}^2}{\Delta m_{31}^2}$ and $\sin^2\theta_{13}$ with our model are two tiny regions close to the top-left corner of the rectangular. In the zoomed box, we have magnified the overlap regions. As mentioned in the text, only the lower overlap region is consistent with all of the experimental data.

These are also simply the columns of mixing matrix $U$ that contain $CP$-violating phases. Comparing Eq. (20) with Eq. (1), we immediately obtain all of the mixing angles $(\theta_{13}, \theta_{12}, \theta_{23})$ and the $CP$-violating phase in terms of $b'$, $B'$, and $\lambda' \equiv \frac{\lambda}{m_0}$ as follows:

$$\sin^2\theta_{13} = \frac{2(1+36B'^2)\lambda'^2}{36(1+2b')^2 + 3(1+36B'^2)\lambda'^2},$$

$$\sin^2\theta_{12} = 1 - \frac{24(1+2b')^2}{36(1+2b')^2 + (1+36B'^2)\lambda'^2},$$

$$\sin^2\theta_{23} = \frac{1}{2} + \frac{6(1+2b')\lambda'}{36(1+2b')^2 + (1+36B'^2)\lambda'^2},$$

$$\tan\delta = 6B'. \tag{21}$$

The first-order corrections to the neutrino masses are obtained from $m_i^{(1)}\delta_{ij} = \langle\nu_i^{(0)}|M_\nu^{c0\dagger}M_\nu' + M_\nu'^\dagger M_\nu^{c0}|\nu_j^{(0)}\rangle$. Therefore, by using Eq. (15) and the first-order of neutrino mass correction, we have

$$|m_1^c|^2 = |m_1^{c0}|^2 - (3b+m_0)\lambda,$$

$$|m_2^c|^2 = |m_2^{c0}|^2 + 2m_0\lambda,$$

$$|m_3^c|^2 = |m_3^{c0}|^2 - (2a+b+m_0)\lambda. \tag{22}$$

As in Eq. (15), $m_1^c$ and $m_3^c$ are complex. Extracting the real and imaginary components of the masses and introducing them as two phases in accordance with Eq. (1), we obtain $\rho = -\arctan\left(\frac{3B}{3b+m_0-\frac{\lambda}{2}}\right)$ and $\sigma = -\arctan\left(\frac{3B}{3b+m_0-\frac{\lambda}{2}}\right) + \arctan\left(\frac{2A+B}{2a+b+m_0-\frac{\lambda}{2}}\right)$. Since $M_\nu^{c0}$ is a symmetric matrix, it could also be used as a Majorana mass matrix, and these phases can be considered as the Majorana phases. In the Dirac case, these phases are transferred to the mass eigenstates. However, in the Majorana case, the phase factors remain and contribute to the $CP$ violation. The requirement $\det(U) = 1$ leads to $\rho + \sigma = 0$ [22]. This, in turn, implies the following:

TABLE II.    The available experimental data for neutrinos for the case of normal mass hierarchy and the predictions of our model. These predictions are obtained from our parameters as shown in Eq. (26). Specifically, we have used $b' \approx -(0.489$–$0.4904)$ and $y \approx (0.948932$–$0.956141)$.

| Parameter | Experimental data | Best experimental fit ($\pm 1\sigma$) | Predictions of our model |
|---|---|---|---|
| $\Delta m_{21}^2 (10^{-5}\ \mathrm{eV}^2)$ | (7.12–8.20) | (7.43–7.81) | (7.12–7.84) |
| $\Delta m_{31}^2 (10^{-3}\ \mathrm{eV}^2)$ | (2.26–2.77) | (2.46–2.61) | (2.23–2.80) |
| $\sin^2 \theta_{12}$ | 0.27–0.37 | 0.30–0.336 | 0.331–0.333 |
| $\sin^2 \theta_{13}$ | 0.016–0.037 | 0.0281–0.0275 | 0.0218–0.0275 |
| $\sin^2 \theta_{23}$ | 0.39–0.64 | 0.400–0.461 | 0.439–0.446 |
|  |  | 0.573–0.635 |  |
| $\delta$ | $\cdots$ | $\cdots$ | 59.21°–59.34° |
| $\sigma = -\rho$ | $\cdots$ | $\cdots$ | 59.44°–59.51° |
| $|J|$ | $\cdots$ | $\cdots$ | 0.029–0.033 |
|  | $\cdots$ | $\cdots$ | $m_1 \approx (0.031625$–$0.035687)$ eV, $\langle m_{\nu_e} \rangle \approx (0.033$–$0.037)$ eV |
| Masses | $\cdots$ | $\cdots$ | $m_2 \approx (0.032732$–$0.036769)$ eV, $\langle m_{\nu_\mu} \rangle \approx (0.043$–$0.048)$ eV |
|  | $\cdots$ | $\cdots$ | $m_3 \approx (0.056849$–$0.063819)$ eV, $\langle m_{\nu_\tau} \rangle \approx (0.046$–$0.051)$ eV |

$$\lambda' = 2(b' + \sqrt{(1 + 2b')^2 + 3B'^2}). \qquad (23)$$

Using Eq. (22), we can write the neutrino mass squared differences as follows:

$$\Delta m_{21}^2 = |m_2^{c0}|^2 - |m_1^{c0}|^2 + 3(b + m_0)\lambda,$$
$$\Delta m_{31}^2 = |m_3^{c0}|^2 - |m_1^{c0}|^2 - 2(a - b)\lambda,$$
$$\Delta m_{32}^2 = |m_3^{c0}|^2 - |m_2^{c0}|^2 - (2a + b + 3m_0)\lambda. \quad (24)$$

A rephasing-invariant measure of $CP$ violation in neutrino oscillation is the universal parameter $J$ [19] given in Eq. (6), and it has a form which is independent of the choice of the Dirac or Majorana neutrinos. Using Eq. (20), the expression for $J$ simplifies to

$$J = -\frac{4(1 + 2b')B'\lambda'}{12 + 48b'(1 + b') + (1 + 36B'^2)\lambda'^2}. \qquad (25)$$

Notice that we have reduced the number of free parameters of our models to just two, i.e., $b'$ and $y$, using all the self-consistency arguments and the complementary ansatz.

## III. COMPARISON WITH EXPERIMENTAL DATA

In this section we compare the experimental data with the results obtained from our model. We do this by mapping two of the constraints obtained from the experimental data onto our parameter space, $b'$ and $y$, as shown in Fig. 1. The two most restricting sets of experimental data come from the values of $\frac{\Delta m_{21}^2}{\Delta m_{31}^2}$ and $\sin^2 \theta_{13}$. The overlap of $\frac{\Delta m_{21}^2}{\Delta m_{31}^2}$ and $\sin^2 \theta_{13}$ and our model is restricted to two tiny regions close to the top-left corner of the parameter space.

First we check the consistency of the rest of the experimental data with the overlap region. We find that the upper overlap region is barely ruled out by the experimental value for $\sin^2 \theta_{23}$, and the lower overlap

region is consistent with all the rest. Having determined the unique overlap region, we cannot only determine all of the parameters of our model but also predict the masses of the neutrinos and the phases. The results for the parameters are as follows:

$$m_0 \approx (3.3\text{–}3.7)10^{-2}\ \mathrm{eV},$$
$$a \approx -(1.71\text{–}1.93)10^{-2}\ \mathrm{eV},$$
$$b \approx -(1.61\text{–}1.81)10^{-2}\ \mathrm{eV},$$
$$B \approx (0.92\text{–}1.04)10^{-2}\ \mathrm{eV},$$
$$\lambda \approx -(1.98\text{–}2.96)10^{-4}\ \mathrm{eV}. \qquad (26)$$

By comparing the values of these parameters, we conclude that the complexification part of the generalization of $M_{\mathrm{FL}}$ accomplished by adding imaginary components denoted by B could not be treated as a perturbation, while the second part of the generalization accomplished by adding $\lambda S^{123}$ can indeed be treated as a perturbation. This is precisely what we have done.

Having determined all of the parameters of our model, we can compare our results with the experimental data. In Table II, we state all of the relevant experimental data presented at $3\sigma$ [2], along with the predictions of our model. As shown in Table II, we have predictions for some physical quantities for which no experimental data exist. We should mention at this point that if we had chosen $S^{123}$ instead of $S^{132}$ to break the $\mu - \tau$ symmetry and generate a nonzero value for $\theta_{13}$, all of our final results and predictions would be unaltered except for $J \to -J$.

An important experimental result for the sum of the three light neutrino masses has just been reported by the Planck measurements of the cosmic microwave background [23], which is

$$\sum m_\nu < 0.23\ \mathrm{eV}(\mathrm{Planck} + \mathrm{WP} + \mathrm{highL} + \mathrm{BAO}). \quad (27)$$

This sum in our model is $\sum m_\nu \approx (0.12\text{–}0.14)$ eV, which is consistent with the above constraint. For the flavor eigenstates, only the expectation values of the masses can be calculated, and they are obtained by the following relation:

$$\langle m_{\nu_i} \rangle = \sum_{j=1}^{3} |U_{ij}|^2 |m_j|, \qquad (28)$$

where $i = e$, $\mu$, and $\tau$. Our predictions for these quantities are also shown in Table II. The Majorana neutrinos can violate lepton number, for example, in neutrinoless double beta decay $(0\nu\beta\beta)$ [24]. Such a process has not yet been observed, and an upper bound has been set for the relevant quantity, i.e., $\langle m_{\nu_{ee}} \rangle$. Results from the first phase of the KamLAND-Zen experiment sets the following constraint: $\langle m_{\nu_{ee}} \rangle < (0.12\text{–}0.25)$ eV at 90% C.L. [25]. Our prediction for this quantity is $\langle m_{\nu_{ee}} \rangle \approx (0.032\text{–}0.036)$ eV, which is consistent with the result of the KamLAND-Zen experiment. We also predict $\langle m_{\nu_e} \rangle \approx (0.033\text{–}0.037)$eV, $\langle m_{\nu_\mu} \rangle \approx (0.043\text{–}0.048)$ eV, and $\langle m_{\nu_\tau} \rangle \approx (0.046\text{–}0.051)$ eV.

## IV. CONCLUSIONS

In this paper, we propose a generalized Friedberg-Lee neutrino mass model in which *CP* violation is possible. Our generalization consists of two steps. First, we add imaginary components to the mass matrix, such that the resulting matrix is symmetric, magic with zero trace. The results are that $\mu - \tau$ symmetry is preserved and nonzero phases including $\delta$ are obtained. However, $\sin\theta_{13}$ remains zero. In the second stage, we break the $\mu - \tau$ symmetry by adding a real matrix, which is the matrix representation of the permutation group element $S^{(123)}$. This part is treated as a perturbation and a nonzero $\theta_{13}$ is generated. The combination of these two generalizations then produces *CP* violation. We use self-consistency arguments and the complementary ansatz to reduce our parameter space to two dimensions. Mapping two sets of the experimental data, i.e., the allowed ranges of $\sin^2\theta_{13}$ and $\frac{\Delta m_{21}^2}{\Delta m_{31}^2}$, onto our allowed region of the parameter space almost pinpoints the values of our parameters. We then check the consistency of all other experimental data with the allowed range. We then predict the mixing angles, the masses of neutrinos, the *CP*-violation parameters $\delta$, $\rho$, $\sigma$, and $J$. All of our findings are shown in Table II along with the latest experimental values. As is evident from Table II, all of our predictions are within the allowed ranges determined by the experiments, and there is an excellent agreement between these values. Our predictions for the neutrino masses and *CP*-violation parameters $\delta$ and $J$ are to tested by future experiments.

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