

Vacuum energy in Kerr-AdS black holes

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We compute the vacuum energy for Kerr black holes with anti-de Sitter (AdS) asymptotics in dimensions $5 \leq D \leq 9$ with all rotation parameters. The calculations are carried out employing an alternative regularization scheme for asymptotically AdS gravity, which considers supplementing the bulk action with counterterms which are a given polynomial in the extrinsic and intrinsic curvatures of the boundary (also known as Kounterterms). The Kerr-Schild form of the rotating solutions enables us to identify the vacuum energy as coming from the part of the metric that corresponds to a global AdS spacetime written in oblate spheroidal coordinates. We find that the zero-point energy for higher-dimensional Kerr-AdS reduces to one of a Schwarzschild-AdS black hole when all the rotation parameters are equal to each other, a fact that is well known in five dimensions. We also sketch a compact expression for the vacuum energy formula in terms of asymptotic quantities that might be useful to extend the computations to higher odd dimensions.

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I. INTRODUCTION

In general relativity, black hole solutions play a fundamental role in the description and the understanding of the gravitational interaction at both macroscopic and microscopic scales. These objects can be described by a given set of parameters, namely, mass, angular momentum, and electric charge. Our interest lies in Kerr solutions, which have mass and angular momentum, i.e., are free to rotate.

From the astrophysics point of view, the observable Universe is made of countless rotating objects, whose geometry is given in a good approximation by a Kerr spacetime. Even though the first black hole solution was proposed in 1916 by Schwarzschild, it took more than 40 years to generalize it to a rotating case. The main difficulty for that was the lack of spherical symmetry and the nondiagonal elements in the metric tensor.

In higher dimensions, the situation is far more complicated. In fact, the number of rotation parameters increases in one every two additional dimensions. For this reason, any type of calculation in this geometry quickly turns very involved as we go up in the spacetime dimension.

Rotating solutions with anti-de Sitter (AdS) asymptotics have become relevant in the context of anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1–3]. Examples of holographic studies which consider this type of solution can be found [4–7]. In all the cases, the properties of the holographic stress tensor in the gravity side (e.g., Weyl anomalies) are matched with the ones defined in a

boundary CFT which lives on a rotating Einstein universe. In the same vein, further insight on this problem was provided in Refs. [8–11].

Quasinormal modes for rotating solutions were studied in Refs. [12,13]. Perturbations of the black hole metric are equivalent to the perturbation of a thermal state on the boundary CFT. The time evolution of the perturbed solution is dual to the time evolution of the thermal state state fluctuations.

Other theoretical developments that involve Kerr solutions identify Hawking-Page transitions [14] with confinement/deconfinement transitions in the strongly coupled regime of the boundary gauge theory [15]. The system also exhibits a gap between the transition lines in both sides of the duality. In Ref. [16] a phase diagram structure similar to a Weiss ferromagnetic system and a van der Waals liquid/gas system for certain critical temperature was found. Sonner studied in Ref. [17] the rotating extension of the idea of a holographic superconductor, in which the model features a superconducting phase transition on the boundary of a Kerr-Newman black hole.

Nonlinear spinning solutions to fluid mechanics were constructed in Refs. [18,19]. The duality is realized through the identification of the stress tensor and the thermodynamic quantities in the fluid side with the boundary stress tensor and thermodynamics of large rotating black holes in the gravity sector.

Thermodynamic instabilities in Kerr solutions from quantum corrections in the partition function were found in Refs. [20–22], as the action and the heat capacity can turn negative.

Extracting holographic quantities from the conformal boundary in AdS gravity requires one to write down the

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Einstein equations for an asymptotic form of the metric [23]. Solving order by order in powers of the radial coordinate, the divergences in the variation of the action are canceled by the addition of local (intrinsic) counterterms. The procedure proves to be satisfactory in many cases but, however, the method in higher dimensions becomes far more involved. As a consequence, there is no general expression for the counterterm series in an arbitrary dimension. The background independence of the counterterm method is reflected in a nonzero value for the energy of the AdS vacuum E_{vac} in odd spacetime dimensions [24]. The total energy $E = M + E_{\text{vac}}$, where M is the Hamiltonian mass, appears also in thermodynamic relations that involve the evaluation of the Euclidean action, in whichever method that does not use background subtraction [25]. The zero-point energy for asymptotically AdS (AAdS) black holes can be identified, within the framework of AdS/CFT, with the Casimir energy of the conformal theory. In particular, in five dimensions, matching the vacuum energy $E_{\text{vac}} = 3\ell^2/32\pi G$ of Schwarzschild-AdS black holes with the Casimir energy of $\mathcal{N} = 4$ super Yang-Mills theory on the boundary provided one of the first realizations of the gauge/gravity correspondence.

Similar computations have been performed for rotating AAdS solutions [4–7].

The value of the vacuum energy also plays a role in the proof of positivity of energy for asymptotically AdS spacetimes in odd dimensions [26,27].

In this article, we extend the results for the vacuum energy in Refs. [4–7] to Kerr-AdS black holes with a maximal number of rotation parameters up to nine spacetime dimensions, using Kounterterm regularization for AdS gravity [28] and exploiting the Kerr-Schild form of the metric for rotating solutions.

This paper is organized as follows: in Sec. II we review Kounterterm regularization for AdS gravity. Once the *extrinsic* counterterms have been introduced, conserved quantities derived within this regularization scheme are revised in Sec. III. In particular, we made use of the separability of the Kounterterm charges in a part that gives the black hole mass and angular momentum, and another one that produces the vacuum energy. In Sec. IV, the metric of Kerr-AdS black holes is cast in Kerr-Schild form in order to isolate the part that contributes to the vacuum energy of rotating solutions. In Sec. V, explicit results up to nine dimensions are shown. Some properties of the zero-point energy for Kerr-AdS are discussed. Finally, the last section is devoted to conclusions and prospects.

II. KOUNTERTERM REGULARIZATION IN ADS GRAVITY

For more than a decade, the AdS/CFT correspondence [1–3] has provided a concrete realization of the long-standing idea of holographic principle. This form of gauge/gravity duality has triggered a growing interest in the community,

as a useful tool to describe strongly coupled systems. Maldacena’s conjecture has gone beyond string theory to be applied in areas as diverse as relativistic hydrodynamics, condensed matter, and quantum chromodynamics.

This duality postulates the equivalence between the partition function of AdS gravity and the one of a boundary CFT, i.e., $Z_{\text{AdS}}(\phi) = Z_{\text{CFT}}(\phi_0)$. Here, we understand that the field ϕ , which lives in the bulk of the spacetime, takes the value ϕ_0 as one approaches the boundary. In the boundary theory, ϕ_0 is interpreted as a source for a pointlike operator \mathcal{O} . In the low-energy limit, the classical gravitational action can be used to compute the partition function of the CFT. Also, physical quantities defined in a finite-temperature field theory can be understood in terms of thermodynamic properties of black holes in the bulk.

For a suitable realization of the gauge/gravity duality, it is necessary to render the gravitational action finite. Within this framework, the convergence of the gravity action for asymptotically AdS spacetimes is achieved carrying out the holographic renormalization program [29–31], which results in the addition of the original action of local (intrinsic) counterterms [24,25] on top of a Gibbons-Hawking term [32,33].

Because of the fact that there is no closed formula for the counterterms in an arbitrary dimension, an alternative series was given in Refs. [28,34]. This proposal, valid for all dimensions, considers the addition of boundary terms which are a polynomial in the extrinsic curvature K_{ij} and the boundary Riemman tensor. By adding this structure at the boundary, one gets a regularized Euclidean action and is able to reproduce the correct black hole thermodynamics in AAdS gravity.

Conserved quantities derived within this framework are particularly useful to deal with solutions with a more complicated structure in the metric, which is the case under investigation in this paper. This is especially relevant in high enough dimensions, as we shall discuss below.

Let us take the action for Einstein gravity with a negative cosmological constant in $D = d + 1$ dimensions

$$I = -\frac{1}{16\pi G} \int_M d^{d+1}x \sqrt{-g} (R - 2\Lambda) + c_d \int_{\partial M} d^d x B_d, \quad (1)$$

where $\Lambda = -\frac{d(d-1)}{2\ell^2}$ and R is the spacetime Ricci scalar. The boundary term B_d is added for the purpose of finiteness of the conserved quantities and, at the same time, it produces a well-posed action principle for AAdS spacetimes.

The spacetime geometry can be described in terms of Gaussian coordinates

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = N^2(r) dr^2 + h_{ij}(r, x) dx^i dx^j, \quad (2)$$

where r is the radial coordinate. The manifold M possesses a single boundary ∂M , which is located at radial infinity.

Indeed, for $r = \infty$, the metric h_{ij} accounts for the intrinsic properties of the boundary, which is parametrized by the coordinates $\{x^i\}$. In turn, the extrinsic properties are given in terms of an outward-pointing spacelike unit normal $n_\mu = (n_r, n_i) = (N, \vec{0})$, as the extrinsic curvature is defined as the Lie derivative of the boundary metric along n , that is,

$$K_{ij} = \mathcal{L}_n h_{ij}. \quad (3)$$

In a more explicit form, the extrinsic curvature is given by

$$K_{ij} := -\frac{1}{2N} h'_{ij}, \quad (4)$$

where prime denotes radial derivative.

For the case of the odd dimensions ($D = 2n + 1$) we are interested in, the boundary term adopts a compact form when expressed with the help of two parametric integrations

$$\begin{aligned} B_{2n} = & 2n\sqrt{-h} \int_0^1 du \int_0^u ds \delta_{[i_1 \dots i_{2n}]_s}^{[j_1 \dots j_{2n}]_s} \\ & \times K_{j_1}^{i_1} \delta_{j_2}^{i_2} \left(\frac{1}{2} \mathcal{R}_{j_3 j_4}^{i_3 i_4} - u^2 K_{j_3}^{i_3} K_{j_4}^{i_4} + \frac{s^2}{\ell^2} \delta_{j_3}^{i_3} \delta_{j_4}^{i_4} \right) \\ & \times \dots \times \left(\frac{1}{2} \mathcal{R}_{j_{2n-1} j_{2n}}^{i_{2n-1} i_{2n}} - u^2 K_{j_{2n-1}}^{i_{2n-1}} K_{j_{2n}}^{i_{2n}} + \frac{s^2}{\ell^2} \delta_{j_{2n-1}}^{i_{2n-1}} \delta_{j_{2n}}^{i_{2n}} \right), \end{aligned} \quad (5)$$

where h is the determinant of the boundary metric, \mathcal{R}_{kl}^{ij} is the boundary Riemann tensor and $\delta_{[i_1 \dots i_{2n}]_s}^{[j_1 \dots j_{2n}]_s}$ is a totally antisymmetric product of $2n$ Kronecker deltas (for conventions, see Appendix A). When expanded, the above expression can be seen as a polynomial in the extrinsic and intrinsic curvatures, where the coefficients are obtained once the integrations in s and u are performed. In that respect, the use of parametric integrations is not a mere formality, but provides an operational tool to derive general expressions for the conserved quantities, regardless of the spacetime dimension. In particular, Kounterterm regularization leads to the only formula for the vacuum energy for AAdS spaces in all odd dimensions, which is covariant with respect to the boundary metric h_{ij} .

The finiteness of the conserved quantities is achieved once the coupling constant of B_{2n} is chosen as

$$c_{2n} := \frac{1}{16\pi G} \frac{(-1)^n \ell^{2n-2}}{2^{2n-2} n(n-1)!}. \quad (6)$$

The above value is singled out by the cancellation of leading-order terms in the asymptotic expansion of the surface term produced by the variation of the action. It is a remarkable fact that this choice of c_{2n} also eliminates the rest of the divergences that appear in the surface term and, subsequently, in the conserved charges. There is no other explanation for this property, other than saying that in

Kounterterm regularization the boundary terms are related to well-known mathematical structures as topological invariants and Chern-Simons densities. Therefore, it is hard to think of a more *geometric* object that can be added to the gravity action for the purpose of regularization.

In addition, there is a partial proof (in even spacetime dimensions) that Kounterterms are able to generate the standard counterterm series upon a suitable expansion of the extrinsic curvature [35]. This conclusion is quite remarkable: holographic renormalization is equivalent to the addition of topological invariants in even dimensions. This also means that the *extrinsic* regularization scheme can be converted into an *intrinsic* one, which is necessary to recover standard holographic quantities in AAdS gravity.

III. KOUNTERTERM CHARGES IN ODD DIMENSIONS

Provided the boundary coincides with the asymptotic region, the Noether procedure leads to conserved quantities associated to a set of asymptotic Killing vectors $\{\xi^i\}$. The conservation of the Noether current $\partial_\mu J^\mu = 0$ implies that J can be written locally as a total derivative which, in turn, implies the existence of a conserved quantity

$$Q[\xi] = \int_{\partial M} d^d x \frac{1}{N} n_\mu J^\mu(\xi). \quad (7)$$

Altogether, it was shown in Ref. [28] that the radial component of the Noether current $J^r = \frac{1}{N} n_\mu J^\mu$ is globally a total derivative on ∂M , that is,

$$Q[\xi] = \int_{\partial M} d^d x \partial_j (\sqrt{-h} \xi^i (q_i^j + q_{(0)i}^j)). \quad (8)$$

Using the Stokes theorem, the above quantity can be written as a surface integral in $(d-1)$ dimensions. In order to do so, we foliate the boundary ∂M in Arnowitt-Deser-Misner form with the coordinates $x^i = (t, y^m)$

$$h_{ij} dx^i dx^j = -\tilde{N}^2(t) dt^2 + \sigma_{mn} (dy^m + \tilde{N}^m dt) (dy^n + \tilde{N}^n dt), \quad (9)$$

$$\sqrt{-h} = \tilde{N} \sqrt{\sigma}. \quad (10)$$

The lapse function \tilde{N} appears in the timelike normal u_i (that generates the foliation) as $u_i = (u_t, u_m) = (-\tilde{N}, \vec{0})$. The tensor σ_{mn} represents the metric of the spatial section at constant time. We denote this surface by the symbol Σ_∞ .

The Noether charge in odd spacetime dimensions is expressed as the sum of two parts

$$Q[\xi] = q[\xi] + q_{(0)}[\xi], \quad (11)$$

where the first integral

$$q[\xi] = \int_{\Sigma_\infty} d^{2n-1}y \sqrt{\sigma} u_j q_i^j \xi^i \quad (12)$$

produces the mass and angular momentum for AAdS spacetimes, with an integrand given by

$$q_i^j = -\frac{1}{2^{n-2}} \delta_{[k_1 \dots i_{2n-1}]^{[j_1 \dots j_{2n-1}]} K_i^k \delta_{j_1}^{i_1} \left[\frac{1}{16\pi G (2n-1)!} \delta_{[j_2 j_3]^{[i_2 i_3]} \times \dots \times \delta_{[j_{2n-2} j_{2n-1}]^{[i_{2n-2} i_{2n-1}]} \right. \\ \left. + n c_{2n} \int_0^1 du \left(R_{j_2 j_3}^{i_2 i_3} + \frac{u^2}{\ell^2} \delta_{[j_2 j_3]^{[i_2 i_3]} \right) \times \dots \times \left(R_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}} + \frac{u^2}{\ell^2} \delta_{[j_{2n-2} j_{2n-1}]^{[i_{2n-2} i_{2n-1}]} \right) \right]. \quad (13)$$

On the other hand, the second part

$$q_{(0)i}[\xi] = \int_{\Sigma_\infty} d^{2n-1}y \sqrt{\sigma} u_j q_{(0)i}^j \xi^i \quad (14)$$

is given in terms of the tensor

$$q_{(0)i}^j = 2n c_{2n} \delta_{[k_1 \dots i_{2n-1}]^{[j_1 \dots j_{2n-1}]} \int_0^1 du u (K_i^k \delta_{j_1}^{i_1} + K_{j_1}^k \delta_i^{i_1}) \left(\frac{1}{2} \mathcal{R}_{j_2 j_3}^{i_2 i_3} - u^2 K_{j_2}^{i_2} K_{j_3}^{i_3} + \frac{u^2}{\ell^2} \delta_{j_2}^{i_2} \delta_{j_3}^{i_3} \right) \\ \times \dots \times \left(\frac{1}{2} \mathcal{R}_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}} - u^2 K_{j_{2n-2}}^{i_{2n-2}} K_{j_{2n-1}}^{i_{2n-1}} + \frac{u^2}{\ell^2} \delta_{j_{2n-2}}^{i_{2n-2}} \delta_{j_{2n-1}}^{i_{2n-1}} \right). \quad (15)$$

Properties of the above formulas are extensively employed in Sec. V, where we compute the vacuum energy for Kerr-AdS metric in odd spacetime dimensions.

In the next section, we review the construction of the Kerr-AdS metric in $D = 2n + 1$ dimensions.

IV. KERR BLACK HOLE METRIC IN ODD SPACETIME DIMENSIONS

The asymptotically flat Kerr spacetime in four dimensions [36] can be obtained as a perturbation to the Minkowski metric that is linear in the parameter M (related to the black hole mass) [37]. The resulting line element adopts the form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{2M}{U} (k_\mu dx^\mu)^2, \quad (16)$$

where k_μ is a null geodesic vector for the seed metric $\eta_{\mu\nu}$ as for the full metric $g_{\mu\nu}$, and the function U is given by

$$U = r + \frac{a^2 z^2}{r^3}, \quad (17)$$

where $r = (0, \infty)$, z is the axis around which the rotation will be defined, and a is a parameter that represents the squashing of the sphere.

In Cartesian coordinates, the explicit form of the deformation $k = k_\mu dx^\mu$ is

$$k = dt + \frac{r(xdx + ydy) + a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r}, \quad (18)$$

and r is defined by the ellipsoidal hypersurface

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1. \quad (19)$$

In a similar way, if one includes a cosmological constant Λ into the gravitational action, the metric takes a linearized form around de Sitter or anti-de Sitter backgrounds $\bar{g}_{\mu\nu}$ (see, e.g., Ref. [38]),

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{2M}{U} k_\mu k_\nu. \quad (20)$$

Once again, k_μ is a null vector for both metrics.

The generalization of the Kerr metric to higher dimensions makes use of the same properties seen above [39–41].

In particular, in odd spacetime dimensions ($D = 2n + 1$) we consider the parametrization for the unit sphere S^{2n-1} , which considers n two-planes, with $2n$ coordinates subjected to a constraint. The coordinates of the i th plane are (u_i, v_i) , whose polar form is given by

$$u_i + iv_i = \hat{\mu}_i e^{i\phi_i}, \quad (21)$$

where we have introduced n azimuthal angles ϕ_i and n direction cosines $\hat{\mu}_i$, which satisfy the constraint

$$\sum_{i=1}^n \hat{\mu}_i^2 = 1. \quad (22)$$

Adding up the time and radial directions in the line element, the global AdS metric in this coordinate set adopts the form

$$d\bar{s}^2 = -\left(1 + \frac{y^2}{l^2}\right) dt^2 + \frac{dy^2}{1 + \frac{y^2}{l^2}} + y^2 \sum_{i=1}^n (d\hat{\mu}_i^2 + \hat{\mu}_i^2 d\phi_i^2). \quad (23)$$

If the dimension of the spacetime is $d + 1$, the number N of independent rotation parameters $\{a_i\}$ corresponds to the number of Casimir invariants of $SO(d)$, that is, $N = \lfloor \frac{d}{2} \rfloor$.

Then, we pass from the sphere parametrization in Eq. (21) to a new set of spheroidal coordinates defined by the transformation

$$\left(1 - \frac{a_i^2}{\ell^2}\right) y^2 \hat{\mu}_i^2 = (r^2 + a_i^2) \mu_i^2, \quad (24)$$

where, once again, the variable μ_i is constrained by the equation

$$\sum_i \mu_i^2 = 1. \quad (25)$$

With this transformation replaced in the metric of global AdS, we can express the line element in terms of the new variables (r, μ_i) , such that the vacuum spacetime (23) is written as

$$\begin{aligned} d\bar{s}^2 = & -W \left(1 + \frac{r^2}{\ell^2}\right) dt^2 + F dr^2 \\ & + \sum_{i=1}^n \frac{r^2 + a_i^2}{\Xi_i} (d\mu_i^2 + \mu_i^2 d\phi_i^2) \\ & - \frac{1}{W(1 + \frac{r^2}{\ell^2})\ell^2} \left(\sum_{i=1}^n \frac{(r^2 + a_i^2)\mu_i d\mu_i}{\Xi_i} \right)^2, \end{aligned} \quad (26)$$

where

$$\Xi_i = 1 - \frac{a_i^2}{\ell^2}. \quad (27)$$

The functions W and F that appear in the metric are given by

$$W \equiv \sum_{i=1}^n \frac{\mu_i^2}{\Xi_i}, \quad F \equiv \frac{r^2}{1 + \frac{r^2}{\ell^2}} \sum_{i=1}^n \frac{\mu_i^2}{r^2 + a_i^2}. \quad (28)$$

This metric obtained as the deformation of global AdS geometry will be used to find the vacuum energy of

Kerr-AdS in the next section. This is justified by the fact that the black hole mass M does not appear in $d\bar{s}^2$ but only in the full metric as $ds^2 = d\bar{s}^2 + \frac{2M}{U} (k_\mu dx^\mu)^2$.

Finally, the explicit form of the perturbation to the deformed vacuum metric $d\bar{s}^2$ is

$$k_\mu dx^\mu = W dt + F dr - \sum_{i=1}^n \frac{a_i \mu_i^2}{\Xi_i} d\phi_i, \quad (29)$$

and where

$$U = \sum_{i=1}^n \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^n (r^2 + a_j^2). \quad (30)$$

The full Kerr-AdS metric is usually expressed in terms of Boyer-Lindquist coordinates which eliminate the components $g_{\mu r}$ with $\mu \neq r$, i.e., no cross terms between dr and the other coordinate differentials. Indeed, it would be convenient, putting the metric in the Gaussian form (2), e.g., to evaluate the mass and angular momenta [42] for rotating black holes from Eqs. (12) and (13). However, in the next section it is argued that, for the purpose of vacuum energy computation, it is enough to consider just the deformation induced by the rotation parameters on the global AdS spacetime, i.e., the line element (26).

In addition, the Kerr-AdS metric can be expressed in Kerr-Schild form, which splits it in two sectors. The first one is the rotating version of global AdS space and the second, a part proportional to the mass parameter. In this way, we can be sure that there will not be missing contributions to the vacuum energy once we switch off the mass. This justifies the fact that, in order to perform the calculations relevant for this paper, we can restrict ourselves to the rotating global AdS metric.

V. VACUUM ENERGY IN KERR-ADS

The deformation of the AdS vacuum defined by Eqs. (26) and (28) preserves the constant-curvature property of global AdS spacetime. This means that the global transformations performed in order to obtain the metric (26) do not modify the local condition

$$R_{\mu\nu}^{\alpha\beta} + \frac{1}{\ell^2} \delta_{[\mu\nu]}^{[\alpha\beta]} = 0. \quad (31)$$

From the argument that follows it is evident that the conserved quantities associated to this part of the metric are identically zero. Indeed, it can be shown that the part $q(\xi)$ of the total charge (11) that produces the mass and angular momentum for AAdS black holes can have its integrand factorized as

$$q_i^j = \frac{nc_{2n}}{2^{n-2}} \delta_{[i_1 i_2 \dots i_{2n}]}^{[j_1 j_2 \dots j_{2n}]} K_i^{i_1} \delta_{j_2}^{i_2} \left(R_{j_3 j_4}^{i_3 i_4} + \frac{1}{\ell^2} \delta_{[j_3 j_4]}^{[i_3 i_4]} \right) \mathcal{P}_{j_5 \dots j_{2n}}^{i_5 \dots i_{2n}}(R, \delta). \quad (32)$$

Here, $\mathcal{P}(R, \delta)$ is a polynomial of $(n-2)$ degrees in the spacetime Riemann tensor R_{kl}^{ij} (its projection at the boundary) and the antisymmetrized Kronecker delta $\delta_{[kl]}^{[ij]}$

$$\mathcal{P}_{j_5 \dots j_{2n}}^{i_5 \dots i_{2n}}(R, \delta) = \sum_{p=0}^{n-2} \frac{D_p}{\ell^{2p}} R_{j_5 j_6}^{i_5 i_6} \dots R_{j_{2(n-p)-1} j_{2(n-p)}}^{i_{2(n-p)-1} i_{2(n-p)}} \delta_{[j_{2(n-p)+1} j_{2(n-p+1)}]}^{[i_{2(n-p)+1} i_{2(n-p+1)}]} \dots \delta_{[j_{2n-1} j_{2n}]}^{[i_{2n-1} i_{2n}]}, \quad (33)$$

with the coefficients of the expansion given by

$$D_p = \sum_{q=0}^p \frac{(-1)^{p-q}}{2q+1} \binom{n-1}{q}. \quad (34)$$

Therefore, any space satisfying the condition (31) globally will possess vanishing charges.

From the explicit form of the full metric, we can notice that M does not appear in $d\bar{s}^2$ in Eq. (20). That means that the parameter M cannot affect the value of the vacuum energy (which is obvious when we think that the vacuum state corresponds to a vanishing mass).

On the other hand, it can be seen that the electric part of the Weyl tensor $\mathcal{E}_j^i \sim n_\mu n^\nu W_{\nu j}^{\mu i}$ of the full Kerr-AdS metric is always proportional to M . In this way, it correctly reproduces the mass and angular momentum from the Ashtekar-Magnon-Das charge definition for AAdS [6,43,44]. Furthermore, the Weyl tensor is—on shell—proportional to the right-hand side of Eq. (31) and, therefore, M should not enter into the expression of $q_{(0)}$.

In summary, we only need the sector in the metric that corresponds to the deformation of global AdS spacetime in order to compute the zero-point energy (14). As the above integral is defined in the limit for $r \rightarrow \infty$, we shall consider the asymptotic expansion of the intrinsic and extrinsic curvatures.

Taking the metric of global AdS in oblate coordinates in Eqs. (26) and (28) and writing down the direction cosines in terms of polar angles (see Appendix C), we see that the squared root of the determinant of the boundary metric behaves as

$$\sqrt{-h} = \tilde{N} \sqrt{\sigma} \sim r^{2n} + \mathcal{O}(r^{2n-2}), \quad (35)$$

where the function \tilde{N} appears in the Arnowitt-Deser-Misner foliation (9).

From explicit computations in the oblate-AdS sector of the Kerr-AdS metric in an arbitrary dimension, one can see that the asymptotic expansion of the extrinsic curvature is

$$K_j^i = -\frac{\delta_j^i}{\ell} + \frac{\ell A_j^i(\theta, \phi)}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right), \quad (36)$$

whereas the intrinsic curvature behaves as

$$\mathcal{R}_{kl}^{ij} \sim \frac{\mathcal{B}_{kl}^{ij}(\theta, \phi)}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right), \quad (37)$$

where A_j^i and \mathcal{B}_{kl}^{ij} are tensor coefficients which do not have radial dependence.

In the expression (15), we have $(n-1)$ terms of the form

$$\begin{aligned} \mathcal{R}_{kl}^{ij} - u^2 (K_k^i k_l^j - K_l^i K_k^j) + \frac{u^2}{\ell^2} \delta_{[kl]}^{[ij]} \\ \sim \frac{1}{r^2} [\mathcal{B}_{kl}^{ij} + u^2 (\delta_k^i A_l^j + \delta_l^j A_k^i - \delta_l^j A_k^i - \delta_k^i A_l^j)] + \mathcal{O}\left(\frac{1}{r^4}\right). \end{aligned} \quad (38)$$

In general, the integration in the continuous parameter u present in the formula for vacuum energy is quite complicated to solve.

However, in any dimension, from explicit computations in the Kerr-AdS metric, one can notice that the leading-order term in the expansion of the intrinsic curvature is the skew-symmetric product of A_j^i with a Kronecker delta, that is,

$$\mathcal{B}_{kl}^{ij} = -(\delta_k^i A_l^j + \delta_l^j A_k^i - \delta_l^j A_k^i - \delta_k^i A_l^j). \quad (39)$$

The above reasoning allows us to factorize the expression (38) in terms of the next-to-leading order in the expansion of the extrinsic curvature as

$$\begin{aligned} \mathcal{R}_{kl}^{ij} - u^2 (K_k^i k_l^j - K_l^i K_k^j) + \frac{u^2}{\ell^2} \delta_{[kl]}^{[ij]} \\ \sim \frac{(u^2 - 1)}{r^2} (\delta_k^i A_l^j + \delta_l^j A_k^i - \delta_l^j A_k^i - \delta_k^i A_l^j) + \mathcal{O}\left(\frac{1}{r^4}\right). \end{aligned} \quad (40)$$

The formula for the zero-point energy (15) also involves the combination

$$\begin{aligned} K_i^k \delta_{j_1}^{i_1} + K_{j_1}^k \delta_i^{i_1} = -\frac{1}{\ell} (\delta_i^k \delta_{j_1}^{i_1} + \delta_{j_1}^k \delta_i^{i_1}) \\ + \frac{\ell}{r^2} (A_i^k \delta_{j_1}^{i_1} + A_{j_1}^k \delta_i^{i_1}) + \mathcal{O}\left(\frac{1}{r^4}\right), \end{aligned} \quad (41)$$

which, when multiplied by the totally antisymmetric Kronecker delta, produces the identical cancellation of

its first term. Just by a simple power-counting argument in the radial coordinate, the formula of the vacuum energy for Kerr-AdS reduces to

$$q_{(0)i}^j = (-2)^{n-1} c_{2n} \delta_{[ki_1 \dots i_{2n-1}]^{[jj_1 \dots j_{2n-1}]} (A_i^k \delta_{j_1}^{i_1} + A_{j_1}^k \delta_i^{i_1}) A_{j_2}^{i_2} \delta_{j_3}^{i_3} \times \dots \times A_{j_{2n-2}}^{i_{2n-2}} \delta_{j_{2n-1}}^{i_{2n-1}} \quad (42)$$

after performing a trivial integration in the parameter u .

Then, the vacuum energy formula (14) in the limit $r \rightarrow \infty$ is written as

$$\begin{aligned} E_{\text{vac}} &= - \int_{\Sigma_\infty} d^{2n-1} y \sqrt{-h} q_{(0)t}^t, \\ &= - \frac{\ell^{2n-1}}{2^{n+3} \pi G n!} \delta_{[p_1 p_2 \dots p_n]^{[n_1 n_2 \dots n_n]} \int_{\Sigma_\infty} d^{2n-1} y \sqrt{-h} \frac{1}{r^{2n}} (A_{n_1}^{p_1} - A_t^t \delta_{n_1}^{p_1}) A_{n_2}^{p_2} \times \dots \times A_{n_n}^{p_n}, \end{aligned} \quad (43)$$

in terms of the of the next-to-leading order quantities in the expansion of both the extrinsic and intrinsic curvatures. Here, the indices $\{n_i, p_i\}$ are restricted to the angular part of the boundary metric, that is, the angles of the sphere S^{2n-1} (see Appendix C). Explicit results up to nine dimensions are given below [45].

A. Five dimensions

As a warm-up we evaluate the five-dimensional version of the rotating AdS vacuum spacetime (26)–(28), and we obtain

$$E_{\text{vac}}^{(5)} = \frac{3\pi\ell^2}{32G} \left(1 + \frac{(\Xi_a - \Xi_b)^2}{9\Xi_a\Xi_b} \right), \quad (44)$$

which is already a standard result in the literature [7,9,10].

We stress the fact that E_{vac} reduces to the one of a static black hole with $R \times S^3$ topology at the boundary, either when the rotation parameters vanish or when they equal ($a = b$). As we shall show below, this feature is also present in higher odd-dimensional Kerr-AdS black holes.

B. Seven dimensions

The sector with $M = 0$ of the Kerr-AdS metric in seven dimensions considers the deformation of global AdS spacetime by the action of three rotation parameters. The formula for the vacuum energy, Eqs. (14) and (15), produces

$$\begin{aligned} E_{\text{vac}}^{(7)} &= - \frac{5\pi^2\ell^4}{128G} \left(1 + \frac{1}{50\Xi_a\Xi_b\Xi_c} ((\Xi_a - \Xi_b)(\Xi_a - \Xi_c)(3\Xi_b + 3\Xi_c - \Xi_a) \right. \\ &\quad \left. + (\Xi_b - \Xi_c)(\Xi_b - \Xi_a)(3\Xi_c + 3\Xi_a - \Xi_b) + (\Xi_c - \Xi_a)(\Xi_c - \Xi_b)(3\Xi_a + 3\Xi_b - \Xi_c)) \right). \end{aligned} \quad (45)$$

The above expression for the zero-point energy can be rewritten in a more compact way as

$$E_{\text{vac}}^{(7)} = - \frac{5\pi^2\ell^4}{128G} \left(1 + \frac{1}{100 \prod_i \Xi_i} \sum_i \sum_j \sum_{k \neq j} (\Xi_i - \Xi_j)(\Xi_i - \Xi_k)(3\Xi_i - 4\Xi_j) \right), \quad (46)$$

where $\Xi = \sum_i \Xi_i$.

In the absence of previous results in the literature to compare with, we take the single-parameter limit in the above expression (Myers-Perry)

$$E_{\text{vac}}^{(7)} = - \frac{\pi^2}{1280\ell^2 G (1 - \frac{a^2}{\ell^2})} (50\ell^6 - 50\ell^4 a^2 + 5\ell^2 a^4 + a^6), \quad (47)$$

from where we see that it coincides with the value computed using a quasilocal stress tensor—properly regularized using counterterm method—by Das and Mann [6], and Awad and Johnson [7].

It is clear that the vacuum energy for Schwarzschild-AdS [46]

$$E_{\text{vac}} = - \frac{5\pi^2\ell^4}{128G} \quad (48)$$

is degenerated because it is the same for seven-dimensional Kerr-AdS with all rotation parameters equal, i.e., $\Xi_a = \Xi_b = \Xi_c$.

C. Nine dimensions

Evaluating the expression of the vacuum energy for a nine-dimensional black hole with the maximal number of rotation parameters is quite demanding from a

computational point of view. Because the formula involves a totally antisymmetric Kronecker delta, the number of calculations increases drastically with the dimension. However, from the discussion above, we notice that Eq. (43) must be integrated only in angular variables, reducing the problem in two dimensions with respect to the one of the spacetime. The result for the zero-point energy in Kerr-AdS in nine dimensions is then given by

$$\begin{aligned}
E_{\text{vac}}^{(9)} = & \frac{\pi^3 \ell^6}{322560G\Xi_a\Xi_b\Xi_c\Xi_d} (15\Xi_a^4 + 15\Xi_b^4 + 15\Xi_c^4 + 15\Xi_d^4 - 55\Xi_a^3\Xi_b - 55\Xi_a^3\Xi_c - 55\Xi_a^3\Xi_d \\
& - 55\Xi_b^3\Xi_a - 55\Xi_b^3\Xi_c - 55\Xi_b^3\Xi_d - 55\Xi_c^3\Xi_a - 55\Xi_c^3\Xi_b - 55\Xi_c^3\Xi_d - 55\Xi_d^3\Xi_a - 55\Xi_d^3\Xi_b \\
& + 55\Xi_d^3\Xi_c + 211\Xi_a^2\Xi_b\Xi_c + 211\Xi_a^2\Xi_b\Xi_d + 211\Xi_a^2\Xi_c\Xi_d + 211\Xi_b^2\Xi_a\Xi_c + 211\Xi_b^2\Xi_a\Xi_d \\
& + 211\Xi_b^2\Xi_c\Xi_d + 211\Xi_c^2\Xi_a\Xi_b + 211\Xi_c^2\Xi_a\Xi_c + 211\Xi_c^2\Xi_b\Xi_d + 211\Xi_d^2\Xi_a\Xi_b + 211\Xi_d^2\Xi_a\Xi_c \\
& + 211\Xi_d^2\Xi_b\Xi_c + 29\Xi_a^2\Xi_b^2 + 29\Xi_a^2\Xi_c^2 + 29\Xi_a^2\Xi_d^2 + 29\Xi_b^2\Xi_c^2 + 29\Xi_b^2\Xi_d^2 + 29\Xi_c^2\Xi_d^2 \\
& + 1569\Xi_a\Xi_b\Xi_c\Xi_d).
\end{aligned} \tag{49}$$

The reader may check, in a straightforward way, that for the case $a = b = c = d$, the above expression has the same property as in five and seven dimensions, as it reduces to the vacuum energy of static spherical black hole

$$E_{\text{vac}}^{(9)} = \frac{35\pi^3 \ell^6}{3072G}. \tag{50}$$

It is evident that there is an equivalent form to Eq. (49) that makes this feature more manifest. Indeed, using all the symmetries under the exchange of rotation parameters, the vacuum energy can be written as

$$\begin{aligned}
E_{\text{vac}}^{(9)} = & \frac{35\pi^3 \ell^6}{3072G} \left[1 + \frac{1}{176400 \prod_l \Xi_l} \sum_i \sum_j \sum_{k \neq j} (\Xi_i - \Xi_j)(\Xi_i - \Xi_k) \right. \\
& \left. \times (120\Xi_i^2 - 366(\Xi_j^2 + \Xi_k^2) + (\Xi_j + \Xi_k)(-2646\Xi_i + 2106(\Xi - \Xi_j - \Xi_k))) \right].
\end{aligned} \tag{51}$$

When we take the limit of a single-parameter rotating black hole ($b = c = d = 0$), E_{vac} adopts the form

$$E_{\text{vac}}^{(9)} = \frac{\pi^3}{21504\ell^2 G(1 - \frac{a^2}{\ell^2})} (245\ell^8 - 245\ell^6 a^2 + 21\ell^4 a^4 + 7\ell^2 a^6 + a^8). \tag{52}$$

At once we notice a different value with respect to the one found for the same solution in Ref. [6]. The origin of this mismatch may be, in fact, that in order to obtain a quasilocal stress tensor, the authors of Ref. [6] performed an integration by parts in the highest-derivative terms of the counterterm series in nine dimensions. This may lead to finite contributions to the vacuum energy that different from ours.

VI. CONCLUSIONS AND PROSPECTS

We have obtained explicit expressions for the vacuum energy for Kerr-AdS black holes, geometry that admits a maximal number of $[(D-1)/2]$ commuting axial

symmetries. The expression up to nine dimensions exhibits an interesting property: the zero-point energy reduces to the one of a static AAdS black hole when all rotation parameters are taken as equal to each other. It would be interesting to understand the implications in the boundary CFT of this vacuum energy degeneracy. It is likely that this fact can be related to a symmetry enhancement that the vacuum solution metric should exhibit in that case (the deformation induced by the rotation parameters is the same in all azimuthal directions).

We have not been able to identify a pattern in E_{vac} for Kerr-AdS, which would allow us to pass from the particular results of the last section to a general formula, valid in any odd dimension. We believe that the explicit expressions we

found can be cast in a more compact form using parametric integrations.

On the other hand, the agreement between the results from Kounterterm charges and the ones obtained by holographic techniques in AdS gravity suggests that Eq. (15) should be a part of the stress tensor $T^{ij}[h] = (2/\sqrt{-h})\delta I_{\text{ren}}/\delta h_{ij}$, defined upon the addition of local counterterms. A direct comparison between both formulas would require, in general, converting extrinsic quantities into intrinsic ones. This can be done considering the expansion of the extrinsic curvature for AAdS spacetimes and noticing that all terms in Eq. (15) up to the relevant order can be expressed as contractions between the Riemann and the Schouten tensors of the boundary metric [47].

A nonzero value for E_{vac} modifies the derivation of the positivity of energy for asymptotically AdS spacetimes, as it has been emphasized in Ref. [27]. The existence of globally defined Killing spinors in a supersymmetry extension of AdS gravity results in a vacuum energy formula given in terms of the coefficients of the Fefferman-Graham expansion of the metric [23]. We hope that the ongoing efforts to compare the Cheng-Skenderis formula to ours are able to provide an answer to this issue.

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APPENDIX A: KRONECKER DELTA OF RANK p

Many of the formulas in this paper are written in a more compact form thanks to the use of the totally antisymmetric Kronecker delta. Such an object of rank p is defined as the determinant

$$\delta_{[\mu_1 \dots \mu_p]}^{[\nu_1 \dots \nu_p]} := \begin{vmatrix} \delta_{\mu_1}^{\nu_1} & \delta_{\mu_1}^{\nu_2} & \dots & \delta_{\mu_1}^{\nu_p} \\ \delta_{\mu_2}^{\nu_1} & \delta_{\mu_2}^{\nu_2} & & \delta_{\mu_2}^{\nu_p} \\ \vdots & & \ddots & \\ \delta_{\mu_p}^{\nu_1} & \delta_{\mu_p}^{\nu_2} & \dots & \delta_{\mu_p}^{\nu_p} \end{vmatrix}. \quad (\text{A1})$$

A contraction of $k \leq p$ indices in the Kronecker delta of rank p produces a delta of rank $p - k$,

$$\delta_{[\mu_1 \dots \mu_k \dots \mu_p]}^{[\nu_1 \dots \nu_k \dots \nu_p]} \delta_{\nu_1}^{\mu_1} \dots \delta_{\nu_k}^{\mu_k} = \frac{(N - p + k)!}{(N - p)!} \delta_{[\mu_{k+1} \dots \mu_p]}^{[\nu_{k+1} \dots \nu_p]}, \quad (\text{A2})$$

where N is the range of indices.

APPENDIX B: GAUSS-NORMAL COORDINATE FRAME

For most of the discussions in the present paper, the relevant components of the Christoffel connection $\Gamma_{\mu\nu}^\alpha$ are expressed in terms of the extrinsic curvature and radial derivatives of the lapse function N as

$$\Gamma_{ij}^r = \frac{1}{N} K_{ij}, \quad \Gamma_{rj}^i = -NK_j^i, \quad \Gamma_{rr}^r = \frac{N'}{N}. \quad (\text{B1})$$

In our conventions, the Riemann tensor is defined as

$$R_{\mu\beta\nu}^\alpha = \partial_\beta \Gamma_{\nu\mu}^\alpha - \partial_\nu \Gamma_{\beta\mu}^\alpha + \Gamma_{\beta\gamma}^\alpha \Gamma_{\nu\mu}^\gamma - \Gamma_{\nu\gamma}^\alpha \Gamma_{\beta\mu}^\gamma, \quad (\text{B2})$$

which leads to the well-known Gauss-Codazzi relations

$$R_{kl}^{jr} = \frac{1}{N} (\nabla_l K_k^j - \nabla_k K_l^j), \quad (\text{B3})$$

$$R_{kr}^{ir} = \frac{1}{N} (K_k^i)' - K_l^i K_k^l, \quad (\text{B4})$$

$$R_{kl}^{ij} = \mathcal{R}_{kl}^{ij}(h) - K_k^i K_l^j + K_l^i K_k^j, \quad (\text{B5})$$

where $\nabla_l = \nabla_l(\Gamma)$ denotes the covariant derivative defined in terms of the Christoffel symbol of the boundary $\Gamma_{jk}^i = \Gamma_{jk}^i(h)$.

APPENDIX C: PARAMETRIZATION OF THE SPHERE S^{2n-1}

In $D = 2n + 1$ dimensions we have $n - 1$ polar angles θ_i , where $0 \leq \theta_i \leq \frac{\pi}{2}$. Altogether, we have n azimuthal angles, $0 \leq \phi_i \leq 2\pi$. The polar angles are related to the direction cosines as

$$\mu_i = \prod_{j=1}^{i-1} \cos \theta_j \sin \theta_i, \quad (\text{C1})$$

or, more explicitly

$$\begin{aligned} \mu_1 &= \sin \theta_1 \\ \mu_2 &= \cos \theta_1 \sin \theta_2 \\ \mu_3 &= \cos \theta_1 \cos \theta_2 \sin \theta_3 \\ &\vdots \\ \mu_{n-1} &= \cos \theta_1 \cos \theta_2 \dots \cos \theta_{n-2} \sin \theta_{n-1} \\ \mu_n &= \cos \theta_1 \cos \theta_2 \dots \cos \theta_{n-2} \cos \theta_{n-1}. \end{aligned} \quad (\text{C2})$$

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