# Self-accelerating massive gravity: Bimetric determinant singularities

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The existence of two metrics in massive gravity theories, in principle, allows solutions where there are singularities in new scalar invariants jointly constructed from them. These configurations occur when the two metrics differ substantially from each other, as in black hole and cosmological solutions. The simplest class of such singularities includes determinant singularities. We investigate whether the dynamics of bimetric massive gravity—where the second metric is allowed to evolve jointly with the spacetime metric—can avoid these singularities. We show that it is still possible to specify nonsingular initial conditions that evolve to a determinant singularity. Determinant singularities are a feature of massive gravity of both fixed and dynamical metric type.

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## I. INTRODUCTION

Massive gravity is a theory with two metrics. In the simplest version, only the usual spacetime metric is dynamical; the second metric is taken to be static and typically flat [1–3]. When the spacetime metric evolves to a point where it deviates far from the second metric, massive gravity enters an interesting regime where singularities in scalar invariants built from the two metrics can arise. By allowing the second metric to evolve with its own dynamics in the so-called bigravity or bimetric massive gravity theory [4], it is possible that the character of these singularities can change.

These issues have been explored in detail for black hole solutions [5–10]. Indeed, the bimetric theory allows a different class of solutions from those of the flat metric theory [11], where the two metrics are simultaneously diagonalizable and horizons coincide. Being static solutions, it is however unclear as to whether dynamical systems evolve into these or other solutions.

A simpler case in which a singularity arises dynamically was studied for the fixed flat metric case in Ref. [12]. Here, the spacetime metric evolves from nonsingular initial conditions to a determinant singularity in unitary gauge where the flat metric is in standard Minkowski form. This implies the presence of a coordinate invariant singularity in the ratio of determinants of the two metrics. Although the theory is formally undefined at this point, one can smoothly join solutions on either side of the singularity with the help of vielbeins, or equivalently Stückel\-berg fields.

In this article, we study the impact of bimetric dynamics on determinant singularities. We begin in Sec. II with a brief review of the bimetric theory and continue in Sec. III with the construction of exact isotropic solutions. We address the determinant singularity in Sec. IV and discuss these results in Sec. V.

## II. BIMETRIC MASSIVE GRAVITY

The Boulware-Deser ghost-free bimetric massive gravity Lagrangian is [4]

$$\mathcal{L}_G = \frac{M_{\rm pl}^2}{2}\sqrt{-\det \mathbf{g}}\left[R - \frac{m^2}{4}\mathcal{U}(\boldsymbol{\gamma}) + \sqrt{\frac{\det \Sigma}{\det \mathbf{g}}}\frac{\mathcal{R}}{\epsilon}\right], \quad (1)$$

where $R$ is the Ricci scalar for the $\mathbf{g}$ metric to which matter is coupled and $\mathcal{R}$ is that of the second metric $\Sigma$. Here $M_{\rm pl} = (8\pi G)^{-1}$ is the reduced Planck mass and $\epsilon$ allows for the second metric to have a different Planck mass. The massive gravity potential term $\mathcal{U}$ is constructed from the square root matrix $\boldsymbol{\gamma}$,

$$(\mathbf{g}^{-1}\Sigma)^\mu{}_\nu \equiv (\boldsymbol{\gamma}^2)^\mu{}_\nu = \gamma^\mu{}_\alpha \gamma^\alpha{}_\nu \quad (2)$$

such that

$$\frac{\mathcal{U}}{4} = \sum_{k=0}^{4} \frac{\beta_k}{k!} F_k, \quad (3)$$

where [1,2,13]

$$\begin{aligned} F_0(\boldsymbol{\gamma}) &= 1,\\ F_1(\boldsymbol{\gamma}) &= [\boldsymbol{\gamma}],\\ F_2(\boldsymbol{\gamma}) &= [\boldsymbol{\gamma}]^2 - [\boldsymbol{\gamma}^2],\\ F_3(\boldsymbol{\gamma}) &= [\boldsymbol{\gamma}]^3 - 3[\boldsymbol{\gamma}][\boldsymbol{\gamma}^2] + 2[\boldsymbol{\gamma}^3],\\ F_4(\boldsymbol{\gamma}) &= [\boldsymbol{\gamma}]^4 - 6[\boldsymbol{\gamma}]^2[\boldsymbol{\gamma}^2] + 3[\boldsymbol{\gamma}^2]^2 + 8[\boldsymbol{\gamma}][\boldsymbol{\gamma}^3] - 6[\boldsymbol{\gamma}^4], \end{aligned} \quad (4)$$

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and [] denotes the trace of the enclosed matrix. To avoid confusion, we refrain from raising and lowering indices where possible, hence $\mathbf{g}^{-1}$ rather than $g^{\mu\nu}$.

Bimetric theories for which the Minkowski metric is a joint solution to both $\mathbf{g}$ and $\Sigma$ are restricted to

$$\begin{aligned}\beta_0 &= -12(1+2\alpha_3+2\alpha_4),\\ \beta_1 &= 6(1+3\alpha_3+4\alpha_4),\\ \beta_2 &= -2(1+6\alpha_3+12\alpha_4),\\ \beta_3 &= 6(\alpha_3+4\alpha_4),\\ \beta_4 &= -24\alpha_4,\end{aligned} \tag{5}$$

and thus we will consider theories parametrized by the graviton mass $m$, the ratio of squared Planck masses $\epsilon$ and the parameters $\{\alpha_3, \alpha_4\}$. Varying the action with respect to each of the metrics gives two Einstein equations,

$$\begin{aligned}R^\mu{}_\nu - \frac{1}{2}R\delta^\mu{}_\nu &= m^2 T^\mu{}_\nu + \frac{1}{M_{\rm pl}^2}T^{(\rm m)\mu}{}_\nu,\\ \mathcal{R}^\mu{}_\nu - \frac{1}{2}\mathcal{R}\delta^\mu{}_\nu &= \epsilon m^2 \mathcal{T}^\mu{}_\nu,\end{aligned} \tag{6}$$

where the potential term supplies an effective stress energy for both metrics, whereas the matter stress energy $T^{(m)}$ is coupled only to $\mathbf{g}$. The construction of $T^\mu{}_\nu$ out of $\boldsymbol{\gamma}$ is given in Eq. (7) of Ref. [14], and the stress tensor source for $\Sigma$ is given by [15]

$$\mathcal{T}^\mu{}_\nu = -\sqrt{\frac{\det \mathbf{g}}{\det \Sigma}}\left[T^\mu{}_\nu + \frac{\mathcal{U}}{8}\delta^\mu{}_\nu\right]. \tag{7}$$

Interestingly, this relation between the stress tensors involves the ratio of metric determinants, which can become singular. Nonetheless, as we shall see next, for self-accelerating solutions both stress tensors are simply constants given by the parameters of the theory.

## III. EXACT BI-ISOTROPIC SOLUTIONS

Exact self-accelerating solutions of bimetric massive gravity can be constructed when the two metrics are simultaneously isotropic,

$$\begin{aligned}g_{\mu\nu}dx^\mu dx^\nu &= -b^2(r,t)dt^2 + a^2(r,t)(dr^2 + r^2 d\Omega^2),\\ \sigma_{ab}dx^a dx^b &= -\beta^2(g,f)df^2 + \alpha^2(g,f)(dg^2 + g^2 d\Omega^2),\end{aligned} \tag{8}$$

where $\sigma_{ab}$ is the representation of $\Sigma$ in the so-called unitary gauge and $f(r,t)$ and $g(r,t)$, not to be confused with the determinant of $\mathbf{g}$, give the transformation between this coordinate system and the one used for $\mathbf{g}$. Note that they represent auxiliary Stückelberg fields

$$\phi^0 = f(t,r), \quad \phi^i = g(t,r)\frac{x^i}{r}, \tag{9}$$

such that the second metric in the same coordinate system as $\mathbf{g}$ is

$$\Sigma_{\mu\nu} = \partial_\mu\phi^a\partial_\nu\phi^b\sigma_{ab}. \tag{10}$$

In general, the number of gravitational degrees of freedom is 7—the five polarization states of one massive graviton, together with two polarizations of one massless graviton. In this representation, the extra polarization states of massive gravity are carried by the Stückelberg fields. While there are four Stückelberg fields, the ghost-free construction eliminates 1 degree of freedom and the assumption of bi-isotropy eliminates 2 more degrees of freedom [16], leaving the pair of Stückelberg fields as a single degree of freedom on top of the 2 usual tensor degrees of freedom of the two metrics.

Bi-isotropy allows us to express the potential as [14,17]

$$\frac{\mathcal{U}}{4} = P_0\left(\frac{\alpha g}{ar}\right) + \sqrt{X}P_1\left(\frac{\alpha g}{ar}\right) + WP_2\left(\frac{\alpha g}{ar}\right), \tag{11}$$

where the $P_n$ polynomials are

$$\begin{aligned}P_0(x) &= -12 - 2x(x-6) - 12(x-1)(x-2)\alpha_3\\ &\quad - 24(x-1)^2\alpha_4,\\ P_1(x) &= 2(3-2x) + 6(x-1)(x-3)\alpha_3 + 24(x-1)^2\alpha_4,\\ P_2(x) &= -2 + 12(x-1)\alpha_3 - 24(x-1)^2\alpha_4.\end{aligned} \tag{12}$$

Here

$$\begin{aligned}X(r,t) &= \left(\frac{\beta}{b}\dot{f} + \mu\frac{\alpha}{a}g'\right)^2 - \left(\frac{\alpha}{b}\dot{g} + \mu\frac{\beta}{a}f'\right)^2,\\ W(r,t) &= \mu\frac{\alpha\beta}{ab}(\dot{f}g' - \dot{g}f')\end{aligned} \tag{13}$$

are related to the $t-r$ block of $\boldsymbol{\gamma}$ as $\sqrt{X} = [\boldsymbol{\gamma}_2]$ and $W = \det\boldsymbol{\gamma}_2$, whereas

$$\boldsymbol{\gamma}^2 = \begin{pmatrix} \frac{\beta^2\dot{f}^2-\alpha^2\dot{g}^2}{b^2} & \frac{\beta^2\dot{f}f'-\alpha^2\dot{g}g'}{b^2} & 0 & 0\\ \frac{\alpha^2\dot{g}g'-\beta^2\dot{f}f'}{a^2} & \frac{-\beta^2 f'^2+\alpha^2 g'^2}{a^2} & 0 & 0\\ 0 & 0 & \frac{\alpha^2 g^2}{a^2 r^2} & 0\\ 0 & 0 & 0 & \frac{\alpha^2 g^2}{a^2 r^2}\end{pmatrix}.$$

The branch choice in the solution to the matrix square root of $\boldsymbol{\gamma}^2$ specifies $\mu = \pm 1$, which remains constant even if $W$ changes sign [12] (cf. [18]). Varying the action with respect to the Stückelberg fields gives the equations of motion for $f$ and $g$. For any bi-isotropic pair of metrics, these equations are exactly solved by $P_1(x_0) = 0$, yielding

$$x_0 = \frac{1 + 6\alpha_3 + 12\alpha_4 \pm \sqrt{1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4}}{3(\alpha_3 + 4\alpha_4)} \tag{14}$$

and

$$\frac{\alpha g}{ar} = x_0. \tag{15}$$

Note that as $\alpha_3 \to -4\alpha_4$ one branch of Eq. (14) remains finite. On both, this consistency condition (15) for self-accelerating solutions requires that the respective radial coordinates are algebraically related.

The stress-energy source for the $\mathbf{g}$ metric is then a cosmological constant [14,17],

$$T^\mu{}_\nu = -\frac{1}{2}P_0(x_0)\delta^\mu{}_\nu. \tag{16}$$

Since this relation holds for any isotropic metric, the interaction potential term acts as a cosmological constant for any isotropic distribution of matter, not just vacuum or homogeneous ones.

Moreover, since

$$\sqrt{\frac{\det \Sigma}{\det \mathbf{g}}} = \det \boldsymbol{\gamma} = x_0^2 W, \tag{17}$$

the stress tensor source to the second metric

$$\begin{aligned} \mathcal{T}^\mu{}_\nu &= -\sqrt{\frac{\det \mathbf{g}}{\det \Sigma}} W \frac{P_2(x_0)}{2} \\ &= -\frac{1}{x_0^2}\frac{P_2(x_0)}{2}\delta^\mu{}_\nu \end{aligned} \tag{18}$$

is also a constant [15,19,20]. Note that the stress tensor remains constant even through a determinant singularity where $\det \mathbf{g}/\det \Sigma \to \infty$. Given the identity

$$\frac{1}{2}P_0(x) + \frac{1}{2}P_2(x) + P_1(x) + (x-1)^2 = 0 \tag{19}$$

and $P_1(x_0) = 0$, if one metric has a positive cosmological constant, the other has a negative one [15] but both metrics may have a negative cosmological constant.

Since there is no matter source to $\Sigma$, the second Einstein equation (6) is then solved by a de Sitter metric in isotropic coordinates

$$\begin{aligned} \alpha(g) &= \frac{1}{1 + \lambda(g/x_0)^2/4}, \\ \beta(g) &= \frac{1 - \lambda(g/x_0)^2/4}{1 + \lambda(g/x_0)^2/4}, \end{aligned} \tag{20}$$

where

$$\lambda = \frac{\epsilon m^2}{6}P_2(x_0). \tag{21}$$

Of course, as $\epsilon \to 0$, so does $\lambda$, and the second metric takes the Minkowski form of the original massive gravity theory in unitary gauge [1,2].

Note that these results are independent of the solution for $f$ which relates unitary or isotropic $\Sigma$ time to isotropic $\mathbf{g}$ time. There are in fact many solutions for this relation that give the same stress tensor and metric structure individually. They are specified by solving the second equation of motion [12,16],

$$P_1'\left(x_0 + \frac{W}{x_0} - \sqrt{X}\right) = 0. \tag{22}$$

Aside from the special parameter choice of $P_1'(x_0) = 0$, where $12\alpha_4 = 1 + 3\alpha_3 + 9\alpha_3^2$, this equation governs the evolution of $f$. Importantly, it remains nonsingular as the determinant $W$ goes to zero. For the special parameter choice, more static solutions exist [21], but the initial value problem in $f$, $g$ is then ill posed [12].

Using Eqs. (13) and (15), we can see that Eq. (22)) is a nonlinear partial differential equation for $f$ whose solutions are specified by boundary conditions such as $f(0,t)$ [16]. Note that for a fixed $\lambda$, both $f$ and $g \propto x_0$, and so a solution for a single set of massive gravity parameters $\alpha_3$, $\alpha_4$ but arbitrary $\lambda$ can be scaled to any choice [16]. The determinant singularity we discuss next is related to a specific choice of $f(0,t)$ in the solution to Eq. (22).

## IV. DETERMINANT SINGULARITY

Given that metric determinants appear in the Einstein equations (6) through (7), it is interesting to examine whether the nature of determinant singularities in fixed metric massive gravity changes when the second metric becomes dynamical. One might expect that a singularity that impacts the equations of motion would be dynamically avoided. We shall see that none of them exhibits singular behavior at a determinant singularity.

In the fixed flat metric theory, we can easily construct solutions that evolve from nonsingular initial conditions to a determinant singularity. By a coordinate transformation, this singularity can be hidden from either metric individually but not both simultaneously. The simplest example is that of an open Friedmann-Robertson-Walker (FRW) universe in the $\mathbf{g}$ metric [22] with a negative cosmological constant term from the interaction potential [12]. Here, the singularity occurs when an initial expansion turns to contraction because of the presence of negative stress energy.

Now let us consider how the dynamics of the second metric alter this singular solution. The open FRW spacetime metric in isotropic coordinates is given by

$$ds^2 = -dt^2 + \left[\frac{a_F(t)}{1 + Kr^2/4}\right]^2 (dr^2 + r^2 d\Omega^2), \tag{23}$$

where the scale factor $a_F$ obeys the ordinary Friedmann equation with spatial curvature $K < 0$,

$$\left(\frac{\dot{a}_F}{a_F}\right)^2 + \frac{K}{a_F^2} = \frac{\rho^{(m)}}{3M_{\rm pl}^2} + \frac{m^2}{6}P_0(x_0). \tag{24}$$

By choosing $\alpha_3$ and $\alpha_4$ appropriately, we can make $P_0 < 0$, and hence the $\mathbf{g}$ metric evolves to a point where $\dot{a}_F = 0$.

We next solve Eq. (22)) for the relationship between the two time coordinates $f$ and $t$. Transforming the isotropic radial coordinate $r$ to the dimensionless angular diameter distance

$$y = \frac{\sqrt{-K}r}{1 + Kr^2/4}, \tag{25}$$

we obtain

$$\begin{aligned} y^2[1 - (\lambda/K)a_F^2]\dot{f}^2 - 2y(1+y^2)\frac{\dot{a}_F}{a_F}\dot{f}\frac{\partial f}{\partial y} \\ - \frac{1+y^2}{a_F^2}[K + y^2(\lambda a_F^2 - \dot{a}_F^2)]\left(\frac{\partial f}{\partial y}\right)^2 \\ = x_0^2 y^2 \frac{K - \lambda a_F^2 + \dot{a}_F^2}{K + \lambda a_F^2 y^2}. \end{aligned} \tag{26}$$

First note that as $\lambda \to 0$, we recover the solution for a fixed flat second metric [12],

$$\lim_{\lambda\to 0} f \equiv f_0 = x_0 a_F \sqrt{\frac{1+y^2}{-K}}, \tag{27}$$

with the boundary condition $f(0,t) \propto a_F(t)$. The determinant singularity appears since both $f_0$ and $g \propto a_F$ and thus $W = 0$ when $\dot{a}_F = 0$ by virtue of Eq. (13).

Now let us check what happens for $\lambda \neq 0$. Since $\lambda = 0$ and $f = f_0$ represent a determinant singularity, the simplest test for whether bimetric dynamics automatically avoids determinant singularities is to solve Eq. (22)) perturbatively for a finite $\lambda/K \ll 1$. Even in this limit, there are many solutions to this equation corresponding to different choices of the perturbed boundary condition $f(0,t)$. The simplest choice is

$$\begin{aligned} f(y,t) = f_0\Big[1 - \frac{1}{6}(-1+2y^2)(\lambda a_F^2/K) \\ + \frac{1}{40}(3 - 4y^2 + 8y^4)(\lambda a_F^2/K)^2 + \ldots\Big]. \end{aligned} \tag{28}$$

Since $\alpha(g)g \propto a_F$, Eq. (13) implies that this solution retains a determinant singularity at $\dot{a}_F = 0$. Other solutions can alter the time at which the determinant becomes singular as a function of radius. Nonetheless, a determinant singularity must appear in all solutions since $f$ remains perturbatively close to $f_0$. $W$ changes sign during the evolution through turnaround and must therefore pass through zero. Although we have assumed $\epsilon \ll 1$ for simplicity, since neither the stress source (18) nor any term in the Stückelberg field equations (22)) becomes singular for $W = 0$, we expect determinant singularities to be allowed even beyond this limit.

## V. DISCUSSION

While the bimetric theory of massive gravity allows the second metric to evolve in response to the first, it does not automatically resolve issues arising from the very existence of two metrics that may evolve to become very different from each other. We have explicitly shown here that it is still possible to construct solutions where a determinant singularity arises from the evolution of nonsingular initial conditions.

This singularity cannot be removed by a coordinate transformation, but the nonsingular equations of motion imply that solutions can be matched on either side of the singularity. The curvature of both metrics remains finite through the singularity, and its presence is hidden from observables in the matter sector. Hence the existence of determinant singularities is a peculiar but perhaps not pathological feature of both fixed and dynamical bimetric massive gravity theories.

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