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Study of field fluctuations and their localization in a thick braneworld generated by gravity nonminimally coupled to a scalar field with the Gauss-Bonnet term

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In this work we study a scenario with a warped five-dimensional smooth braneworld with four-dimensional (4D) Minkowski geometry built from bulk scalar matter nonminimally coupled to gravity with an additional Gauss-Bonnet term. We present exact solutions for the full braneworld configuration in contrast to previous results where only approximate solutions were constructed due to the highly non-linear character of the relevant differential equations. These solutions allow us to study the necessary conditions for the finiteness of the 4D Planck mass and, additionally, enable us to perform a more rigorous analysis of 4D gravity localization compared to approximate approaches. It is remarkable that all the constructed braneworld configurations lead to standard 4D gravity localization since they contain a localized massless tensor mode (the graviton). We also analyze the localization properties of scalar, vector, and tensor fluctuation modes for the constructed field configurations. We show that for the considered backgrounds, only the massless tensor mode, i.e., the 4D graviton, is localized on the brane, while the vector and scalar modes are not confined to the brane.

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I. INTRODUCTION

Braneworld models [1–9] are an interesting alternative to the standard Kaluza-Klein compactification. In order to protect the standard four-dimensional (4D) physics of some unobserved effects, in the Kaluza-Klein paradigm extra dimensions are compactified to a tiny size (for a 4D observer this is equivalent to a very high energy regime). The braneworld scenario has another approach; in this alternative the physics of our 4D world is compatible with the existence of infinite extra dimensions. In this approach, we consider that the Standard Model (SM) fields are trapped on a 4D hypersurface, called a 3-brane (our Universe). In contrast with ordinary matter, gravity and exotic matter can reside in the whole higher-dimensional manifold (bulk). Although the gravitational field can propagate through all dimensions, one of the first requirements to have realistic braneworld models is to recover standard 4D gravity on the brane. For example, if the induced geometry on the brane is flat, we need to localize a 4D graviton on our brane.

The braneworld models are classified with respect to their width in thin and thick branes. In scenarios with a thin braneworld [7–9], the curvature scalars are singular at the location of the branes, which is a direct consequence of the null width of the branes. In spite of that complication, the so-called Israel-Lanczos junction conditions [10–12] make harmless such singularities when we study the localization of gravity and matter as well as when computing several effective parameters of the model, as for instance the 4D Planck mass and the 4D cosmological constant.

Considering that ordinary matter fields are completely confined on a region with null width is just an approximation. The phenomenology of the Standard Model is well known at the electroweak energy scale m_{EW}; therefore, although the standard matter cannot move freely through the extra dimensions, it is possible that they can access to distances $r \leq m_{\mathrm{EW}}^{-1} \sim 10^{-19}$ m without contradictions with the Standard Model [5]. In other words, there are more realistic alternatives to thin brane configurations; they are known, generically, as thick braneworlds or domain

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walls. When the brane has a nontrivial width, there is no Israel-Lanczos junction condition, and this might be a mathematical advantage when the action of the model is complicated (when it contains, for instance, too many matter fields, nonminimal couplings, higher order curvature terms, etc).

Thick branes might be constructed by using different procedures. One of the most evident examples consists in replacing the delta functions in the action by nonsingular source functions. Another simple method is by using self-interacting scalar fields minimally coupled to gravity as done in [13–23]. A more elaborated procedure to generate a thick brane configuration is by introducing a nonminimal coupling between gravity and matter. This kind of interactions arises in several contexts; e.g., it appears naturally in cosmology, the Brans-Dicke theory, supergravity, and in all the known effective low-energy string theory models (see [24] and references therein). Within the framework of thin braneworlds, the Randall-Sundrum scenario has been modified by considering bulk self-interacting scalar fields nonminimally coupled to gravity, for instance, in [25–30]. Thick brane generalizations of this kind of models have been considered in [31,32].

The perturbative stability of these nonminimally coupled brane configurations was explored in Refs. [27,31]. Namely, in [27] the authors performed a stability analysis of a perturbed trivial scalar field and obtained instability regions characterized by certain value of the nonminimal coupling parameter in the conformal limit. Furthermore, in [31] it was shown that when the scalar field is nontrivial, the instability region completely disappears under linear perturbations for any value of the nonminimal coupling parameter. Later on localization of gravity and various matter fields was considered in this kind of braneworld configurations in [32], where deformed branes were also obtained.

On the other hand, it is well known that the standard Einstein-Hilbert action can be supplemented by higher-order curvature corrections without generating, in the equations of motion, terms containing three or higher order derivatives of the metric with respect to the space-time coordinates [33]. The particular combination which satisfies this requirement is known as Gauss-Bonnet (GB) invariant. Although in 4D this term has a topological origin which does not contribute to the classical equations of motion[1], in dimensions higher than four the GB invariant has a nontrivial contribution to the dynamical equations [35,36]. In our work the space-time has five dimensions; on it the Gauss-Bonnet term takes the following form

$$\mathcal{R}^2_{\rm GB} = R^{ABCD}R_{ABCD} - 4R^{AB}R_{AB} + R^2, \qquad (1)$$

where indices in capital roman letters A, B, C, D range over the five bulk dimensions.

Another property of the GB combination in dimensions higher than four is that it leads to a ghost-free theory, furthermore this invariant is present in different higher dimensional models. For instance, in string theory the GB-term appears in the first string tension correction to the (tree-level) effective action [37–43].

In the context of braneworld scenarios, the influence of higher curvature terms in scalar-field-generated thick brane models has been studied in Refs. [44–54]. In particular, within the framework of thick braneworld scenarios generated by a self-interacting minimally coupled scalar field, the GB invariant has been studied in connection with the localization properties of the various modes of the geometry in Ref. [44] (see also, for instance, [53]). A next step towards modifying the above scalar-tensor models of thick braneworlds, would be to assume a nonminimal coupling of the scalar field with the curvature Ricci scalar [55–57], which means that the higher-dimensional gravitational coupling is point dependent. In other words, the gravitational interactions in the bulk are jointly propagated by the higher-dimensional graviton and by the scalar field. The latter drives the strength of gravity by giving a local dynamics to the gravitational coupling. In 4D this kind of coupling is motivated from requiring compatibility with the Mach principle [58].

In this work we develop further previous results obtained in [59], where a 5D thick braneworld is modeled by a smooth scalar domain wall nonminimally coupled to gravity with a Gauss-Bonnet term on the bulk. The field equations that describe the thick braneworld dynamics are difficult when the nonminimal coupling and the Gauss–Bonnet term are turned on simultaneously. For this reason, in the articles [28] and [59] the authors were able to obtain just approximate solutions to the field equations when both effects are present. Therefore, one of the purposes of our research is to obtain exact solutions for a 5D thick braneworld configuration which accounts for both a nonminimal coupling of the scalar field to gravity and the Gauss-Bonnet invariant on the bulk.

On the other hand, the geometry of the braneworld models we shall consider possesses a broken 5D Poincaré symmetry, but preserves the 4D Poincaré one. As a consequence of both these facts it follows that the fluctuations of the system can be classified into tensor, vector and scalar sectors with respect to the symmetry group $SO(3,1)$, and all of these sectors possess nontrivial dynamics that, in principle, can influence the 4D phenomenology of the system.

A careful analysis of the localization of tensor, vector and scalar fluctuations of the fields that generate a braneworld configuration needs to have exact solutions at hand, otherwise we would need to make assumptions on the behavior of either the warp factor or the scalar field in order

[1]In scenarios with Ads_4 the 4D Gauss-Bonnet term produces nontrivial results in the conserved quantities of the theory [34].

to get sensible results (see Sec. VII for details). Notwithstanding, there is no guarantee of the fulfillment of these assumptions within a given scalar-tensor system. For instance, the assumption of certain behavior of the geometrical entities of the model could lead to a divergent behaviour of the scalar field and, hence, a divergent character of the effective 4D Planck mass, rendering an ill model where there is no localization of gravity. There is no need to mention that the complete and detailed analysis of the localization properties of *all* the tensor, vector and scalar perturbations of the scalar-tensor system that generates the braneworld model is very involved. Thus, another aim of this article is the rigorous study of the localization properties of *all* the sectors of gravity and matter field fluctuations for the considered scenario with exact field configurations, generalizing previously obtained results in which only the tensor sector was taken into account (see [28,59]).

In other words, as an application of the obtained exact braneworld field configurations we shall perform a study of the conditions under which we recover (i) a well-defined effective 4D Planck mass, (ii) a localized tensor zero mode fluctuation (graviton) that accounts for the observed 4D general relativity theory, (iii) the presence of vector and scalar fluctuations that do not considerably alter the low-energy phenomenology observed in 4D. This is because general relativity predicts that in empty 4D Minkowski space-time, the only relevant fluctuations come from the tensor sector (from the massless mode). Thus, although in our scenario the scalar and vector sectors are nontrivial, their effects must be suppressed for a 4D observer on the brane.[2] The delocalized character of nontensor modes provides a mechanism that protects the low energy physics of the flat brane from these unwanted effects. In order to achieve these aims, we shall consider just positive values for the nonminimal coupling function, a condition which guarantees an effective positive 4D Planck mass.

The paper is organized as follows: in Sec. II we present the action of the braneworld configuration along with the field equations for the proposed metric ansatz. In Sec. III we obtain the expression for the effective 4D Planck mass under dimensional reduction starting from the original 5D action. We then consider, in Sec. IV, a positive valued nonminimal coupling function for the scalar field by imposing the condition $L > 0$, which guarantees a positive effective 4D Planck mass. In particular, we consider a concrete function that offers a clear dependence of the effective 4D Planck mass on the nonminimal coupling parameter, a fact that in turn allows us to easily compare to this magnitude in the minimally coupled case, when this parameter vanishes. Moreover, this form of the function L also makes evident that the scalar field must be bounded, restricting the universe of mathematically available solutions to those which are physically meaningful field configurations from the 4D point of view. We also choose a suitable warp factor for the metric within this section. We further impose the conditions for the solutions of the scalar field to possess a mirror symmetry along the extra dimension in Sec. V. In this way we study some general properties that we wish to be present in our model, essentially, to have a positive and finite 4D coupling constant (the Planck mass), to consider regular warp factors that avoid the presence of curvature singularities, and to select physically simple solutions among all the obtained scalar field configurations. In Sec. VI we construct exact solutions for the highly nonlinear differential equations for the scalar field, a necessary step for providing a rigorous study of the consistency and localization properties of the metric fluctuations of our braneworld configuration. Moreover, these solutions are essential for analyzing the positiveness of the 4D Planck mass (an indispensable property of any viable theory). In Sec. VII we further analyze the perturbations of the geometry which can be classified into scalar, vector and tensor sectors according to the 4D Poincaré symmetry group. We establish that both the scalar and vector sectors are not localized on the brane in contrast to the 4D tensor massless mode. We finally summarize our results and conclusions at the end of the paper.

II. THE MODEL

Let us explore a thick braneworld described by the following 5D action (a similar set up is studied in references [31,44]),

$$S = -\int d^5x\sqrt{|g|}\left[\frac{L}{2\kappa}R - \frac{1}{2}(\nabla\varphi)^2 + V + \alpha'\mathcal{R}^2_{\rm GB}\right], \qquad (2)$$

where the constant $\alpha' > 0$ and $\kappa \simeq 1/M^3$, M—the 5D Planck mass. Besides, φ is a real scalar field and $V = V(\varphi)$ its self-interaction potential. The quantity $L = L(\varphi)$ describes the nonminimal coupling between the scalar field φ and the Einstein-Hilbert term and $\mathcal{R}^2_{\rm GB}$ is the 5D Gauss-Bonnet term (1).

In the scalar-tensor gravity theory (2), 15 degrees of freedom g_{AB}, plus the scalar field φ, propagate the gravitational interaction. Hence, this is not a pure geometrical theory of gravity. In particular, the metric coefficients define the geodesic motion of test particles, while the scalar field determines locally the strength of gravity by means of the effective gravitational coupling $\propto \kappa/L(\varphi)$.

The way the scalar field φ is coupled to the curvature in the present theory, deserves an independent comment. Actually, in this paper, just as a matter of necessary simplicity, we have considered explicit coupling of φ to the curvature scalar R, but not to the Gauss-Bonnet invariant. Intuition, instead, dictates that the scalar field should

[2]This reasoning is confirmed by the fact that in a 5D thick braneworld model generated by a scalar field minimally coupled to gravity, the corrections to the Newton's law coming from the scalar modes are more suppressed than the corrections arising from the tensor modes [60].

couple in a similar fashion to R and to $\mathcal{R}^2_{\rm GB}$, since both contribute towards the curvature of space-time. This will be the subject of forthcoming work. We want to underline that this "asymmetric" coupling has nothing to do with physical requirements, but just with simplicity of further mathematical handling since, as we shall see further, the relevant equations of motion are highly nonlinear.

The Einstein's field equations that come from the action (2) take the following form:

$$LR_{AB} + \epsilon\mathcal{Q}_{AB} = \kappa\tau_{AB} + \nabla_A\nabla_B L + \frac{1}{3}g_{AB}\Box L, \tag{3}$$

where

$$\mathcal{Q}_{AB} = \frac{1}{3}g_{AB}\mathcal{R}^2_{\rm GB} - 2RR_{AB} + 4R_{AC}R^C_B + 4R^{CD}R_{ACBD} - 2R_{ACDE}R_B^{CDE} \tag{4}$$

is the Lanczos tensor representing the Gauss-Bonnet corrections to the Einstein's field equations, $\epsilon = 2\alpha'\kappa$, $\Box \equiv g^{CD}\nabla_C\nabla_D$, and the stress-energy tensor τ_{AB}, corresponding to the scalar matter content on the bulk, is defined as usual,

$$\tau_{AB} = \partial_A\varphi\partial_B\varphi - \frac{2}{3}g_{AB}V(\varphi).$$

The remaining terms in the right-hand side (rhs) of (3) come from the nonminimal coupling of the scalar field to the curvature scalar.

Furthermore, the equation that describes the dynamics of the scalar field (Klein-Gordon equation) can be written as follows,

$$\Box\varphi + \frac{1}{2\kappa}RL_\varphi + \frac{dV}{d\varphi} = 0, \tag{5}$$

where $L_\varphi = dL/d\varphi$. As it is done, for instance, in Ref. [61], the geometry of our braneworld model is described by a warped metric in conformally flat coordinates,

$$ds^2 = a^2(w)[\eta_{\mu\nu}dx^\mu dx^\nu - dw^2], \tag{6}$$

where we use the signature $(+----)$, $\eta_{\mu\nu}$ is the 4D Minkowski metric, the variable w is the extra coordinate and all the dimensions are of infinite extend. Due to the Einstein field equations, the simplicity of our metric ansatz (6) implies that the field φ depends only on the fifth coordinate w. Thus, the Einstein and Klein-Gordon equations (3) and (5) give rise to

$$V + \frac{3\mathcal{H}L_\varphi\varphi'}{\kappa a^2} + \frac{1}{2\kappa a^2}(\varphi''L_\varphi + \varphi'^2L_{\varphi\varphi}) - \frac{3}{2\kappa a^2}\left[\frac{4\epsilon}{a^2}\mathcal{H}^2(\mathcal{H}^2 + \mathcal{H}') - (\mathcal{H}' + 3\mathcal{H}^2)L\right] = 0, \tag{7}$$

$$\frac{1}{\kappa}L_\varphi\varphi'' - \frac{2}{\kappa}\mathcal{H}L_\varphi\varphi' + \left(1 + \frac{L_{\varphi\varphi}}{\kappa}\right)\varphi'^2 - \frac{3}{\kappa}(\mathcal{H}^2 - \mathcal{H}')q = 0, \tag{8}$$

$$\varphi'' + 3\mathcal{H}\varphi' - \frac{dV}{d\varphi}a^2 - \frac{2L_\varphi}{\kappa}(3\mathcal{H}^2 + 2\mathcal{H}') = 0. \tag{9}$$

Here a prime denotes derivative with respect to the extra coordinate w (for instance, $\phi' = d\phi/dw$); in addition to this,

$$\mathcal{H} \equiv \frac{a'}{a}, \quad \text{while } q \equiv L - \frac{4\epsilon}{a^2}\mathcal{H}^2.$$

As one can straightforwardly check, the three equations (7)–(9) are not independent; only two of them are.

III. PLANCK MASSES

In this section we shall derive the 4D effective coupling constant of our model, the Planck mass, starting from the 5D action (2) through a mechanism called dimensional reduction. In general, the 4D effective theory can be obtained by integrating the 5D action with respect to the fifth coordinate w. In order to do that let us consider a generalization of the 5D line–element (6) where the 4D Minkowski metric $\eta_{\mu\nu}$ is replaced by an arbitrary 4D metric $\tilde{g}_{\mu\nu}(x)$[3]:

$$ds^2 = a^2(w)[\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu - dw^2], \quad \text{where } x \equiv \{x^\mu\}. \tag{10}$$

When performing the integration with respect to the extra dimension, the 5D fundamental theory is reduced to a 4D Einstein-Hilbert effective action plus the corrections that come from the scalar matter and higher curvature terms of the bulk. In this section we will focus in the analysis of the 4D Planck mass, therefore we only need to extract the 4D Einstein-Hilbert effective action after the dimensional reduction of the original 5D action,

$$S_4 \simeq M^2_{\rm Pl}\int d^4x\sqrt{|\tilde{g}_4|}\tilde{R}_4 + \cdots, \tag{11}$$

where the subscript 4 labels quantities computed with respect to 4D metric $\tilde{g}_{\mu\nu}(x)$ and $M_{\rm Pl}$ is the effective 4D Planck mass.

[3]In general, the dimensional reduction of the original 5D action must render a nontrivial 4D Einstein-Hilbert term. Therefore, such a generalization of the 4D metric is needed because the curvature scalar corresponding to 4D Minkowski space-time vanishes. The simplest way to do that it is to replace $\eta_{\mu\nu}\mapsto\eta_{\mu\nu} + \delta g_{\mu\nu}(x)$, where $\delta g_{\mu\nu}(x)$ is a perturbation of the 4D Minkowski geometry.

A method for explicitly finding the 4D effective theory, in particular the 4D Einstein-Hilbert action, starting from the 5D one consists in interpreting the warp factor as a conformal function as follows [62],

$$\tilde{g}_{AB}\mapsto g_{AB}=a^2(w)\tilde{g}_{AB}(x)=a^2(w)\begin{pmatrix}\tilde{g}_{\mu\nu}(x) & 0\\ 0 & -1\end{pmatrix}, \quad (12)$$

and rewriting the 5D action in terms of the quantities defined with respect to $\tilde{g}_{AB}(x)$. Equation (11) asserts the splitting of the quantity $\tilde{g}_{AB}(x)$ into $\tilde{g}_{\mu\nu}(x)$ and $g_{ww}=-1$.

In order to obtain the Einstein-Hilbert part of the 4D effective action, it is only necessary to consider the above conformal transformation on the terms $-L(\phi)R/2\kappa$ and $\alpha'\mathcal{R}^2_{\rm GB}$ in (2). The following expressions display these quantities after the conformal transformation,

$$R=a^{-2}(\tilde{R}-8\tilde{\Box}\vartheta-12(\tilde{\nabla}\vartheta)^2), \quad (13)$$

$$\begin{aligned}\mathcal{R}^2_{GB}=\frac{1}{3a^4}\{&\tilde{\mathcal{R}}^2_{GB}-12\tilde{R}(\tilde{\nabla}\vartheta)^2-24\tilde{R}\,\tilde{\Box}\,\vartheta\\&+48\tilde{R}^{AB}(\tilde{\nabla}_A\tilde{\nabla}_B\vartheta-\tilde{\nabla}_A\vartheta\tilde{\nabla}_B\vartheta)\\&+72[(\tilde{\Box}\vartheta)^2+(\tilde{\nabla}\vartheta)^4-(\tilde{\nabla}_A\tilde{\nabla}_B\vartheta)(\tilde{\nabla}^A\tilde{\nabla}^B\vartheta)]\\&+144(\tilde{\nabla}_A\tilde{\nabla}_B\vartheta)(\tilde{\nabla}^A\vartheta)(\tilde{\nabla}^B\vartheta)+144(\tilde{\nabla}\vartheta)^2\tilde{\Box}\vartheta\},\end{aligned} \quad (14)$$

where $\vartheta=\ln a$. The terms with a tilde are defined with respect to the metric $\tilde{g}_{AB}(x)$. The next step consists in using the relations (12)–(14) in order to obtain the effective 4D Ricci scalar. Therefore, after substituting (13) and (14) into (2) and integrating the prefactor of the 4D Ricci scalar with respect to the fifth dimension, the relation between 4D and 5D Planck masses can be written as follows,

$$\begin{aligned}M^2_{\rm Pl}&\simeq M^3\int_{-\infty}^{\infty}a^3(w)\left[L+\frac{4\epsilon}{a^2}(\mathcal{H}^2+2\mathcal{H}')\right]dw\\&=M^3\int_{-\infty}^{\infty}a^3(w)q dw+8M^3\epsilon[a']^{\infty}_{-\infty}.\end{aligned} \quad (15)$$

It is evident that the $M_{\rm Pl}$ explicitly depends on the nonminimal coupling function L as well as on the Gauss-Bonnet corrections to the Einstein-Hilbert term parametrized by ϵ.

This quantity should be positive and finite for a consistent theory. Moreover, it should reproduce the effective gravitational couplings that we observe in our 4D world, if we wish to recover 4D gravity on the brane.[4] Later on we shall see that these conditions are fulfilled for a wide class of solutions of our scalar-tensor model.

[4]In particular, if the nonminimal coupling funtion $L(\varphi)$ increases with the same rapidity as (or faster than) the factor $a^{-3}(w)$, then the effective 4D Planck mass $M_{\rm Pl}$ will diverge, leading to a theory with unphysical 4D gravitational couplings.

IV. BACKGROUND GEOMETRY AND NON–MINIMAL COUPLING

Although our scenario does not impose any restriction on the functional form of $L(\varphi)$, apart of being positive, here we shall consider the simple nontrivial coupling function [25] (see also [31]):

$$L(\phi)=1-\frac{\xi}{2}\phi^2. \quad (16)$$

The choice of the coupling function $L(\varphi)$ in the above expression is inspired in 4D phenomenology [58,63–66]. The parameter ξ characterizes the strength of the nonminimal coupling between scalar matter and gravity. Besides, if assume small values of the coupling parameter—as it is implicit in the present paper— and bounded scalar, this choice reflects the fact that only small deviation from the case $\xi=0$ is being considered. When $\xi=0$, the scenario is described by a bulk scalar field minimally coupled to gravity (it was explored in Ref. [44]). This particular form of $L(\varphi)$ is very simple, however, it still gives us an idea of the effects of the nonminimal coupling on the thick brane, furthermore, in contrast with other more complicated ansatze, it is not too hard to solve its associated field equations. Notwithstanding, it is worth noticing that one can consider different nonminimal coupling functions $L(\varphi)$ which still are positive and finite that do not restrict the scalar field and lead to interesting results like deformed braneworld configurations as in [32]. The important point here is to avoid getting negative or divergent values for the integral $M^2_{\rm Pl}\sim\int L(\phi(w))a^3(w)dw$ since this would imply an effective 4D coupling constant that does not reproduce the observed gravitational interactions of our world.

Positivity of L leads to φ^2 being bounded from above:

$$L>0\Rightarrow\phi^2<\frac{2}{\xi}\Rightarrow|\phi|<\sqrt{\frac{2}{\xi}} \quad (17)$$

and we will impose this restriction by hand on the field configurations obtained below in order to ensure a positive and finite value for the effective 4D Planck mass.

In this work, besides, we shall consider a regular geometry of the form (6), which interpolates between two asymptotically AdS_5 space-times, depicted by the following warp factor [44]:

$$a(w)=\frac{a_0}{\sqrt{1+b^2w^2}}, \quad (18)$$

where the width of the thick brane is $1/b$, and the quantity a_0 is dimensionless and related to the radius of the asymptotic AdS_5 space-time. All of the resulting quadratic curvature invariants constructed with this warp factor are regular and asymptotically constant.

Once we have introduced the nonminimally coupling function (16) and the warp factor (18), we can proceed to solve the field equations (7)–(9). From the mathematical point of view, these differential equations are conveniently treated in terms of a dimensionless variable. Therefore, we shall perform the following change of variable $w \mapsto v = bw$, where v is dimensionless. Thus, by using (16) and (18), Eq. (8) can be rewritten as:

$$\xi\varphi\varphi'' + \frac{2\xi v}{1+v^2}\varphi\varphi' + (\xi-\kappa)\varphi'^2 - \frac{3\xi}{2}\frac{\varphi^2}{(1+v^2)^2} = -\frac{3}{(1+v^2)^2} + \frac{4\epsilon b^2}{a_0^2}\frac{3v^2}{(1+v^2)^3}, \tag{19}$$

where, now, the prime denotes derivative with respect to the dimensionless variable v. It is difficult to find the general solution to the above equation, however, several interesting (particular) exact solutions can be found. Here we shall split the analysis into three separated cases corresponding to: i) a scalar field minimally coupled to gravity and including a 5D Gauss-Bonnet invariant ($\xi = 0$ and $\epsilon \neq 0$) [44], ii) a scalar field nonminimally coupled to curvature without the Gauss-Bonnet term ($\xi \neq 0$ and $\epsilon = 0$), and iii) the more general situation when both nonlinear effects are present, i.e., when $\xi \neq 0$ and $\epsilon \neq 0$. Once a given particular (exact) solution for the scalar field $\varphi = \varphi(v)$ is found, one hence can write the self-interaction potential as a function of the (dimensionless) extra-dimensional coordinate v:

$$V(v) = \frac{3b^2}{2\kappa a_0^2}\left\{\frac{4\epsilon b^2}{a_0}\frac{v^2(2v^2-1)}{(1+v^2)^2} + \frac{1-4v^2}{1+v^2} + \xi\left[(1+v^2)(\varphi\varphi''+\varphi'^2) - 2v\varphi\varphi' - \left(\frac{1-4v^2}{1+v^2}\right)\frac{\varphi^2}{2}\right]\right\}. \tag{20}$$

The above expression is just a rewriting of Eq. (7) for the case of interest in this paper, i.e., for the nonminimally coupling function (16) and the warp factor (18).

It is worth mentioning that only when we have $\varphi(v)$ and $a(v)$ at hand, we can perform the integration of (15) and compute the effective 4D Planck mass $M_{\rm Pl}^2$. Moreover, once $\varphi(v)$ and $V(\varphi(v))$ are given as functions of the dimensionless variable v, the components of the stress energy tensor $\tau_{AB} = \tau_{AB}(v)$, are also known functions of the dimensionless extra-dimensional variable.

V. MIRROR SYMMETRY

An important remark on the particular solutions we are looking for is related to the symmetries of the metric coefficients. As mentioned, in the general case when there is a nonminimal coupling between the scalar field and the curvature scalar ($\xi \neq 0 \Rightarrow L = L(\phi) \neq 1$), gravity is propagated both by g_{AB} and by φ. Hence, one should naively expect that the (real) scalar field, $\varphi(v)$, will respect the same symmetries as the metric functions g_{AB}. In the remaining part of this section we shall show that, as a matter of fact, while the latter is a mandatory requirement for the case when $\xi = 0$ (in the minimal coupling case), this property is not present in the general case in which $\xi \neq 0$.

In the present case the metric coefficients in (6) inherit the "mirror" symmetry

$$v \mapsto -v \Rightarrow g_{AB}(v) = g_{AB}(-v), \tag{21}$$

distinctive of the warp factor (18). In consequence, the tensors R_{AB}, and $\mathcal{Q}_{AB}$, in the left-hand side (lhs) of Einstein's field equations (3)

$$R_{AB}(v) = R_{AB}(-v), \qquad \mathcal{Q}_{AB}(v) = \mathcal{Q}_{AB}(-v), \tag{22}$$

also respect invariance under (21).

For the minimal coupling case ($\xi = 0 \Rightarrow L = 1$), since the Einstein's field equations read,

$$R_{AB} + \epsilon\mathcal{Q}_{AB} = \kappa\tau_{AB}, \tag{23}$$

the invariance (22) will entail that, necessarily, $\tau_{AB}(v) = \tau_{AB}(-v)$, which means, in turn, that

$$V(v) = V(-v), \qquad \varphi(v) = \pm\varphi(-v), \tag{24}$$

where, in the last equality, only one of the two signs, "+", or "−", is to be chosen at once. In other words, φ can be either even or odd, under (21).

In the general case $\xi \neq 0$ ($L = L(\varphi) \neq 1$), on the contrary, since the coupling function L is multiplying the Ricci tensor in the lhs of (3), mirror symmetry (21),(22) is respected only if the additional requirement

$$L(\varphi(v)) = L(\varphi(-v)), \tag{25}$$

is fulfilled.[5] In the case studied in this paper, since L is quadratic in φ (see Eq. (16)), the latter requirement will entail, again, that, under (21), φ can be either even, or odd.

In case φ does not show any obvious symmetry under (21),

$$\varphi(v) \neq \pm\varphi(-v) \Rightarrow L(\varphi(v)) \neq L(\varphi(-v)),$$

the coupling function L counteracts the symmetry displayed by R_{AB} and, in consequence, the lhs of (3)—the pure geometrical part of Einstein's equations—is not invariant under (21) anymore. Hence, if $\xi \neq 0$, mathematical consistency of the solutions does not impose any mirror symmetry requirement on φ. This can be only an independent *ad hoc* requirement.

[5]In our analysis we took into account the fact that the operators $\nabla_A\nabla_B$, and $\Box$ in the rhs of Eq. (3), are invariant under (21). Recall that, in the present case, there is functional dependence on the extra-coordinate v only.

Notwithstanding, in the present paper the symmetry requirements (21),(22),(25) will be used to select physically simple solutions among the set of all possible exact solutions we will be able to find. Consequently, we shall take into consideration only those φ-configurations which are either even $(\phi(v) = \phi(-v))$ or odd $(\varphi(v) = -\varphi(-v))$ under $v \mapsto -v$. While for the minimally coupled case it is legitimate, as shown above, in the case when $\xi \neq 0$, it is not required by mathematical consistency but by symmetric aesthetic instead.

VI. EXACT SOLUTIONS

In this section we shall exactly solve the differential Eq. (19) for certain special cases and present graphics of the corresponding profiles of the self–interaction potential given by (20).

As already mentioned, here we shall split the study into three particular cases:

(A) $\epsilon \neq 0$ and $\xi = 0$—minimal coupling.
(B) $\xi \neq 0$ and $\epsilon = 0$—nonminimal coupling.
(C) $\epsilon \neq 0$ and $\xi \neq 0$—the general case.

In each case we will rely on a number of mathematical assumptions so that, after achieving considerable simplification, particular exact solutions can be found.

A. Minimal coupling case ($\epsilon \neq 0$ and $\xi = 0$)

If one sets $\xi = 0$ then (19) adopts the following form,

$$\varphi'^2 = \frac{3}{\kappa(1+v^2)^2} - \frac{4\epsilon b^2}{\kappa a_0^2}\frac{3v^2}{(1+v^2)^3}, \tag{26}$$

and can be easily solved. Actually, let us first change to the variable,

$$\tau = \arctan(v) \Leftrightarrow v = \tan(\tau), \qquad \text{hence } \sin^2\tau = \frac{v^2}{1+v^2}, \tag{27}$$

so that the derivatives of φ with respect to v and τ are related by

$$(1+v^2)\frac{d}{dv} = \frac{d}{d\tau} \Leftrightarrow (1+v^2)\varphi' = \dot{\varphi}, \tag{28}$$

where overdots stand for derivatives with respect to τ. Notice that $v \in]-\infty, +\infty[\Rightarrow \tau \in]-\pi/2, \pi/2[$. In terms of this new variable, Eq. (26) transforms into the following first-order equation,

$$\dot{\varphi} = \pm\sqrt{\frac{3}{\kappa}}\sqrt{1 - \frac{4\epsilon b^2}{a_0^2}\sin^2\tau}, \tag{29}$$

which can be integrated in quadratures to give

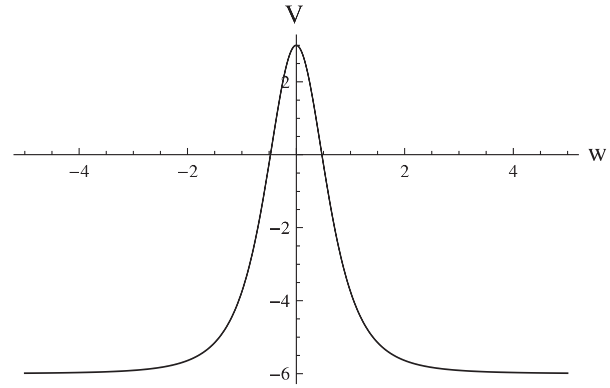

FIG. 1. The self–interaction potential for the case $\epsilon \neq 0$ and $\xi = 0$. In this figure we set $k = 1$, $a_0 = 1$, $b = 1$ and $\kappa = 1/2$.

$$\varphi^{\pm}(\tau) = \pm\sqrt{\frac{3}{\kappa}}E(\sin\,\tau, k) + \varphi_0^{\pm}, \tag{30}$$

where

$$E(\sin\,\tau, k) = \int_0^\tau \sqrt{1 - k^2\sin^2\zeta}\,d\zeta, \qquad k = \sqrt{\frac{4\epsilon b^2}{a_0^2}} \tag{31}$$

is the incomplete elliptic integral of the second kind [67], $\varphi_0^{\pm}$ are arbitrary integration constants, and the $\pm$ signs represent different branches of the solution.

The particular case when $a_0 = \sqrt{4\epsilon}b \Rightarrow k = 1$ was studied in [61], where $\phi^{\pm}(\tau) = \pm\sqrt{3/\kappa}\,\sin\,\tau + \phi_0^{\pm}$, or, in terms of the dimensionless variable v [see Eq. (27)],

$$\varphi^{\pm}(v) = \pm\sqrt{\frac{3}{\kappa}}\frac{v}{\sqrt{1+v^2}} + \varphi_0^{\pm}. \tag{32}$$

As before, the $\pm$ signs describe two possible branches of the solution. The scalar field solutions (30),(32), do not respect invariance under (24) unless the constants $\varphi_0^{\pm}$ are set to zero, $\varphi_0^{\pm} = 0$.[6] Hence, in order to meet the wished symmetry requirements mentioned in the Sec. V, the particular exact solutions of Eq. (19), for the minimally interacting case ($\xi = 0$), are

$$\varphi^{\pm}(v) = \pm\sqrt{\frac{3}{\kappa}}E\left(\frac{v}{\sqrt{1+v^2}}, k\right), \qquad k \neq 1, \tag{33}$$

$$\varphi^{\pm}(v) = \pm\sqrt{\frac{3}{\kappa}}\frac{v}{\sqrt{1+v^2}}, \qquad k = 1. \tag{34}$$

The profile of the self-interaction potential V is shown in Fig. (1). As one can see, V is asymptotically constant and

[6] Recall that $E(\sin\,\tau, k) = -E(-\sin\,\tau, k)$.

negative. This result is consistent with the fact that our geometry (6) is asymptotically AdS_5.

Thus, the 4D Planck mass corresponding to the minimally coupled case with Gauss-Bonnet term is

$$M_{\rm Pl}^2 = \frac{a_0^3}{2b\kappa} - \frac{4a_0 b\epsilon}{3\kappa} = \frac{a_0^3}{2b\kappa}\left(1 - \frac{2k^2}{3}\right). \qquad (35)$$

This is a finite quantity where the correction coming from the Gauss-Bonnet invariant is encoded in the second term of the rhs.

B. Nonminimal coupling case ($\xi \neq 0$ and $\epsilon = 0$)

Now when $\epsilon = 0$ in (19) we obtain the following differential equation,

$$\varphi\varphi'' + \frac{2v}{1+v^2}\varphi\varphi' + \left(1 - \frac{\kappa}{\xi}\right)\varphi'^2 - \frac{3}{2}\frac{\varphi^2}{(1+v^2)^2} = -\frac{3}{\xi(1+v^2)^2}. \qquad (36)$$

If we make the same replacement as before $v \mapsto \tau = \arctan v$ [see Eq. (27)] when $\epsilon = 0$, then Eq. (36) can be rewritten in the form

$$\varphi\ddot{\varphi} + \left(\frac{\xi - \kappa}{\xi}\right)\dot{\varphi}^2 = \frac{3}{2}\varphi^2 - \frac{3}{\xi}. \qquad (37)$$

Then, without loss of generality, one can make the following assumption,

$$\dot{\varphi}^2 = h(\varphi) \Rightarrow \ddot{\varphi} = \frac{1}{2}\frac{dh(\varphi)}{d\varphi}, \qquad (38)$$

so that Eq. (37) can be written as a first-order differential equation,

$$\frac{dh(\varphi)}{d\varphi} + 2\lambda\frac{h(\varphi)}{\varphi} = 3\varphi - \frac{6}{\xi\phi}, \qquad (39)$$

where

$$\lambda = \frac{\xi - \kappa}{\xi}. \qquad (40)$$

The following expression for $h(\varphi)$ solves Eq. (39),

$$h(\varphi) = \frac{3\varphi^2}{2(1+\lambda)} - \frac{3}{\xi\lambda} + C\varphi^{-2\lambda}, \qquad (41)$$

where C is an arbitrary constant.

Here, for the sake of simplicity of mathematical handling, we set $C = 0$ in Eq. (41). Then, after substituting back (41) into (38), one is left with the following first-order differential equation:

$$\dot{\varphi} = \pm\sqrt{\frac{3}{2(1+\lambda)}}\sqrt{\varphi^2 - \frac{2(1+\lambda)}{\xi\lambda}}. \qquad (42)$$

Equation (42) can be integrated in quadratures,

$$\pm\int \frac{d\varphi}{\sqrt{\varphi^2 - \frac{2}{\xi}\left(\frac{2\xi-\kappa}{\xi-\kappa}\right)}} = \sqrt{\frac{3\xi}{2(2\xi-\kappa)}}\tau + C_0, \qquad (43)$$

where we have returned to the original parameters ξ and κ through (40), and C_0 is an integration constant which in the following calculations we fix to meet the imposed symmetry requirement (24). Hence, depending on the interval in ξ parameter, one will obtain different particular exact solutions,

1. Case $0 < \xi < \kappa/2$

$$\varphi^{\pm}(\tau) = \pm\sqrt{\frac{2(\kappa - 2\xi)}{\xi(\kappa-\xi)}}\cos\left(\sqrt{\frac{3\xi}{2(\kappa-2\xi)}}\tau\right), \qquad (44)$$

or, in terms of the original dimensionless variable [see Eq. (27)],

$$\varphi^{\pm}(v) = \pm\varphi_{01}\cos(\beta \arctan v), \qquad (45)$$

where, for compactness of writing, we have introduced the following constants:

$$\varphi_{01} = \sqrt{\frac{2(\kappa-2\xi)}{\xi(\kappa-\xi)}}, \qquad \beta = \sqrt{\frac{3\xi}{2(\kappa-2\xi)}}. \qquad (46)$$

The bound (17) on φ^2 leads to

$$\varphi_{01}^2 = \frac{2(\kappa-2\xi)}{\xi(\kappa-\xi)} < \frac{2}{\xi} \Rightarrow \kappa - 2\xi < \kappa - \xi,$$

which, for $0 < \xi < \kappa/2$, is always fulfilled. Hence, positivity of $L(\varphi)$ does not impose any additional constraint on the parameter ξ.

On the other hand, for the above profile of the field the self–interaction potential is constant and negative at $w \to \infty$ (see Fig. 2).

The corresponding effective 4D Planck mass for this case reads

$$M_{\rm Pl}^2 = \frac{a_0^3}{2b(\kappa-\xi)}\left[1 + \frac{(\kappa-2\xi)^2\cos(\beta\pi)}{\kappa(8\xi-\kappa)}\right], \qquad (47)$$

which is finite and positive since $L > 0$ within the interval $0 < \xi < \kappa/2$.

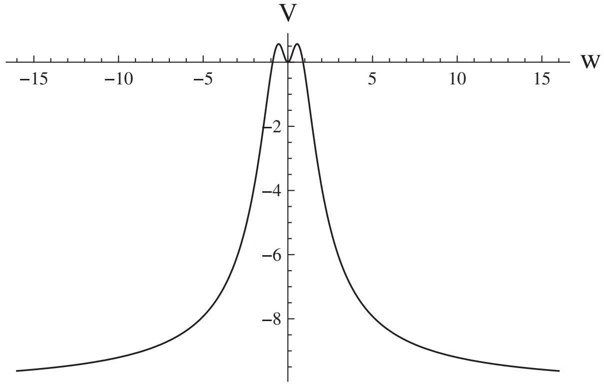

FIG. 2. V corresponding to solution (45). In this figure we set $\xi = 1/10$, $a_0 = 1$, $b = 1$ and $\kappa = 1/2$.

2. Case $\kappa/2 < \xi < \kappa$

$$\varphi^{\pm}(\tau) = \pm\sqrt{\frac{2(2\xi-\kappa)}{\xi(\kappa-\xi)}}\sinh\left(\sqrt{\frac{3\xi}{2(2\xi-\kappa)}}\tau\right). \tag{48}$$

This solution can be expressed in the language of the dimensionless variable $v = bw$,

$$\varphi^{\pm}(v) = \pm\varphi_{02}\,\sinh(\beta\,\arctan\,v), \tag{49}$$

where

$$\varphi_{02} = \sqrt{\frac{2(2\xi-\kappa)}{\xi(\kappa-\xi)}}.$$

The positive character of L within the above mentioned interval, $\kappa/2 < \xi < \kappa$, renders the following condition for any value of τ,

$$\sinh^2\left(\sqrt{\frac{3\xi}{2(2\xi-\kappa)}}\tau\right) < \left(\frac{\kappa-\xi}{2\xi-\kappa}\right). \tag{50}$$

In principle, this inequality can be fulfilled around the point where τ vanishes,

$$0 < \left(\frac{\kappa-\xi}{2\xi-\kappa}\right). \tag{51}$$

Notwithstanding, when τ approaches the values $\pm\frac{\pi}{2}$, the inequality (50) becomes

$$\sinh^2\left(\sqrt{\frac{3\xi}{2(2\xi-\kappa)}}\frac{\pi}{2}\right) < \left(\frac{\kappa-\xi}{2\xi-\kappa}\right). \tag{52}$$

It turns out that this inequality can never be satisfied within the interval $\kappa/2 < \xi < \kappa$. This conclusion can be made from the behavior of L within the above mentioned interval: while the lhs of the inequality exponentially diverges as $\xi \mapsto \frac{\kappa^+}{2}$ and evolves towards the value $\sinh^2\left(\sqrt{\frac{3}{2}}\frac{\pi}{2}\right)$ as $\xi \mapsto \kappa^-$, the rhs linearly diverges as $\xi \mapsto \frac{\kappa^+}{2}$ and vanishes as ξ approaches the other end of the interval, i.e., when $\xi \mapsto \kappa^-$. Thus, the nonminimal coupling function L is not definite positive and possesses regions where it is negative within the interval $\kappa/2 < \xi < \kappa$.[7]

Since $L > 0$ is a very strong condition, in principle it could be possible to still have a positive Planck mass for a more restricted interval of ξ. This fact indicates that one must compute the effective 4D Planck mass (relaxing for a while the $L > 0$ condition) in order to see whether the correct gravitational couplings can still be recovered in 4D for the aforementioned solution of the model within the interval $\kappa/2 < \xi < \kappa$,

$$M_{\rm Pl}^2 = \frac{a_0^3}{2b\kappa}\left[\frac{\kappa}{(\kappa-\xi)} - \frac{(2\xi-\kappa)^2}{(\kappa-\xi)}\frac{\cosh\,(\beta\pi)}{(8\xi-\kappa)}\right]. \tag{53}$$

One must further impose the finiteness and positivity of the effective 4D Planck mass, requirements that lead to the following condition for $\kappa/2 < \xi < \kappa$:

$$\frac{\kappa(8\xi-\kappa)}{(2\xi-\kappa)^2} > \cosh\,(\beta\pi). \tag{54}$$

This inequality cannot be fulfilled when ξ lies within the interval $\kappa/2 < \xi < \kappa$: the rhs of the inequality diverges exponentially as $\xi \mapsto \frac{\kappa^+}{2}$ and approaches the value $\cosh(\sqrt{\frac{3}{2}}\pi) \approx 23.45$ when $\xi \mapsto \kappa^-$, while the lhs diverges quadratically as $\xi \mapsto \frac{\kappa^+}{2}$ and tends to 7 when $\xi \mapsto \kappa^-$. Thus, the opposite claim actually holds: when ξ lies between $\kappa/2$ and κ, then $M_{\rm Pl}^2 < 0$, yielding a negative 4D effective coupling constant for gravitational interactions, i.e., to repulsive 4D gravity.

3. Case $\xi > \kappa$

$$\varphi^{\pm}(\tau) = \pm\sqrt{\frac{2(2\xi-\kappa)}{\xi(\xi-\kappa)}}\cosh\left(\sqrt{\frac{3\xi}{2(2\xi-\kappa)}}\tau\right), \tag{55}$$

or, in terms of $v = bw$,

$$\varphi^{\pm}(v) = \pm\varphi_{02}\,\cosh(\beta\,\arctan\,v). \tag{56}$$

However, positivity of the coupling function $L(\varphi)$ imposes the following nonalgebraic constraint on ξ and κ:

[7]It is worth noticing that one can consider that $L(\varphi) > 0$ implies that the extra dimension is compact. In this case the Planck mass is finite. However, since we are considering an unbounded extra dimension, we shall not consider this situation here.

$$\cosh^2\left(\frac{\pi}{2}\sqrt{\frac{3\xi}{2(2\xi-\kappa)}}\right) < \left(\frac{\xi-\kappa}{2\xi-\kappa}\right). \quad (57)$$

This inequality is not compatible with the above assumed condition $\xi > \kappa$ since then, the lhs is always greater than one, while the rhs is less than the unity. It turns out that under the restriction $\xi > \kappa$, the function L is negative along the whole extra dimension, giving rise a negative 4D effective Planck mass and, hence, to a repulsive gravity as in the previous case.

All the nonminimally coupled scalar field solutions (45), (49), and (56) respect the "mirror" symmetry (24)–(25); moreover, the expressions (45) and (56) are symmetric under $v \mapsto -v$: $\phi^{\pm}(v) = \phi^{\pm}(-v)$, whereas solution (49) is odd under such a symmetry: $\phi^{\pm}(v) = \phi^{\mp}(-v)$. However, from the above presented exact field configurations, just the solution (45) renders a physically viable model that can correctly describe the 4D effective gravity of our world.

C. General case: $\xi \neq 0$, $\epsilon \neq 0$

In this case we shall construct solutions that involve both the nonminimal coupling of the scalar field to gravity and the presence of the Gauss-Bonnet term. Despite the fact of the highly nonlinear character of the differential equation (19), we were able to obtain some exact solutions when there exists some relationship between the parameters of our model.

1. Particular solution: case $\lambda = -1$

If one performs the following change of variable $y = \text{arcsinh}\, v$ and redefine the field $\varphi = \psi^{1/(1-\lambda)}$, then (19) can be easily solved for the particular case when $\lambda = -1$ (see (40) for reference) or, equivalently, $\kappa = 2\xi$, since in this case we get a second order linear differential equation.

Let us see how this comes about: by changing the variable $v = \sinh y$, hence, $\cosh y = \sqrt{1+v^2}$, from (19) we get the following equation for φ:

$$\varphi(\ddot{\varphi} + \tanh y\dot{\varphi}) - \lambda(\dot{\varphi})^2 - \frac{3}{2}\,\text{sech}^2 y\varphi^2 = \frac{3}{\xi}\,\text{sech}^2 y\left(\frac{4\epsilon b^2}{a_0^2}\tanh^2 y - 1\right), \quad (58)$$

where overdots now mean derivatives with respect to y. By further performing the following substitution $\varphi = \psi^{1/(1-\lambda)}$ we get

$$\ddot{\psi} + \tanh y\dot{\psi} - \frac{3}{2}(1-\lambda)\text{sech}^2 y\psi = \frac{3}{\xi}(1-\lambda)\text{sech}^2 y(k^2\tanh^2 y - 1)\psi^{\frac{\lambda+1}{\lambda-1}}, \quad (59)$$

where $k^2 = \frac{4\epsilon b^2}{a_0^2}$. By choosing the particular case $\lambda = -1$ we actually get the linear differential equation,

$$\ddot{\psi} + \tanh y\dot{\psi} - 3\,\text{sech}^2 y\psi = \frac{6}{\xi}\,\text{sech}^2 y\,(k^2\tanh^2 y - 1). \quad (60)$$

This equation has the following real solution,

$$\psi = \varphi^2 = \frac{14 - 10k^2 + 6k^2\,\text{sech}^2 y}{7\xi} + C_1\cosh(A) + C_2\sinh(A), \quad (61)$$

where $A = 2\sqrt{3}\arctan(\tanh\frac{y}{2})$, and C_1 and C_2 are arbitrary constants. By going back to the dimensionless variable v we get the solution for the field φ,

$$\varphi = \sqrt{\frac{14-10k^2}{7\xi} + \frac{6k^2}{7\xi(1+v^2)} + C_1\cosh(A) + C_2\sinh(A)}, \quad (62)$$

where now $A = 2\sqrt{3}\arctan(\frac{\sqrt{v^2+1}-1}{v})$.

This solution fulfills the symmetry requirements (24)–(25) when $C_2 = 0$. Similar to previous cases, V is asymptotically constant as is shown in Fig. 3.

The positive nature of the radicand in the solution (62) restricts the values C_1 can adopt from below,

$$C_1 > \frac{2(2k^2-7)}{7\xi}, \quad (63)$$

whereas by imposing the $L > 0$ requirement, the integration constant C_1 is bounded from above,

$$C_1 < \frac{10k^2}{7\xi}\,\text{sech}\left(\frac{\sqrt{3}\pi}{2}\right), \quad (64)$$

leading to the following constraint,

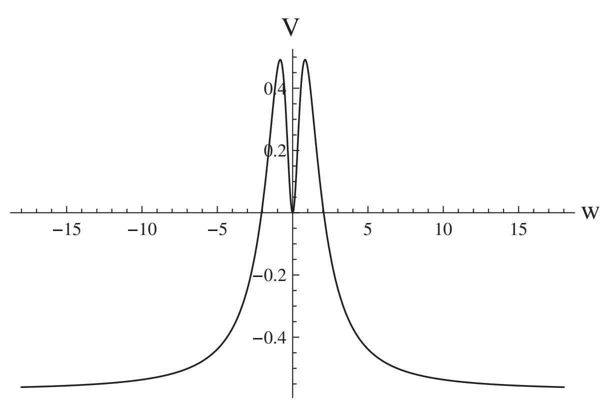

FIG. 3. V associated to solution (61). We set $C_1 = 0$, $C_2 = 0$, $\xi = 1/4$, $a_0 = 1$, $b = 1$, $\epsilon = 1/3$ and $\kappa = 1/2$.

$$\frac{2(2k^2-7)}{7\xi} < C_1 < \frac{10k^2}{7\xi}\,\mathrm{sech}\left(\frac{\sqrt{3}\pi}{2}\right), \tag{65}$$

which holds for arbitrary ξ and restricts the values of k through $k^2 < \frac{7}{2-5\,\mathrm{sech}(\frac{\sqrt{3}}{2}\pi)}$. Note that, in principle, C_1 can be negative.

When computing the 4D effective Planck mass according to (15), we could not perform the integration for $C_1 \neq 0$ and, hence, we have set this constant to zero. For this special case the Planck mass adopts the form

$$M_{\mathrm{Pl}}^2 = \frac{a_0^3}{21b\kappa}(9a_0^2k^2 - 28\epsilon b^2) = \frac{2a_0^3k^2}{21b\kappa}, \tag{66}$$

which is positive definite and finite as it should be for a well defined 4D effective theory that reproduces the gravitational interactions of our world.

2. Solutions for arbitrary λ

Let us recall that if we perform the coordinate transformation $x\mapsto\tau = \arctan\, v$ (see Eq. (27)) for the case in which the Gauss-Bonnet term is nontrivial, then the Eq. (19) adopts the form

$$\varphi\ddot{\varphi} + \left(\frac{\xi-\kappa}{\xi}\right)\dot{\varphi}^2 = \frac{3}{2}\varphi^2 - \frac{3}{\xi}(1-k^2\sin^2\tau), \tag{67}$$

which will be from now on the relevant differential equation to be solved.

Let us now consider more general solutions in which the value of the parameter λ is arbitrary (see (40)). When dealing with Eq. (67), we can perform the following transformation in order to reduce the order of this differential equation,

$$\dot{\varphi}(\tau) = \varphi(\tau)f(\tau), \tag{68}$$

where $f(\tau)$ is an integrable function of τ. We further can redefine the scalar field φ as follows

$$\varphi^2 = \psi(\tau). \tag{69}$$

Thus, Eq. (67) transforms into

$$\frac{1}{2}f\dot{\psi} + \left(\dot{f} + \lambda f^2 - \frac{3}{2}\right)\psi = -\frac{3}{\xi}(1-k^2\sin^2\tau). \tag{70}$$

With the aid of these two transformations, the second order differential equation (67) can be reduced to a system of first order differential equations, namely, to a nonlinear first order differential equation (general Riccati equation) for $f(\tau)$ and a linear one for the new function $\psi(\tau)$,

$$\dot{f} + \lambda f^2 - \frac{3}{2} = \frac{1}{2}fH(\tau), \tag{71}$$

$$\dot{\psi} + H(\tau)\psi = -\frac{3(1-k^2\sin^2\tau)}{\xi f}, \tag{72}$$

where $H(\tau)$ is a function of τ. Solving this system is more easy than solving the original differential equation (67) if we make a suitable choice of the function $H(\tau)$.

By setting $H(\tau) = 2c_1/f$, with $c_1 = \mathrm{const.}$ in the general Riccati equation (71) [this is equivalent to setting the prefactor of ψ to a constant in Eq. (70)] we get the following solution for f,

$$f = -\sqrt{\frac{l}{\lambda}}\tan\,[\sqrt{l\lambda}(\tau-\tau_0)], \tag{73}$$

where $2l = -(2c_1+3)$ and τ_0 is an arbitrary phase. We further substitute this solution into the linear equation (72) for $\psi(\tau)$ and proceed to solve it. The general solution for this equation with arbitrary l possesses a lengthy expression and is given in terms of products of several hypergeometric and exponential functions combined with a sine at certain power. For the sake of simplicity we shall restrict to the case in which $l = 1/\lambda$, getting the following solution,

$$\psi = \frac{6\lambda[4+3\lambda-(2+3\lambda)k^2\sin^2(\tau-\tau_0)]}{(4+\lambda)(2+\lambda)} + c_2\sin^{-(2+3\lambda)}(\tau-\tau_0), \tag{74}$$

where c_2 is an arbitrary constant. Once we have a solution for ψ we can get back to the original field $\varphi = \sqrt{\psi}$ according to (69). This solution will be subject to the relation (68) which implies that $c_2 = 0$, leading to

$$\varphi = \sqrt{\frac{6\lambda[4+3\lambda-(2+3\lambda)k^2\sin^2(\tau-\tau_0)]}{(4+\lambda)(2+\lambda)}}. \tag{75}$$

In this case the $L > 0$ and real φ conditions translate into the following restrictions for λ and k,

$$a)\quad 0 < \frac{3\lambda\xi[(4+3\lambda)-(2+3\lambda)k^2]}{(4+\lambda)(2+\lambda)} < 1 \quad \text{if} \quad \frac{\lambda(2+3\lambda)}{(4+\lambda)(2+\lambda)} < 0, \tag{76}$$

$$b)\quad 0 < \frac{3\lambda\xi(4+3\lambda)}{(4+\lambda)(2+\lambda)} < 1 \quad \text{whenever} \quad \frac{\lambda(2+3\lambda)}{(4+\lambda)(2+\lambda)} > 0, \tag{77}$$

where, in principle, λ can adopt negative values.

By making use of the formula (15), the 4D effective Planck mass for the solution (75) adopts the form

$$M_{\rm Pl}^2 = \frac{a_0^3}{b\kappa}\left[1 - \frac{k^2}{3} + \lambda\xi\frac{(2+3\lambda)k^2 - 3(4+3\lambda)}{(4+\lambda)(2+\lambda)}\right]. \quad (78)$$

A further relation between k and λ simplifies the solution for φ. For instance, when $k^2 = \frac{4+3\lambda}{2+3\lambda}$, we get a sine or cosine function depending on the phase constant τ_0. It should be pointed out that both the original differential equation (67) and (72) are invariant under a suitable simultaneous rescaling of the field φ and the parameter ξ. This fact can be used to arrive at the following particular solutions for the field φ:

$$a)\quad \varphi = \sqrt{\frac{3}{\xi|\lambda|}}\sin(\arctan\, v),$$
$$k^2 = 1 - \frac{5}{2|\lambda|}\quad \text{with } \lambda < -5/2, \quad (79)$$

$$b)\quad \varphi = \sqrt{\frac{6}{5\xi}}\cos(\arctan\, v),$$
$$k^2 = 1 + \frac{2\lambda}{5}\quad \text{with } \lambda > -5/2. \quad (80)$$

In Fig. 4 we show the profile of the V for the solution (80).[8] Again, it is constant and negative at $w \to \infty$.

The 4D Planck mass for solution (79) reads

$$M_{\rm Pl}^2 = \frac{6a_0^3}{5b\kappa}\left(1 - \frac{4}{9}k^2\right), \quad (81)$$

whereas for solution (80) it is

$$M_{\rm Pl}^2 = \frac{3a_0^3}{5b\kappa}\left(1 - \frac{5}{9}k^2\right). \quad (82)$$

There is another solution of the same type that is valid for the special value $\lambda = -11/8$, which implies a concrete proportional relation between ξ and κ according to (40),

$$c)\quad \varphi = B\cos(2\ \arctan\, v) + C, \quad (83)$$

where the following constants

$$B_\pm = \pm\sqrt{\frac{3}{154\xi}\left[7(2-k^2) \pm 2\sqrt{4k^4 + 49(1-k^2)}\right]},$$
$$C_\pm = \pm\frac{\sqrt{33}k^2}{\sqrt{14\xi\left[7(2-k^2) \pm 2\sqrt{4k^4+49(1-k^2)}\right]}},$$

[8] For solution (79) of the scalar field the qualitative behavior of V is similar.

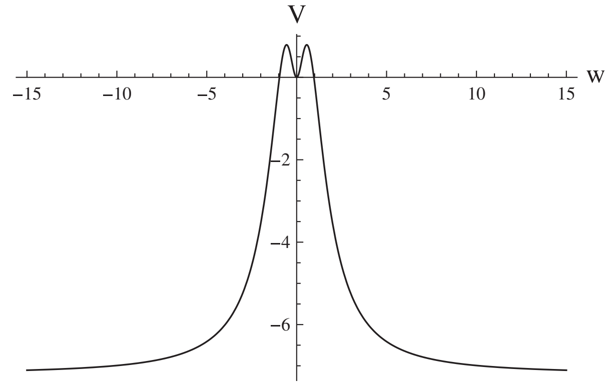

FIG. 4. V associated to solution (80). We set $\xi = 1/3$, $a_0 = 1$, $b = 1$, $\epsilon = 1/5$ and $\kappa = 1/2$.

must have the same sign (provided that the radicand is positive) since one can show that both parameters ξ and k can be expressed as

$$\xi = \frac{6}{11B^2 + 14BC + 3C^2},$$
$$k^2 = \frac{28BC}{11B^2 + 14BC + 3C^2},$$

implying that the left-hand sides of these equalities must be positive.

One can look for more complex solutions than the ones displayed here. However, the relations that define the involved integration constants become more and more lengthy and difficult their physical interpretation or viability.

From the above obtained exact solutions of this section, the field configurations (62),(79) and (80) meet the symmetry conditions (24)–(25) and lead to viable 4D effective theories that reproduce the correct gravitational couplings of our world.

VII. GRAVITATIONAL FLUCTUATIONS

In order to study the localization properties of gravity within the framework of braneworlds, a rigorous analysis of gravitational fluctuations is needed.

In 4D standard cosmology, the fluctuations of the geometry can be classified into scalar, vector, and tensor modes with respect to the three–dimensional rotation group $SO(3)$ [68,69]. This fact makes more feasible the study of the metric fluctuations because at the linear level the dynamical equations of the scalar, vector, and tensor modes are decoupled.

Within the braneworld models considered in our work, where the 4D geometry is Poincaré invariant, the fluctuations of the metric may be also classified into scalar, vector, and tensor modes with respect to the transformations of the $SO(3,1)$ symmetry group (see [70] for details).

Let us consider the fluctuations of both the metric and the scalar field around the gravitational background specified in (6) and the field equations (7)–(9). In other words, the perturbed geometry has the following form

$$ds_p^2 = [a^2(w)\eta_{AB} + H_{AB}(x,w)]dx^A dx^B, \quad \text{where } x \equiv \{x^\mu\}, \tag{84}$$

while the fluctuation of the scalar field is

$$\varphi_p = \varphi(w) + \chi(x,w). \tag{85}$$

The functions $H_{AB}(x,w)$ and $\chi(x,w)$ are the metric and the field fluctuations, respectively. The index p denotes perturbed quantities.

By taking into account the 4D Poincaré symmetry of our background metric (6), the fluctuations can be written as

$$H_{AB} = H_{AB}^{(S)} + H_{AB}^{(V)} + H_{AB}^{(T)}, \tag{86}$$

where

$$H_{AB}^{(S)} = a^2(w)\begin{pmatrix} 2(\eta_{\mu\nu}\psi + \partial_\mu\partial_\nu E) & \partial_\mu C \\ \partial_\mu C & 2\zeta \end{pmatrix}, \tag{87}$$

$$H_{AB}^{(V)} = a^2(w)\begin{pmatrix} \partial_\mu f_\nu + \partial_\nu f_\mu) & D_\mu \\ D_\mu & 0 \end{pmatrix}, \tag{88}$$

$$H_{AB}^{(T)} = a^2(w)\begin{pmatrix} 2h_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}. \tag{89}$$

The upper indices S, V and T denote the scalar, vector and tensor parts of the fluctuations, respectively. The tensor $h_{\mu\nu}(x,w)$ is transverse and traceless with respect to the 4D Minkowski metric $\eta_{\mu\nu}$, in other words

$$h_\mu^\mu = 0, \qquad \partial_\nu h_\mu^\nu = 0. \tag{90}$$

Also, the vectors $f_\mu(x,w)$ and $D_\mu(x,w)$ are divergence free,

$$\partial^\mu f_\mu = 0, \qquad \partial^\mu D_\mu = 0. \tag{91}$$

The four remaining functions $\psi(x,w)$, $E(x,w)$, $C(x,w)$ an $\zeta(x,w)$ are scalars with respect to 4D Poincaré transformations.

The relations (87)–(91) tell us that, apparently, we only have 15 independent degrees of freedom of the metric fluctuations. Moreover, the covariance of our setup implies that the gravitational perturbation theory has some unphysical gauge degrees of freedom [68,69]. In our case we can make use of this gauge freedom completely by fixing 5 of the above 15 degrees of freedom.

A simple choice that completely fixes the above mentioned gauge freedom is the longitudinal gauge given by

$$E = 0, \qquad C = 0, \qquad f_\mu = 0. \tag{92}$$

Thus, we finally have only 10 independent degrees of freedom.

A. Tensor modes

As quoted in [59] the equation for the evolution of the tensor fluctuation modes is

$$\Psi_{\mu\nu}'' - \frac{\left(\sqrt{s}\right)''}{\sqrt{s}}\Psi_{\mu\nu} - \frac{r}{s}\Box^\eta\Psi_{\mu\nu} = 0, \tag{93}$$

where $\Psi_{\mu\nu} = \sqrt{s(w)}h_{\mu\nu}$ with $s(w) = a^3 q$. In addition, $r(w) = a^3(q + \frac{(q-L)'}{2\mathcal{H}})$ and $\Box^\eta$ denotes the d'Alembertian with respect to the metric η.

In order to study the mass spectrum of these fluctuations let us consider the following separation of variables $\Psi_{\mu\nu}(x,w) = \vartheta(w)\epsilon_{\mu\nu}(x)$. In terms of this new variables the equation (93) splits into

$$\Box^\eta\epsilon_{\mu\nu}^m + m^2\epsilon_{\mu\nu}^m = 0, \tag{94}$$

$$\vartheta_m'' - \frac{\left(\sqrt{s}\right)''}{\sqrt{s}}\vartheta_m + m^2\frac{r}{s}\vartheta_m = 0, \tag{95}$$

where $\epsilon_{\mu\nu}^m(x)$ is a 4D tensor mode with mass m; on the other hand, $\vartheta_m(w)$ is the 5D profile of the field $\Psi_{\mu\nu}$ and characterizes its localization properties. Equation (95) can be interpreted as a Sturm-Liouville eigenvalue problem with the associated norm,

$$\langle\vartheta|\vartheta\rangle = \int_{-\infty}^{\infty}\frac{r}{s}\vartheta^2 dw. \tag{96}$$

The zero mode $\epsilon_{\mu\nu}^0(x)$ is the massless 4D graviton. This mode is localized on the brane if $\langle\vartheta_0|\vartheta_0\rangle = \int_{-\infty}^{\infty}\frac{r}{s}\vartheta_0^2 dw$ is finite. It is not difficult to show from (95) that $\vartheta_0 = \sqrt{s}$ when $m = 0$, then, the localization condition for the massless graviton is

$$\langle\vartheta_0|\vartheta_0\rangle = \int_{-\infty}^{\infty} a^3 q dw - 4\epsilon[a']_{-\infty}^{\infty} + 8\epsilon\int_{-\infty}^{\infty}\frac{a'^2}{a}dw. \tag{97}$$

As one can observe the above expression is quite similar to (15). Therefore, for the geometry described in(18), if the 4D massless graviton is localized on the brane, then the 4D Planck mass is finite.

Let us investigate the localization properties of the massless graviton in the backgrounds studied in Sec. VI.

By substituting the expression of the warp factor into (97) we obtain the following result:

$$\langle\vartheta_0|\vartheta_0\rangle_{\mathrm{GB}} = \frac{2a_0^3}{b}\left(1+\frac{k^2}{3}\right). \tag{98}$$

The above expression tells us that the 4D graviton is localized on the brane. This result generalizes the particular situation considered in [44] where $\xi = 0$, $\epsilon \neq 0$ and $k = 1$ for an arbitrary value of k, showing that the zero mass graviton is also localized on the brane.

In the following we shall consider the opposite case where $\xi \neq 0$ and $\epsilon = 0$. The normalization condition depends of the behavior of the integrand $a^3 L(\varphi)$. With regard to the study of gravity localization, the backgrounds defined in (45) and (49) are similar because both have a regular and finite nonminimal coupling function $L(\varphi)$. Therefore, the convergence of (97) is completely determined by the a^3 behavior when $w \to \infty$. For the warp factor (18) $a^3 \sim 1/|w|^3$ at infinity along the fifth dimension. This fact implies again that the massless tensor mode is localized on the brane, since the third term in (97) is finite.

Finally, let us study the general case ($\xi \neq 0$ and $\epsilon \neq 0$). Some exact solutions for this general situation are shown in (62),(79),(80) and (83). The norm for the zero mass mode takes the following form,

$$\langle\vartheta_0|\vartheta_0\rangle = \int_{-\infty}^{\infty} a^3(L-1)dw + \langle\vartheta_0|\vartheta_0\rangle_{\mathrm{GB}},$$

where $\langle\vartheta_0|\vartheta_0\rangle_{\mathrm{GB}}$ is the norm written in (98). On the other hand, the function L associated to (62),(79),(80) and (83) is finite along the extra dimension, thus, like in the previous cases the normalization condition depends on the behavior of a^3 at infinity. Hence, we conclude that the zero tensor mode is normalized. In summary, all our background solutions recover the standard 4D graviton on the brane for the case when both the nonminimal coupling and the Gauss–Bonnet term are present in the model.

B. Vector modes

In contrast with the tensor sector, the simultaneous presence of the Gauss–Bonnet term and the nonminimal coupling interaction makes much more difficult the study of vector modes. Thus, in this work we will study each case separately and will leave the general case for a future investigation.

1. Vector modes with the Gauss-Bonnet term only

In this subsection we analyze the case where the nonminimal effects are negligible ($\xi = 0$ and $\epsilon \neq 0$). Using (84),(88),(91) and substituting them in (3) we obtain the equations for the vector modes on the longitudinal gauge,

$$(D^\nu)' + \left(\frac{q'}{q} + 3\mathcal{H}\right)D^\nu = 0, \tag{99}$$

$$\Box^\eta D_\mu = 0, \tag{100}$$

where the first relation is a constraint and defines the profile of D_μ along the extra dimension. Furthermore, since there is no mass term in the above second equation, it shows that there is only a massless mode, the graviphoton, with no massive vector fluctuations.

In the case of vector modes we do not have a Sturm-Liouville eigenvalue problem, in consequence, the issue of defining the norm is more involved than for the tensor sector. Therefore, it is more appropriate to use the perturbed version of the action (2) up to second order with respect to the vector fluctuations [44,61]. This perturbed action can be written as follows,

$$\delta^{(2)}S_V = \int d^4x dw \frac{1}{2}(\eta^{\alpha\beta}\partial_\alpha \mathcal{D}^\mu \partial_\beta \mathcal{D}_\mu), \tag{101}$$

where $\mathcal{D}_\mu = a^{3/2}\sqrt{q}D_\mu$ is called the "canonical normal mode." The zero mode associated to (100) is localized on the brane if $\delta^{(2)}S_V$ is finite. By making a suitable variable separation,

$$\mathcal{D}^\mu = \frac{v^\mu(x)}{a^{3/2}(w)\sqrt{q(w)}}, \tag{102}$$

the zero mass mode takes the form

$$\Box^\eta v^\mu(x) = 0, \tag{103}$$

where $v^\mu(x)$ is the 4D part of the vector sector of fluctuations.

The norm of this mode reads

$$\langle\mathcal{D}^\mu|\mathcal{D}_\mu\rangle = \int_{-\infty}^{\infty}\frac{dw}{a^3 q}. \tag{104}$$

When studying the localization properties of the zero mass tensor mode, we learned that it is necessary to have a convergent integral of $a^3 q$. Furthermore, the norm of the massless vector mode has an opposite behavior compared to the tensor mode one. Therefore, if one wishes the 4D graviton to be localized on the brane, the vector sector (represented by its zero mass mode) will necessarily not be localized on it.

2. Vector modes with the nonminimal coupling only

In this subsection we shall study the situation when $\epsilon = 0$ and $\xi \neq 0$. One way to obtain the mass spectrum of the vector fluctuations consists in passing from the action (2) (with $\epsilon = 0$) to the Einstein frame and then, perform the perturbation analysis. In order to do that we apply a conformal transformation as follows:

$$\bar{g}_{AB} = L^{2/3} g_{AB}.$$

In the new frame the background action can be written as

$$S \mapsto S_{EF} = \int_{M_5} d^5x \sqrt{|\bar{g}|} \left\{ -\bar{R} + \frac{1}{2}(\bar{\nabla}\sigma)^2 - \bar{V}(\sigma) \right\}, \quad (105)$$

where all quantities with an overline are associated to the metric $\bar{g}_{AB}$, σ is the new scalar field and $\bar{V}$ is its self-interaction potential. The relationship between the old and new variables is

$$d\sigma = \sqrt{\frac{1}{L} + \frac{8}{3}\left(\frac{L_\varphi}{L}\right)^2} d\varphi, \quad (106)$$

$$\bar{V} = \frac{V}{L^{5/3}}. \quad (107)$$

Moreover, the general fluctuations on the Einstein frame are related with the old quantities as follows:

$$\bar{H}_{AB} = \frac{2a^2}{3L^{1/3}} \chi L_\varphi \eta_{AB} + L^{2/3} H_{AB}. \quad (108)$$

The first term in the above expression does not contribute to the vector sector since it belongs to the scalar sector, then it is easy to obtain that $\bar{D}_\mu = D_\mu$. Thus, the equations for the vector modes are

$$(\bar{D}^\nu)' + 3\bar{\mathcal{H}}\bar{D}^\nu = 0,$$
$$\Box^\eta \bar{D}_\mu = 0,$$

where $\bar{\mathcal{H}} = \bar{a}'/\bar{a}$ and $\bar{a} = aL^{1/3}$.

Similarly to the previous case, let us define a new vector variable $\bar{\mathcal{D}}_\mu = \bar{a}^{3/2}\bar{D}_\mu$. Therefore, the norm of the massless mode takes the form

$$\langle \bar{\mathcal{D}}^\mu | \bar{\mathcal{D}}_\mu \rangle = \int_{-\infty}^{\infty} \frac{dw}{\bar{a}^3} = \int_{-\infty}^{\infty} \frac{dw}{a^3 L}. \quad (109)$$

The localization properties of this case are similar to those of the previous one. The localization of the massless tensor mode implies the delocalization of the vector sector of fluctuations. In other words, if the effective Planck mass is finite (or the 4D massless graviton is localized on the brane), then the vector sector is not confined to the brane.

In principle, it still can have observable effects in the 4D phenomenology since its projection to the brane can be nonvanishing. Notwithstanding, this delocalization phenomenon of the vector modes implies that they have little influence in the 4D low energy physics, at least less influence than the tensor sector.

C. Scalar modes

In the longitudinal gauge the scalar fluctuations of the geometry can be expressed as

$$H_{AB}^{(S)} = 2a^2(w) \begin{pmatrix} \eta_{\mu\nu}\psi & 0 \\ 0 & \zeta \end{pmatrix}.$$

Furthermore, the matter provides an extra degree of freedom χ to the scalar sector. Hence, it follows that we have three independent scalar degrees of freedom. One can obtain the dynamical equations of the scalar fluctuations by perturbing the Einstein and Klein–Gordon equations with respect to the scalar sector. These equations adopt the following form

$$4q\psi'' + 4\psi'\left[q + \frac{q'}{\mathcal{H}} + \frac{\varphi' L_\varphi}{3}\right] + 8\zeta\left[\mathcal{H}'L + \frac{\varphi'^2}{8} - \frac{4\epsilon\mathcal{H}^2}{a^2}(2\mathcal{H}' - \mathcal{H}^2) + \frac{(\varphi' L_\varphi)'}{3}\right] + 4\zeta'\left[\frac{\varphi' L_\varphi}{3} + \mathcal{H}q\right]$$
$$+ q\Box^\eta\zeta + \frac{8\epsilon}{a^2}(\mathcal{H}' - \mathcal{H}^2)\Box^\eta\psi + \frac{1}{3}\frac{dV}{d\varphi}a^2\chi + \varphi'\chi' + 4\mathcal{H}'\chi L_\varphi + \frac{1}{3}\left[4(\chi L_\varphi)'' - \Box^\eta(\chi L_\varphi)\right] = 0, \quad (110)$$

$$q\psi'' + \psi'\left[7\mathcal{H}L - \frac{4\epsilon\mathcal{H}}{a^2}(2\mathcal{H}' + 5\mathcal{H}^2) + \frac{7\varphi' L_\varphi}{3}\right] + \zeta'\left[\frac{\varphi' L_\varphi}{3} + \mathcal{H}q\right]$$
$$+ 2\zeta\left[(3\mathcal{H}^2 + \mathcal{H}')L - \frac{8\epsilon\mathcal{H}^2}{a^2}(\mathcal{H}' + \mathcal{H}^2) + 2\mathcal{H}\varphi' L_\varphi + \frac{(\varphi' L_\varphi)'}{3}\right] + 2\mathcal{H}(\chi L_\varphi)'$$
$$+ \frac{1}{3}\frac{dV}{d\varphi}a^2\chi - q\Box^\eta\psi + (\mathcal{H}' + 3\mathcal{H}^2)\chi L_\varphi + \frac{1}{3}\left[(\chi L_\varphi)'' - \Box^\eta(\chi L_\varphi)\right] = 0, \quad (111)$$

$$q\zeta - 2\psi\left[L - \frac{4\epsilon\mathcal{H}'}{a^2}\right] - \chi L_\varphi = 0, \tag{112}$$

$$\frac{\varphi'\chi}{2} + 3q(\psi' + \mathcal{H}\zeta) + (\chi L_\varphi)' - \mathcal{H}\chi L_\varphi + L'\zeta = 0. \tag{113}$$

$$\chi'' + 3\mathcal{H}\chi' - \Box^\eta\chi + \varphi'[4\psi' + \zeta'] + 2\zeta(\varphi'' + 3\mathcal{H}\varphi') - \frac{\partial^2 V}{\partial\varphi^2}a^2\chi - \frac{1}{2}\Big\{2(2\mathcal{H}' + 3\mathcal{H}^2)\chi L_{\varphi\varphi} + L_\varphi[\Box^\eta\zeta - 3\Box^\eta\psi + 4\zeta(2\mathcal{H}' + 3\mathcal{H}^2) + 4(\psi'' + \mathcal{H}[4\psi' + \zeta'])]\Big\} = 0. \tag{114}$$

Equations (110)–(111) come from the perturbed Einstein equations, while the expressions (112) and (113) are constraint equations. Finally, Eq. (114) represents the fluctuated Klein-Gordon equation. Then, there is only one independent scalar degree of freedom. As in the vector sector, in this case we will study separately the effects of the Gauss-Bonnet term and the nonminimal coupling on the scalar modes.

1. Scalar modes with the Gauss-Bonnet term only

Although some aspects of this case were studied in [44], it is helpful to consider some of its details. The master equation of the system can be obtained by using the constraints (112),(113) and the dynamical equations (110), (111) and (114),

$$\Phi'' - z\left(\frac{1}{z}\right)'\Phi - \left(1 + \frac{q'}{\mathcal{H}q}\right)\Box^\eta\Phi = 0, \tag{115}$$

where $\Phi = \frac{a^{3/2}q}{\phi'}\psi$ and $z = \frac{a^{3/2}\phi'}{\mathcal{H}}$. In order to study the mass spectrum of Φ, let us assume that

$$\Box^\eta\Phi = -m^2\Phi.$$

Thus, the Eq. (115) can be written as

$$\Phi'' - z\left(\frac{1}{z}\right)''\Phi + \left(1 + \frac{q'}{\mathcal{H}q}\right)m^2\Phi = 0. \tag{116}$$

The associated norm for the above eigenvalue problem is

$$\langle\Phi|\Phi\rangle = \int_{-\infty}^{\infty}\left(1 + \frac{q'}{\mathcal{H}q}\right)\Phi^2 dw. \tag{117}$$

The scalar massless mode is localized on the brane if its norm is finite. In other words,

$$\langle\Phi_0|\Phi_0\rangle = \int_{-\infty}^{\infty}\left(1 + \frac{q'}{\mathcal{H}q}\right)\Phi_0^2 dw < \infty, \tag{118}$$

where $\Phi_0 = \frac{1}{z}$. Let us apply this general analysis to the background solution (30). It is not hard to show that when $w \to \infty$,

$$\left(1 + \frac{q'}{\mathcal{H}q}\right)\Phi_0^2 \sim \begin{cases} |w|^7, & \text{when } k = 1 \\ |w|^5, & \text{when } k \neq 1 \end{cases}. \tag{119}$$

Therefore, in both cases the massless scalar mode is not localized on the brane.

2. Scalar modes with the nonminimal coupling only

Similar to the vector modes case, the analysis of the scalar sector without the Gauss-Bonnet term is easier in the Einstein frame. In this case the metric fluctuations can be expressed as follows,

$$\bar{H}^{(S)}_{AB} = 2\bar{a}^2\begin{pmatrix} \eta_{\mu\nu}\bar{\psi} & 0 \\ 0 & \bar{\zeta} \end{pmatrix}, \tag{120}$$

where $\bar{\psi} = \psi + \frac{\chi L_\phi}{3L}$ and $\bar{\zeta} = \zeta - \frac{\chi L_\varphi}{3L}$. On the other hand, the scalar field fluctuation in the Einstein frame $\bar{\chi}$ is related to χ as follows,

$$\bar{\chi} = \sqrt{\frac{1}{L} + \frac{8}{3}\left(\frac{L_\phi}{L}\right)^2}\,\chi.$$

By considering the above expression and the Eqs. (110)–(114) when $\epsilon = 0$, we obtain the following equation for the scalar sector of fluctuations,

$$\bar{\Phi}'' - \bar{z}\left(\frac{1}{\bar{z}}\right)'\bar{\Phi} + m^2\bar{\Phi} = 0, \tag{121}$$

where $\bar{\Phi} = \frac{\bar{a}^{3/2}}{\sigma'}\bar{\Psi}$ and $\bar{z} = \frac{\bar{a}^{3/2}\sigma'}{\bar{\mathcal{H}}}$. Thus, the norm of zero mode reads

$$\langle\bar{\Phi}_0|\bar{\Phi}_0\rangle = \int_{-\infty}^{\infty}\bar{\Phi}_0^2 dw = \int_{-\infty}^{\infty}\frac{1}{\bar{z}^2}dw. \tag{122}$$

The localization properties of the solutions (45) and (49) are similar in the sense that their massless mode has the same behavior at infinity, i.e.,

$$\frac{1}{\bar{z}^2} \sim |w|^5, \quad \text{when} \quad w \to \infty.$$

The above result tells us that the zero mass scalar mode is not localized on the brane.

Therefore, the results of the subsections VII B and VII C tell us that both the vector and the scalar fluctuations modes behave in a quite similar way when regarding their localization properties, both are delocalized from the brane if 4D gravity is localized on it or, equivalently, if the 4D Planck mass is finite.

VIII. CONCLUSIONS

We present a scenario with a thick braneworld model built by a scalar field nonminimally coupled to the Einstein-Hilbert term. Furthermore, there is a Gauss-Bonnet term in the bulk. We obtain the effective 4D Planck mass coming from the dimensional reduction of the setup (2) and study its properties. As a consequence of the nonminimal coupling between the scalar matter and gravity, $M_{\rm Pl}$ depends explicitly on the profile of the bulk scalar field φ. Therefore, in order for us to be able to interpret our physical results in closed form we need to construct exact solutions for our braneworld configuration.

Despite the highly nonlinear nature of the relevant differential equation for the scalar field, we were able to obtain several exact particular solutions, contrary to previous studies that implement approximate solutions and/or particular cases that consider just one of the two effects.

In order to elucidate the physical effects of the inclusion of the nonminimal coupling and of the Gauss-Bonnet term, we considered three cases in which we switched on/off the corresponding parameters. In all of these cases we were able to construct exact solutions that render an effective Planck mass positive and finite by imposing a restriction on the nonminimal coupling function $L(\phi)$ for certain values of the parameter space of these solutions. However, there are also exact field configurations for which the 4D Planck mass is negative, yielding braneworld models that cannot physically describe the gravitational interactions of our world.

We also studied the whole set of gravitational fluctuations—classified into scalar, vector and tensor modes with respect to the 4D Poincaré symmetry group—for our braneworld model. For all the considered backgrounds, the massless zero mode of the tensor fluctuations is localized on the brane. This fact allows us to recover 4D gravity in our world. In contrast to this, the massless scalar and vector modes are delocalized.

Although the scalar and vector sectors have a nonzero projection on the brane, due to the delocalization phenomenon it is not strange that for our scenario they do not have sensible observable effects at low energies. As we commented above, this last assertion is reinforced by the fact that in a simple scenario, the corrections to Newton's law coming from the scalar modes are more suppressed than the corrections coming from the tensor modes. Therefore, the above result suggests that in gravitational experiments it is more probable to detect first the effects of the tensor sector rather than the effects of the scalar and vector ones. It is noteworthy that the delocalization phenomenon on thick branes also appears when the nonminimal coupling and the Gauss-Bonnet are not present in our scenario [61]. Then the above observation suggests us that in thick braneworlds with 4D Poincaré invariant geometries, the decoupling of the vector and scalar sectors of the 4D phenomenology is a robust effect of this kind of scenarios.

As a further step it will be interesting to study the stability of the field configurations within the framework of the present braneworld model when both nonlinear effects are switched on. It is not completely clear if the delocalization of the vector and scalar sectors is maintained in other class of solutions not studied here, this seems to be a very suggestive direction. Another interesting topic is to consider a nonminimal coupling of the bulk scalar field not only with the Einstein-Hilbert piece of the Lagrangian but, also, with the Gauss-Bonnet term. Investigation of these issues is left for future work.

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