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Modular invariant regularization of string determinants and the Serre GAGA principle

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Since any string theory involves a path integration on the world-sheet metric, their partition functions are volume forms on the moduli space of genus g Riemann surfaces $\mathcal{M}_g$, or on its super analog. It is well known that modular invariance fixes strong constraints that in some cases appear only at higher genus. Here we classify all the Weyl and modular invariant partition functions given by the path integral on the world-sheet metric, together with space-time coordinates, b-c and/or β-γ systems, that correspond to volume forms on $\mathcal{M}_g$. This was a long standing question, advocated by Belavin and Knizhnik, inspired by the Serre GAGA principle and based on the properties of the Mumford forms. The key observation is that the Bergman reproducing kernel provides a Weyl and modular invariant way to remove the point dependence that appears in the above string determinants, a property that should have its superanalog based on the super Bergman reproducing kernel. This is strictly related to the properties of the propagator associated to the space-time coordinates. Such partition functions $\mathcal{Z}[\mathcal{J}]$ have well-defined asymptotic behavior and can be considered as a basis to represent a wide class of string theories. In particular, since noncritical bosonic string partition functions $\mathcal{Z}_D$ are volume forms on $\mathcal{M}_g$, we suggest that there is a mapping, based on bosonization and degeneration techniques, from the Liouville sector to first order systems that may identify $\mathcal{Z}_D$ as a subclass of the $\mathcal{Z}[\mathcal{J}]$. The appearance of b-c and β-γ systems of any conformal weight shows that such theories are related to W algebras. The fact that in a large N 't Hooft-like limit two-dimensional W_N minimal models conformal field theories are related to higher spin gravitational theories on AdS_3, suggests that the string partition functions introduced here may lead to a formulation of higher spin theories in a string context.

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I. INTRODUCTION

Despite the great progress in string theory, finding a finite one in four dimensions is still an open question. Essentially, there is one candidate, superstring theory but it must be compactified. Other possible string theories are those in noncritical dimensions. However, such theories are elusive; in essence, the proper way to treat the Liouville measure is still unknown.

Each string theory involves a path integration on the world-sheet metric so that the corresponding partition functions are volume forms on the moduli space of genus g Riemann surfaces $\mathcal{M}_g$. Other formulations involve super Riemann surfaces, so that the partition functions correspond to volume forms on the supermoduli space of super Riemann surfaces. However, at least for low genus, it has been shown that there is a projection to a volume form on $\mathcal{M}_g$. Other formulations of superstring theories suggest that it may exist a mechanism, possibly involving a rearrangement of the elementary fields, that may project the theory to $\mathcal{M}_g$ even in higher genus.

One of the main results here is the classification of all the possible Weyl and modular invariant string partition functions given by the path integral on the world-sheet metric, together with space-time coordinates, b-c and/or β-γ systems, that correspond to volume forms on $\mathcal{M}_g$. This also provides the way to test modular invariance that, as well known, fixes strong constraints that may appear only at higher genus. It should be stressed that the classification may also lead to uncover new symmetries underlying string theories. The investigation is based on two key properties of the Bergman reproducing kernel, namely its Weyl and modular invariance. In this way, it is possible to remove the point dependence that appears in the string determinants.

The role of modular invariance has been first noticed in [1]. The Polyakov formulation of string theories [2] led to a considerable progress in the covariant calculations of string partition functions and amplitudes [3–16] where modular invariance is a basic issue.

The need of classifying the string partition functions corresponding to volume forms on $\mathcal{M}_g$ is also suggested by the strictly related approaches to investigate superstring perturbation theory. The first one is to consider the original theory trying to derive, step by step from first principles, its measure on $\mathcal{M}_g$ or on the supermoduli space. This is essentially due to D'Hoker and Phong in the case of genus two [17]. Very recently Witten has proposed a systematic approach to the formulation on supermoduli of superstring perturbation theory [18].

Another approach is to change the elementary fields of the theory and, again, derive the corresponding measure on

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$\mathcal{M}_g$. This is mainly due to the Berkovits pure spinor formulation [19,20].

Another way is to match the natural constraints of the theory with the constraints imposed by the geometry of $\mathcal{M}_g$. This has been used to guess the form of the four-point function at arbitrary genus, leading to a result [21] which is also in agreement with more recent investigations related to the R^4 nonrenormalization theorem in $N = 4$ supergravity [22]. A similar ideology led to guess the structure of the Neveu-Schwarz-Ramond (NSR) partition function at any genus [17,23], culminating with the Grushevsky ansatz [24].

Of course the above is a schematic view, as the three approaches are strictly related, and each of them contributes to the other.

From the above lessons we learn that the basic physical motivation to consider the possible volume forms on $\mathcal{M}_g$ is modular invariance. It is just modular invariance that implies that the string partition functions correspond to a volume form $\mathcal{M}_g$. In this respect, considering the case of higher genus Riemann surfaces is an essential ingredient to understand the structure of a given theory. There are other important issues, for example the problem of treating the zero mode insertions in the path integral is quite different in the case of the sphere and the torus with respect to the case of negatively curved Riemann surfaces. As a consequence, several questions, such as the one of modular invariant regularization of the standard combination of string determinants, may not appear in genus zero and one. This is essentially due to Riemann-Roch theorem telling us that the space of zero modes of a conformal field of a given weight may be zero dimensional on the sphere and nontrivial for $g \geq 2$. Similarly, since the torus is flat, it is a special case as all the zero modes essentially correspond to the constant.

Another reason to study volume forms on $\mathcal{M}_g$ for $g \geq 2$ goes back to the Friedan-Shenker analytic approach to two-dimensional (2D) conformal field theories (CFTs) [9]. Modular invariance is again the key issue. Explicit examples are the ones by Gaberdiel and Volpato [25]. They have shown that higher genus vacuum amplitudes of a meromorphic conformal field theory uniquely determine the affine symmetry of the theory. In particular, the vacuum amplitudes of the $E_8 \times E_8$ theory and the $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory differ at genus 5. The fact that the discrepancy only arises at rather high genus is just a consequence of the modular properties of higher genus amplitudes.

Another explicit realization of the Friedan-Shenker approach concerns just the NSR superstring. In particular, it has been shown in [26] that there exists a natural choice of the local coordinate at the node on degenerate Riemann surfaces that greatly simplifies the computations. This makes clear the power of such an approach as now one may derive, at any genera, consistency relations involving the amplitudes and the measure. As a result chiral superstring amplitudes can be obtained by factorizing the higher genus chiral measure induced by considering suitable degeneration limits of Riemann surfaces. Even in such investigations modular invariance is the key symmetry.

Classifying string partition functions corresponding to volumes forms on $\mathcal{M}_g$ may also lead to uncover new symmetries. String theories essentially concern the bosonic and supersymmetric ones. The first is affected by the tachyon, whereas the superstring, although free of such singularities, still needs more than four dimensions and one has to compactify the extra dimensions. In principle, it may happen that there are other string theories with some underlying hidden symmetry. Investigating such a question requires the preliminary basic step of classifying all forms on $\mathcal{M}_g$ satisfying the main properties that a string theory should have. Let us summarize them.

1. Since each string theory involves the path integration over the world-sheet metric, it should be a modular invariant $(3g-3, 3g-3)$ form, i.e. a volume form on $\mathcal{M}_g$.
2. Such forms should correspond to determinants of Laplacians associated to the space-time coordinates to b-c and/or β-γ systems of any conformal weight.
3. The combination of such determinants should be Weyl invariant.

Satisfying such conditions is essentially equivalent to require that the partition functions

$$\int_{\mathcal{M}_g} \mathcal{Z}[\mathcal{J}] = \int DgDXD\Psi \, \exp(-S[X] - S[\Psi]), \tag{1.1}$$

correspond to volume forms on $\mathcal{M}_g$. Here $S[X]$ is the Polyakov action in

$$D = 26 + 2\sum_{k\in\mathcal{I}} n_k c_k \tag{1.2}$$

dimensions, where $c_k = 6k^2 - 6k + 1$ is (minus) $1/2$ the central charge of the nonchiral (b-c) β-γ system of weight k. $\mathcal{I}$ is the set of conformal weights $k \in \mathbb{Q}$, $\mathcal{J}$ the set of $n_k \in \mathbb{Z}/2$. $D\Psi$ denotes the product on $k \in \mathcal{I}$ of $|n_k|$ copies of the nonchiral measures, including the zero mode insertions, of weight k b-c systems for $n_k > 0$, or β-γ systems for $n_k < 0$. $S[\Psi]$ is the sum of the corresponding nonchiral b-c and β-γ actions. We will see that there exists a Weyl and modular invariant regularization of string determinant that eliminates the points dependence due to the insertion of zero modes. This will lead to a consistent definition of $\mathcal{Z}[\mathcal{J}]$ as volume form on $\mathcal{M}_g$. One of the main consequences of the present investigation is that the partition functions $\mathcal{Z}[\mathcal{J}]$ include a class of finite strings, even in four dimensions [27].

The content of the paper is as follows. In Sec. II we review the partition function of b-c and β-γ systems. In particular, we will consider the problem of treating the point dependence due to the zero mode insertions. In Sec. III we will introduce the way to eliminate the point dependence of the zero mode insertions in the path integral, preserving the modular and Weyl symmetries. In Sec. IV we will consider the bosonic string partition function $\mathcal{Z}_D$ in noncritical dimension D. In particular, since $\mathcal{Z}_D$ is a volume form on $\mathcal{M}_g$, its behavior at the boundary of $\mathcal{M}_g$ fixes some conditions that may be reproduced by $\mathcal{Z}[\mathcal{J}]$ for some $\mathcal{J}$. We suggest that there is a map from the non-Gaussian measures on diffeomorphisms and the Liouville fields to the Gaussian one that leads, via bosonization techniques, to represent the Liouville sector by means of first-order systems. We will also show that W algebras naturally arise in our construction. Interestingly, this may lead to represent higher spin fields in a string context.

Although the prescription introduced in Sec. III is essentially the only well-defined recipe for any Riemann surface, there is a related approach which is defined on canonical curves. These curves are the ones of genus two and the nonhyperelliptic compact Riemann surfaces with $g > 2$. In Sec. V we show that instead of integrating with $B^{1-n}(z_j, \bar{z}_j)$ each pair $b(z_j)\bar{b}(z_j)$ of the zero mode insertions, one may divide them by the determinant of $B^{(n)}(z_j, \bar{z}_k)$, denoting the n-fold Hadamard product of $B(z_j, \bar{z}_k)$. This implies that the ratio of determinants of Laplacians corresponding to the path integral on the world-sheet metric, together and on space-time coordinates and to b-c and/or β-γ systems, become volume forms on the moduli space of canonical curves $\hat{\mathcal{M}}_g$. We will show that det $B^{(n)}(z_j, \bar{z}_k)$ is expressed in terms of the recently introduced vector-valued Teichmüller modular forms [28]. We will also consider the Chern classes associated to our construction and see their relation with the tautological classes arising in 2D topological gravity.

Section VI is devoted to further developments and to the conclusions. In the Appendix we introduce the mapping to the single indexing used in Sec. V.

II. PARTITION FUNCTION OF FIRST ORDER SYSTEMS

A. Some notation

Let C be a Riemann surface of genus $g \geq 2$ and denote by $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ a symplectic basis of $H_1(C, \mathbb{Z})$. Let $\{\omega_i\}_{1\leq i\leq g}$ be the basis of $H^0(K_C)$ with the standard normalization $\oint_{\alpha_i} \omega_j = \delta_{ij}$ and $\tau_{ij} = \oint_{\beta_i} \omega_j$ the Riemann period matrix. Set $\tau_2 \equiv \mathrm{Im}\tau$. The basis of $H_1(C, \mathbb{Z})$ is determined up to the transformation

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto \begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix} = \begin{pmatrix}D & C\\ B & A\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \quad \gamma \equiv \begin{pmatrix}A & B\\ C & D\end{pmatrix} \in \Gamma_g,$$

which induces the following transformation on the period matrix:

$$\tau \mapsto \gamma \cdot \tau = (A\tau + B)(C\tau + D)^{-1}.$$

We will also consider the Deligne-Mumford compactification of the moduli space of genus g stable curves with n-punctures $\bar{\mathcal{M}}_{g,n}$. It turns out that $\bar{\mathcal{M}}_g$ is a projective variety with compactification divisor

$$D = \bar{\mathcal{M}}_g \setminus \mathcal{M}_g = D_0, \ldots, D_{[g/2]}. \tag{2.1}$$

A curve belongs to $D_{k>0} \cong \bar{\mathcal{M}}_{g-k,1} \times \bar{\mathcal{M}}_{k,1}$ if it has one node separating it into two components of genus k and g-k. The locus in $D_0 \cong \bar{\mathcal{M}}_{g-1,2}$ consists of surfaces that become, on removal of the node, genus g-1 double punctured surfaces. Surfaces with multiple nodes lie in the intersections of the D_k.

The compactified moduli space $\bar{\mathcal{M}}_{g,n}$ of stable curves with n-punctures is defined in an analogous way to $\bar{\mathcal{M}}_g$. The important point now is that the punctures never collide with the node. In particular, the configurations of two colliding punctures are stabilized by considering them as the limit in which the n-punctured surface degenerates into a $(n-1)$-punctured curve and the thrice punctured sphere. Consider the Riemann theta function with characteristics

$$\theta\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](z,\tau) = \sum_{k\in\mathbb{Z}^g} e^{\pi i^t(k+a)\tau(k+a)+2\pi i^t(k+a)(z+b)},$$

where $z \in \mathbb{C}^g$ and a, $b \in \mathbb{R}^g$. If δ', $\delta'' \in \{0, 1/2\}^g$, then $\theta[\delta](z,\tau) := \theta\left[\begin{smallmatrix}\delta'\\ \delta''\end{smallmatrix}\right](z,\tau)$ has definite parity in z $\theta[\delta](-z,\tau) = e(\delta)\theta[\delta](z,\tau)$, where $e(\delta) := e^{4\pi i^t\delta'\delta''}$. There are 2^{2g} different characteristics of definite parity. By Abel Theorem each one of such characteristics determines the divisor class of a spin bundle $L_\delta \simeq K_C^{\frac{1}{2}}$, so that we may call them spin structures. There are $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd spin structures. Let ν be a nonsingular odd characteristic. The holomorphic 1-differential

$$h_\nu^2(p) = \sum_1^g \omega_i(p)\partial_{z_i}\theta[\nu](z)_{|_{z=0}},$$

$p \in C$, has $g - 1$ double zeros. The prime form,

$$E(z,w) = \frac{\theta[\nu](w-z,\tau)}{h_\nu(z)h_\nu(w)},$$

is a holomorphic section of a line bundle on $C \times C$, corresponding to a differential form of weight $(-1/2, -1/2)$ on $\tilde{C} \times \tilde{C}$, where $\tilde{C}$ is the universal cover of C. It has a first order zero along the diagonal of $C \times C$. In particular, if t is a local coordinate at $z \in C$ such that $h_\nu = dt$, then

$$E(z,w) = \frac{t(w) - t(z)}{\sqrt{dt(w)}\sqrt{dt(z)}}(1 + \mathcal{O}((t(w) - t(z))^2)).$$

Note that $I(z + {}^t\alpha n + {}^t\beta m) = I(z) + n + \tau m$, m, $n \in \mathbb{Z}^g$, and

$$E(Z + {}^t\alpha n + {}^t\beta m, w) = \chi e^{-\pi i^t m\tau m - 2\pi i^t m I(z-w)} E(Z, w),$$

where $\chi = e^{2\pi i({}^t\nu' n - {}^t\nu'' m)} \in \{-1, +1\}$, m, $n \in \mathbb{Z}^g$. We will also consider the prime form $E(z, w)$ and the multivalued $g/2$-differential $\sigma(z)$ on C with empty divisor, satisfying the property

$$\sigma(z + {}^t\alpha n + {}^t\beta m) = \chi^{-g} e^{\pi i(g-1)^t m\tau m + 2\pi i^t m\mathcal{K}^z} \sigma(z),$$

where χ is and $\mathcal{K}^z$ the vector of Riemann constants. Such conditions fix $\sigma(z)$ only up to a factor independent of z; the precise definition, to which we will refer, can be given, following [29], on the universal covering of C (see also [30]).

B. String determinants and Mumford forms

Consider the covariant derivative ∇_z^{-n} acting on $-n$-differentials and its adjoint ∇_{1-n}^z. If $\rho \equiv 2g_{z\bar{z}}$ is the metric tensor in local complex coordinates, that is $ds^2 = 2g_{z\bar{z}}dzd\bar{z}$, then

$$\nabla_z^{-n} = \rho^{-n}\partial_z \rho^n, \qquad \nabla_{1-n}^z = \rho^{-1}\partial_{\bar{z}}.$$

We will consider the determinants of such operators and of the Laplacian $\Delta_{1-n} = \nabla_z^{-n}\nabla_{1-n}^z$ acting on $1-n$ differentials. Set

$$c_n = 6n^2 - 6n + 1.$$

Let $\varphi_1^n, \ldots, \varphi_{N_n}^n$, $N_n = h^0(K_C^n)$, be a basis of $H^0(K_C^n)$. The partition function of the nonchiral b-c system is [12]

$$|\det \varphi_j^n(z_k)|^2 \frac{\det' \Delta_{1-n}}{\det \mathcal{N}_n} = \int Db D\bar{b} Dc D\bar{c} \prod_j b(z_j)\bar{b}(z_j) \times e^{-\frac{1}{2\pi}\int_C \sqrt{g} b\nabla_{1-n}^z c + \mathrm{c.c.}} = |Z_n(z_1, \ldots, z_{N_n})|^2 e^{-c_n S_L(\rho)}, \quad (2.2)$$

where $S_L(\rho)$ is the Liouville action and

$$(\mathcal{N}_n)_{jk} = \int_C \bar{\varphi}_j^n \rho^{1-n} \varphi_k^n.$$

Multiplying (2.2) by $\prod_1^{N_n} \rho^{1-n}(z_j, \bar{z}_j)$ and integrating over C^{N_n} by using the generalization of (3.3) in [21] yields

$$\det' \Delta_{1-n} = \int Db D\bar{b} Dc D\bar{c} \prod_1^{N_n} \int_{C^{N_n}} \rho^{1-n}(z_i, \bar{z}_i) b(z_i)\bar{b}(z_i) \times e^{-\frac{1}{2\pi}\int_C \sqrt{g} b\nabla_{1-n}^z c + \mathrm{c.c.}}, \quad (2.3)$$

where we absorbed in the measure a numerical factor. It turns out that for $n \neq 1$ [12]

$$Z_n(z_1, \ldots, z_{N_n}) = Z_1[\omega]^{-1/2} \theta\left(\sum_i z_i - (2n-1)\Delta\right) \prod_{i<j} E(z_i, z_j) \prod_i \sigma^{2n-1}(z_i), \quad (2.4)$$

and

$$Z_1(z_1, \ldots, z_g) = Z_1[\omega]^{-1/2} \frac{\theta(\sum_i z_i - w - \Delta)\prod_{i<j} E(z_i, z_j)\prod_i \sigma(z_i)}{\sigma(z)\prod_i E(z_i, w)}. \quad (2.5)$$

It turns out that

$$Z_1[\omega] = \frac{Z_1(z_1, \ldots, z_g)}{\det \omega_j(z_k)}$$

can be formally considered as the partition function of a chiral scalar. We have

$$Z_1^{3/2}[\omega] = \frac{\theta(\sum_i z_i - w - \Delta)\prod_{i<j} E(z_i, z_j)\prod_i \sigma(z_i)}{\det \omega_j(z_k)\sigma(z)\prod_i E(z_i, w)}.$$

Also note that

$$Z_n[\varphi^n] = \frac{Z_n(z_1, \ldots, z_{N_n})}{\det \varphi_i^n(z_j)} \quad (2.6)$$

is independent of the points. One may easily check that the $g/2$-differential σ, the carrier of the gravitational anomaly, is missing in

$$F_{g,n}[\varphi^n] = \frac{Z_n[\varphi^n]}{Z_1[\omega]^{c_n}}. \quad (2.7)$$

This corresponds to the fact that such a ratio defines the Mumford form of degree n. More precisely, consider the $(3g-2)$-dimensional complex space $\mathcal{C}_g$, called universal curve, built by placing over each point of $\mathcal{M}_g$ the corresponding curve C. Consider the map π projecting $\mathcal{C}_g$ to $\mathcal{M}_g$. Denote by $L_n = R\pi_*(K^n_{\mathcal{C}_g/\mathcal{M}_g})$ the vector bundle on $\mathcal{M}_g$ of rank N_n with fiber $H^0(K_C^n)$ at the point of $\mathcal{M}_g$ representing C. Let $\lambda_n = \det L_n$ be the determinant line bundle. The Mumford isomorphism is [31]

$$\lambda_n \cong \lambda_1^{\otimes c_n}.$$

It turns out that, for each n, the Mumford form is

$$\mu_{g,n} = F_{g,n}[\varphi^n] \frac{\varphi_1^n \wedge \cdots \wedge \varphi_{N_n}^n}{(\omega_1 \wedge \cdots \wedge \omega_g)^{c_n}}. \tag{2.8}$$

Therefore, $\mu_{g,n}$ is the unique, up to a constant, holomorphic section of $\lambda_n \otimes \lambda_1^{-c_n}$ nowhere vanishing on $\mathcal{M}_g$. The fact that the bosonic string measure is essentially given by the Mumford form $\mu_{g,2}$ has been first observed by Manin [6]. For $n = 2$ its expression in terms of theta functions has been given in [6] whereas $\mu_{g,n}$ has been obtained in [12,7,29]. See also [32] for a related investigation.

By (2.2,2.3) and (2.6) it follows that

$$\det{}'\Delta_{1-n} = |Z_n[\varphi^n]|^2 \det \mathcal{N}_n e^{-c_n S_L(\rho)}, \tag{2.9}$$

and the modulo square analog of (2.8) is

$$\frac{\det{}'\Delta_{1-n}}{(\det{}'\Delta_0)^{c_n}} = |F_{g,n}[\varphi^n]|^2 \frac{\det \mathcal{N}_n}{(\mathcal{N}_0 \det \mathcal{N}_1)^{c_n}}. \tag{2.10}$$

C. β-γ systems

The above description extends to the case of β-γ systems giving the inverse of the determinants with respect to the case of the corresponding b-c systems. Since the treatment of zero modes is more subtle, let us shortly review the known results. Following [13] and its notation, it turns out that the correlators of the chiral β-γ system of weight $3/2$,

$$G(y_i; w_j; x_k) = \int [D\beta D\gamma]_\delta e^{-S(\beta,\gamma)} \prod_{l=1}^{m} \gamma(y_l) \prod_{j=1}^{m-2g+2} \delta(\gamma(w_j)) \prod_{k=1}^{m+1} \Theta(\beta(x_k)),$$

can be expressed in terms of theta functions as

$$G(y_i; w_j; x_k) = \frac{\prod_{l=1}^{m} Z_{\frac{3}{2},\delta}(\sum x_k - \sum w_j - y_l - 2\Delta)}{\prod_{l=1}^{m+1} Z_{\frac{3}{2},\delta}(\sum_{k\neq l} x_k - \sum w_j - 2\Delta)},$$

where $Z_{\frac{3}{2},\delta}$ denotes $Z_{3/2}[\varphi^{3/2}]$ with now the theta function in (2.4) having characteristic δ. The chiral partition function corresponds to $m = 2g - 2$. In this case, taking $y_i = x_i$, $i = 1, \ldots, 2g - 2$, such an expression reduces to

$$G(x_i; 0; x_i) = \frac{1}{Z_{\frac{3}{2},\delta}(\sum_1^{2g-2} x_i - 2\Delta)},$$

which is just the inverse of the corresponding expression for the chiral b-c system. Repeating the construction in the nonchiral case, and for any n, it can be seen that the partition function for the nonchiral β-γ system is just the inverse of (2.9).

D. Modular invariance and zero modes

Let us consider again Eq. (2.10). It shows that under a Weyl transformation the ratio of Laplacian determinants

$$\frac{\det{}'\Delta_{1-n}}{(\det{}'\Delta_0)^{c_n}}$$

has the same transformation properties of

$$\frac{\det \mathcal{N}_n}{(\mathcal{N}_0 \det \mathcal{N}_1)^{c_n}}.$$

This means that the anomalous transformation under Weyl rescaling of the two ratios is a kind of residual anomaly which follows from the definition of the partition function. Actually, there is some degrees of freedom in treating the zero modes, and one may also choose

$$\frac{\det{}' \Delta_{1-n}}{\det \mathcal{N}_n},$$

rather than $\det{}' \Delta_{1-n}$. However, this still gives a residual ambiguity, due to the choice of the basis of the zero modes $\varphi_1^n, \ldots, \varphi_{N_n}^n$. To discuss such a question, it is instructive to recall how the bosonic partition function in the critical dimension is obtained.

First, the moduli part of the measure on the world-sheet metric in the path integral reduces to

$$\frac{\det{}'\Delta_{-1}}{\det \mathcal{N}_2} |\wedge^{\max} \varphi_j^2|^2.$$

Note that this is different from the partition function for a nonchiral b-c system of weight 2, where $|\wedge^{\max}\varphi_j^2|^2$ is replaced by $|\det \varphi_j^2(z_k)|^2$. By (2.9)

$$\frac{\det{}'\Delta_{-1}}{\det \mathcal{N}_2} |\wedge^{\max} \varphi_j^2|^2 = |Z_2[\varphi^2]|^2 e^{-13 S_L(\rho)} |\wedge^{\max} \varphi_j^2|^2.$$

The scalar integration gives $(\det{}' \Delta_0/\mathcal{N}_0)^{-13}$ so that the critical bosonic string measure on $\mathcal{M}_g$ is

$$\mathcal{Z}_{\rm Pol} = \left(\frac{\det'\Delta_0}{\mathcal{N}_0}\right)^{-13} \frac{\det'\Delta_{-1}}{\det \mathcal{N}_2} |\wedge^{\max}\varphi_j^2|^2 = \left|\frac{Z_2[\varphi^2]}{Z_1[\omega]^{13}}\right|^2 \frac{|\wedge^{\max}\varphi_j^2|^2}{(\det \mathcal{N}_1)^{13}}. \quad (2.11)$$

Comparing such an expression with Eq. (2.7) we get the precise relation with the Mumford form of degree 2

$$\mathcal{Z}_{\rm Pol} = |F_{g,2}[\varphi^2]|^2 \frac{|\wedge^{\max}\varphi_j^2|^2}{(\det \tau_2)^{13}}, \quad (2.12)$$

where we used $\mathcal{N}_1 = \tau_2$.

III. STRING PARTITION FUNCTIONS AS VOLUME FORMS ON $\mathcal{M}_g$

As discussed in the Introduction, since any possible string theory would involve a path integration on the world-sheet metric, a central question is to classify determinants of Laplacians associated to the space-time coordinates, b-c and/or β-γ systems of any conformal weight, corresponding to volume forms on $\mathcal{M}_g$. The combination of such determinants should be Weyl and modular invariant.

Note that independence on the choice of the basis of $H^0(K_C^2)$ in (2.11), and therefore the absence of a source of modular anomaly, is due to the fact that the metric measure leads to a term $\det \mathcal{N}_2$ at the denominator, whose dependence on the choice of the basis is balanced by $|\wedge^{\max}\varphi_j^2|^2$. This means that apparently it is not possible to define volume forms on $\mathcal{M}_g$ considering the ratio of Laplacians of determinants, and therefore partition functions on the world sheet with b-c systems, unless they come, as in the case of the critical bosonic string, as an integration on metrics. In the case of $n \neq 2$, this would imply considering metrics on some vector bundle. Let us explicitly illustrate the problem. According to (2.2) and (2.10) the partition function of $2c_n$ scalars and a b-c system of weight n would give

$$\left(\frac{\det'\Delta_0}{\mathcal{N}_0}\right)^{-c_n} \frac{\det'\Delta_{1-n}}{\det \mathcal{N}_n} |\det \varphi_j^n(z_k)|^2 = \left|\frac{Z_n[\varphi^n]}{Z_1[\omega]^{c_n}}\right|^2 \frac{|\det \varphi_j^n(z_k)|^2}{(\det \mathcal{N}_1)^{c_n}}, \quad (3.1)$$

whose structure is different from the one of the critical bosonic string (2.11). For arbitrary n the term $|\wedge^{\max}\varphi_j^n|^2$ is replaced by $|\det \varphi_j^n(z_k)|^2$. They both guarantee independence from the choice of the basis of $H^0(K_C)$, and therefore modular invariance. On the other hand, it is commonly believed that removing the apparently harmless dependence on the points in (3.1) may lead to a modular or a Weyl anomaly. In this section we will show that this common belief is due to an undue identification between positive (1,1)-forms and path-integral metric. In particular, depending on the context, the same positive definite (1,1)-form may correspond or not to the path-integral metric. Such an apparent ambiguity is quite evident once one notes that the ratio of any two positive definite (1,1)-forms define a possible Weyl transformation, and this, of course, does not imply that all positive definite (1,1)-forms should be Weyl transformed.

In the following we use a positive definite (1, 1)-form to integrate on C^{N_n} the zero mode insertions. It should be observed that even if such a form has the same properties of a metric, it is explicitly constructed in terms of Weyl and modular invariant quantities. As a consequence, depending on the context, it can be seen as a metric, so that getting the Weyl factor, or as a Weyl invariant (1, 1)-form. In this way the Weyl invariant ratios of regularized string determinants correspond to (0, 0)-forms on $\mathcal{M}_g$ that, multiplied by the Polyakov measure, define volume forms on $\mathcal{M}_g$. Such a recipe is consistently defined on any Riemann surface.

A. The fiber and the Weyl transformations

Let $\mathcal{I}$ be the set of conformal weights $k \in \mathbb{Q}$. Set $\mathcal{J} = \{n_k \in \mathbb{Z}/2 | k \in \mathcal{I}\}$ and let $D\Psi$ be the product on $k \in \mathcal{I}$ of $|n_k|$ copies of the nonchiral measures, including the zero mode insertions, of weight k b-c systems for $n_k > 0$, or β-γ systems for $n_k < 0$. We denote by $S[\Psi]$ the sum of the corresponding nonchiral b-c and β-γ actions.

In (1.1) the Weyl invariant string partition functions corresponding to integrals of $\mathcal{Z}[\mathcal{J}]$ over $\mathcal{M}_g$ have been introduced. The $\mathcal{Z}[\mathcal{J}]$ correspond to the Polyakov partition function $\mathcal{Z}_{\rm Pol}$ times a rational function of determinants of Laplacians. Namely

$$\mathcal{Z}[\mathcal{J}] = \mathcal{Z}_{\rm Pol} \prod_{k\in\mathcal{I}} \mathcal{Z}_k^{n_k}, \quad (3.2)$$

where, tentatively,

$$\mathcal{Z}_n \sim \left(\frac{\det'\Delta_0}{\mathcal{N}_0}\right)^{-c_n} \frac{\det'\Delta_{1-n}}{\det \mathcal{N}_n} |\det \varphi_j^n(z_k)|^2, \quad (3.3)$$

with the right-hand side of (3.3) coinciding with the following partition function:

$$\int DXDbD\bar{b}DcD\bar{c} \prod_j b(z_j)\bar{b}(z_j) \exp\left(-S[X] - \frac{1}{2\pi}\int_C \sqrt{g} b \nabla_{1-n}^z c + \text{c.c.}\right), \quad (3.4)$$

where now $S[X]$ is the Polyakov action in $2c_n$ dimensions.

If it were not for the dependence on the points, the right-hand side of (3.3) would be the good definition for such a combination of partition functions. The reason is that it satisfies the condition $D = 2c_n$ that guarantees the invariance of $\mathcal{Z}_n$ under Weyl transformations

$$g \longrightarrow e^{\sigma} g.$$

As we will see, the precise definition of $\mathcal{Z}_n$ requires a modification of the standard treatment of the zero modes that will lead to a point independent, Weyl and modular invariant regularization of such ratio of determinants.

In the previous section we saw that the string determinants are strictly related to the Mumford forms. In particular,

$$|F_{g,n}[\varphi^n]|^2 = \left(\frac{\det' \Delta_0}{\mathcal{N}_0 \det \mathcal{N}_1}\right)^{-c_n} \frac{\det' \Delta_{1-n}}{\det \mathcal{N}_n}. \qquad (3.5)$$

Until now, the only Mumford form which appeared in string theory is $\mu_{g,2}$, that is the one defining the Polyakov measure. On the other hand, to express (3.2) as well-defined quantities on $\mathcal{M}_g$, requires one to find what is the precise correspondence between the modulo square of Mumford forms $\mu_{g,n}$ and $\mathcal{Z}_n$. This has been an open question since the times of the covariant formulation of string theories. In particular, Belavin and Knizhnik stressed that the Mumford forms are Weyl anomaly free [7]. They also observed that since the holomorphic structure of $\mathcal{M}_g$ is an algebraic structure, it follows that any holomorphic quantity on $\mathcal{M}_g$, such as $\mu_{g,2}$, is an algebraic object. This is essentially a consequence of the Serre GAGA principle [33] that led to the following conjecture [7].

Multiloop amplitudes (and not only vacuum amplitudes) in any conformally invariant string theory (such as the bosonic in $D = 26$ or the superstring in $D = 10$) can be expressed in terms of algebraic objects (functions or sections of holomorphic bundles) on the moduli space of Riemann surfaces. Quantum geometry is therefore the complex geometry of the space $\bar{\mathcal{M}}_g$.

In this context, it was suggested in [34] that noncritical strings may be formulated in terms of Mumford forms. In spite of its geometrical and physical elegance, the Belavin-Knizhnik conjecture has not been developed so far. There are several reasons for that. We will see that such reasons are strictly related and admit a natural physical solution leading to a modular invariant regularization of the string determinants.

An obvious reason why apparently the Mumford forms of degree $n \neq 2$ should not play a role in string theory is that only $|\mu_{g,2}|^2$ defines a volume form on $\mathcal{M}_g$. This question was in debate during the eighties. To map $|\mu_{g,n}|^2$ to volume forms on $\mathcal{M}_g$ requires solving the problem of the fiber, that is, loosely speaking, replacing the wedge products of n-differentials by scalar quantities. Let us recall where the point is.

First, note that $|\wedge^{\max}\varphi_j^1|^2$ and $\det \int_C \bar{\varphi}_j^1 \wedge \varphi_k^1$ have the same modular transformations. In particular, the Hodge fiber $|\wedge^{\max}\omega_i|^2$ maps to

$$\det \frac{1}{2i}\int_C \bar{\omega}_j \wedge \omega_k = \det \tau_2,$$

where $\tau_2 = \mathrm{Im}\tau$, with $\tau_{jk} = \oint_{\beta_j}\omega_k$ the Riemann period matrix.

Even the wedge products $|\wedge^{\max}\varphi_i^2|^2$ appearing in the Polyakov string are not a problem; since they represent the infinitesimal volume elements on $\mathcal{M}_g$, it is just the term coming from the path integration on the metric. However, it is commonly believed that integrating on C^{N_n} in the case of $|\wedge^{\max}\varphi_i^n|^2$, $n \neq 2$ (or even $n = 2$, if one wants to reduce $|\mu_{g,2}|^2$ to a scalar quantity) leads to a Weyl anomaly. The reason is that $|\det \varphi_j^n(z_k)|^2$ requires a metric to be integrated. However, the metric with respect to which one has to consider the Weyl transformations is the one on which one integrates in the path integral and these concern only the metric defining the Laplacians and the associated zero mode matrices $(\mathcal{N}_n)_{ij}$. This means that, in principle, one can multiply $|\det \varphi_j^n(z_k)|^2$ by the product of any $(1-n, 1-n)$ form in z_k and then integrating over C^{N_n} without worrying about any Weyl anomaly. Nevertheless, it is clear that this would lead to a considerable ambiguity. In this respect note that making a Weyl transformation requires one to identify which ones of the (1, 1)-forms in a given expression correspond to the metric or are defined in a metric dependent way. The question then is to find a positive definite (1, 1)-form which is defined in a Weyl and modular invariant way.

B. Weyl and modular invariant integration

A key observation is that since the kernel of $\partial_{\bar{z}}$ is metric independent, it follows that the space of zero modes $H^0(K_C^n)$ is Weyl invariant. We then introduce the $(1, 1)$-forms

$$B_Y(z, \bar{z}) = \sum_1^g \varphi_j^1(z) Y_{jk} \bar{\varphi}_k^1(z),$$

and will integrate $\prod_{k=1}^{N_n} B_Y^{1-n}(z_k, \bar{z}_k)|\det \varphi_j^n(z_k)|^2$ on C^{N_n}. To fix Y we use modular invariance. To this end, we use the fact that the dependence on the modular transformations of the integration on C is entirely given by the transformation properties of the integrand. This means that $B_Y(z, \bar{z})$ is modular invariant and positive definite if $Y_{jk}^{-1} = \frac{1}{2i}\int_C \varphi_j^1 \wedge \bar{\varphi}_k^1$. With this choice, $B_Y(z, \bar{w})$ coincides with

$$B(z, \bar{w}) = \sum_1^g \omega_j(z)(\tau_2^{-1})_{jk}\bar{\omega}_k(w), \qquad (3.6)$$

which is the Bergman reproducing kernel [29]. It should be stressed that the above investigation does not imply that

$B(z,\bar{z})$ cannot be considered as a metric on C. This is a general fact that holds for any positive definite (1, 1)-form. In particular, any ratio between two positive definite (1, 1)-forms defines a possible Weyl transformation. In other words, considering a reference metric tensor ρ, there exists a Weyl transformation such that $e^{\sigma}\rho = B$. On the other hand, this does not mean that under a Weyl transformation one should transform all possible (1, 1)-forms. For example, if an expression contains the term ρB, under the Weyl transformation $e^{\sigma}\rho = B$ would transform to B^2, whose inverse is ρB. We will perform a similar transformation that will simplify the expressions of $\mathcal{Z}_n$ [see Eq. (3.11)].

It is instructive to recall that the Bergman reproducing kernel also appears in the two point function of a scalar field

$$f(z,w) = \langle X(z)X(w)\rangle.$$

This is the Green function for the scalar Laplacian, so that it satisfies the following equations:

$$\int_C \sqrt{g} f(z,w) = 0,$$
$$\partial_z\partial_{\bar{z}} f(z,w) = -\pi\delta(z-w) + \frac{\pi}{A}\sqrt{g(z)},$$
$$\partial_z\partial_{\bar{w}} f(z,w) = \pi\delta(z-w) - \pi B(z,\bar{w}),$$

where $A = \int_C \sqrt{g}$. This shows that the building block of the string correlation functions naturally selects a Weyl and modular invariant kernel, providing another way to show that the Bergman reproducing kernel (3.6) is intrinsically defined; it depends only on the complex structure of C and on the points z and w. One may say that it is essentially the only way to select two points on C in a way which is anomaly free. In particular, since $B(z,\bar{z})$ is a positive definite (1, 1)-form, it can be used to integrate the zero modes in a Weyl and modular invariant way. Multiplying (2.2) by $\prod_1^{N_n} B^{1-n}(z_j,\bar{z}_j)$ and integrating over C^{N_n} leads to

$$X_n := \det \mathcal{M}_n \frac{\det' \Delta_{1-n}}{\det \mathcal{N}_n} = \int Db D\bar{b} Dc D\bar{c} (b\bar{b})_n \times \exp\left(-\frac{1}{2\pi}\int_C \sqrt{g} b \nabla^z_{1-n} c + \text{c.c.}\right), \tag{3.7}$$

where

$$(b\bar{b})_n = \int_{C^{N_n}} \prod_j B^{1-n}(z_j,\bar{z}_j) b(z_j)\bar{b}(z_j),$$

and

$$(\mathcal{M}_n)_{jk} = \int_C \bar{\varphi}_j^n(z) B^{1-n}(z,\bar{z}) \varphi_k^n(z).$$

The string determinants are then

$$\mathcal{Z}_n = X_n \int DX e^{-S[X]}. \tag{3.8}$$

Furthermore, we now have the precise identification of the string determinants in (3.3)

$$\mathcal{Z}_n = \left(\frac{\det' \Delta_0}{\mathcal{N}_0}\right)^{-c_n} \frac{\det \mathcal{M}_n \det' \Delta_{1-n}}{\det \mathcal{N}_n}. \tag{3.9}$$

Note that $\det \mathcal{M}_1 = \det \mathcal{N}_1 = \det \tau_2$. It should be stressed that the above prescription is equivalent to map $|\mu_{g,n}|^2$ to the (0, 0)-forms

$$\mathcal{Z}_n = |F_{g,n}|^2 \frac{\det \mathcal{M}_n}{(\det \tau_2)^{c_n}}, \tag{3.10}$$

which is equivalent to map the modulo square of the wedge products in the Mumford to (0, 0)-forms, that is

$$\frac{|\wedge^{\max} \varphi_j^n|^2}{|\wedge^g \omega_j|^{2c_n}} \to \frac{\det \mathcal{M}_n}{(\det \tau_2)^{c_n}}.$$

The above results show that if $D = 2c_n$, then the corresponding partition function admits a natural Weyl and modular invariant regularization which is point independent.

There is a remarkable mechanism that simplifies the expression of $\mathcal{Z}_n$ in (3.9). Namely, since $|F_{g,n}|^2$ in (3.5) is Weyl invariant, we can just choose as a metric the Bergman metric, that is

$$\rho(z,\bar{z}) = B(z,\bar{z}).$$

In this way

$$\det \mathcal{N}_n = \det \mathcal{M}_n, \qquad \mathcal{N}_0 = \int_C B = g.$$

The result is that the basis φ_j^n does not appear at all and $\mathcal{Z}_n$ assumes the simplified form

$$\mathcal{Z}_n = (\det' \Delta_{B,0})^{-c_n} \det' \Delta_{B,1-n}, \tag{3.11}$$

where the Laplacians are defined with respect to the Bergman metric. In particular, with this choice

$$\mathcal{Z}[\mathcal{J}] = (\det' \Delta_{B,0})^{-13-\sum_{k\in\mathcal{I}} n_k c_k} \frac{\det' \Delta_{B,-1}}{\det \mathcal{M}_2} \times \prod_{k\in\mathcal{I}} (\det' \Delta_{B,1-n})^{n_k} |\wedge^{3g-3} \varphi_j^2|^2, \tag{3.12}$$

that, by (3.10), is

$$\mathcal{Z}[\mathcal{J}] = \frac{|F_{g,2}|^2}{(\det \tau_2)^{13}} \prod_{k\in\mathcal{I}} \left(\frac{|F_{g,k}|^2 \det \mathcal{M}_k}{(\det \tau_2)^{c_k}} \right)^{n_k} |\wedge^{3g-3} \varphi_j^2|^2, \tag{3.13}$$

which, by (2.4,2.5) and (2.7) provides the expression of $\mathcal{Z}[\mathcal{J}]$ in terms of theta functions.

IV. $\mathcal{Z}[\mathcal{J}]$ NONCRITICAL STRINGS AND W ALGEBRAS

A. Noncritical strings

The typical singularities of string theories arise when some handle of the Riemann surface is pinched. Such a degenerate surface belongs to the Deligne-Mumford boundary $\partial\bar{\mathcal{M}}_g$. The standard example is the tachyon singularity. Let us consider the singularity structure at $\partial\bar{\mathcal{M}}_g$ associated to the Mumford forms for any n. The tachyon singularity of the critical bosonic string corresponds to $n = 2$.

In [29] Fay derived the singular behavior of the Mumford forms at the Deligne-Mumford boundary. He used Bers-like basis $\varphi_t^n = \{\varphi_{i,t}^n\}_{i\in I_{N_n}}$ for $H^0(K_C^n)$. It turns out that in the case of separating degeneration

$$F_{g,n}[\varphi_t^n] \sim t^{-n(n-1)/2} \frac{E(a,b)^{n-n^2}}{(2\pi i)^{(2n-1)^2}} F_{g-1,n}[\varphi^n], \tag{4.1}$$

where a, b are two points identified on the smooth genus $g-1$ curve. In the case of degeneration corresponding to a reducible singular curve obtained by identifying points on two smooth curves of genus g_1 and $g-g_1$, we have

$$F_{g,n}[\varphi_t^n] \sim \epsilon t^{-n(n-1)/2} F_{g-g_1,n}[\varphi^n] F_{g_1,n}[\varphi^n], \tag{4.2}$$

where ϵ is a fixed $(2g-2)$th root of unity. The tachyon singularity of the critical bosonic string corresponds to $n = 2$. The above asymptotic analysis is just a consequence of the Grothendieck-Riemann-Roch theorem and of the Mumford formula.

Let us consider the measure on the world-sheet metric. This includes the integration on the diffeomorphisms and on the Liouville field $D_g v^z D_g v^{\bar{z}} D_g \sigma$. It is well known that such measures are not Gaussian. This is the problem of quantizing Liouville theory. Let us consider the measure on the diffeomorphisms. Since

$$\langle v, v\rangle_{g=e^\sigma \hat{g}} = \int_C \sqrt{\hat{g}} \hat{g}_{ab} e^{2\sigma} v^a v^b,$$

it follows that $\mathrm{Vol}_g(\mathrm{Diff}(\Sigma))$ depends on σ. In critical string theory it is assumed that such a dependence can be absorbed into $D_g\sigma$ and then one drops the $D_g v^z D_g v^{\bar{z}}$ term. However for $D \neq 26$ such a procedure still needs to be fully understood. To overcome such a question we consider the bosonic partition function in noncritical dimensions

$$\int_{\mathcal{M}_g} \mathcal{Z}_D = \int DgDX \exp(-S[X]), \tag{4.3}$$

where $S(X)$ is the Polyakov action in D dimensions, so that $\mathcal{Z}_{\mathrm{Pol}} = \mathcal{Z}_{26}$. Of course, like $\mathcal{Z}[\mathcal{J}]$, even $\mathcal{Z}_D$ must be a well-defined volume form on $\mathcal{M}_g$. Since $\mathcal{Z}_D$ should be a volume form on $\mathcal{M}_g$, the central charge of the Liouville sector is

$$c_L = 26 - D. \tag{4.4}$$

This is the reason why (4.4) has the same structure of (1.2). This suggests considering

$$c_L = -\sum_{n_k\in\mathcal{I}} 2n_k c_k. \tag{4.5}$$

By means of a semiclassical analysis it should be possible to check the behavior of $\mathcal{Z}_D$ when the Riemann surface degenerates, that is at the Deligne-Mumford boundary $\partial\bar{\mathcal{M}}_g$. This fixes some condition on $\mathcal{J}$ in such a way that $\mathcal{Z}[\mathcal{J}]$ has the same behavior of $\mathcal{Z}_D$. This would select the first order systems as possible candidates to represent the Liouville partition functions. This means that there is a mechanism, related to the bosonization of first order systems, mapping the non-Gaussian measures to the Gaussian ones of the b-c and β-γ systems, as suggested in [34].

It is instructive to understand what happens in the case in which the Liouville sector can be represented by a single first order system. Let us first consider the case of the β-γ system. This means

$$c_L = 12k^2 - 12k + 2, \tag{4.6}$$

that is the weight of the corresponding β-γ system is

$$k = \frac{3 \pm \sqrt{81 - 3D}}{6}. \tag{4.7}$$

Rational values of k with integer D are obtained for

$$D = 0, 15, 24, 27,$$

corresponding to

$$k = 2, 3/2, 1, 1/2.$$

Similarly, in the case of b-c systems one gets

$$c_L = -12k^2 + 12k - 2, \tag{4.8}$$

$$k = \frac{3 \pm \sqrt{3D - 75}}{6}, \tag{4.9}$$

and rational values of k with integer D correspond to

$$D = 25 + 3n^2,$$

with

$$k = \frac{1}{2}(n+1),$$

$n = 0, 1, 2, \ldots$. The above can be generalized to real and even complex values of n. From the point of view of the b-c and β-γ systems this is always possible just because the action contains terms such as $b\bar{\partial}c$ and $\beta\bar{\partial}\gamma$, which are well-defined (1, 1)-forms even for $n \in \mathbb{C}$. A related aspect has been considered in [35] where a general method to absorb the spin fields in b-c systems of real weight was introduced. Subsequently, complex powers of line bundles in connection with string scattering amplitudes have been considered by Voronov [36]. Such an extension of first order systems to real and, more generally, complex weights is of considerable interest and should be further investigated.

B. W algebras and volume forms on $\mathcal{M}_g$

There is a nice interpretation of the Mumford forms that should be further investigated. A reason why the Polyakov partition function leads to $|\mu_{g,2}|^2$ is that the world-sheet metric is deformed by the Beltrami differentials; these are the dual of $H^0(K_C^2)$. It follows that $|\mu_{g,n}|^2$ should be associated to a theory containing the path integration on a metric deformed by the generalized Beltrami differentials introduced [37]. These are the dual of $H^0(K_C^n)$. In particular, using the single indexing introduced in the Appendix, one may consider the map

$$\omega_i^{(n)} \mapsto \frac{1}{2\pi i} d\tau_i^{(n)}$$

that defines the tangent space to the moduli space associated to the holomorphic n-differentials, that is the moduli space of vector bundles on Riemann surfaces. This is like the Kodaira-Spencer map sending $\omega_i^{(2)}$ to $\frac{1}{2\pi i} d\tau_i^{(2)}$. It follows that $|\mu_{g,n}|^2$ should correspond to

$$\int Dg^{(n)} D\Phi e^{-S[\varphi]}.$$

More generally, one should consider partition functions such as

$$\int \prod_{k\in\mathcal{I}} Dg^{(k)} D\Phi e^{-S[\Phi]}, \tag{4.10}$$

where $g^{(k)}$ are the metrics associated to $H^0(K_C^k)$, whose dual spaces are the generalized Beltrami differentials, and $S[\Phi]$ some conformal action leading to an anomaly $2\sum_{k\in\mathcal{I}} c_k$. Understanding the field content of (4.10) should lead to formulate a class of conformal field theories. In this respect note that the generalized Beltrami differentials are related to the chiral split for the higher order diffeomorphism anomalies. The Wess-Zumino conditions correspond to the cocycle identities (see Sec. 3.4 of [37]).

Since such theories are associated to W algebras, it would be interesting to investigate a possible relation with higher spin theories. In particular, note that in a large N 't Hooft-like limit 2D W_N minimal models CFTs are related to higher spin gravitational theories on AdS_3 [38]. This may suggest the existence of a formulation of higher spin theories in a string context.

V. VOLUME FORMS ON $\mathcal{M}_g$ AND HADAMARD PRODUCT

Another interesting possibility to use the Bergman reproducing kernel to remove the point dependence due to the zero mode insertions is to consider the determinant of the Hadamard n-fold product of $B(z_i, \bar{z}_j)$. Although such a recipe may lead, depending on the set $\mathcal{J}$, to zeroes or singularities on the hyperelliptic Riemann surfaces, it defines volume forms on $\mathcal{M}_g$ with interesting properties.

A. Hadamard product of the Bergman kernel

Let us first shortly review a result in [30]. Here we use the single index notation defined in the Appendix. In the following we consider the matrix $B^{\circ n}(z_j, \bar{z}_k)$ whose j, kth entry is $(B(z_j, \bar{z}_k))^n$. This is the n-fold Hadamard product of $B(z_j, \bar{z}_k)$. Furthermore, we will consider the determinant of $B^{\circ n}(z_j, \bar{z}_k)$, with the indices j, k ranging from 1 to N_n. For each positive integer n defines $I_n = \{1, \ldots\ldots, n\}$. For all $z_i, w_i \in C$, $i \in I_{N_n}$, we define

$$K_n = \frac{\det B^{\circ n}(z_i, \bar{z}_j)}{|\det \varphi_j^n(z_k)|^2} |\kappa[\varphi^n]|^2. \tag{5.1}$$

It can be proved that [30]

$$K_n = \sum_{\substack{i_{N_n} > \cdots > i_1 = 1 \\ j_{N_n} > \cdots > j_1 = 1}}^{M_n} \kappa[\omega_{i_1}^{(n)}, \ldots, \omega_{i_{N_n}}^{(n)}] \times \frac{|\tau_2^{-1} \cdots \tau_2^{-1}|_{j_1 \ldots j_{N_n}}^{i_1 \ldots i_{N_n}}}{\prod_{k=1}^{N_n} \chi_{i_k} \chi_{j_k}} \bar{\kappa}[\omega_{j_1}^{(n)}, \ldots, \omega_{j_{N_n}}^{(n)}], \tag{5.2}$$

where $|A \cdots A|_{j_1 \cdots j_m}^{i_1 \cdots i_m}$ denotes the minors of $(A \cdots A)$

$$|A \cdots A|_{j_1 \cdots j_m}^{i_1 \cdots i_m} = \det_{\substack{i \in i_1, \ldots, i_m \\ j \in j_1, \ldots, j_m}} (A \cdots A)_{ij},$$

$i_1, \ldots, i_m, j_1, \ldots, j_m \in I_{M_n}$, with $m \in I_{M_n}$.

B. Zero modes, det $B^{\circ n}(z_j, \bar{z}_k)$ and volume forms on $\mathcal{M}_g$

In the following we investigate a way to absorb the point dependence due to the insertion of the zero modes which is related to the one introduced in Sec. III. This may lead to zeros or singularities on the hyperelliptic Riemann surfaces of genus greater than two. The moduli space of compact Riemann surfaces that does not contain such Riemann surfaces is called the moduli space of canonical curves $\hat{\mathcal{M}}_g$.

Let us set

$$\kappa[\varphi^1] = \frac{1}{Z_1[\varphi^1]^{\frac{3}{2}}},$$

and, for $n > 1$

$$\kappa[\varphi^n] = \frac{1}{Z_1[\omega]^{\frac{3}{2}} Z_n[\varphi^n]}.$$

Note that $\det B(z_j, z_k) = |\det \omega_j(z_k)|^2 / \det \tau_2$, so that

$$K_1 = \frac{|\kappa[\omega]|^2}{\det \tau_2}.$$

Also note that replacing $|\omega_1 \wedge \cdots \wedge \omega_g|^2$ in $|\mu_{g,n}|^2$ by $\det \tau_2$ does not break modular invariance. Together with the Kodaira-Spencer map $\omega_i \omega_j \to d\tau_{ij}/(2\pi i)$, this is what one does in passing from the Mumford form $|\mu_{g,2}|^2$ to the Polyakov measure. What is less obvious is the analog of $\det \tau_2$ when one considers the wedge products $\varphi_1^n \wedge \cdots \wedge \varphi_{N_n}^n$. First notice that due to the term $\det \varphi_i^n(z_j)$ in $\kappa[\varphi^n]$, the Mumford forms are independent of the choice of the bases $\varphi_1^n, \ldots.., \varphi_{N_n}^n$. On the other hand, using τ_{ij} as moduli parameters naturally requires one to use N_n elements $\omega_i^{(n)}$ as basis of $H^0(K_C^n)$, a fact that led to the concept of vector-valued Teichmüller modular forms [28,39]. Therefore, one has to consider $\omega_{i_1}^{(n)} \wedge \cdots \wedge \omega_{i_{N_n}}^{(n)}$. In order to have a volume form for $n = 2$, one has to consider the Kodaira-Spencer map[1]

$$|\omega_{i_1}^{(2)} \wedge \cdots \wedge \omega_{i_{3g-3}}^{(2)}|^2 \to |d\tau_{i_1} \wedge \cdots \wedge d\tau_{i_{3g-3}}|^2.$$

However, as it will be clear below, we can also consider a map involving K_2. Let us stress that, thanks to the term $\det \omega_i^{(2)}(z_j)$ in the denominator of $\mu_{g,2}$, when $d\tau_{i_1} \wedge \cdots \wedge d\tau_{i_{3g-3}}$ vanishes in some subspace of $\mathcal{M}_g$, e.g. in the hyperelliptic loci of genus $g \geq 3$, this is balanced by the vanishing of $\det \omega_i^{(2)}(z_j)$.

The situation is different when looking for the analog of the map $|\omega_1 \wedge \cdots \wedge \omega_g|^2 \to \det \tau_2$ in the case of $\omega_{i_1}^{(n)} \wedge \cdots \wedge \omega_{i_{N_n}}^{(n)}$, even in the case $n = 2$. The answer is to replace the modulo square of Mumford forms building blocks by K_n. In particular,

$$\left|\frac{\kappa[\omega]}{\omega_1 \wedge \cdots \wedge \omega_g}\right|^2 \to K_1,$$

and, for $n \neq 1$

$$\left|\frac{\kappa[\varphi^n]}{\varphi_1^n \wedge \cdots \wedge \varphi_{N_n}^n}\right|^2 \to K_n,$$

so that

$$|\mu_{g,n}|^2 = \left|\frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\varphi^n]} \frac{\varphi_1^n \wedge \cdots \wedge \varphi_{N_n}^n}{(\omega_1 \wedge \cdots \wedge \omega_g)^{c_n}}\right|^2 \tag{5.3}$$

maps to the nonchiral analog

$$V_n(\tau) = \frac{K_1^{(2n-1)^2}}{K_n} \frac{1}{(\det \tau_2)^{2n(n-1)}}, \tag{5.4}$$

which is a (0, 0)-form on $\mathcal{M}_g$. Equation (5.2) shows that the building blocks of V_n are just the vector-valued Teichmüller modular forms introduced in [28]

$$[i_{N_n+1}, \ldots, i_{M_n} | \tau] = \epsilon_{i_1, \ldots, i_{M_n}} \frac{\kappa[\omega_{i_1}^{(n)}, \ldots, \omega_{i_{N_n}}^{(n)}]}{\kappa[\omega]^{(2n-1)^2}}, \tag{5.5}$$

$i_1, \ldots.., i_{M_n} \in \{1, \ldots.., M_n\}$, and that define the string measures [39].

Recall that $\mathcal{I}$ denotes the set of conformal weights $k \in \mathbb{Q}$ and $\mathcal{J}$ is the set of $n_k \in \mathbb{Z}/2$. We consider the volume forms on $\mathcal{M}_g$

$$V[\mathcal{J}] = \mathcal{Z}_{\text{Pol}} \prod_{k \in \mathcal{I}} V_k^{n_k}. \tag{5.6}$$

Note that since $\kappa[\omega_{i_1}^{(n)}, \ldots.., \omega_{i_{N_n}}^{(n)}]$ vanishes on the hyperelliptic loci with $g \geq 3$ [28], by (5.2) also K_n vanishes there. Therefore, $V[\mathcal{J}]$, depending on the set $\mathcal{J}$, may be vanishing or singular in such loci. Consider

$$V[-1_2] = K_2(\tau_2^{-1}) \left|\frac{d\tau_{i_1} \wedge \cdots \wedge d\tau_{i_{3g-3}}}{\kappa[\omega_{i_1}^{(2)}, \ldots, \omega_{i_{3g-3}}^{(2)}]}\right|^2.$$

This can be very explicitly expressed up to $g = 4$. For $g = 2$ and $g = 3$ we have

$$V[-1_2] = \frac{|d\tau_1 \wedge d\tau_2 \wedge d\tau_3|^2}{(\det \tau_2)^3},$$

$$V[-1_2] = \frac{|d\tau_1 \wedge \cdots \wedge d\tau_6|^2}{(\det \tau_2)^4}.$$

[1]Note that such two terms have the same transformation properties under $\text{Sp}(2g, \mathbb{Z})$.

In the case of genus four

$$V[-1_2] = \sum_{\substack{i_9 > \cdots i_1 = 1 \\ j_9 > \cdots > j_1 = 1}}^{10} S_{4p}(\tau) \frac{|\tau_2^{-1}\tau_2^{-1}|^{i_1 \ldots i_9}_{j_1 \ldots j_9}}{\prod_{k=1}^9 \chi_{i_k}\chi_{j_k}} \bar{S}_{4q}(\tau) \left| \frac{d\tau_1 \wedge \cdots \wedge \widehat{d\tau_k} \wedge \cdots \wedge d\tau_{10}}{S_{4k}(\tau)} \right|^2, \quad (5.7)$$

where $p = I_{10} \backslash \{i_1, \ldots, i_9\}$ and $q = I_{10} \backslash \{j_1, \ldots, j_9\}$ (note that here we are using the single indexing notation introduced in the Appendix) and

$$S_{4ij}(Z) = \frac{1 + \delta_{ij}}{2} \frac{\partial F_4(Z)}{\partial Z_{ij}},$$

with

$$F_g = 2^g \sum_{\delta \text{even}} \theta^{16}[\delta](0, Z) - \left(\sum_{\delta \text{even}} \theta^8[\delta](0, Z) \right)^2.$$

F_4 is the Schottky-Igusa form, and has the property of vanishing only on the Jacobian, so that it provides the effective solution of the Schottky problem. It is immediate to see that $V[-1_2]$ is the volume form on $\mathcal{M}_g$ induced by the Siegel metric on the Siegel upper half-space. Its expression for any genus, but without the use of theta constants, was given in [30] (see also [40,41]).

From the above findings it follows that

$$V_n = Y_n \int DX e^{-S[X]}, \quad (5.8)$$

where $S[X]$ is the Polyakov action in $2c_n$ dimensions and

$$Y_n = \int Db D\bar{b} Dc D\bar{c} \frac{\prod_i b(z_i) \bar{b}(z_i)}{\det B^{\circ n}(z_j, \bar{z}_k)} \times \exp\left(-\frac{1}{2\pi} \int_C \sqrt{g} b \nabla^z_{1-n} c + \text{c.c.} \right). \quad (5.9)$$

By (2.2) and (5.1), it follows that

$$Y_n = \frac{|\kappa[\varphi^n]|^2}{K_n} \frac{\det' \Delta_{1-n}}{\det \mathcal{N}_n}. \quad (5.10)$$

Furthermore, by (5.6) we have the following Weyl anomaly free partition functions:

$$\int_{\mathcal{M}_g} V[\mathcal{J}] = \int Dg DX D\Psi \exp(-S[X] - S[\Psi]), \quad (5.11)$$

where now $S[X]$ is the Polyakov action in $D = 26 + 2\sum_{k \in \mathcal{I}} n_k c_k$ dimensions. $D\Psi$ is the product on $k \in \mathcal{I}$ of $|n_k|$ copies of the measure of weight k b-c systems, including the zero modes insertion, if $n_k > 0$, or β-γ systems if $n_k < 0$. $S[\Psi]$ denotes the sum of the corresponding nonchiral actions.

C. Curvature forms

Both X_n in (3.7) and Y_n in (5.10) provide an enumeration of the Laplacian of determinants whose normalization eliminates the dependence on the choice of the basis of $H^0(K_C^n)$. We saw that whereas X_n naturally arises by looking for a Weyl and modular invariant determinant regularization, in the case of Y_n, the hyperelliptic loci may be in their divisor. Such a property, and the structure of both X_n and Y_n, suggest that they represent key quantities to investigate the geometry of $\bar{\mathcal{M}}_g$.

Let us go back to the space $\bar{\mathcal{M}}_g$. It turns out that the components D_k of Deligne-Mumford boundary, introduced in (2.1), provide, together with the divisor associated to Weil-Petersson class $[\omega_{WP}]/2\pi^2$, a basis for $H_{6h-8}(\bar{\mathcal{M}}_g, \mathbb{Q})$. Consider the universal curve $\mathcal{C}\bar{\mathcal{M}}_{g,n}$ over $\bar{\mathcal{M}}_{g,n}$, built by placing over each point of $\bar{\mathcal{M}}_{g,n}$ the corresponding curve. Note that $\bar{\mathcal{M}}_{g,1}$ can be identified with $\mathcal{C}\bar{\mathcal{M}}_g$. More generally $\bar{\mathcal{M}}_{g,\underline{n}}$ can be identified with $\mathcal{C}_n(\bar{\mathcal{M}}_g) \backslash \{\text{sing}\}$ where $\mathcal{C}_n(\bar{\mathcal{M}}_g)$ denotes the n-fold fiber product of the n copies $\mathcal{C}_{(1)}\bar{\mathcal{M}}_g, \ldots, \mathcal{C}_{(n)}\bar{\mathcal{M}}_g$ of the universal curve over $\bar{\mathcal{M}}_g$ and $\{sing\}$ is the locus of $\mathcal{C}_n(\bar{\mathcal{M}}_g)$ where the punctures come together.

Denote by $K_{\mathcal{C}/\mathcal{M}}$ the cotangent bundle to the fibers of $\mathcal{C}\bar{\mathcal{M}}_{g,n} \to \bar{\mathcal{M}}_{g,n}$, built by taking all the spaces of (1, 0)-forms on the various Σ and pasting them together into a bundle over $\mathcal{C}\bar{\mathcal{M}}_{g,n}$. Let C be a curve in $\bar{\mathcal{M}}_{g,n}$. Consider the cotangent space $T^*C_{|_{z_i}}$. It varies holomorphically with z_i giving a holomorphic line bundle $\mathcal{L}_{(i)}$ on $\bar{\mathcal{M}}_{g,n}$. Considering the z_i as sections of the universal curve $\mathcal{C}\bar{\mathcal{M}}_{g,n}$ we have $\mathcal{L}_{(i)} = z_i^*(K_{\mathcal{C}/\mathcal{M}})$.

Let us consider the Witten intersection numbers [42]

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\bar{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_{(1)})^{d_1} \wedge \cdots \wedge c_1(\mathcal{L}_{(n)})^{d_n}, \quad (5.12)$$

which are nonvanishing only if $\sum d_i = 3g - 3 + n$. These are related to the Mumford tautological classes [43]

$$\kappa_l = \pi_*(c_1(\mathcal{L})^{l+1}) = \int_{\pi^{-1}(p)} c_1(\mathcal{L})^{l+1}, \quad (5.13)$$

$p \in \bar{\mathcal{M}}_g$, where $\mathcal{L}$ is the line bundle whose fiber is the cotangent space to the one marked point of $\bar{\mathcal{M}}_{g,1}$ and π: $\bar{\mathcal{M}}_{g,1} \to \bar{\mathcal{M}}_g$ is the projection forgetting the puncture. The κ's correlation functions $\langle\kappa_{s_1}\cdots\kappa_{s_n}\rangle = \langle\wedge_{i=1}^n\kappa_{s_i},\bar{\mathcal{M}}_g\rangle$, which are nonvanishing only if $\sum_i s_i = 3g-3$, are related to and τ's correlators. For example performing the integral over the fiber of $\pi:\bar{\mathcal{M}}_{g,1}\to\bar{\mathcal{M}}_g$,

$$\langle\tau_{3g-2}\rangle = \int_{\bar{\mathcal{M}}_{g,1}} c_1(\mathcal{L})^{3g-2} = \int_{\bar{\mathcal{M}}_g}\kappa_{3g-3} = \langle\kappa_{3g-3}\rangle. \quad (5.14)$$

It is useful to express the κ's correlators in the form [42]

$$\langle\kappa_{d_1-1}\cdots\kappa_{d_n-1}\rangle = \int_{\mathcal{C}_n(\bar{\mathcal{M}}_g)} c_1(\hat{\mathcal{L}}_{(1)})^{d_1}\wedge\cdots\wedge c_1(\hat{\mathcal{L}}_{(n)})^{d_n}, \quad (5.15)$$

where $\hat{\mathcal{L}}_{(i)} = \pi_i^*(K_{\mathcal{C}_{(i)}/\mathcal{M}})$ and $\pi_i\colon\mathcal{C}_n(\bar{\mathcal{M}}_g)\to\mathcal{C}_{(i)}\bar{\mathcal{M}}_g$ is the natural projection. Then notice that $\mathcal{C}_n(\bar{\mathcal{M}}_g)$ and $\bar{\mathcal{M}}_{g,n}$ differ for a divisor at infinity only. This is the unique difference between $\langle\kappa_{d_1-1}\cdots\kappa_{d_n-1}\rangle$ and $\langle\tau_{d_1}\cdots\tau_{d_n}\rangle$. This leads to relations for arbitrary correlators.

In [44] Wolpert proved that the first tautological class corresponds to the Weil-Petersson two-form

$$\kappa_1 = \omega_{WP}/\pi^2.$$

By well-known results on ω_{WP}, it follows that

$$\kappa_1 = \frac{6i}{\pi}\bar\partial\partial\log\frac{\det\tau^{(2)}}{\det'\Delta_{\hat{g},0}}, \quad (5.16)$$

where $\Delta_{\hat{g},0}$ denotes the Laplacian with respect to the Poincaré metric and ∂, $\bar\partial$ are the holomorphic and antiholomorphic components of the external derivative $d=\partial+\bar\partial$ on $\mathcal{M}_g$. This implies that κ_1 can be seen as the curvature form in $\mathcal{M}_g$ of the Hodge line bundle $(\lambda_1;\langle,\rangle_Q)$ endowed with the Quillen norm

$$\langle\omega,\omega\rangle_Q = \frac{\det\tau^{(2)}}{\det'\Delta_{\hat{g},0}}, \quad (5.17)$$

where $\omega\equiv\omega_1\wedge\cdots\wedge\omega_g$, that is

$$\kappa_1 = 12c_1(\lambda_1). \quad (5.18)$$

A natural question is whether this is a signal for the existence of more general relations between determinant of Laplacians and the tautological classes. This would also imply a relationship between 2D gravity, topological theories and the string determinants.There is an obstruction to extend Eq. (5.18) to the tautological classes of higher degree. The reason is that in general the Quillen norm depends on the choice of the basis of $H^0(K_C^n)$. The exception is just (5.17) since in this case there is a canonical choice, that is $\omega_1,\ldots,\omega_g$, which is the one defining the Hodge bundle. On the other hand, it has been shown in [28] that there are natural bases for $H^0(K_C^n)$ outside the hyperelliptic locus, the $\omega_i^{(n)}$'s. In the case $n=2$ this led to basic results on the Polyakov measure [39], which in fact corresponds to the coefficients of the quadrics describing the world sheet in the projective space $\mathbb{P}^{g-1}$. This suggests defining

$$\langle i_{N_n+1},\ldots,i_{M_n},i_{N_n+1},\ldots,i_{M_n}\rangle = \frac{\det\mathcal{N}_n(i_{N_n+1},\ldots,i_{M_n})}{\det'\Delta_{\hat{g},1-n}}, \quad (5.19)$$

with

$$\det\mathcal{N}_n(i_{N_n+1},\ldots,i_{M_n}) = \det\int_C\bar\omega_j^{(n)}\hat\rho^{1-n}\omega_k^{(n)},$$

where $\hat\rho\equiv 2\hat{g}_{z\bar z}$ is the Poincaré metric tensor in local complex coordinates and the determinant is taken on the matrix's indices running in the set $\{i_1,\ldots,i_{N_n}\}$. Note that for $n=1$ (5.19) coincides with the Quillen norm. One may immediately check that it holds

$$|\mu_{g,n}|^2 = \frac{\langle\omega,\omega\rangle_Q^{c_n}}{\langle i_{N_n+1},\ldots,i_{M_n},i_{N_n+1},\ldots,i_{M_n}\rangle}\frac{|\omega_{i_1}^{(n)}\wedge\cdots\wedge\omega_{i_{N_n}}^{(n)}|^2}{|\omega_1\wedge\cdots\wedge\omega_g|^{2c_n}}. \quad (5.20)$$

On the other hand, the problem of the independence on the choice of the basis has been one of the main points of our initial investigation. This led us to introduce X_n and Y_n, which in fact do not depend on the choice of the basis $H^0(K_C^n)$. Therefore both X_n and Y_n provide an intrinsic way to define new curvature forms

$$\sigma_n = c_1(X_n^{-1}[\hat{g}]), \quad (5.21)$$

$$\nu_n = c_1(Y_n^{-1}[\hat{g}]), \quad (5.22)$$

where $X_n[\hat{g}]$ and $Y_n[\hat{g}]$ denote X_n and Y_n with Δ_{1-n} and $\mathcal{N}_n$ evaluated with respect to the Poincaré metric. We conclude this section observing that related structures have been considered in [45].

VI. FURTHER DIRECTIONS AND CONCLUSIONS

We repeatedly saw that the key step in our construction concerns the manifestly Weyl and modular invariant structure of the Bergman reproducing kernel. It is constructed in terms of one of the bases of $H^0(K_C)$. We used it in two ways. In the first one $B^{1-n}(z,\bar z)$ has been used to integrate the zero modes of the b field of weight n. Such a prescription is well defined on any Riemann surface. We also considered the determinant of the Hadamard n-fold product of $B(z_i,\bar z_j)$. This is a modular invariant quantity which is proportional to $|\det\varphi_j(z_k)|^2$ but is independent of the choice of the $\varphi_1,\ldots,\varphi_{N_n}$. As such it provides the tool to absorb, in a modular invariant way, the dependence on the points due to the insertion of the zero modes in the path integral. Since

$\det B^{\circ n}(z_i,\bar z_j)$ vanishes on the hyperelliptic Riemann surfaces, it may happen that, depending on the set $\mathcal{J}$, the resulting partition function vanishes or has singularities there. Both $B^{1-n}(z,\bar z)$ and $\det B^{\circ n}(z_i,\bar z_j)$ are related to the space of symmetric powers of $H^0(K_C)$. The latter led to the concept of vector-valued Teichmüller modular forms [28] which provide the building blocks for the Mumford forms [28]. In [39] it has been shown that such forms can be expressed in terms of $K_n = M_n - N_n$ forms vanishing on the Jacobian, thus extending to any genus the expression for the Polyakov measure for $g = 4$ conjectured by Belavin-Knizhink [7] and by Morozov [11]. This also suggested formulating the bosonic string on the Siegel upper half-space, a matter related to the problem of characterizing the Jacobian locus, i.e. the Schottky problem. In [39] it was also shown that such vector-valued Teichmüller modular forms appear in constructing the superstring measure and in the Grushevsky ansatz [17,21,23,24,46–55].

In [21] it was suggested that the pure spinor Berkovits formulation of superstring theory [19,20] may be related to the Schottky problem. The reason is that the conditions of pure spinors are reminiscent of the relations for the quadrics

$$\sum_{i,j=1}^{g} C_{ij}^k \omega_i \omega_j = 0,$$

$k = 1,\ldots,K_2$, describing C in $\mathbb{P}^{g-1}$. It has been shown in [28] that the vector-valued Teichmüller modular forms, i.e. the building blocks of the string measures, provide a suitable combination of the coefficients of such quadrics. In particular, it turns out that the vector-valued Teichmüller modular forms are just the determinants of such coefficients [39].

We have seen that the Mumford forms relate basic aspects in string theory, such as modular and Weyl invariance, to the geometry of $\mathcal{M}_g$. Until now, the unique Mumford form of interest for string theories has been $\mu_{g,2}$, the one of the bosonic string. On the other hand, we have seen that even the other Mumford forms lead, by their nonchiral extension, to partition functions which are volume forms on $\mathcal{M}_g$. In the Berkovits construction, there are several fields leading to sections of λ_n and, due to the scalars, to powers of the Hodge bundle. In general, the invariance under Weyl and modular transformations provides strong constraints, in particular the one that follows from metric integration requires the partition function to be a volume form on $\mathcal{M}_g$. A further analysis of the Berkovits approach may show a relation to the partition functions introduced here. In this respect, it should be observed that the extension to the case of fields with fractional weight, essentially reduces to the problem of adding the dependence on the spin structures.

As we said, our construction is of interest also in superstring perturbation theories. In this respect, let us recall that a considerable step in finding the superstring measure is the Grushevsky ansatz [24], which has been successful in many respects. It satisfies quite stringent constraints up to genus four. Recently, Dunin-Barkowski, Sleptsov and Stern proved that the Grushevsky ansatz may satisfy such conditions up to genus five [55]. In [39] it has been observed that it is natural to believe that the phenomenon appearing at genus four, i.e. the Schottky-Igusa form F_4 defines both the bosonic and superstring measures, generalizes to higher genus. The reason is that since F_4 vanishes only on the Jacobian and therefore characterizes it, one should expect that the superstring measure, like the bosonic one [39], continues to be characterized by the forms vanishing on the Jacobian. Since these increase with the genus, they are $K_2 = M_2 - N_2 = (g-2)(g-3)/2$, one should expect that the extension of Grushevsky's ansatz should involve all of them, not just one. In particular, in genus five, one should expect three forms. Such an observation is somehow related to the very interesting result by Codogni and Shepherd-Barron, namely that it does not exist a stable Schottky form [54], so that, at least, one cannot expect that the extension of Grushevsky's ansatz may involve only one form; this should happen already at the genus five.

There is one more reason for that. One of the main results in recent work on superstring perturbation theory [18] is that, at least for $g \geq 5$, the moduli space of super Riemann surfaces does not map to the moduli space of Riemann surfaces with a spin structure. This result, and the appearance of more forms just from $g = 5$, may suggest the existence of some way to overcome the problems in treating the super period matrix and related geometrical quantities. In turn, this may be related with the fact that Grushevsky's ansatz involves fractional powers of forms that seem unlikely that could be well defined on the Jacobian with the increasing of the genus.

A related aspect has been considered in [35] where a general method to absorb the spin fields in b-c systems of real weight was introduced. This may suggest considering a suitable extension of the nonchiral analog of the Mumford forms to real weight. From the point of view of the b-c and β-γ systems this is always possible just because the action contains terms such as $b\bar\partial c$ and $\beta\bar\partial\gamma$, which are well-defined (1, 1)-forms even for $n \in \mathbb{C}$.

Let us conclude by observing that some of the geometry underlying the present construction has an interesting application to Seiberg-Witten theory [56,57], which will be considered elsewhere.

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APPENDIX $\mathrm{Sym}^n H^0(K_C)$

In the following we introduce a single indexing to denote quantities such as $\omega_i\omega_j$, $i, j = 1, \ldots, g$, by $\omega_i^{(2)}$, $i = 1, \ldots, g(g+1)/2$.More generally, one may consider the basis $\tilde{\omega}_1^{(n)}, \ldots, \tilde{\omega}_{M_n}^{(n)}$, $M_n = \binom{g+n-1}{n}$, of $\mathrm{Sym}^n H^0(K_C)$ whose elements are symmetrized tensor products of n-tuples of vectors of the basis $\omega_1, \ldots, \omega_g$, taken with respect to an arbitrary but fixed ordering. The image $\omega_i^{(n)}$, $i = 1, \ldots, M_n$, of $\tilde{\omega}_i^{(n)}$ under ψ: $\mathrm{Sym}^n H^0(K_C) \to H^0(K_C^n)$ is surjective for $g = 2$ and for C nonhyperelliptic of genus $g > 2$. For each $n \in \mathbb{Z}_{>0}$, set $I_n = \{1, \ldots, n\}$. Let us fix the index ordering and introduce some notation as in [40].

Let V be a g-dimensional vector space and denote by

$$\mathrm{Sym}^n V \ni \eta_1 \cdot \eta_2 \cdots \eta_n = \sum_{s\in\mathcal{P}_n} \eta_{s_1} \otimes \eta_{s_2} \otimes \cdots \otimes \eta_{s_n},$$

the symmetrized tensor product of an n-tuple $(\eta_1, \ldots, \eta_n)$ of elements of V. It is useful to fix an isomorphism $\mathbb{C}^M \to \mathrm{Sym}^2\mathbb{C}^g$ and, more generally, an isomorphism $\mathbb{C}^{M_n} \to \mathrm{Sym}^n\mathbb{C}^g$, $n \in \mathbb{Z}_{>0}$.

Let A: $\mathbb{C}^M \to \mathrm{Sym}^2\mathbb{C}^g$, $M \equiv M_2$ be the isomorphism $A(\tilde{e}_i) = e_{1_i} \cdot e_{2_i}$, with $\{\tilde{e}_i\}_{i\in I_M}$ the canonical basis of $\mathbb{C}^M$ and

$$(1_i, 2_i) = \begin{cases} (i,i), & 1 \le i \le g, \\ (1, i-g+1), & g+1 \le i \le 2g-1, \\ (2, i-2g+3), & 2g \le i \le 3g-3, \\ \vdots & \vdots \\ (g-1, g), & i = g(g+1)/2, \end{cases}$$

so that $1_i 2_i$ is the ith element in the M-tuple $(11, 22, \ldots, gg, 12, \ldots, 1g, 23, \ldots..)$. In general, one can define an isomorphism A: $\mathbb{C}^{M_n} \to \mathrm{Sym}^n\mathbb{C}^g$, with $A(\tilde{e}_i) = (e_{1_i}, \ldots, e_{\mathfrak{n}_i})$, by fixing the n-tuples $(1_i, \ldots, \mathfrak{n}_i)$, $i \in I_{M_n}$, in such a way that $1_i \le 2_i \le \ldots \le \mathfrak{n}_i$.

Let $\mathcal{P}_n$ be the group of permutations of n elements. For each vector $u = {}^t(u_1, \ldots, u_g) \in \mathbb{C}^g$ and matrix $A \in M_g(\mathbb{C})$, set

$$\underbrace{u \ldots u_i}_{n\ \text{times}} = \prod_{\mathfrak{m}\in\{1,\ldots,\mathfrak{n}\}} u_{\mathfrak{m}_i}, \qquad \underbrace{(A \cdots A)}_{n\ \text{times}}{}_{ij} = \sum_{s\in\mathcal{P}_n} \prod_{\mathfrak{m}\in\{1,\ldots,\mathfrak{n}\}} A_{\mathfrak{m}_i s(\mathfrak{m})_j},$$

$i, j \in I_{M_n}$. In particular, let us define

$$\chi_i \equiv \chi_i^{(n)} = \prod_{k=1}^{g} \Bigg(\sum_{\mathfrak{m}\in\{1,\ldots,\mathfrak{n}\}} \delta_{k\mathfrak{m}_i} \Bigg)! = (\delta \cdots \delta)_{ii},$$

$i \in I_{M_n}$ [we will not write the superscript (n) when it is clear from the context], where δ denotes the identity matrix, so that, for example,

$$\chi_i^{(2)} = 1 + \delta_{1_i 2_i},$$
$$\chi_i^{(3)} = (1 + \delta_{1_i 2_i} + \delta_{2_i 3_i})(1 + \delta_{1_i 3_i}).$$

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