String theory and emergent AdS geometry in higher spin field theories

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We analyze the Weyl invariance constraints on higher spin vertex operators in open superstring theory describing massless higher spin gauge field excitations in d-dimensional space-time. We show that these constraints lead to low-energy equations of motion for higher spin fields in AdS space, with the leading-order β function for the higher spin fields producing Fronsdal's operator in AdS_{d+1}, despite that the higher spin vertex operators are originally defined in flat background. The correspondence between the β function in string theory and AdS_{d+1} Fronsdal operators in space-time is found to be exact for $d = 4$, while for other space-time dimensions, it requires modifications of manifest expressions for the higher spin vertex operators. We argue that the correspondence considered in this paper is the leading order of a more general isomorphism between Vasiliev's equations and equations of motion of extended open string field theory, generalized to include the higher spin operators.

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I. INTRODUCTION

Both string theory and higher spin gauge theories have been immensely active and fascinating fields for many years. These two cutting-edge fields are in fact deeply connected to each other. At this point, our understanding of this connection is very far from being complete, still leaving many profound and conceptual questions unanswered. In the meantime the interplay between higher spins and strings appears of crucial relevance to fundamental questions such as underlying reasons behind the AdS/CFT conjecture holography principle, origin of space-time geometry, and others (for an incomplete list of references, see Refs. [1–19]).

There exists a number of examples linking string and higher spin dynamics. It is well known that massless higher spin modes appear in the tensionless limit of string theory as the massive vertex operators carrying spin $s \sim m^2$ (where m is the mass). This correspondence has been explored in a number of insightful papers (e.g., see Refs. [20,21] for some reviews). This approach has many obvious advantages (it is, in principle, straightforward to construct higher spin vertex operators in the massive sector both in bosonic and superstring theory), however it faces a number of difficulties as well, many of them related to the fact that, in general, the tensionless limit of string theory is the difficult one to describe. In particular, it seems hard to recover the full set of Stuckelberg symmetries when the vertex operators technically become massless, as $\alpha' \to \infty$. The vertex operators constructed in this approach can be used to describe metriclike Fronsdal fields (rather than the framelike gauge fields in Vasiliev's theory [22–27]); therefore fundamental space-time symmetries, related to higher-spin currents are not manifest in this approach. Moreover, while this approach allows to understand the structure of vertex operators in flat space-time, it is known to be difficult to extend it to the anti-de Sitter (AdS) case (e.g., Ref. [28]) since straightforward quantization of strings in the AdS background is not known beyond the semiclassical limit [29]. At the same time, AdS geometry appears to be a pertinent and crucial ingredient in constructing consistently interacting higher spin theories. Apart from crucial relevance to AdS/CFT correspondence [4,5], understanding higher spin dynamics in AdS space is of special interest to us since it is the AdS geometry that circumvents the limitations of Coleman and Mandula's theorem [30,31], leaving the possibility of constructing consistent higher spin interactions at all orders, following the Vasiliev equations [32].

In our previous papers, we showed that, apart from the tensionless limit, higher spin vertex operators describing emissions of higher spin gauge fields by an open string also can be constructed at an arbitrary tension value but at noncanonical ghost pictures [33,34]. These operators are related to global symmetries present in noncritical superstring theory, including hidden AdS isometries and their higher spin extensions, that can be classified using the formalism of ghost cohomologies [34]. The generators inducing these symmetries do not mix with standard Poincare generators, in this sense describing "symmetries of different world," coexisting with our world within "larger" superstring theory, which includes picture-dependent operators existing at nonzero ghost numbers, for which the ghost dependence cannot be removed by picture changing (these states and related

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symmetries, however, have no effect on standard string perturbation theory).

The hidden symmetry generators can be conveniently classified in terms of ghost cohomologies H_n. For the sake of completeness, we briefly remind the reader of the definition of H_n, for which the properties were analyzed in a number of previous works (e.g., see Refs. [33,34]) For each positive $n > 0$ H_n is defined as a set of physical [Becchi-Rouet-Stora-Tyutin (BRST) closed and BRST nontrivial] vertex operators existing at minimal positive picture n and above, annihilated by inverse picture-changing transformation at minimal picture n (with the picture transformations above the picture n generated by usual direct and inverse picture changings). For each negative $n \leq -3$, and below H_n is defined as a set of physical vertex operators existing at minimal positive picture n and below (i.e., $n-1$, $n-2$, etc.) annihilated by direct picture-changing transformation at minimal negative picture n (with the picture transformations above the picture n generated by usual direct and inverse picture changings). The cohomologies of positive and negative orders are isomorphic according to $H_n \sim H_{-n-2} (n \geq 1)$. Also, H_0 by definition consists of all picture-independent operators (existing at all picture representations) while H_{-1} and H_{-2} are empty. Thus, all conventional string theory operators (such as a photon, a graviton, or Poincare generators) are the elements of H_0 or $H_0 \otimes H_0$. The generators inducing AdS transvections in the larger string theory are the elements of $H_1 \sim H_{-3}$ while the massless closed string vertex operator of spin 2 bilinear in transvections, $H_1 \otimes H_1 \sim H_{-3} \otimes H_{-3}$, describes gravitational fluctuations around the AdS vacuum (see below). Massless open string operators of spin $s \geq 3$ describing framelike gauge fields in Vasiliev's theory are the elements of $H_{s-2} \sim H_{-s}$; their explicit construction will be given below. The fusion rules describing operator products between the vertices of different H_s have the same structure as the higher spin algebras in AdS space; in other words, the operator product expansion (OPE) algebra in the larger string theory constitutes one (very convenient) realization of AdS higher spin algebras.

Given the global symmetry generators, it is then straightforward to construct the appropriate vertex operators in open and closed string theories describing emissions of massless particles of various spins (with the open string physical vertex operators being objects linear in the symmetry generators and the closed string operators being bilinear in the symmetry generators). The purpose of this work is to analyze how AdS geometry emerges in the β-function equations for the massless higher spin modes in superstring theory. In the leading order, the Weyl invariance constraints on the higher spin vertex operators lead to low-energy equations of motion for massless higher spin fields defined by the Fronsdal operator in AdS space-time. The AdS structure of the Fronsdal operator (with the appropriate masslike terms) emerges, despite the fact that the higher spin vertex operators are initially defined in the flat background in superstring theory. The appearance of the AdS geometry is directly related to the ghost cohomology structure of the higher spin vertices and is detected through the off-shell analysis of the two-dimensional scale invariance of the vertex operators for higher spins. It is crucial that, in order to see the emergent AdS geometry, one must go off shell, e.g., to analyze the scale invariance of the operators in $2+\epsilon$ dimensions so that the trace $T_{z\bar{z}}$ of the stress-energy tensor generating two-dimensional Weyl transformations is no longer identically zero. Namely, it is the off-shell analysis of the operators at nonzero H_n that allows us to catch cosmological-type terms in low-energy equations of motion while the on-shell constraints on the operators (such as BRST conditions) do not detect them, only leading to standard Pauli–Fierz equations for massless higher spins in flat space. This is a strong hint that, from the string-theoretic point of view, the appropriate framework to analyze the higher spin interactions is the off-theory, i.e., string field theory, with the string field theory (SFT) equations of motion, $Q\Psi = \Psi \star \Psi$, related to Vasiliev's equations in unfolding formalism. It is important to stress, however, that Vasiliev's equations must be related to the enlarged, rather than ordinary SFT, with the string field Ψ extended to higher ghost cohomologies. Higher spin interactions in AdS should then be deduced from the off-shell string field theory computations involving higher spin vertex operators for Vasiliev's framelike fields on the world sheet boundary, with the appropriate insertions of $T_{z\bar{z}}$ in the bulk controlling the cosmological constant dependence. The rest of this paper is organized as follows. In the next section, we review the hidden AdS isometries in noncritical superstring theory and construction of the vertex operator in $H_1 \otimes H_1$ based on these isometries, describing gravitational fluctuations around the underlying AdS background, and appearance of the cosmological term in its beta function as a result of the off-shell scale-invariance condition. Next, we analyze the Weyl invariance of higher spin operators for massless spin s fields in Vasiliev's formalism, constructed in $H_{s-2} \sim H_{-s}$ In this work we mostly limit ourselves to the peculiar case of the higher spins that are polarized and propagating along the AdS boundary (which nevertheless is the limit relevant for holography) while also making some comments regarding the bulk-dependent case. The Weyl invariance, in the leading order, leads to the low-energy equations of motion for the higher spins, determined by Fronsdal's operator in AdS space. In the concluding section, we outline the higher order extension of this calculation (currently in progress) in order to establish the isomorphism between higher spin vertices in AdS space and off-shell amplitudes in the extended string field theory with $T_{z\bar{z}}$ insertions. The ultimate aim of this program is to explore the conjectured isomorphism between equations of extended SFT and Vasiliev equations that describe higher spin interactions in an unfolded approach.

II. HIDDEN ADS ISOMETRIES AND GRAVITONS IN ADS

The starting point is noncritical superstring theory in flat d-dimensional space-time, with the action given by

$$
\begin{aligned}
S_{\mathrm{RNS}} &= S_{\mathrm{matter}} + S_{bc} + S_{\beta\gamma} + S_{\mathrm{Liouville}} \\
S_{\mathrm{matter}} &= -\frac{1}{4\pi}\int d^2z(\partial X_m\bar{\partial}X^m + \psi_m\bar{\partial}\psi^m + \bar{\psi}_m\partial\bar{\psi}^m) \\
S_{bc} &= \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \bar{b}\partial\bar{c}) \\
S_{\beta\gamma} &= \frac{1}{2\pi}\int d^2z(\beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}) \\
S_{\mathrm{Liouville}} &= -\frac{1}{4\pi}\int d^2z(\partial\phi\bar{\partial}\phi + \bar{\partial}\lambda\lambda + \partial\bar{\lambda}\bar{\lambda} + \mu_0 e^{B\phi}(\lambda\bar{\lambda} + F)),
\end{aligned} \tag{1}
$$

where $X^m(m = 0, \ldots d-1)$ are the space-time coordinates and ψ^m are their world sheet superpartners, $b = e^{-\sigma}$, $c = e^{\sigma}$ are reparametrization ghosts, $\gamma = e^{\varphi-\chi} \equiv e^{\varphi}\eta$ and $\beta = e^{\chi-\varphi}\partial\chi \equiv \partial\xi e^{-\varphi}$ aresuperconformal ghosts, ϕ, λ, F are components of the super Liouville field, and the Liouville background charge is $Q = B + B^{-1} = \sqrt{\frac{9-d}{2}}$. The action (1) is obviously invariant under global Poincare symmetries generated by

$$
\begin{aligned}
P^m &= \oint dz\partial X^m(z) \\
P^{mn} &= \oint dz(\partial X^{[m}X^{n]} + \psi^m\psi^n).
\end{aligned} \tag{2}
$$

The standard physical vertex operators in superstring theory are the objects that are the elements of H_0, linear in Poincare generators P (for open strings) or bilinear (for closed strings), up to multiplication by the exponent field $\sim e^{ipX}$. For example, the photon operator is $V_m \sim \oint dz\Pi_m e^{ipX}$, and the graviton is $V_{mn} \sim \int d^2z\Pi_m\bar{\Pi}_n e^{ipX}(z,\bar{z})$, where $\Pi_m = \partial X_m + i(p\psi)\psi_m$ at picture 0, $\Pi_m = e^{-\varphi}\psi_m$ at picture -1, etc. The crucial point is that, apart from obvious Poincare symmetries of flat space-time, the action (1) also has nonlinear global symmetries realizing hidden AdS isometry algebra. Namely, as a warm-up example, it is straightforward to check the invariance of Eq. (1) under

$$
\begin{aligned}
\delta_\alpha X_m &= \alpha(\partial(e^{\varphi}\psi_m) + 2e^{\varphi}\partial\psi_m) \\
\delta_\alpha\psi_m &= -\alpha(e^{\varphi}\partial^2X_m + 2\partial(e^{\varphi}\partial X_m)) \\
\delta_\alpha\gamma &= \alpha e^{\varphi}(\psi_m\partial^2X^m - 2\partial\psi_m\partial X^m) \\
\delta_\alpha b &= \delta_\alpha c = \delta_\alpha\beta = 0
\end{aligned} \tag{3}
$$

as well as under the dual version of these transformations, given by replacing $\varphi \to -3\varphi$ in the transformation laws for X and ψ; vanishing variations of b, c, and γ ghosts; and the transformation of the β ghost given by

$$
\delta\beta = \partial\xi e^{-4\varphi}\sum_{k=0}^{2}P^{(k)}_{-3\varphi}\partial^{(2-k)}F_{\frac{5}{2}}, \tag{4}
$$

where α is a global bosonic infinitesimal parameter, the polynomials $P^{(n)}_f = e^{-f(z)}\frac{d^n}{dz^n}e^{f(z)}$ are the conformal weight n operators if $f(z)$ is linear in the ghost fields φ, χ, and σ and $F_{\frac{5}{2}}$ is a dimension-$\frac{5}{2}$ primary field: $F_{\frac{5}{2}} = \psi_m\partial^2X^m - 2\partial\psi_m\partial X^m$. The generators of Eqs. (3) and (4) are easily constructed to be given by

$$
T^{(+1)} = \oint dz e^{\varphi}F_{\frac{5}{2}}(z) \tag{5}
$$

for Eq. (3) and

$$
T^{(-3)} = \oint dz e^{-3\varphi}F_{\frac{5}{2}}(z) \tag{6}
$$

for Eq. (4). The operator (6) is BRST invariant and nontrivial while the operator (5) is not, as it does not commute with the supercurrent terms of Q_{brst}. To make Eq. (5) BRST invariant, one has to modify it with $b-c$ ghost-dependent terms according to the homotopy K transformation $T \to L = K\circ T$, where, in general, T is an operator given by an integral of dimension-one primary field V, not commuting with Q_{brst},

$$
T = \oint dzV(z),
$$

and the transformation is defined as

$$
\begin{aligned}
K\circ T = T &+ \frac{(-1)^N}{N!}\oint\frac{dz}{2i\pi}(z-w)^N :K\partial^N W:(z) \\
&+ \frac{1}{N!}\oint\frac{dz}{2i\pi}\partial_z^{N+1}[(z-w)^N K(z)]K\{Q_{\mathrm{brst}}, U\},
\end{aligned} \tag{7}
$$

where w is some arbitrary point on the world sheet, U and W are the operators defined according to

$$[Q_{\mathrm{brst}}, V(z)] = \partial U(z) + W(z), \tag{8}$$

$$K = ce^{2\chi - 2\varphi} \tag{9}$$

is the homotopy operator satisfying

$$\{Q_{\mathrm{brst}}, K\} = 1,$$

and N is the leading order of the operator product

$$K(z_1)W(z_2) \sim (z_1 - z_2)^N Y(z_2) + O((z_1 - z_2)^{N+1}). \tag{10}$$

In the case of the symmetry generator (5), we have $N = 2$. It is straightforward to check that, with the definition Eq. (7), the operator $K\circ T$ is BRST invariant. The homotopy transformation (7) is straightforward to generalize for closed string operators, multiplying it by antiholomorphic transformation, so that the invariant closed string operator is $K\bar{K}\circ \int d^2 z\ldots$. The important property of the K transformation is the homomorphism relation preserving the OPE structure constants so that, up to BRST-exact terms, the OPE structure constants of BRST-invariant operators $K\circ T_1$ and $K\circ T_2$ can be read off the OPE of the noninvariant operators T_1 and T_2, with the appropriate K transform (7) of the right-hand side (see Ref. [34] for the proof). Given Eqs. (6)–(10), the dual symmetry generators $T^{(-3)}$ and $K\circ T^{(+1)}$ belong to isomorphic cohomologies H_{-3} and H_1 note that both $T^{(+1)}$ and $K\circ T^{(+1)}$ generate global symmetries in space-time; however, while the noninvariant operator $T^{(+1)}$ generates the symmetry transformations (3) that do not involve the ghost fields b, c, and β, the invariant operator $K\circ T^{(+1)}$ generates the extended (complete) version of Eq. (3), which involves all the ghost fields. Given the definitions (5), (7), the extended space-time transformations are straightforward to construct; we will not present their manifest form here for the sake of brevity. It shall be sufficient to note that, due to the homomorphism property [34], the K-transformed symmetry generators satisfy the same symmetry algebra relations as the abbreviated noninvariant operators, such as Eq. (12). As a simple analogy of the above, one can think of the noninvariant symmetry generators $\sim \oint dz\psi^m\psi^n$ inducing the truncated global symmetries of (1) satisfying correct commutation relations for Lorentz rotations in space-time. However, the abbreviated noninvariant generators only act on ψ and not on X. To make them invariant, one has to add extra terms proportional to $\int \partial X^{[m}X^{n]}$ so that the invariant rotation generator satisfies the same symmetry algebra but now acts on both X and ψ. We now turn to the question of symmetry algebras satisfied by the generators of the same type as Eqs. (5)–(7). First of all, it is straightforward to check that, up to BRST-exact terms, these operators all commute with Poincare generators (2). The geometrical meaning of the hidden symmetries (3), (5)–(7) becomes clearer if one considers the vector analogs of these transformations given by

$$\begin{aligned}
L^m &= K\circ \oint dz e^{\varphi}(\lambda\partial^2 X^m - 2\partial\lambda\partial X^m)\\
L^+ &= K\circ \oint dz e^{\varphi}(\lambda\partial^2\phi - 2\partial\lambda\partial\phi)\\
L^{mn} &= K\circ \oint dz\psi^m\psi^n\\
L^{+m} &= K\circ \oint dz\lambda\psi^m.
\end{aligned} \tag{11}$$

Then, with some effort involving tedious picture-changing transformations [34], one can show that the operators (11) realize the AdS_{d+1} isometry algebra:

$$\begin{aligned}
[L^m, L^n] &= -L^{mn}\\
[L^+, L^m] &= -L^{+m}\\
[L^m, L^{np}] &= -\eta^{mn}L^p + \eta^{mp}L^n\\
[L^+, L^{mn}] &= 0\\
[L^{+m}, L^{np}] &= -\eta^{mn}L^{+p} + \eta^{mp}L^{+n}\\
[L^{mn}, L^{pq}] &= \eta^{mp}L^{nq} + \cdots
\end{aligned} \tag{12}$$

It is worth mentioning that the minus signs on the RHS of the first two commutators in Eqs. (11) and (12) appear in a rather nontrivial way, through a process of cumbersome OPE calculations [34]. The radial coordinate of the underlying AdS_{d+1} space related to the isometry algebra (12) naturally coincides with the Liouville direction [35–37]. The operators of AdS transvections L^m and L^+ are the elements of H_1 and can also transformed into isomorphic H_{-3} cohomology by replacing $K\circ \oint dz \to \oint dz$, $e^{\varphi} \to e^{-3\varphi}$. The next step is to construct physical vertex operators based on isometry generators (3), (5)–(7). Obviously the object of particular interest is the spin-2 operator in the closed string sector, bilinear in transvection generators (11), (12), with appropriate momentum-dependent extra terms to ensure BRST invariance. As above, for our purposes, we shall limit ourselves to excitations polarized and propagating along the AdS boundary (which in our case is simply orthogonal to the Liouville direction). The construction in $H_{-3}\otimes H_{-3}$ cohomology leads to the following expression for the operator:

$$\begin{aligned} V_{s=2} &= G_{mn}(p)\int d^2z e^{-3\varphi-3\bar{\varphi}} R^m \bar{R}^n e^{ipX}(z,\bar{z}) \\ R^m &= \bar{\lambda}\partial^2 X^m - 2\partial\lambda\partial X^m \\ &\quad + ip^m\left(\frac{1}{2}\partial^2\lambda + \frac{1}{q}\partial\phi\partial\lambda - \frac{1}{2}\lambda(\partial\phi)^2 + (1+3q^2)\lambda\left(3\partial\psi_p\psi^p - \frac{1}{2q}\partial^2\phi\right)\right) \\ m &= 0,\ldots,d-1, \end{aligned} \tag{13}$$

where G_{mn} is symmetric. The operator on $H_1 \otimes H_1$ is constructed likewise by replacing $\int d^2z \to K\bar{K}\circ\int d^2z$ and $-3\varphi \to \varphi$, $-3\bar{\varphi} \to \bar{\varphi}$. Provided that $k^2 = 0$, 0 it is straightforward to check its BRST invariance with respect to the flat space BRST operator

$$Q_{\mathrm{brst}} = \oint dz\left(cT - bc\partial c - \frac{1}{2}\gamma\psi_m\partial X^m - \frac{1}{4}b\gamma^2\right) \tag{14}$$

as well as the linearized diffeomorphism invariance since the transformation $G^{mn}(p) \to G^{mn}(p) + p^{(m}\epsilon^{n)}$ shifts holomorphic and antiholomorphic factors of $V_{s=2}$ by terms BRST exact in small Hilbert space [34]. To identify this symmetric massless spin-2 state with gravitational fluctuations, however, one needs to analyze the low-energy equations of motion for G_{mn}, for which the leading order is given by the Weyl constraints. We will address this question in the next section.

III. FLAT VS ADS GRAVITONS: WEYL INVARIANCE AND COHOMOLOGY STRUCTURES

As an instructive example, in this section we shall consider in detail the scale-invariance constraints on the operator (13) of $H_{-3} \otimes H_{-3}$ and compare them to those for the ordinary graviton in superstring theory (1). To see the difference, let us first recall the most elementary example–the graviton in bosonic string theory given by

$$V = G_{mn}\int d^2z \partial X^m \bar{\partial} X^n e^{ipX}. \tag{15}$$

The condition $[Q, V] = 0$ leads to constraints, $p^2 G_{mn}(p) = p^m G_{mn}(p) = 0$, related to linearized Ricci tensor contributions to the graviton's β function. The complete linearized contribution to the graviton's β function, however, is given by $\beta_{mn} = R_{mn}^{\mathrm{linearized}} + 2\partial_m\partial_n D$ (where D is the space-time dilaton) with the last term particularly accounting for the $\sim e^{-2D}$ factor in the low-energy effective action. This term in fact is *not* produced by any of the on-shell (BRST) constraints on the graviton vertex operator; to recover it, one has to analyze the *off-shell* constraints related to the Weyl invariance. Namely, the generator of the Weyl transformations is given by the $T_{z\bar{z}}$ component of the stress energy, which is identically zero on shell in $d = 2$ but nonzero in $d = 2 + \epsilon$. The leading-order contribution to the *β-function* of the closed string vertex operator V is determined by the coefficient in front of $\sim\frac{1}{|z-w|^2}$ in the midpoint OPE of $T_{z\bar{z}}(z,\bar{z})$ and $V(w,\bar{w})$ leading to logarithmic divergence in the integral $\sim\int d^2z \int d^2w T_{z\bar{z}}(z,\bar{z})V(w,\bar{w})$. In bosonic string theory, one has

$$T_{z\bar{z}} \sim -\partial X_m \bar{\partial} X^m + \partial\sigma\bar{\partial}\sigma + \partial\bar{\partial}(\ldots), \tag{16}$$

skipping the full-derivative part proportional to the two-dimensional Laplacian related to background charge, as it leads to contact terms in the OPE with V, not contributing to its β function. Using Eqs. (15) and (16), one easily calculates

$$\begin{aligned} &\int d^2z \int d^2w T_{z\bar{z}}(z,\bar{z})V(w,\bar{w}) \\ &\sim G_{mn}(p)\int d^2z\int d^2w \frac{1}{|z-w|^2} e^{ipX}\left(\frac{z+w}{2},\frac{\bar{z}+\bar{w}}{2}\right)\{p^2\partial X^m\bar{\partial}X^n - \frac{1}{2}\left(p^m p_s \partial X^s \bar{\partial} X^n + p^n p_s \partial X^m \bar{\partial} X^s\right) \\ &\quad + \frac{1}{4}\eta^{mn} p_s p_t \partial X^s \bar{\partial} X^t\}\left(\frac{z+w}{2},\frac{\bar{z}+\bar{w}}{2}\right) \\ &\sim \ln\Lambda[p^2 G_{mn}(p) - \frac{1}{2}\left(p^s p_m G_{ns}(p) + p^s p_n G_{ms}(p)\right) + 2p_m p_n D(p)]\int d^2\zeta \partial X^m \bar{\partial} X^n e^{ipX}(\zeta,\bar{\zeta}), \end{aligned} \tag{17}$$

where we introduced $\ln \Lambda = \int \frac{d^2\xi}{|\xi|^2}$, $\zeta = z + w$ and identified the dilaton with the trace of the space-time metric: $D(p) \sim \eta^{st} G_{st}(p)$.

For conventional reasons and in order not to introduce too many letters in this paper, we adopt the same notation, Λ, for both the world sheet cutoff and the cosmological constant in space-time. However, we hope that the distinction between those will be very clear to a reader from the context; in particular, in this paper the cutoff shall always appear in terms of logs, while all expressions in the cosmological constants are either linear or polynomial.

The coefficient in front of the integral thus determines the leading-order contribution to the graviton's β function. The first three terms in this coefficient simply give linearized Ricci tensor while the last one proportional to the trace of the space-time metric determines the string coupling dependence. All these terms contain two space-time derivatives, and obviously no cosmological-type contributions appear. The calculation analogous to Eq. (17) is of course similar in superstring theory, producing the similar answer. However, in comparison with the bosonic string, the superstring case also contains some instructive subtlety, which is useful to observe for future calculations. That is, consider the graviton operator in superstring theory at the canonical $(-1,-1)$ picture:

$$V^{(-1,-1)} = G_{mn}(p) \int d^2 z e^{-\varphi-\bar{\varphi}} \psi^m \bar{\psi}^n e^{ipX}(z,\bar{z}). \quad (18)$$

The generator of Weyl transformations in superstring theory is

$$\begin{aligned} T_{z\bar{z}} &= T_X + T_\psi + T_{b-c} + T_{\beta-\gamma} + T_{\text{Liouv}} \\ &= \frac{1}{2}\partial X_m \bar{\partial} X^m - \frac{1}{2}(\bar{\partial}\psi_m \psi_m + \partial\bar{\psi}_m \bar{\psi}^m) \\ &\quad + \frac{1}{2}\partial\sigma\bar{\partial}\bar{\sigma} - \frac{1}{2}\partial\varphi\bar{\partial}\bar{\phi} + \frac{1}{2}\partial\chi\bar{\partial}\bar{\chi} + \partial\bar{\partial}(\cdots). \end{aligned} \quad (19)$$

The contribution of T_X to the scale transformation of Eq. (18) is again easily computed to give $\sim p^2 G_{mn}(p) \ln \Lambda V^{(-1,-1)}$, i.e., the gauge fixed linearized Ricci tensor (with the gauge condition $p^m G_{mn} = 0$ imposed by invariance under transformations of $V^{(-1,-1)}$ by worldsheet superpartners of T_X, namely, $G_{+\bar{z}}$ and G_{-z}). To compute the contribution from T_ψ, it is convenient to bosonize ψ according to

$$\begin{aligned} \psi_1 \pm i\psi_2 &= e^{\pm i\phi_1} \\ &\cdots \\ \psi_{d-1} \pm i\psi_d &= e^{\pm i\phi_{\frac{d}{2}}} \end{aligned} \quad (20)$$

(for simplicity we can assume the number d of dimensions even without loss of generality) Then the stress-energy tensor for ψ is

$$T = -\frac{1}{2}(\bar{\partial}\psi_m \psi^m + \partial\bar{\psi}_m \bar{\psi}^m) = \sum_{i=1}^{\frac{d}{2}} \partial\phi_i \bar{\partial}\bar{\phi}^i. \quad (21)$$

Writing $\psi_1 = \frac{1}{2}(e^{i\phi_1} - e^{-i\phi_1})$, it is easy to compute the contribution of T_ψ to the β function:

$$\begin{aligned} &\int d^2 z T_\psi(z,\bar{z}) G_{mn}(p) \int d^2 w e^{-\varphi-\bar{\varphi}} \psi^m \bar{\psi}^n e^{ipX}(w,\bar{w}) \\ &= \frac{1}{2} \ln \Lambda G_{mn}(p) \int d^2\zeta e^{-\varphi-\bar{\varphi}} \psi^m \bar{\psi}^n e^{ipX}(\zeta,\bar{\zeta}) \\ &\equiv \frac{1}{2} \ln \Lambda V^{(-1,-1)}. \end{aligned} \quad (22)$$

Note the scale transformation by T_ψ contributes the term proportional to $\sim \frac{1}{2} G_{mn}$ with no derivatives, i.e., a "cosmological-type" term. The cosmological term is of course absent in the overall graviton's β function as the contribution (22) is precisely cancelled by the scale transformation of the ghost part of $V^{(-1,-1)}$ by $T_{\beta\gamma} = -\frac{1}{2}|\partial\varphi|^2 + \partial\bar{\partial}(\ldots)$,

$$\begin{aligned} &\int d^2 z T_{\beta\gamma}(z,\bar{z}) G_{mn}(p) \int d^2 w e^{-\varphi-\bar{\varphi}} \psi^m \bar{\psi}^n e^{ipX}(w,\bar{w}) \\ &= -\frac{1}{2} \ln \Lambda V^{(-1,-1)}, \end{aligned} \quad (23)$$

with the minus sign related to that of the φ-ghost field in the trace of the stress-energy tensor. So the absence of the cosmological term in the graviton's β function in superstring theory is in fact the result of the smart cancellation between the Weyl transformations of the matter and the ghost factors of the graviton operator at $(-1,-1)$ canonical picture (despite that the final answer—the absence of the overall cosmological term— may seem obvious) The same result of course applies to the graviton operator (18) transformed to any other ghost picture since it is straightforward to check that both Γ and Γ^{-1} are Weyl invariant, up to BRST-exact terms. The absence of cosmological (or mass-like) terms in the Weyl transformation laws is actually typical for any massless operators of H_0 or $H_0 \otimes H_0$; at nonzero pictures it is the consequence of the cancellation of Weyl transformations for the matter and the ghosts, as was demonstrated above. This observation is of importance since, as it will be shown below, this matter-ghost cancellation does not occur for operators of nonzero H_n's, in particular, for the spin-2 operator (13) in closed string theory and for massless operators for higher spin fields of Vasiliev type in the open string sector. Namely, we will show that for the operator (13) the scale-invariance constraints lead to the cosmological term, while for massless higher spin fields, the similar constraints lead to the emergence of AdS geometry in Fronsdal's operator in the low-energy limit. We start by analyzing the scale transformation of the spin operator (13) by T_X. The canonical picture for the operator (13) is $(-3,-3)$. To deduce the transformation law for the operator (13), it is sufficient to consider the momentum-independent part $\sim R_0^m \bar{R}_0^n$ of the matter factor $\sim R^m \bar{R}^n$ in (13),

$$R^m = R_0^m + ik^m(\ldots) \qquad R_0^m = \lambda\partial^2 X^m - 2\partial\lambda\partial X^m, \tag{24}$$

and similarly for $\bar{R}^m$. Then the straightforward application of T_X to

$$G_{mn}(p)\int d^2 w e^{-3\varphi-3\bar{\varphi}} R_0^m \bar{R}_0^n e^{ipX}(w,\bar{w})$$

gives

$$\begin{aligned}
&\int d^2 z T_X(z,\bar{z}) G_{mn}(p)\int d^2 w e^{-3\varphi-3\bar{\varphi}} R_0^m \bar{R}_0^n e^{ipX}(w,\bar{w})\\
&= \ln\Lambda \times G_{mn}(p)\int d^2\zeta\Big\{-\frac{1}{2}p^2 e^{-3\varphi-3\bar{\varphi}} R_0^m \bar{R}_0^n e^{ipX}(\zeta,\bar{\zeta})] - \frac{i}{8}p^m\partial^2(e^{-3\varphi}\lambda e^{ipX}(\zeta))e^{-3\bar{\varphi}}\bar{R}_0^n e^{ipX}(\bar{\zeta})\\
&\quad + \frac{i}{2}p^m\partial\partial(e^{-3\varphi}\partial\lambda e^{ipX}(\zeta))e^{-3\bar{\varphi}}\bar{R}_0^n e^{ipX}(\bar{\zeta}) + (c.c.; m\leftrightarrow n)]\Big\}\\
&= \ln\Lambda G_{mn}(p)\int d^2\zeta\left(-\frac{1}{2}p^2\delta_q^n + \frac{1}{2}p^m p_q\right)e^{-3\varphi-3\bar{\varphi}} R_0^m \bar{R}_0^q e^{ipX} + \cdots,
\end{aligned} \tag{25}$$

where we dropped BRST-exact terms and only kept terms contributing to the G_{mn}'s β function, skipping those relevant to β functions of the space-time fields other than G_{mn}. In addition, for simplicity we skipped the dilaton-type contributions involving the trace of G_{mn}; it is, however, straightforward to generalize the computation to include the dilaton, accounting for the standard factor of e^{-2D} in the effective action. Comparing the transformation laws (17) and (25), one easily concludes that the contribution of T_X transformation to the G_{mn} β function results in the linearized Ricci tensor $R_{mn}^{\mathrm{linearized}}$. Next, consider the contributions from $T_\lambda = -\frac{1}{2}(\partial\bar{\lambda}\bar{\lambda} + \bar{\partial}\lambda\lambda)$ and $T_{\beta\gamma}$ to β_{mn}. The analysis is similar to the one for the ordinary graviton operator (18)–(23); however, the crucial difference is that this time there is no cancellation between transformations due to the world sheet matter (Liouville) fermion and the $\beta-\gamma$ ghost, observed above. As previously stated, the transformation of Eq. (25) by T_λ contributes

$$\begin{aligned}
&G_{mn}(p)\int d^2 z T_\lambda(z,\bar{z})\int d^2 w e^{-3\varphi-3\bar{\varphi}} R_0^m \bar{R}_0^n e^{ipX}(w,\bar{w})\\
&\quad = \frac{1}{2}\ln\Lambda G_{mn}(p)\int d^2\zeta e^{-3\varphi-3\bar{\varphi}} R_0^m \bar{R}_0^n e^{ipX}(w,\bar{w}).
\end{aligned} \tag{26}$$

On the other hand, the transformation by $T_{\beta-\gamma}$ produces

$$\begin{aligned}
&G_{mn}(p)\int d^2 z T_{\beta-\gamma}(z,\bar{z})\int d^2 w e^{-3\varphi-3\bar{\varphi}} R_0^m \bar{R}_0^n e^{ipX}(w,\bar{w})\\
&\quad = -\frac{9}{2}\ln\Lambda G_{mn}(p)\int d^2\zeta e^{-3\varphi-3\bar{\varphi}} R_0^m \bar{R}_0^n e^{ipX}(w,\bar{w}),
\end{aligned} \tag{27}$$

where we used the OPE $|\partial\varphi|^2(z,\bar{z})e^{-3\varphi-3\bar{\varphi}}(w,\bar{w}) \sim \frac{9}{|z-w|^2}e^{-3\varphi-3\bar{\varphi}}(w,\bar{w})$. Unlike the case of the ordinary graviton, the cosmological-type contributions from the scale transformations of the ghost and the matter part of the operator (13) no longer cancel each other. As a result, the overall cosmological term $\sim(\frac{9}{2}-\frac{1}{2})G_{mn}$ appears in the β function of Eq. (28) for which the leading order is now given by

$$\beta_{mn} = R_{mn}^{\mathrm{linearized}} - 8G_{mn} \tag{28}$$

(with the extra factor of 2 related to the normalization of the Ricci tensor). Note that Eq. (28) is written in the units implying $\alpha' = 1$ (absorbed in the momentum normalization in vertex operators). Recovering the α' dependence, it is easy to see that the cosmological constant appearing in Eq. (28) is related to the AdS radius of the order of the string length, i.e., the scale where one expects higher spin symmetry enhancement to occur. In this sense the appearance of higher spin vertex operators, discussed in the next section, is natural since their zero momentum limits should correspond to generators of higher spin symmetry algebra. The emergence of the cosmological term in Eq. (28) is thus closely related to the ghost cohomology structure of the operator (13), i.e., to the fact that the canonical picture for this operator is $(-3,-3)$ while the standard $(-1,-1)$ picture representation of the "ordinary" graviton does not exist for Eq. (13).

Collecting Eqs. (25)–(27), this altogether allows us to identify the space-time massless spin-2 G_{mn} field emitted by $H_{-3}\otimes H_{-3}$ with the gravitational fluctuations around the AdS vacuum. In fact, this is not a surprise since the operator (13) was originally built as a bilinear of the generators (11), (12) realizing transvections in AdS. The next step is to generalize the above arguments to the vertex operators for the massless higher spin fields (with $s \geq 3$), which are also the elements of nonzero cohomologies $H_{s-2} \sim H_{-s}$. In analogy with the mechanism generating the cosmological term in Eq. (28), we expect that the scale-invariance analysis of these operators shall also lead to the appearance of the masslike terms in their β functions (although the operators by themselves are massless). We shall attempt to show that the "masslike" terms are in fact related to the AdS geometry couplings of the higher spin fields, adding up to appropriate AdS Fronsdal operators in their low-energy equations of motion in the leading order.

IV. HIGHER SPIN OPERATORS: WEYL INVARIANCE AND β FUNCTIONS

In this section we extend the analysis of the previous sections to vertex operators describing massless higher spin excitations in open superstring theory. The space-time fields emitted by these operators correspond to symmetric higher spin gauge fields in Vasiliev's framelike formalism. The main result of this section is that the leading order of the β function for the higher spin operators gives the low-energy equations of motion determined by Fronsdal operator in the AdS space, despite the fact that the operators are initially defined around the flat background. As in the case of the AdS graviton considered in the previous section, the information about the AdS geometry is encrypted in the ghost cohomology structure of the operators. In the framelike formalism [22–27], [38–41], a symmetric higher spin gauge field of spin s is described by collection of two-row fields $\Omega^{s-1|t} \equiv \Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_t}(x)$ with $0 \leq t \leq s-1$ and the rows of lengths $s-1$ and t. The only truly dynamical field of those is $\Omega^{s-1|0}$ while the fields with $t \neq 0$, called the extra fields, are related to the dynamical one through generalized zero torsion constraints,

$$\Omega^{s-1|t} \sim D^{(t)}\Omega^{s-1|0}, \tag{29}$$

where $D^{(t)}$ is a certain order t linear differential operator preserving the symmetries of the appropriate Yang tableaux. There are altogether $s-1$ constraints for the field of spin s. As for the dynamical $\Omega^{s-1|0}$ field (symmetric in all the a indices), it splits into two diagrams with respect to the manifold m index. Assuming the appropriate pullbacks, the one-row symmetric diagram describes the dynamics of the *metriclike* symmetric Fronsdal field of spin s while the two-row component of $\Omega^{s-1|0}$ can be removed by appropriate gauge transformation. In the language of string theory, the higher spin s operators are the elements of $H_{s-2} \sim H_{-s}$. The on-shell (Pauli–Fierz type) constraints on these space-time fields follow from the BRST-invariance constraints on the vertex operators while the gauge transformations correspond to shifting the vertex operators by BRST-exact terms (see Ref. [42] for a detailed analysis). The zero torsion constraints (29) relating $\Omega^{s-1|t}$ gauge fields with different t follow from the cohomology constraints on their vertex operators $V_{s-1|t}$, that is, by requiring that all these vertex operators belong to the same cohomology $H_{s-2} \sim H_{-s}$ (there are, however, certain subtleties with this scheme arising at $t = s-1$ or $t = s-2$ tjat were discussed in Ref. [42] for the $s = 3$ case). The on-shell (Pauli–Fierz type) constraints on these space-time fields follow from the BRST-invariance constraints on the vertex operators while the gauge transformations correspond to shifting the vertex operators by BRST-exact terms (see Ref. [42] for a detailed analysis). Furthermore, it turns out that the vertex operators $V_{s-1|0}$ generating the $\Omega_m^{a_1\ldots a_{s-1}}$ dynamical fields in space-time are only physical when $\Omega^{s-1|0}$ are fully symmetric one-row fields (describing Fronsdal's metriclike tensors for symmetric fields of spin s) while the operators for the two-row ($s-1$, 1) fields are typically the BRST commutators in the small Hilbert space, and therefore the space-time fields are pure gauge [42]. This altogether constitutes the dictionary between vertex operators in superstring theory (extended to higher ghost cohomologies). Finally, the zero torsion constraints (29) relating $\Omega^{s-1|t}$ gauge fields with different t follow from the cohomology constraints on their vertex operators $V_{s-1|t}$, that is, by requiring that all these vertex operators belong to the same cohomology $H_{s-2} \sim H_{-s}$ (with some subtleties at $t = s-1$ or $t = s-2$, mentioned above). The zero torsion and cohomology constraints involving the $t = s-1$ and $s-2$ cases are very interesting and deserve separate consideration; however, we shall not discuss them in this paper for the sake of brevity.

To understand the meaning of the cohomology constraints, it is useful to recall first a much simpler example known from the conventional Ramond–Ramond sector of closed superstring theory. Namely, the relation between cohomology and zero torsion constraints can be thought of as a symmetric higher spin generalization of a more elementary and familiar example of standard Ramond–Ramond vertex operators in closed critical superstring theory. It is well known that the canonical picture representation for the Ramond–Ramond operators is given by

$$\begin{aligned} V_{RR}^{(-\frac{1}{2},-\frac{1}{2})} &= F_{\alpha\beta}(p) \int d^2 z e^{-\frac{\varphi}{2}-\frac{\bar{\varphi}}{2}} \Sigma^\alpha \bar{\Sigma}^\beta e^{ipX}(z,\bar{z}) \\ F_{\alpha\beta}^{(p)} &\equiv \gamma_{\alpha\beta}^{m_1\ldots m_p} F_{m_1\ldots m_p}, \end{aligned} \tag{30}$$

where $F_{\alpha\beta}^{(p)}$ is the Ramond–Ramond p-form field strength (contracted with ten-dimensional gamma matrices). Note that since the operator (30) is the source of the field strength (the derivative of the gauge potential) it does not carry Ramond–Ramond (RR) charge (which instead is carried by a corresponding Dp-brane). The operator (30) exists at all the pictures and is the element of $H^{(-\frac{1}{2},-\frac{1}{2})}$ cohomology (which is the superpartner of $H^{(0,0)}$ consisting of all picture-independent physical states). It is, however, possible to construct the vertex operator that couples to Ramond–Ramond gauge potential rather than field strength. The canonical picture for such an operator is $(-\frac{3}{2},-\frac{1}{2})$ (or equivalently $(-\frac{1}{2},-\frac{3}{2})$, with the explicit expression given by

$$U_{RR}^{(-\frac{1}{2},-\frac{3}{2})} = A_{\alpha\beta}(p-1)\int d^2 z e^{-\frac{\varphi}{2}-\frac{3\bar\varphi}{2}}\Sigma^\alpha\bar\Sigma^\beta e^{ipX}(z,\bar z)$$
$$A_{\alpha\beta}^{(p-1)} \equiv \gamma_{\alpha\beta}^{m_1\ldots m_p}A_{m_1\ldots m_{p-1}}, \tag{31}$$

where generically $\mathcal{A}$ is arbitrary. The U operator (31) is generally not the picture-changed version of the V operator (30) nor is it the element of $H^{(-\frac{1}{2},-\frac{1}{2})}$ for general A. To relate $U_{RR}^{(-\frac{1}{2},-\frac{3}{2})}$ to $V_{RR}^{(-\frac{1}{2},-\frac{1}{2})}$ of Eq. (30) by the picture changing,

$$V_{RR}^{(-\frac{1}{2},-\frac{1}{2})} = :\Gamma U_{RR}^{(-\frac{3}{2},-\frac{1}{2})}, \tag{32}$$

one has to impose the constraint $\mathcal{F} = d\mathcal{A}$ that ensures that U is the physical operator of $H^{(-\frac{1}{2},-\frac{1}{2})}$. Thus, the cohomology constraint in U leads to the standard relation between the gauge potential and the field strength. Similarly, the generalized $H_{s-2}\sim H_{-s}$-cohomology constraints on higher spin operators $V_{s-1|t}$ for $\Omega^{s-1|t}$ space-time fields lead to generalized zero torsion constraints (29). Note that for $0\le t\le s-3$, the canonical pictures for $V_{s|t}$ are $2s-t-5\sim t+3-2s$ with the cohomology constraints $V_{s|t}\in H_{s-2}\sim H_{-s}$ inducing the chain (29) of zero torsion relations.

We are now prepared to analyze the scale-invariance constraints for open string vertex operators describing the Vasiliev-type higher spin fields in space-time. It turns out that for massless fields of spin s the canonical picture representation is especially simple for the field with $t=s-3$, that is, for $\Omega^{s-1|s-3}$. The explicit vertex operator expression for this field is given by

$$\begin{aligned} V_{s-1|s-3} &= \Omega_m^{a_1\cdots a_{s-1}|b_1\ldots b_{s-3}}(p)\oint dz e^{-s\varphi-s\bar\varphi} \\ &\quad\times \psi^m\partial\psi_{b_1}\partial^2\psi_{b_2}\ldots\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}\ldots\partial X_{s-1}e^{ipX} \\ &\sim \Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}(p)K\circ\int d^2 z e^{(s-2)\varphi-(s-2)\bar\varphi} \\ &\quad\times \psi^m\partial\psi_{b_1}\partial^2\psi_{b_2}\ldots\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}\ldots\partial X_{s-1}e^{ipX}. \end{aligned}\tag{33}$$

For $s=3$, this immediately gives the operator for the Fronsdal field considered in Refs. [33,42]. The on-shell conditions on $\Omega^{s-1|s-3}$ to ensure the BRST invariance of Eq. (33) are not difficult to obtain using the BRST charge (14). The commutation with the T_X component of the stress-energy part of $Q_{\rm brst}$ leads to the tracelessness of Ω in the a indices, that is, $\Omega_{ma}^{aa_1\ldots a_{s-3}|b_1\ldots b_{s-3}}=0$, which is the well-known constraint on framelike fields and to the second Pauli–Fierz constraint of transversality: $p_a\Omega_m^{aa_1\ldots a_{s-2}|b_1\ldots b_{s-3}}(p)=0$. The commutation with the T_ψ part of $Q_{\rm brst}$, given by $-\frac{1}{2}\oint dzc\partial\psi_p\psi^m$, requires the symmetry of Ω in the b indices, as it is easy to see from the OPE between T_ψ and $S_{mb_1\ldots b_{s-3}}=\psi_m\partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}}$—the latter is the primary field of dimension $h_\psi=\frac{1}{2}(s-2)^2$ only if S is symmetric and traceless in all indices. While the symmetry in the b indices is another standard familiar constraint in the framelike formalism, the symmetry and tracelessness of m with respect to the b indices is an extra condition on Ω that can be obtained by partial fixing of the gauge symmetries of Ω. Given that the above conditions are fulfilled, the commutation with the supercurrent part of $Q_{\rm brst}$ produces no new constraints; however, there is one more condition coming from the H_{-s}-cohomology constraint on $V_{s-1|s-3}$, that is,

$$:\Gamma V_{s-1|s-3} := 0. \tag{34}$$

This constraint further requires the vanishing of the mixed trace over any pair of $(a,b)=$ indices: $\eta_{ab}\Omega_m^{aa_1\ldots a_{s-2}|bb_1\ldots b_{s-4}}=0$. Fortunately, the gauge symmetry of Ω is more than powerful enough to absorb this extra constraint as well. Finally, we are left to consider the BRST nontriviality conditions on Eq. (33). First of all, the nontriviality constraint, $V_{s-1|s-3}\neq\{Q_{\rm brst}, W_{s-1|s-3}\}$, where W is some operator in small Hilbert space, requires either

$$\eta_a^m\Omega_m^{aa_1\ldots a_{s-2}|b_1\ldots b_{s-3}}\neq 0 \tag{35}$$

or

$$p^m\Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}\neq 0 \tag{36}$$

since otherwise, generically, there exist operators

$$\begin{aligned} W_{s-1|s-3} &\sim \Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}\sum_{k=0}^{s-1}\oint dz e^{\chi-(s-1)\varphi}\partial\chi \\ &\quad\times\partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}\ldots\partial X_{s-1} \\ &\quad\times\partial^{s-1-k}X_m G^{(k)}(\varphi,\chi)e^{ipX} \end{aligned}\tag{37}$$

commuting with the stress tensor part of $Q_{\rm brst}$ while, at the same time, the commutators of the supercurrent part of $Q_{\rm brst}$ with $W_{s-1|s-3}$ are proportional to $V_{s-1|s-3}$: $\{Q_{\rm brst},W_{s-1|s-3}\}=\alpha_s V_{s-1|s-3}$, where α_s are some numbers (generically, nonzero) and $G^{(k)}(\varphi,\chi)$ are polynomials in derivatives of φ and χ of conformal dimension k (generically, inhomogeneous in degree and quite cumbersome) such that

$$e^{\chi-(s+1)\varphi}\partial^{s-1-k}X_m G^{(k)}(\varphi,\chi)\partial\chi$$

is a primary field (this is a rather stringent constraint, which, nevertheless, typically has nontrivial solutions for generic s; e.g., see Ref. [33] for some concrete examples).

For this reason, unless one of the nontriviality conditions, Eq. (35) or Eq. (36), holds, the operators $V_{s-1|s-3}$ are BRST exact in small Hilbert space; however, if either Eq. (35) or Eq. (36) are satisfied, the W operators do not commute with the stress-energy tensor part of Q_{brst}, and therefore their overall commutators with Q_{brst} no longer produce $V_{s-1|s-3}$ with the latter now being in BRST cohomology and physical. However, it is easy to see that out of two possible nontriviality conditions (35), (36), it is the second one (36) that must be chosen since the first one clearly violates the H_{-s}-cohomology condition (34). This immediately entails the gauge transformations for the Ω field,

$$\Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}} \to \Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}} + p_m \Lambda^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}, \qquad (38)$$

that shift $V_{s-1|s-3}$ by BRST-trivial terms irrelevant for amplitudes and lead to well-known vast and powerful gauge symmetries possessed by the higher spin fields. Note that, although all the above analysis has been performed for the operators at negative cohomologies (which are simpler from the technical point of view), all the above results directly apply to the corresponding operators at isomorphic positive H_{s-2} cohomologies since the explicit isomorphism between negative and positive cohomologies is BRST invariant [33].

To complete our analysis of BRST on-shell constraints on the higher spin operators of $H_{s-2} \sim H_{-s}$, we shall comment on the only remaining possible source of BRST triviality for $V_{s-1|s-3}$ coming from operators proportional to the ghost factor $\sim e^{2\chi-(s+2)\varphi}$. All the hypothetical operators in the small Hilbert space with such a property are given by

$$U_{s-1|s-3} = \Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}} \oint dz c\partial\xi\partial^2\xi e^{-(s+2)\varphi} \times R^{(2s-2)}(\varphi,\chi,\sigma)\psi^m \partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}} \times \partial X_{a_1}\ldots\partial X_{s-1} e^{ipX}, \qquad (39)$$

where $R^{(2s-2)}$ is the conformal dimension $2s-2$ polynomial in derivatives of φ, χ, and σ (again, homogeneous in conformal weight but not in degree). Indeed, the commutator of the matter supercurrent part of Q_{brst}, given by $-\frac{1}{2}\oint dw\gamma\psi_m\partial X^m$ with $U_{s-1|s-3}$, is zero since the leading order of the OPE between $\gamma\psi_m\partial X^m(w)$ and the integrand of $U_{s-1|s-3}$ at a point z is nonsingular, that is, proportional to $(z-w)^0$, as is easy to check. At the same time, the commutator of $U_{s-1|s-3}$ with the ghost supercurrent part of Q_{brst}, given by $-\frac{1}{4}b\gamma^2$, is nonzero and is proportional to $V_{s-1|s-3}$,

$$\{Q_{\mathrm{brst}}, U_{s-1|s-3}\} = \lambda_s V_{s-3} \qquad (40)$$

(where λ_s are certain numbers), provided that the coefficient σ_{2s-2} in front of the leading OPE order of $R^{(2s-2)}$ and $b\gamma^2$ is nonzero:

$$R^{(2s-2)}(z){:}b\gamma^2{:}(w) \sim \frac{\sigma_{2s-2}b\gamma^2(w)}{(z-w)^{2s-2}} + O(z-w)^{2s-3}\sigma_{2s-2} \neq 0. \qquad (41)$$

Then, provided that the conditions

$$\lambda_s \neq 0 \qquad (42)$$

and

$$\sigma_{2s-2} \neq 0 \qquad (43)$$

are both satisfied, the operator $V_{s-1|s-3}$ could be trivial only if the stress-tensor part of Q_{brst} commuted with $U_{s-1|s-3}$, which is only possible if (given the on-shell conditions on Ω described above)

$$G_s(z) = {:}c\partial\xi\partial^2\xi e^{-(s+2)\varphi}R^{(2s-2)}(\varphi,\chi,\sigma){:}(z) \qquad (44)$$

is a primary field. That is, the OPE of G_s with the full ghost stress-energy tensor,

$$T_{gh} = \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}(\partial\varphi)^2 + \frac{3}{2}\partial^2\sigma + \frac{1}{2}\partial^2\chi - \partial^2\varphi, \qquad (45)$$

is generically given by

$$T_{gh}(z)G_s(w) = \sum_{k=0}^{2s-1}\frac{y_k Y^{(-\frac{s^2}{2}-s+k)}(w)}{(z-w)^{2s+2-k}} + \frac{(s-\frac{1}{2}s^2)G_s(w)}{(z-w)^2} + \frac{\partial G_s(w)}{(z-w)} + O(z-w)^0, \qquad (46)$$

where y_k are numbers and $Y^{(-\frac{s^2}{2}-s+k)}$ are operators of conformal dimensions $-\frac{s^2}{2}-s+k$. So the $V_{s-1|s-3}$ operators are trivial only if the constraints

$$y_k = 0 \qquad k = 0,\ldots,2s-1 \qquad (47)$$

are fulfilled simultaneously with the conditions (42), (43). Clearly, for s large enough, the constraints (42), (43), (47) are altogether too restrictive, leaving no room for any possible choice of $R^{(2s-2)}(\varphi,\chi,\sigma)$, so the operators are nontrivial [of course, provided that Eq. (36) holds as well]. To see this note that for any large s and given k in the sum (46) the number of independent operators $Y^{(-\frac{s^2}{2}-s+k)}$ is of the order of $\sim\frac{d}{dk}(\frac{e^{a\sqrt{2s-k}}}{\sqrt{2s-k}})$, where a is a certain constant, since the number of conformal weight n polynomials is of the order of the number of partitions of n, which, in turn, is given by the Hardy–Ramanujan asymptotic formula for large n. Summing over k, it is clear that the number of constraints (47) on $G^{(2s-2)}$ is asymptotically of the order of $\frac{e^{a\sqrt{s}}}{\sqrt{s}}$ while the number of independent terms in $R^{(2s-2)}$ is of the order of $\frac{e^{a\sqrt{s}}}{s}$, so the number of constraints (47) exceeds the number of possible operators $U_{s-1|s-3}$ by the factor of the order of $\sqrt{s}$. Therefore, all the operators (33) with large spin values are BRST nontrivial, provided that Eq. (36) is satisfied. For the lower values of s, however, the constraints (42), (43), (47) have to be analyzed separately. For $s=3$, 4 it can be shown that the constraints (43), (47) lead to polynomials satisfying $\lambda_s=0$, so the appropriate higher spin operators are physical. For $5\leq 10$ direct numerical analysis shows the incompatibility of the conditions (42), (43), (47), with the number of constraints exceeding the number of operators of the type (39) posing a potential threat of BRST triviality, showing that operators with spins greater than 4 are physical as well.

With the on-shell BRST conditions pointed out, the next step is to analyze the scale-invariance (off-shell) constraints on the operators (33). It is instructive to start with the $s=3$ case since for $s=3$ $\Omega_{s-1|s-3}$ is precisely the Fronsdal field. Similarly to the closed string case, the Weyl transformation of $V_{s-1|s-3}$ is determined by the OPE coefficient in front of the $\sim|z-\tau|^{-2}$ term in the operator product $\lim_{z,\bar{z}\to\tau}T^{z\bar{z}}(z,\bar{z})V_{s-1|s-3}(\tau)$, where τ is on the world sheet boundary and, as previously mentioned, the ϵ-expansion setup is assumed, so $T^{z\bar{z}}\neq 0$. Starting from the transformation by $T_X=-\frac{1}{2}|\partial\vec{X}|^2$, we have

$$\int d^2zT_X^{z\bar{z}}(z,\bar{z})\Omega_m^{a_1a_2}(p)\oint d\tau e^{-3\varphi}\psi^m\partial X_{a_1}\partial X_{a_2}e^{ipX}(\tau)$$
$$\sim\ln\Lambda\times\oint d\tau e^{-3\varphi}\psi^m\partial X_{a_1}\partial X_{a_2}e^{ipX}(\tau)[-p^2\Omega_m^{a_1a_2}(p)+2p_tp^{(a_1}\Omega_m^{a_2)t}-p^{a_1}p^{a_2}\Omega_m'], \tag{48}$$

where we introduced $\Omega_m'\equiv\eta_{a_1a_2}\Omega_m^{a_1a_2}$ (similarly, using Fronsdal's notations, the "prime" will stand for contraction of a pair of fiber 0 indices for any other higher spin field below). This gives the part of the leading-order contribution to the spin-3 β function proportional to Fronsdal's operator in flat space. The analysis of the contributions by $T_\psi^{z\bar{z}}$ and by $T_{\beta-\gamma}^{z\bar{z}}$ is analogous to the one performed in the previous section for the AdS graviton operator (13), and the result is

$$\int d^2z(T^{(}_{\psi}z,\bar{z})+T^{(}_{\beta-\gamma}z,\bar{z}))\Omega_m^{a_1a_2}(p)\oint d\tau e^{-3\varphi}\psi^m\partial X_{a_1}\partial X_{a_2}e^{ipX}(\tau)\sim-8\ln\Lambda\Omega_m^{a_1a_2}\times\oint d\tau e^{-3\varphi}\psi^m\partial X_{a_1}\partial X_{a_2}e^{ipX}(\tau), \tag{49}$$

where the coefficient in front of Ω ensures that the overall normalization of Eq. (49) is consistent with that of Eq. (48). As in the case of the cosmological term appearing in the graviton's β function (28), the appearance of the masslike term in the spin-3 β function (48), (49) is due to the noncancellation of the corresponding terms in the Weyl transformation laws for the matter and for the ghost parts, which in turn is the consequence of the $H_{-3}\sim H_1$-cohomology coupling of the spin-3 operator. The term (49) in the β function is *not*, however, a mass term. Namely, combined together, the contributions (48), (49) give the low-energy equations of motion for a massless spin-3 field, corresponding to the special case of Fronsdal's operator in the AdS space acting on a spin-3 field that is polarized along the AdS boundary and is propagating along the boundary. The correspondence between Eq. (49) and the masslike term in Fronsdal's operator in AdS_{d+1} is exact for $d=4$; to make the correspondence precise for $d\neq 4$ requires some modification of the operators of the type (33) (see the discussion below for the general spin case).

The next step is to generalize this simple calculation to the general spin value and to calculate the β functions of the framelike fields (33). The vertex operators (33) do not generate Fronsdal's fields for $s\geq 4$ (but rather the derivatives of Fronsdal's fields), and explicit expressions for $V_{s-1|t}$-operators for $0\leq t\leq s-4$, following from cohomology constraints, are generally quite complicated. For example, the manifest form of operators for $\Omega_{s-1|s-4}$ fields at the canonical $(-s-1)$ picture is given by

$$V_{s-1|s-4}=\Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-4}}(p)\oint dze^{-(s+1)\varphi}\psi_m\partial\psi^{b_1}\ldots\partial^{s-4}\psi^{b_{s-4}}$$
$$\times\sum_{k=0}^{2s-3}T^{(2s-3-k)}(\varphi)\left[\sum_{j=1}^{k-1}a_j\partial^jX_q\partial^{k-j}X^q+b_j\partial^{j-1}\psi_q\partial^{k-j}\psi^q\right], \tag{50}$$

where a_j and b_j are certain coefficients and $T^{(2s-3-k)}(\varphi)$ are again certain conformal dimension $2s-3-k$ inhomogeneous polynomials in the derivatives of φ. The coefficients and the polynomial structures must be chosen to ensure that the integrand of Eq. (50) is a primary field of dimension 1, and the picture-changing transformation of the operator (50) is nonzero, producing an operator at picture $-s$ and at cohomology H_{-s}, so that the hohomology condition on Eq. (50) produces the zero torsionlike condition relating the framelike fields in Vasiliev's formalism,

$$
\begin{aligned}
:\Gamma V_{s-1|s-4} &:= V_{s-1|s-3} + \{Q_{\rm brst}, \ldots\} \\
\Omega^{s-1|s-3}(p) &\sim p\Omega^{s-1|s-4}(p),
\end{aligned}
\tag{51}
$$

so that the transformation of $V_{s-1|s-4}$ produces the vertex operator proportional to $V_{s-1|s-3}$ with the space-time field $\Omega^{s-1|s-3}(p) \sim p\Omega^{s-1|s-4}(p)$ given by a certain first-order differential operator acting on $\Omega^{s-1|s-4}(p)$. The explicit structure of this operator (giving one of the zero-curvature constraints) is determined by the details of the picture changing; for example, one of the contributions to Eq. (50) from the picture transformation of the $k=0$ term in Eq. (50) results from the OPE contributions,

$$
\begin{aligned}
e^{\varphi}(z)e^{-(s+1)\varphi}(w) &\sim (z-w)^{s+1}e^{-s\varphi}\left(\frac{z+w}{2}\right) + \cdots \\
\partial X^q(z)e^{ipX}(w) &\sim (z-w)^{-1} \times (-ip^q)e^{ipX}\left(\frac{z+w}{2}\right) + \cdots \\
e^{\varphi}(z)T^{(2s-3)}(\varphi)(w) &\sim (z-w)^{3-2s}e^{\varphi}\left(\frac{z+w}{2}\right) + \cdots,
\end{aligned}
\tag{52}
$$

so the leading OPE order of the product of the picture-changing operator $\Gamma \sim -\frac{1}{2}e^{\varphi}\psi_q\partial X^q + \cdots$ with $V_{s-1|s-4}$ is $\sim(z-w)^{3-s}$, so to obtain the normally ordered contribution, relevant to the picture-changing transformation (51), one has to expand the remaining field $\psi_q(z)$ of Γ up to the order of $s-3$ around the midpoint $\frac{z+w}{2}$, which altogether produces the result proportional to $V_{s-1|s-3}$, with the space-time field proportional to the space-time derivative of $\Omega^{s-1|s-4}$ [as it is clear from the second OPE in Eq. (52)]. There are of course many other terms in the OPE between Γ and $V_{s-1|s-4}$, but, provided that all the coefficients and the polynomial structures in Eq. (50) are chosen correctly, they all give the result proportional to $V_{s-1|s-3}$, up to BRST-exact terms and with the zero torsion condition

$$
\Omega^{s-1|s-3} \sim \partial\Omega^{s-1|s-4}
$$

controlled by the picture-changing procedure. The explicit expressions for the operators with $t = s-5, s-6, \ldots$ and, ultimately, for $t=0$ (Fronsdal's field) are increasingly complicated for general s. However, to deduce the Weyl-invariance constraints on massless vertex operators for Fronsdal's fields of spin s, we do not actually need to know the explicit expressions for $V_{s-1|0}$. The key point here is the mutual independence of the Weyl transformations and the cohomology constraints on the vertex operators. That is, the cohomology constraints relate Fronsdal's operator at a canonical $3-2s$ picture and the operator for the $\Omega^{s-1|s-3}$ extra field through

$$
\Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}(p)\oint dz e^{-s\varphi}\psi^m\partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}\ldots\partial X_{a_{s-1}}e^{ipX} = \Omega_m^{a_1\ldots a_{s-1}}:\Gamma^{s-3}:\oint dz U^{m(-2s+3)}_{a_1\ldots a_{s-1}}(p),
\tag{53}
$$

where U is the integrand of the vertex operator for Fronsdal's field. Since Γ is BRST and Weyl invariant, the relation (53) allows us to deduce the low-energy equations of motion for Fronsdal's fields by studying the Weyl transformations of the operators (33), which are much simpler. The transformations of $V_{s-1|s-3}$ by $T_X^{z\bar{z}}$ and $T_{\beta-\gamma}^{z\bar{z}}$ are computed similarly to the spin-3 case considered above. One easily finds

$$\begin{aligned}&\int d^2zT_X^{z\bar{z}}(z,\bar{z})\Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}(p)\oint d\tau e^{-s\varphi}\psi^m\partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}\ldots\partial X_{a_{s-1}}e^{ipX}\\&\sim\ln\Lambda\oint d\tau e^{-s\varphi}\psi^m\partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}\ldots\partial X_{a_{s-1}}e^{ipX}\\&\quad\times[-p^2\Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}(p)+p_t\Sigma_1(a_1|a_2\ldots a_{s-1})p^{a_1}\Omega_m^{a_2\ldots a_{s-1}t|b_1\ldots b_{s-3}}\\&\quad-\frac{1}{2}\Sigma_2(a_{s-2},a_{s-1}|a_1,\ldots,a_{s-3})p^{a_{s-1}}p^{a_{s-2}}(\Omega'_m)^{a_1\ldots a_{s-3}|b_1\ldots b_{s-3}}\end{aligned}\tag{54}$$

and

$$\begin{aligned}&\int d^2zT_{\beta-\gamma}^{z\bar{z}}(z,\bar{z})\Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}(p)\oint d\tau e^{-s\varphi}\psi^m\partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}\ldots\partial X_{a_{s-1}}e^{ipX}\\&\sim-s^2\Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}(p)\ln\Lambda\oint dze^{-s\varphi}\psi^m\partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}\ldots\partial X_{a_{s-1}}e^{ipX}.\end{aligned}\tag{55}$$

Here $\Sigma_1(b|a_1...a_n)$ and $\Sigma_2(b_1,b_2|a_1....a_n)$ are Fronsdal's symmetrization operations [43], acting on free indices, e.g., $\Sigma_p(a_1,\ldots,a_p|b_1,\ldots,b_s)T^{aa_1\ldots a_p}H_a^{b_1\ldots b_s}$, where H is symmetric, symmetrizes over $a_1,\ldots a_p;b_1,\ldots,b_s$.

To compute the Weyl transform of the ψ part, it is again helpful to use the bosonization relations (20), (21). Since the bosonized ϕ_i fields carry no background charges [as it is clear from the stress-energy tensor (21)], the coefficient in front of the $|z-\tau|^2$ term in the OPE of $T_\psi^{z\bar{z}}(z,\bar{z})$ and $V_{s-1|s-3}(\tau)$ coincides with the conformal dimension of the ψ-factor, $\psi^m\partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}}$, which is equal to $\frac{1}{2}(s-2)^2$, so

$$\begin{aligned}&\int d^2zT_\psi^{z\bar{z}}(z,\bar{z})\Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}(p)\oint d\tau e^{-s\varphi}\psi^m\partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}\ldots\partial X_{a_{s-1}}e^{ipX}\\&\sim(s-2)^2\Omega_m^{a_1\ldots a_{s-1}|b_1\ldots b_{s-3}}(p)\ln\Lambda\oint dze^{-s\varphi}\psi^m\partial\psi_{b_1}\ldots\partial^{s-3}\psi_{b_{s-3}}\partial X_{a_1}\ldots\partial X_{a_{s-1}}e^{ipX}+\cdots\end{aligned}\tag{56}$$

(again, with no factor of $\frac{1}{2}$ due to the normalization chosen for the kinetic term). The last identity is true as long as the ψ factor is a primary field, i.e., the appropriate on-shell conditions are imposed on Ω. It is not difficult to see, however, that the contributions due to the off-shell part are generally proportional to space-time derivatives of Ω and its traces, multiplied by higher spin operators that are not of the form (33), so these contributions are irrelevant for β functions of the higher spin fields of Vasiliev's type (instead, they contribute to the low-energy equations of motion of more complicated higher spin fields, such as those with mixed symmetries; so these contributions may become important in various generalizations of Vasiliev's theory). Collecting Eqs. (54)–(56) and using the cohomology constraint (53), we deduce that the leading-order β function for the massless Fronsdal fields of spin s is

$$\begin{aligned}\beta_m^{a_1\ldots a_{s-1}}&=-p^2\Omega_m^{a_1\ldots a_{s-1}}(p)+\Sigma_1(a_1|a_2,\ldots a_{s-1})p_tp^{a_1}\Omega_m^{a_2\ldots a_{s-1}t}\\&\quad-\frac{1}{2}\Sigma_2(a_{s-2},a_{s-1}|a_1,\ldots,a_{s-3})p^{a_{s-1}}p^{a_{s-2}}(\Omega'_m)^{a_1\ldots a_{s-3}}-4(s-1)\Omega_m^{a_1\ldots a_{s-1}}.\end{aligned}\tag{57}$$

The appearance of the masslike terms is related to the emergence of the curved geometry already observed in Eq. (28). Namely, vanishing of the β function (57) gives, in the leading order, the low-energy effective equations of motion on Ω given by

$$\hat{F}_{\mathrm{AdS}}\Omega = 0, \tag{58}$$

where $\hat{F}_{\mathrm{AdS}}$ is Fronsdal's operator in AdS_{d+1} space (exactly for $d=4$ and with some modifications in other dimensions) for which the action is restricted on higher spin fields Ω polarized along the AdS boundary. Indeed, the explicit expression for Fronsdal's operator in AdS_{d+1} [43], acting on symmetric spin-s fields polarized along the boundary is

$$\begin{aligned}(\hat{F}_{\mathrm{AdS}}\Omega)^{a_1\ldots a_s} &= \nabla_A\nabla^A\Omega^{a_1\ldots a_s} - \Sigma_1(a_1|a_2\ldots a_s)\nabla_t\nabla^{(a_1}\Omega^{a_2\ldots a_s t)}\\ &\quad + \frac{1}{2}\Sigma_2(a_1,a_2|a_3,\ldots,a_s)\nabla^{a_1}\nabla^{a_2}(\Omega')^{a_3\ldots a_s} - m_\Omega^2\Omega^{a_1\ldots a_s} + 2\Sigma_2\Lambda g^{a_1a_2}(\Omega')^{a_3\ldots a_s}\\ m_\Omega^2 &= -\Lambda(s-1)(s+d-3),\end{aligned} \tag{59}$$

where $A=(a,\alpha)$ is the AdS_{d+1} space-time index (with the Latin indices being along the boundary and α being the radial direction)

The cosmological constant in our units is fixed $\Lambda=-4$ to make it consistent with the Weyl transform of the AdS graviton operator (13). In what follows, we shall ignore the last term in this operator since, in the string theory context, it is related to the higher-order (cubic) contributions to the β function, which are beyond the leading-order Weyl invariance constraints. For the remaining part, consider the box (∇^2) of Ω first. It is convenient to use the Poincare coordinates for AdS:

$$ds^2 = \frac{R^2}{y^2}(dy^2 + dx_a dx^a). \tag{60}$$

With the Christoffel symbols,

$$\Gamma^y_{a_1a_2} = -\Gamma^y_{yy}\delta_{a_1a_2} = -\frac{1}{y}\delta_{a_1a_2}, \tag{61}$$

one easily computes

$$\nabla_A\nabla^A\Omega_{a_1\ldots a_s}(x) \equiv (\nabla_a\nabla^a + \nabla_y\nabla^y)\Omega_{a_1\ldots a_s}(x) = (\partial_a\partial^a - \Lambda s(s+d))\Omega_{a_1\ldots a_s}. \tag{62}$$

Substituting Eq. (62) into the AdS Fronsdal operator in the momentum space (with the Fourier transformed boundary coordinates) gives

$$\begin{aligned}(\hat{F}_{\mathrm{AdS}}\Omega(p))^{a_1\ldots a_s} &= -p^2\Omega_m^{a_1\ldots a_{s-1}}(p) + \Sigma_1(a_1|a_2\ldots a_{s-1})p_t p^{a_1}\Omega_m^{a_2\ldots a_{s-1}t}\\ &\quad - \frac{1}{2}\Sigma_2(a_{s-2},a_{s-1}|a_1,\ldots a_{s-3})p^{a_{s-1}}p^{a_{s-2}}(\Omega'_m)^{a_1\ldots a_{s-3}} + \Lambda(s+3-d)\Omega_m^{a_1\ldots a_{s-1}}.\end{aligned} \tag{63}$$

Thus, the β functions for the $V_{s-1|0}$ vertex operators coincide with AdS Fronsdal operators precisely for the AdS_5 case ($d=4$). For other values of d, the string theoretic calculation of the masslike factor $m_\Omega^2 \sim \Lambda(s-1)$ is still proportional to s, but there is a discrepancy proportional to $d-4$. This discrepancy can always be cured, however, by suitable modification of the ψ part of the vertex operators of the type (33). This modification typically involves the shift of the canonical picture of the operator for Fronsdal's field from $2s-3$ to $2s-3+|d-4|$ and is somewhat tedious, but straightforward, with the explicit form depending on d. However, the shift does not change the order of the cohomology, which is still $H_{s-2} \sim H_{-s}$ for each value of s. The Regge-style behavior (57) of the masslike terms in Fronsdal operators is thus the consequence of the cohomology structure of the higher spin vertices in the larger string theory.

V. CONCLUSIONS

We have shown that the massless higher spin operators (33), although initially constructed around the flat background in d dimensions, lead to the low-energy higher spin dynamics in the underlying AdS_{d+1} space, the presence of which is initially hinted at by the hidden symmetries of the superstring action (3), (11), (12) and by the cosmological terms appearing in the β function of the spin-2 operator (13) identified with the gravitational fluctuations around the AdS vacuum. In this paper we limited ourselves to the special case of vertex operators, describing the space-time higher spin fields polarized (and propagating) along the AdS boundary. The generalization to the bulk case involves switching on the Liouville mode in expressions for the operators, which accounts for the radial AdS direction. This generalization shall be important to perform since hopefully it shall reveal interesting interplay between AdS geometry and the Liouville central charge in various dimensions [44] as well as nontrivial relations between Liouville structure constants and those of higher spin algebra in various dimensions. Another important direction to explore is related to the higher order corrections to the β functions of the higher spin operators, mixing the Weyl transformations with the higher-order vertex operator contributions in the sigma model (1). One obvious complication that can be seen immediately is that the cohomology argument (53), allowing us to deduce the β functions for the Fronsdal fields in the leading order by studying those for the extra field operators in the framelike formalism, is no longer valid at higher orders, with the contributions to the β functions no longer being linear. At the same time, manifest expressions for vertex operators for Fronsdal's higher fields are generally too complicated to work with in a straightforward way, unless some structural algorithm may be found. One could still hope though that, with certain modifications, the cohomology argument (53) could still work at higher orders, allowing one to compute the low-energy couplings of Fronsdal's fields by using the extra field operators for which the structure is far simpler. We hope to elaborate on the higher order contributions in the near future, with the work currently in progress. The off-shell arguments considered in this paper strongly suggest that the most natural string-theoretic framework for understanding the structure of the higher spin interactions at higher orders is the cubiclike string field theory, extended to ghost cohomologies of higher orders, containing the higher spin operators. The relevant objects to compute in such an approach are the off-shell correlators

$$A_N \sim \langle T_{z\bar{z}} \ldots T_{z\bar{z}} V_{s_1} \ldots V_{s_N} \rangle, \tag{64}$$

with Vasiliev-type fields being on the world sheet boundary and the Weyl generators inserted in the bulk. The insertions of Weyl generators account for the AdS curvature effects in higher spin interactions, with the number of the insertions corresponding to the order in the cosmological constant. In general, this is not an easy computation to get through; however, at the first nontrivial order in cosmological constant Λ (with only one T insertion in SFT correlators), the formalism of Sen–Zwiebach type of open superstring field theory [45] (extended to higher cohomologies) can hopefully be used, at least for the fields of Vasiliev's type. The key point here is that equations of extended superstring field theory $\sim Q\Psi \sim \Psi \star \Psi$ hold the information about higher spin couplings at all orders similarly to Vasiliev's equations. In fact, the isomorphism between extended SFT and Vasiliev's equations may ultimately be a correct language to understand higher spin holography in general. Extended string field theory, as we may hope further, could be an efficient approach to understand the dynamics and geometrical aspects of multiparticle generalizations and of quantum higher spin field theories [7] in general. Testing β functions for higher spin fields through string field theory at higher orders, to establish their consistency with higher spin interactions in AdS should thus provide a nontrivial check of the conjectured isomorphism between equations of Vasiliev and the formalism of extended SFT. If this isomorphism holds, one can hope that the string field theory formalism will provide an efficient tool to explore the higher spin holography in general. It would be particularly interesting to compare the SFT computation of higher spin couplings with those performed in [10] by using Vasiliev's equations. We hope to be able to elaborate on these issues in future works. Another separate question of interest is whether, in addition to the AdS isometry (3), the action (1) also possesses any extra compact isometries that one could interpret as R symmetry. While the generators inducing the isometry (3) are the elements of $H_1 \sim H_{-3}$, the isometries that could be interpreted in terms of R symmetries may be contained in higher order cohomologies, such as $H_2 \sim H_{-4}$ and $H_3 \sim H - 5$. Examples of such isometries were particularly considered in Ref. [46], where they were interpreted in terms of extra space-time dimensions. Identifying the isometries of this type with the R symmetries would be particularly important to ensure that the dual CFT theories are supersymmetric.

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