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Wilson line approach to gravity in the high energy limit

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We examine the high energy (Regge) limit of gravitational scattering using a Wilson line approach previously used in the context of non-Abelian gauge theories. Our aim is to clarify the nature of the Reggeization of the graviton and the interplay between this Reggeization and the so-called eikonal phase which determines the spectrum of gravitational bound states. Furthermore, we discuss finite corrections to this picture. Our results are of relevance to various supergravity theories, and also help to clarify the relationship between gauge and gravity theories.

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I. INTRODUCTION

The structure of scattering amplitudes in both gauge and gravity theories continues to attract significant attention, due to a wide variety of phenomenological and formal applications. Although superficially very different from each other, there is mounting evidence that gauge and gravity theories may be related to each other in intriguing ways. Such developments motivate the need to study aspects of amplitudes in a variety of theories using a common language, and to compare and contrast phenomena in gauge theories with their gravitational counterparts.

This paper studies one such phenomenon, that of four-point scattering of massive (and massless) particles in the so-called *Regge limit*, in which the center-of-mass energy far exceeds the momentum transfer. The properties of amplitudes in this limit have been studied for a long time, for example in the context of strong interactions predating the advent of QCD (see e.g. Refs. [1–3] and references therein). Their asymptotic high energy behavior is dictated by singularities in the complex angular momentum plane, which may take the form of poles or cuts. Simple poles give rise to a powerlike growth of scattering amplitudes with the center-of-mass energy:

$$\mathcal{A} \sim \left(\frac{s}{-t}\right)^{\alpha(t)}, \tag{1.1}$$

where we have defined the Mandelstam invariants[1]

$$s = (p_1 + p_2)^2; \qquad t = (p_1 - p_3)^2; \qquad u = (p_1 - p_4)^2. \tag{1.2}$$

These satisfy the momentum conservation constraint

$$s + t + u = \sum_{i=1}^{4} m_i^2 \tag{1.3}$$

in terms of the particle momenta $\{p_i\}$ and masses $\{m_i\}$, and we label particles as shown in Fig. 1. The function $\alpha(t)$ in Eq. (1.1) is known as the *Regge trajectory*, whose physical origin is the exchange of a family of particles in the t-channel. Reggeization has also been studied within the context of perturbative quantum field theory, for both scalar and (non)-Abelian gauge theories [4–28]. There, Regge behavior of amplitudes follows after first demonstrating that elementary constituents themselves *Reggeize*. For, example, one may show in QCD that the Feynman gauge propagator in the Regge limit is dressed according to

$$-\frac{\eta_{\mu\nu}}{k^2} \to -\frac{\eta_{\mu\nu}}{k^2}\left(\frac{s}{-t}\right)^{\tilde{\alpha}(t)}, \tag{1.4}$$

where $\tilde{\alpha}(t)$ is related in a straightforward way to the Regge trajectory $\alpha(t)$. The gluon and quark trajectories in QCD are known to two-loop order [22–27]. At one-loop order, they are given by

$$\tilde{\alpha}^{(1)}(t) = \frac{\alpha_s(\mu^2)}{2\pi}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\frac{C_R}{\epsilon}, \tag{1.5}$$

in $d = 4 - 2\epsilon$ dimensions, where C_R is the quadratic Casimir operator in the appropriate representation, and μ the renormalization scale. Note that this is purely infrared singular (up to scale-related logarithms). Apart from the particle-dependent Casimir, there is a universal coefficient, which may be written in terms of the one-loop cusp anomalous dimension [29,30]. The latter quantity controls the ultraviolet renormalization of Wilson line operators, and this connection will become clear in what follows. At

[1]In this paper, we use the metric convention $(+,-,-,-)$.

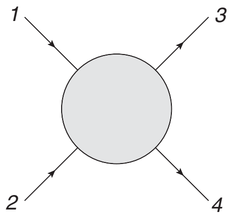

FIG. 1. Particle labels used throughout for $2 \to 2$ scattering.

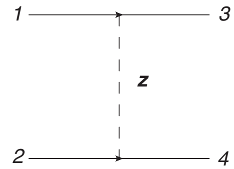

FIG. 2. The Regge limit as two Wilson lines separated by a transverse distance $\vec{z}$.

two-loop order, the Regge trajectories of the quark and gluon are no longer purely infrared singular, also involving finite terms.

A convenient formalism for studying Reggeization was introduced in Refs. [29,30], and is based upon the fact that in the Regge limit of $2 \to 2$ scattering, the incoming particles glance off each other, such that the outgoing particles are highly forward, and essentially do not recoil. They can therefore only change by a phase, and for this phase to have the right gauge-transformation properties to form part of a scattering amplitude, it must correspond to a Wilson line operator. Thus, scattering in the forward limit can be described as two Wilson lines separated by a transverse distance (or *impact parameter*) $\vec{z}$, a situation depicted in Fig. 2. The ultraviolet behavior of the Wilson line correlator reproduces the infrared singularities of the scattering amplitude, and thus the infrared singular parts of the gluon Regge trajectory. Ultraviolet renormalization of Wilson lines is governed by the cusp anomalous dimension, hence the connection between this quantity and the Regge trajectory noted above.

The relationship between infrared singularities and the Regge limit was studied further recently in Refs. [31,32], which used a conjectured formula for the all-order infrared (IR) singularity structure of QCD (the *dipole formula* of Refs. [33–35], itself motivated by explicit two-loop calculations [36–38]) to show that Reggeization occurs generically up to next-to-leading logarithmic order in $s/(-t)$, for any allowable t-channel exchange. The corresponding Regge trajectory is dictated by the cusp anomalous dimension, as already noted in Refs. [29,30], and involves the quadratic Casimir operator in the representation of the exchanged particle. Beyond this logarithmic order, the authors of Refs. [31,32] noted a breakdown of simple Regge pole behavior, associated with a color operator which has also been linked to the breakdown of collinear factorization in certain circumstances [39–41]. A breakdown of simple Regge pole behavior at this order is consistent with previous two-loop calculations of quark and gluon scattering [42], and is likely to signal the appearance of Regge cuts associated with multi-Reggeon exchange [18,43]. The analysis of Refs. [31,32] also used the Regge limit to constrain possible corrections to the QCD dipole formula (known to break already for massive particles [44–46]). Such corrections may potentially occur at three-loop order, and have also been investigated in Refs. [35,47–50] (see also Refs. [51–58] for recent work on understanding IR singularities at higher orders).

Although much is known about the Regge limit of gauge theories in perturbative gauge theory, the situation in gravity is more confused[2]. The one-loop Regge trajectory of the graviton was first derived in Refs. [59–61], within the context of both Einstein-Hilbert gravity and its supersymmetric extensions. One of the authors of the present paper (H. J. S.) argued in Ref. [62] (based on earlier studies employing analyticity arguments [63–67]) that Reggeization of the graviton in $\mathcal{N} = 8$ supergravity follows from that of the gluon in $\mathcal{N} = 4$ super-Yang-Mills theory [68,69] (see also Refs. [42,70]) as a consequence of the well-known Kawai-Lewellen-Tye relations [71] relating scattering amplitudes in the two theories. This was considered from a Feynman diagrammatic point of view in Refs. [72–74], which also discussed the potential structure of Regge cut contributions in supergravity.

There has recently been a rekindled interest in the Regge limit of gravity. A chief motivation is the study of gravitational scattering in the transplanckian regime [75] (see also Refs. [76–81]). This regime allows one to explore conceptual questions of quantum gravity, such as the existence, or otherwise, of a gravitational S-matrix [82,83]. An interesting feature is that high energy scattering in gravity is dominated by long-distance rather than short-distance behavior, a fact which is ultimately traceable to the dimensionality of the gravitational coupling constant, and the masslessness of the graviton. The lack, or otherwise, of ultraviolet renormalizability ceases to be problematic in this limit, and one may show that the long-distance behavior is insensitive to the amount of supersymmetry. However, some confusion remained in Ref. [75] about the role of graviton Reggeization, and the interplay between this and the so-called *eikonal phase* which appears at high energy [84], and which is associated with the formation of gravitational bound states. This confusion is in part related to the fact that the graviton Regge trajectory is linear in the squared momentum transfer t [59–62,72], and thus becomes kinematically subleading in the strict Regge limit $s/(-t) \to \infty$. The issue of graviton Reggeization is also

[2]Throughout this paper, we use the term *gauge theory* to refer only to Abelian and non-Abelian gauge symmetries acting on internal degrees of freedom, rather than on spacetime degrees of freedom as in gravity.

complicated by double logarithmic contributions in $s/(-t)$, which have been discussed at length in Ref. [85].

The Regge limit has also been the focus of studies which aim to relate the properties of gauge and gravity theories. Examples include Refs. [86,87], which use the high energy limit to probe the all-order validity of the proposed double copy structure between gauge and gravity theories [88–90]. As the work of Refs. [29,30] (and, subsequently, Refs. [31,32]) makes clear, the Regge limit can be at least partially understood in terms of soft gluon physics and Wilson lines. The soft limit of gravity was first considered in Ref. [91], and has recently been more extensively studied in Refs. [92–98]. The latter papers seek to cast the gravitational behavior in terms of contemporary gauge theory language, and thus to expose common physics in the soft limits of both theories. This includes the introduction of Wilson line operators for soft graviton emission [92,93], whose vacuum expectation values give rise to a gravitational *soft function*, the UV singularities of which correspond to the IR singularities of a scattering amplitude. As in QCD, this function exponentiates. Unlike QCD, however, the gravitational soft function has the special property of being *one-loop exact*, meaning that there are no higher loop corrections to the exponent [92–94].

The aim of this paper is to examine the Regge limit of (super-)gravity using Wilson lines, using a similar approach to the QED/QCD case of Refs. [29,30]. There are a number of motivations for doing so. First, the analysis presents an interesting application of the gravitational Wilson line operators of Refs. [92,93]. Second, the calculation provides a common language for Reggeization in both gauge and gravity theories, which is particularly elegant in revealing common features of the two cases (such as the appearance of relevant quadratic Casimir operators in Regge trajectories). Third, the Wilson line calculation ties together a number of previous results in gravity in a particularly transparent fashion, and helps to clarify some of the confusions inherent in the existing literature (such as the interplay between the eikonal phase and Reggeization of the graviton). We will also discuss the impact of infrared-finite corrections to the scattering amplitude, using one- and two-loop results in a variety of supergravity theories [99–106].

The structure of the paper is as follows. In Sec. II, we review the approach of Refs. [29,30] to the Regge limit in terms of Wilson lines, with some slight differences to which we draw attention. In Sec. III we carry out a similar calculation in quantum gravity, using the Wilson line operators of Refs. [92,93], and compare the results with the QCD case. In Sec. IV, we examine the impact of finite terms in various supergravity theories on the interpretation of the scattering amplitude in the Regge limit. In Sec. V, we apply the Wilson line approach to multigraviton scattering, for any number of gravitons. Finally, in Sec. VI we discuss our results before concluding. Certain technical details are collected in the appendices.

II. WILSON LINES AND REGGEIZATION IN QCD

In this section, we review the approach of Refs. [29,30] for describing the forward limit of $2 \to 2$ scattering in QCD in terms of a pair of Wilson lines separated by a transverse distance. Much of this calculation is very similar to the gravity case considered in the next section, and thus examining the QCD case first allows a detailed comparison between gauge and gravity theories. Unless otherwise stated, we will consider the scattering of massive particles, where for convenience we assume a common mass m. The Regge limit we consider is then given by

$$s \gg -t \gg m^2. \tag{2.1}$$

Note that one has to make a choice here as to how to order the scales t and m^2, as is inevitable when one introduces a mass scale. It is useful to have such a mass scale, however, especially when we consider the gravity case.

The Regge limit corresponds to a high center-of-mass energy, with comparatively negligible momentum transfer. This corresponds to highly-forward scattering, such that the incoming particles barely glance off each other. Using the momentum labels of Fig. 1, the Mandelstam invariants are given by Eq. (1.2), and momentum conservation can be expressed by Eq. (1.3). It is clear that in the forward limit the incoming particles do not recoil in the transverse direction, and thus can only change by a phase due to their interaction. As remarked in the introduction, this suggests that one may model the two incoming particles (together with their outgoing counterparts) by Wilson lines, which are separated by a transverse distance $\vec{z}$. The latter is a two-vector which is orthogonal to the beam direction, corresponding to the *impact parameter* or distance of closest approach. This setup is shown in Fig. 2. In principle we need only specify a single direction for each Wilson line. However, it is useful to keep the notion of which part of each Wilson line is incoming and which outgoing, and thus we keep labels for each particle as shown in the figure.

Let us now consider the quantity[3]

$$\tilde{M} = \int d^2\vec{z} e^{-i\vec{z}\cdot\vec{q}} \langle 0|\Phi(p_1, 0)\Phi(p_2, z)|0\rangle, \tag{2.2}$$

where we define the Wilson line operator

$$\Phi(p, z) = \mathcal{P} \exp\left[ig_s p^\mu \int_{-\infty}^{\infty} ds A_\mu(sp + z)\right]. \tag{2.3}$$

The argument of Φ describes the contour of the Wilson line, in terms of a momentum p and a constant offset z. The exponent contains the non-Abelian gauge field A_μ, where the $\mathcal{P}$ symbol denotes path ordering of color generators

[3]Note that we use a tilde to denote the momentum-space Fourier transform of a position-space amplitude.

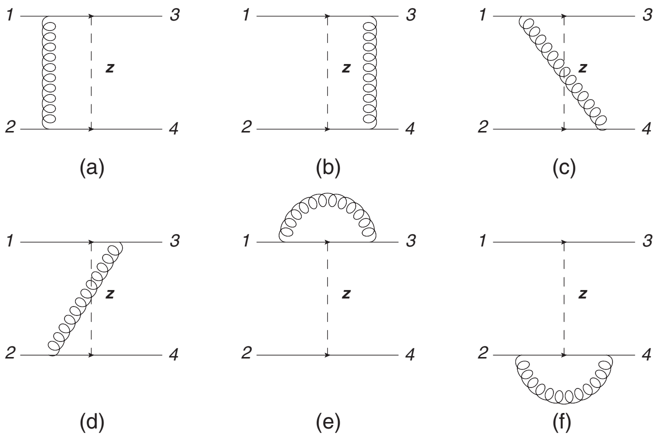

FIG. 3. One-loop diagrams entering the calculation of the Wilson line vacuum expectation value of Eq. (2.2).

along the Wilson line contour. We then see that Eq. (2.2) involves a vacuum expectation value of two Wilson lines along directions p_1 and p_2, separated by the 4-vector z. As discussed above, this separation will only have nonzero transverse components, such that $z^2 = -\vec{z}^2$. Were this separation to be absent, the vacuum expectation value in Eq. (2.2) would correspond exactly to the Regge limit of the soft function. As is well known, this soft function is exactly zero in dimensional regularization, as it involves cancellations between UV and IR poles. The former are associated with shrinking gluon emissions toward the cusp formed by the Wilson lines at the origin. The presence of the separation vector $\vec{z}$ thus means that the UV poles are absent (i.e. there is then no cusp). In other words, $\vec{z}$ acts as a UV regulator (see Ref. [29] for a prolonged discussion of this point).

Equation (2.2) constitutes a two-dimensional Fourier transform of the Wilson line expectation value, from position to momentum space. The two-momentum $\vec{q}$ is conjugate to the impact parameter $\vec{z}$, and in fact satisfies $\vec{q}^2 = -t$ in the center-of-mass frame. This is because in the extreme forward limit, the 4-momentum transfer

$$q = p_1 - p_3 \tag{2.4}$$

(which will be conjugate to the 4-separation z) has zero light-cone components

$$q^{\pm} = \frac{1}{\sqrt{2}}(q^0 \pm q^3), \tag{2.5}$$

so that $q = (0, \vec{q}, 0)$. Our task is now to calculate the quantity of Eq. (2.2), and show that it indeed contains known properties of the eikonal scattering amplitude.

The full set of one-loop diagrams to be calculated is shown in Fig. 3. In the following, we will use the Catani-Seymour notation $\mathbf{T}_i$ to denote a color generator on leg i [107,108]. Using the position space gluon propagator (see e.g. Ref. [109])

$$D_{\mu\nu}(x-y) = -\eta_{\mu\nu}\frac{\Gamma(d/2-1)}{4\pi^{d/2}}[-(x-y)^2]^{1-d/2} \tag{2.6}$$

in $d = 4 - 2\epsilon$ dimensions, diagram (a) gives

$$\begin{aligned} M_a^{(1)} &= \frac{g_s^2\Gamma(1-\epsilon)\mu^{2\epsilon}}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_2(p_1\cdot p_2)\int_{-\infty}^{0}ds\int_{-\infty}^{0}dt[-(sp_1-tp_2)^2+\vec{z}^2]^{\epsilon-1} \\ &= \frac{g_s^2\Gamma(1-\epsilon)\mu^{2\epsilon}}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_2\cosh\gamma_{12}\int_0^{\infty}ds\int_0^{\infty}dt[-s^2-t^2+2st\cosh\gamma_{12}+\vec{z}^2]^{\epsilon-1} \end{aligned} \tag{2.7}$$

where $p_i^2 = m^2$, the cusp angle γ_{ij} is defined via

$$\cosh\gamma_{ij} = \frac{p_i\cdot p_j}{m^2} \tag{2.8}$$

and in the second line of Eq. (2.7) we redefined $s \to -s/m$, $t \to -t/m$. We also introduced the dimensional regularization scale μ. Next, one may set $s \to \sqrt{\vec{z}^2}s$, $t \to \sqrt{\vec{z}^2}t$, followed by $t \to st$, so that Eq. (2.7) becomes

$$\begin{aligned} M_a^{(1)} &= \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_2(\mu^2\vec{z}^2)^{\epsilon}\cosh\gamma_{12}\int_0^{\infty}ds\int_0^{\infty}dts[s^2(-1-t^2+2t\cosh\gamma_{12})+1]^{\epsilon-1} \\ &= \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_2(\mu^2\vec{z}^2)^{\epsilon}\cosh\gamma_{12}\int_0^{\infty}dt\left[\frac{[s^2(-1-t^2+2t\cosh\gamma_{12})+1]^{\epsilon}}{2\epsilon(-1-t^2+2t\cosh\gamma_{12})}\right]_0^{\infty}. \end{aligned} \tag{2.9}$$

We see that this result is well defined for $\epsilon < 0$. This is to be expected, given that the transverse separation $\vec{z}$ acts as a UV regulator, and ϵ acts as an IR regulator, leaving

$$M_a^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_2(\mu^2\vec{z}^2)^\epsilon\frac{1}{2\epsilon}\cosh\gamma_{12} \times\int_0^\infty\frac{dt}{1+t^2-2t\cosh\gamma_{12}}. \tag{2.10}$$

Completing the square in the denominator and substituting $t = u\sinh\gamma_{12} + \cosh\gamma_{12}$, one obtains

$$M_a^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_2(\mu^2\vec{z}^2)^\epsilon\frac{1}{2\epsilon}\coth\gamma_{12} \times\int_{-\coth\gamma_{12}}^\infty\frac{du}{u^2-1}. \tag{2.11}$$

Carefully implementing the $i\epsilon$ prescription in the propagator, one evaluates the integral to obtain

$$\int_{-\coth\gamma_{12}}^\infty\frac{du}{u^2-1} = i\pi-\gamma_{12}, \tag{2.12}$$

yielding

$$M_a^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_2(\mu^2\vec{z}^2)^\epsilon\frac{1}{2\epsilon}(i\pi-\gamma_{12})\coth\gamma_{12}. \tag{2.13}$$

The calculation of diagram (b) in Fig. 3 is very similar, and yields

$$M_b^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_3\cdot\mathbf{T}_4(\mu^2\vec{z}^2)^\epsilon\frac{1}{2\epsilon}(i\pi-\gamma_{34})\coth\gamma_{34}. \tag{2.14}$$

Diagrams (c) and (d) are different, because they involve the exchange of a gluon between an incoming and outgoing leg, rather than between a pair of both ingoing (or both outgoing) legs. It is relatively straightforward to trace the effect of this in the above calculation; the effect is to switch the sign of the lower limit of the u integral in Eq. (2.12), which then evaluates to γ_{ij}. Thus

$$M_c^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_4(\mu^2\vec{z}^2)^\epsilon\frac{1}{2\epsilon}\gamma_{14}\coth\gamma_{14}, \tag{2.15}$$

$$M_d^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_2\cdot\mathbf{T}_3(\mu^2\vec{z}^2)^\epsilon\frac{1}{2\epsilon}\gamma_{23}\coth\gamma_{23}. \tag{2.16}$$

Diagram (e) yields

$$M_e^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)\mu^{2\epsilon}}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_3(p_1\cdot p_3) \times\int_{-\infty}^0 ds\int_0^\infty dt[-(sp_1-tp_3)^2]^{\epsilon-1}, \tag{2.17}$$

which can be obtained from Eq. (2.7) by relabeling of external momenta and setting the transverse separation $\vec{z}$ to zero. Setting $s\to -s/m$ and $t\to t/m$, this becomes

$$\begin{aligned} M_e^{(1)} &= \frac{g_s^2\Gamma(1-\epsilon)\mu^{2\epsilon}}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_3\cosh\gamma_{13} \times\int_0^\infty ds\int_0^\infty dt[-s^2-t^2-2st\cosh\gamma_{13}]^{\epsilon-1} \\ &= \frac{g_s^2\Gamma(1-\epsilon)\mu^{2\epsilon}}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_3\cosh\gamma_{13} \times\int_0^\infty ds\, s^{2\epsilon-1}\int_0^\infty dt[-1-t^2-2t\cosh\gamma_{13}]^{\epsilon-1}, \end{aligned} \tag{2.18}$$

where in the second line we have rescaled $t\to ts$. One sees that the s integral contains both a UV and an IR pole. This is to be expected, given that there is no transverse separation between particles 1 and 3, which acted as a UV regulator in the previous diagrams. One must introduce a counterterm for the UV pole, which amounts to keeping only the IR pole in Eq. (2.18). Alternatively, one may simply introduce a UV cutoff, and here we will use the same cutoff that we have already used, returning to this point later. Noting that, after the various rescalings we have performed, s has dimensions of length, we may define the s-integral above via

$$\int_0^\infty ds\, s^{2\epsilon-1} \to \int_{\sqrt{\vec{z}^2}}^\infty ds\, s^{2\epsilon-1} = -\frac{(\vec{z}^2)^\epsilon}{2\epsilon}. \tag{2.19}$$

Then Eq. (2.18) becomes

$$M_e^{(1)} = -\frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_3(\mu^2\vec{z}^2)^\epsilon\frac{1}{2\epsilon}\cosh\gamma_{13} \times\int_0^\infty dt[-1-t^2-2t\cosh\gamma_{13}]^{\epsilon-1}. \tag{2.20}$$

The remaining integral over t is finite as $\epsilon\to 0$, in which case it is evaluated similarly to Eq. (2.10) above to give[4]

$$M_e^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_1\cdot\mathbf{T}_3(\mu^2\vec{z}^2)^\epsilon\frac{1}{2\epsilon}\gamma_{13}\coth\gamma_{13} + \mathcal{O}(\epsilon^0). \tag{2.21}$$

Likewise, diagram (f) gives

$$M_f^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\mathbf{T}_2\cdot\mathbf{T}_4(\mu^2\vec{z}^2)^\epsilon\frac{1}{2\epsilon}\gamma_{24}\coth\gamma_{24} + \mathcal{O}(\epsilon^0). \tag{2.22}$$

Let us now combine all diagrams and take the Regge limit (2.1). In this limit, we have

[4]Here and in subsequent equations, we keep an overall ϵ-dependent factor, which contributes finite terms that are removed upon renormalizing the Wilson line correlator in the $\overline{\text{MS}}$ scheme.

$$\gamma_{ij} = \cosh^{-1}\left(\frac{p_i \cdot p_j}{m^2}\right) \underset{p_i \cdot p_j \gg m^2}{\longrightarrow} \log\left(\frac{2p_i \cdot p_j}{m^2}\right), \quad (2.23)$$

and thus, also approximating $(p_i + p_j)^2 \simeq 2p_i \cdot p_j$,

$$\gamma_{12}, \gamma_{34} \to \log\left(\frac{s}{m^2}\right), \qquad \gamma_{14}, \gamma_{23} \to \log\left(-\frac{u}{m^2}\right),$$
$$\gamma_{13}, \gamma_{24} \to \log\left(-\frac{t}{m^2}\right). \quad (2.24)$$

Furthermore, in the Regge limit one has $s \simeq -u$, so that

$$\gamma_{12}, \gamma_{34}, \gamma_{14}, \gamma_{23} \to \log\left(\frac{s}{m^2}\right),$$
$$\gamma_{13}, \gamma_{24} \to \log\left(-\frac{t}{m^2}\right). \quad (2.25)$$

Finally, $\coth(\gamma_{ij}) \to 1$, so that the Regge limit of the sum of diagrams (a)-(f) gives

$$\sum_i M_i^{(1)} = \frac{g_s^2 \Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \frac{(\mu^2 \vec{z}^2)^\epsilon}{2\epsilon} \left\{ i\pi[\mathbf{T}_1 \cdot \mathbf{T}_2 + \mathbf{T}_3 \cdot \mathbf{T}_4] + \log\left(\frac{s}{m^2}\right)[-\mathbf{T}_1 \cdot \mathbf{T}_2 - \mathbf{T}_3 \cdot \mathbf{T}_4 + \mathbf{T}_1 \cdot \mathbf{T}_4 + \mathbf{T}_2 \cdot \mathbf{T}_3] + \log\left(-\frac{t}{m^2}\right)[\mathbf{T}_1 \cdot \mathbf{T}_3 + \mathbf{T}_2 \cdot \mathbf{T}_4] \right\} + \mathcal{O}(\epsilon^0). \quad (2.26)$$

We may simplify this expression further by introducing the color operators

$$\mathbf{T}_s^2 = (\mathbf{T}_1 + \mathbf{T}_2)^2, \qquad \mathbf{T}_t^2 = (\mathbf{T}_1 - \mathbf{T}_3)^2 \quad (2.27)$$

whose eigenstates are pure s- and t-channel exchanges, and the corresponding eigenvalue in each case is the quadratic Casimir operator appropriate to the representation of the exchanged particle. Using these, together with color conservation $\mathbf{T}_1 + \mathbf{T}_2 = \mathbf{T}_3 + \mathbf{T}_4$, we obtain

$$\mathbf{T}_1 \cdot \mathbf{T}_2 + \mathbf{T}_3 \cdot \mathbf{T}_4 = \frac{1}{2}\left[(\mathbf{T}_1 + \mathbf{T}_2)^2 + (\mathbf{T}_3 + \mathbf{T}_4)^2 - \sum_{i=1}^4 C_i\right] = \mathbf{T}_s^2 - \frac{1}{2}\sum_{i=1}^4 C_i, \quad (2.28)$$

$$-\mathbf{T}_1 \cdot \mathbf{T}_2 - \mathbf{T}_3 \cdot \mathbf{T}_4 + \mathbf{T}_1 \cdot \mathbf{T}_4 + \mathbf{T}_2 \cdot \mathbf{T}_3 = (\mathbf{T}_1 - \mathbf{T}_3) \cdot (\mathbf{T}_4 - \mathbf{T}_2) = \mathbf{T}_t^2, \quad (2.29)$$

$$\mathbf{T}_1 \cdot \mathbf{T}_3 + \mathbf{T}_2 \cdot \mathbf{T}_4 = -\frac{1}{2}\left[(\mathbf{T}_1 - \mathbf{T}_3)^2 + (\mathbf{T}_2 - \mathbf{T}_4)^2 - \sum_{i=1}^4 C_i\right] = -\mathbf{T}_t^2 + \frac{1}{2}\sum_{i=1}^4 C_i, \quad (2.30)$$

where C_i is the quadratic Casimir operator in the representation of external particle i. Using these in Eq. (2.26) yields

$$\sum_i M_i^{(1)} = \frac{g_s^2 \Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \frac{(\mu^2 \vec{z}^2)^\epsilon}{2\epsilon} \left[i\pi \mathbf{T}_s^2 + \mathbf{T}_t^2 \log\left(\frac{s}{-t}\right) + \frac{1}{2}\left(\log\left(-\frac{t}{m^2}\right) - i\pi\right)\sum_{i=1}^4 C_i \right] + \mathcal{O}(\epsilon^0). \quad (2.31)$$

Note that we have not included self-energy diagrams associated with any of the incoming or outgoing particles. These would contribute constant terms which will not concern us in what follows.

Some further comments are in order regarding the above calculation, and how it differs from that presented in Refs. [29,30]. Here, we separated the incoming and outgoing branch of each Wilson line, and included diagrams in which a gluon is absorbed and emitted from the same line, using the same ultraviolet cutoff as for the diagrams in which a gluon spans both Wilson lines. Had we used a different cutoff, this would have contributed an additional logarithmic dependence, beginning only at $\mathcal{O}(\epsilon^0)$ level. Our motivation for including the additional diagrams was so as to be able to combine terms to generate logarithms of $s/(-t)$, as opposed to the calculation of Refs. [29,30], which instead considers[5] logarithms of s/m^2. The choice made here allows us to more easily make contact with the case of massless external particles studied in Refs. [31,32], as the mass dependence has canceled in the color non-diagonal terms. As $m \to 0$, an additional (collinear)

[5]References [29,30] also consider the alternative Regge limit $s, m^2 \gg |t|$, rather than the choice made in Eq. (2.1). The diagrams which we include here do not contribute logarithms of s/m^2 in that paper, so they can be neglected.

singularity appears in Eq. (2.31), here appearing as a logarithm of the mass. Were one to use dimensional regularization to regulate both soft and collinear singularities, Eq. (2.31) would have a double pole in ϵ in the massless limit. Because the above calculation includes soft information only, it also misses hard collinear contributions, which appear in the full amplitude as (hard) jet functions divided by *eikonal jets* [110–113]. Such contributions are irrelevant to the discussion of the Regge trajectory [31,32], and will not bother us in gravity, where collinear singularities are absent [91,94]. Apart from the different collinear regulator, and the lack of hard collinear terms, Eq. (2.31) agrees with the result found in Refs. [31,32] by taking the Regge limit of the QCD dipole formula at one loop.

Note that only the first two contributions in the square bracket of Eq. (2.31) have a nontrivial color structure when acting on the color structure of the hard interaction. The final term is color-diagonal, involving only quadratic Casimir operators. Let us interpret the various contributions in more detail. We know that the soft function exponentiates. Thus, we may exponentiate Eq. (2.31) to obtain

$$\exp\left\{\frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\vec{z}^2)^\epsilon}{2\epsilon}\left[i\pi\mathbf{T}_s^2+\mathbf{T}_t^2\log\left(\frac{s}{-t}\right)+\frac{1}{2}\left(\log\left(-\frac{t}{m^2}\right)-i\pi\right)\sum_{i=1}^4 C_i\right]\right\}. \tag{2.32}$$

In the Regge limit, the term involving $\log(s/-t)$ dominates, and the above combination reduces to

$$\left(\frac{s}{-t}\right)^{K\mathbf{T}_t^2}, \qquad K=\frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\vec{z}^2)^\epsilon}{2\epsilon}. \tag{2.33}$$

For a Born interaction dominated by a t-channel exchange in the Regge limit (which is usually the case), this operator acts to Reggeize the exchanged particle. That is, it leads to an amplitude with the behavior

$$\left(\frac{s}{-t}\right)^{J+KC_R}, \tag{2.34}$$

where C_R is the quadratic Casimir associated with the exchanged particle, in representation R of the gauge group, and J is the spin of the particle [which leads to an appropriate power of $(s/-t)$ in the Born amplitude].

However, there are cases in which $C_R=0$. An example is electron scattering in QED, in which the Born amplitude is dominated by t-channel exchange of the photon, which has zero squared charge. Then the $i\pi$ terms in Eq. (2.32) give

$$\exp\left[\frac{i\alpha}{\epsilon}(\mu^2\vec{z}^2)^\epsilon\right], \qquad \alpha=\frac{e^2\Gamma(1-\epsilon)}{4\pi^{1-\epsilon}}\coth\gamma_{12}, \tag{2.35}$$

where e is the electron charge, and we have restored the full dependence on the cusp angle $\gamma_{12}=\gamma_{34}$. Since

$$\coth\gamma_{12}=\coth\left[\cosh^{-1}\left(\frac{p_1\cdot p_2}{m^2}\right)\right]=\frac{s-2m^2}{\sqrt{s(s-4m^2)}}, \tag{2.36}$$

one has

$$\alpha=\frac{e^2}{4\pi}\frac{s-2m^2}{\sqrt{s(s-4m^2)}}+\mathcal{O}(\epsilon). \tag{2.37}$$

Equation (2.35) constitutes the QED equivalent of the gravitational *eikonal phase* discussed in Ref. [84]. Expanding in ϵ gives

$$\begin{aligned}\exp\left[\frac{i\alpha}{\epsilon}(\mu^2\vec{z}^2)^\epsilon\right]&=\exp\left[i\frac{\alpha}{\epsilon}+i\alpha\log(\mu^2z^2)+\mathcal{O}(\epsilon)\right]\\&=(\mu^2\vec{z}^2)^{i\alpha}e^{i\alpha/\epsilon}.\end{aligned} \tag{2.38}$$

One can then carry out the Fourier transform of Eq. (2.2) to obtain (at this order)

$$\begin{aligned}\tilde{M}&=\int d^2\vec{z}e^{-i\vec{q}\cdot\vec{z}}(\mu^2\vec{z}^2)^{i\alpha}e^{i\alpha/\epsilon}\\&=\frac{4\pi i\alpha}{t}e^{i\alpha/\epsilon}\left(\frac{-t}{4\mu^2}\right)^{-i\alpha}\frac{\Gamma(1+i\alpha)}{\Gamma(1-i\alpha)}\end{aligned} \tag{2.39}$$

where we have taken a Hankel transform of order zero, and recalled that $t=-\vec{q}^2$. This has poles in the plane of the Mandelstam invariant s, stemming from the Γ function in the numerator i.e. when

$$i\alpha=-N, \quad N=1,2,\ldots \tag{2.40}$$

Then Eq. (2.37) implies that the physical poles of the scattering amplitude are at

$$s=2m^2\left[1-\left(1+\frac{e^4}{16\pi^2N^2}\right)^{-1/2}\right]. \tag{2.41}$$

Given that poles in s of a scattering amplitude represent bound states, Eq. (2.41) represents the spectrum of s-channel states produced in electron scattering (i.e. positronium). Indeed, the above calculation reproduces Eq. (17) of Ref. [114].

In this section, we have introduced the Wilson line formalism of Ref. [29] for examining the Regge limit in QCD and QED. In particular, we have seen two effects emerge:

(i) If the Born interaction is dominated by a t-channel exchange in the Regge limit, then this particle Reggeizes at leading log order, with a trajectory which depends on the quadratic Casimir in the appropriate representation of the gauge group.

(ii) There is a pure phase term, the *eikonal phase*, which is associated with the formation of s-channel bound states.

As is well known, the first of these contributions arises at the Feynman diagram level from vertical ladder graphs, and the second arises from horizontal ladder graphs. Which of these is kinematically leading in the Regge limit depends in the present case on the squared charge of the particle being exchanged in the t-channel. If this is nonzero, the Reggeization term dominates. If, however, the squared charge is zero (as in the case of the photon), then the eikonal phase is the dominant effect.

Things get more complicated beyond leading logarithmic order in $(s/-t)$. One must include higher order contributions to the soft function, as well as include the possibility of cross talk between the eikonal phase and Reggeization terms. This becomes especially cumbersome in QCD, due to the fact that the color operators associated with the eikonal phase and Reggeization terms do not commute. This has already been noted in Refs. [31,32], where it was identified with a lack of simple Regge pole behavior at next-to-next-to-leading-log order.

Having seen how things work in QED and QCD, we examine the case of gravity in the following section.

III. WILSON LINE APPROACH FOR GRAVITY

In the previous section, we have reviewed the Wilson line approach for examining the Regge limit of gauge theory scattering amplitudes in some detail. The case of gravitational scattering can be obtained quite straightforwardly from the above results. Note that we here discuss explicitly the case of Einstein-Hilbert gravity. As we will see, this will also have features in common with supersymmetric extensions.

Let us first recall the form[6] of the gravitational Wilson line operator [92,93] (see also Ref. [104])

$$\Phi_g(p,z) = \exp\left[i\frac{\kappa}{2}p^\mu p^\nu \int_{-\infty}^{\infty} ds\, h_{\mu\nu}(sp+z)\right], \quad (3.1)$$

where $\kappa = \sqrt{32\pi G_N}$ in terms of Newton's constant G_N. We will use the de Donder gauge graviton propagator[7]

$$D_{\mu\nu,\alpha\beta}(x-y) = P_{\mu\nu,\alpha\beta}\frac{\Gamma(d/2-1)}{4\pi^{d/2}}[-(x-y)^2]^{1-d/2},$$
$$P_{\mu\nu,\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\nu\alpha}\eta_{\mu\beta}-\frac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta}\right). \quad (3.2)$$

By analogy with the gauge theory case, we now wish to calculate the amplitude

$$\tilde{M}_g = \int d^2\vec{z}e^{-i\vec{z}\cdot\vec{q}}\langle 0|\Phi_g(p_1,0)\Phi_g(p_2,z)|0\rangle, \quad (3.3)$$

i.e. a pair of gravitational Wilson lines separated by a transverse distance $\vec{z}$. The diagrams will be the same as those of Fig. 3. Given that the denominator structure of the propagator (3.2) is the same as that of (2.6), we do not have to recalculate any of the kinematic integrals. All that changes in each diagram is the overall prefactor of $p_i \cdot p_j$, obtained by contracting two eikonal Feynman rules with the gluon propagator. In the gravity case this will be replaced by

$$p_i^\mu p_i^\nu P_{\mu\nu,\alpha\beta}p_j^\alpha p_j^\beta = (p_i\cdot p_j)^2 - \frac{1}{d-2}m^4. \quad (3.4)$$

The result for diagram (a) is then

$$M_{g,a}^{(1)} = -\left(\frac{\kappa}{2}\right)^2\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}(\mu^2\vec{z}^2)^\epsilon \times\left[p_1\cdot p_2 - \frac{m^4}{2(1-\epsilon)}\frac{1}{p_1\cdot p_2}\right]\frac{1}{2\epsilon}(i\pi-\gamma_{12})\coth\gamma_{12}. \quad (3.5)$$

In the Regge limit, neglecting terms of $\mathcal{O}(m^2/s, m^2/t)$, this may be obtained from Eq. (2.13) by replacing

$$g_s \to \frac{\kappa}{2}, \qquad \mathbf{T}_i \to p_i \quad (3.6)$$

and switching the overall sign. The other diagrams are similar, so the sum of gravitational diagrams in the Regge limit may be obtained by making the replacements (as $m \to 0$)

$$g_s \to \frac{\kappa}{2}, \quad \mathbf{T}_s^2 \to s, \quad \mathbf{T}_t^2 \to t, \quad C_i \to 0 \quad (3.7)$$

and switching the overall sign in Eq. (2.31), yielding

$$\sum_i M_{g,i}^{(1)} = -\left(\frac{\kappa}{2}\right)^2\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\vec{z}^2)^\epsilon}{2\epsilon} \times\left[i\pi s + t\log\left(\frac{s}{-t}\right)\right] + \mathcal{O}(\epsilon^0) \quad (3.8)$$

where we have dropped the $\mathcal{O}(m^2)$ terms which vanish in the Regge limit. Note that the logarithmic dependence on the mass has completely canceled in the sum over diagrams due to the absence of collinear divergences in gravity [91]. Thus one would expect the same result for the scattering of strictly massless particles. Furthermore, one-loop exactness tells us that there are no perturbative corrections to Eq. (3.8).

[6] The factor of two error in Eq. (3.2) of Ref. [92] has been corrected in v3.

[7] Note that this differs by a factor of -2 from that used in Ref. [96], which can be traced to our use of $\kappa = \sqrt{32\pi G_N}$ and metric $(+,-,-,-)$ in the present paper, rather than $\kappa = \sqrt{16\pi G_N}$ and metric $(-,+,+,+)$ in Ref. [96].

Two terms occur in the soft function in the Regge limit: an $i\pi s$ eikonal phase term and a $t\,\log(s/-t)$ term, which will Reggeize the graviton. Expanding Eq. (3.8) in ϵ and exponentiating the result gives

$$e^{-i\pi s K_g/\epsilon}\left(\frac{s}{-t}\right)^{-K_g t/\epsilon}(\mu^2\vec{z}^2)^{-K_g[i\pi s+t\,\log(s/-t)]},$$
$$K_g=\left(\frac{\kappa}{2}\right)^2\frac{\Gamma(1-\epsilon)}{8\pi^{2-\epsilon}}. \tag{3.9}$$

When acting on the Born interaction [which is $\mathcal{O}(s^2)$], the powerlike term in $(s/-t)$ corresponds to Reggeization of the graviton with a trajectory

$$\alpha_g(t)=2-\frac{tK_g}{\epsilon}. \tag{3.10}$$

We see that the one-loop perturbative Regge trajectory in gravity is infrared singular (up to scale logarithms), as is known to be the case in QCD and QED. However, this trajectory is linear in the Mandelstam invariant t, reproducing the results of Refs. [59–62,72]. In the present approach, however, the comparison with QCD appears in a particularly elegant fashion. As discussed in Refs. [31,32], we expect Reggeization to occur at leading logarithmic order if the tree-level interaction is dominated by the t-channel exchange of a particle with a nonvanishing squared charge. The Regge trajectory is then infrared singular at one loop, and contains the quadratic Casimir associated with the exchanged particle. Here exactly the same mechanism occurs for the graviton, and the relevant gravitational quadratic Casimir is the squared four-momentum, which in this case is simply the Mandelstam invariant t.

The eikonal phase term, which in QCD involved a quadratic Casimir operator for s-channel exchanges, now contains the Mandelstam invariant s. This in turn implies that Reggeization of the graviton is kinematically suppressed with respect to the eikonal phase in the strict Regge limit of $s/(-t)\to\infty$. In Feynman diagram terms: horizontal ladders and crossed ladders (which build up the eikonal phase as discussed in Ref. [84]) win out over vertical ladders (which build up the Reggeized graviton). Nevertheless, both effects are present and clearly show up in the Wilson line calculation.

As has already been commented above, in hindsight we could have obtained the gravity result of Eq. (3.8) from the QCD case of Eq. (2.31) without detailed calculation, by the simple replacements of Eq. (3.7). The final replacement corresponds to the setting to zero of quadratic Casimir operators associated with the external legs, here a consequence of having considered massless particles in the gravity case (indeed, as discussed in Ref. [97], this is one way of appreciating the cancellation of collinear divergences in gravity). The replacements (3.7) are consistent, at least in general, with what one would expect from the double copy procedure of Refs. [88–90]. In addition to the coupling constant replacement, color operators are replaced by their momentum counterparts which, in Feynman diagram language, is equivalent to the replacement of color factors by kinematic numerators. The double copy was considered in more detail in this context in Ref. [86], which also discussed the relationship between shock waves in both gauge and gravity theories. The latter point can also be understood in the language of Wilson lines, as we briefly describe in Appendix A.

As in the QED case, one may carry out the Fourier transform over the impact parameter. After substituting Eq. (3.9) into Eq. (3.3), one obtains (at this order)

$$\tilde{M}_g=\frac{-4\pi K_g e^{-i\pi s K_g/\epsilon}}{t}\left(\frac{s}{-t}\right)^{-K_g t/\epsilon}$$
$$\times\left[i\pi s+t\,\log\left(\frac{s}{-t}\right)\right]\times\left(\frac{-t}{4\mu^2}\right)^{K_g[i\pi s+t\,\log(s/-t)]}$$
$$\times\frac{\Gamma[1-K_g(i\pi s+t\,\log(s/-t))]}{\Gamma[1+K_g(i\pi s+t\,\log(s/-t))]}. \tag{3.11}$$

The ratio of Euler gamma functions no longer constitutes a pure phase. Also, it now gives rise to cuts in the s plane, rather than poles. By standard Regge theory arguments, the high-energy behavior of an amplitude $A(s,t)$ is related to its analytically continued partial wave coefficients $F(t,j)$, where the angular momentum j has become a complex variable, by (see e.g. Ref. [115])

$$F(t,j)=\int_1^\infty ds\, s^{-j-1}A(s,t). \tag{3.12}$$

Thus, cuts in the s-plane give rise to Regge cuts in the complex angular momentum plane. Note that such cuts will only appear if both the Reggeization and eikonal phase term are kept, thus they are due to a cross talk between these two contributions. This is consistent with the results of Refs. [31,32], which demonstrated a breakdown of Regge pole behavior at three loop order in QCD, associated with the presence of both a Reggeization and an eikonal phase term. This was assumed to herald the arrival of Regge cut contributions at this order in perturbation theory. For example, the color factor associated with the non-Regge-pole-like contribution was nonplanar, and consistent with Feynman diagrams which lead to cuts [4]. Here we see directly that cross talk between the eikonal phase and Reggeization terms leads to cutlike behavior. It is interesting to remark that gravity theories provide a simpler testing ground for such ideas, lying somewhere between Abelian and non-Abelian gauge theories in terms of complication: although multigraviton vertices are present (unlike an Abelian gauge theory), there is no noncommuting color structure. There may well be other problems in QCD whose conceptual structure is simplified by examining a gravitational analogue.

If we neglect the Reggeization term, and restore full mass dependence in the eikonal phase term, Eq. (3.11) becomes

$$\tilde{M}_g = -\frac{4\pi i G(s)}{t} e^{-iG(s)/\epsilon} \left(\frac{-t}{4\mu^2}\right)^{iG(s)} \frac{\Gamma[1-iG(s)]}{\Gamma[1+iG(s)]}, \quad (3.13)$$

where[8]

$$G(s) = G_N \left(\frac{s^2 - 4m^2 s + 2m^4}{\sqrt{s(s-4m^2)}}\right) \quad (3.14)$$

which essentially agrees with the eikonal amplitude in Refs. [76,84]. The Euler gamma function then gives rise to poles in the amplitude, corresponding to the spectrum of bound states discussed in Sec. IV of Ref. [84].

In this section, we have seen that both a Reggeization term and an eikonal phase term are present in gravity. However, the Regge trajectory of the graviton is linear in t, and hence the eikonal phase dominates in the strict Regge limit. Cross-talk between the eikonal phase and Reggeization terms is associated with Regge cut behavior.

The above analysis was carried out in Einstein-Hilbert (nonsupersymmetric) gravity. However, it also applies to the four-graviton amplitude in supergravity, if one dresses only the tree-level hard interaction with the eikonal calculation discussed here. This is because the leading infrared singularity at each order in perturbation theory arises from the Born amplitude dressed only by graviton emissions between the external legs (the highest spin objects in the theory). There are no corrections to the gravitational soft function, as dictated by one-loop exactness [91–94]. However, subleading IR singularities (and infrared finite parts) will arise in the amplitude from higher order contributions to the hard interaction, which are sensitive to the additional matter content, and hence the degree of supersymmetry.

The only information that we have used about the hard interaction in the above calculation is that the Wilson lines are separated by a transverse distance. This means that we have no control over finite parts of the amplitude. One would think this is irrelevant to the issue of graviton Reggeization at one-loop order, as the perturbative Regge trajectory is purely infrared singular at this order. However, the finite terms do lead to complications, as we discuss in the following section.

IV. INFRARED-FINITE CONTRIBUTIONS IN SUPERGRAVITY

In the previous sections, we have reviewed the Regge limit of QCD from a Wilson line point of view, and applied this same reasoning to gravity. Use of this common language showed a number of similarities between the two theories: namely the presence of both a Reggeization and eikonal phase term, and an infrared singular Regge trajectory at one loop that contained the relevant quadratic Casimir operator. These facts, by themselves, lead to the fact that the eikonal phase is kinematically dominant in gravity, and subdominant in QCD. In this section, we discuss another important difference between the QCD and gravity cases: in the latter, Reggeization is interrupted even at one loop by the presence of double log terms of the same order in $x \equiv -t/s$ in the infrared finite part of the amplitude. Let us begin by considering the one-loop amplitude.

A. One-loop results

We here consider $\mathcal{N} = M$ supergravity, where $4 \leq M \leq 8$. One-loop results were obtained in Refs. [99–101,105]. Following Ref. [106], we write the one-loop four-graviton amplitude as

$$\mathcal{M}_4^{(1),\mathcal{N}=M} = \left(\frac{\kappa}{8\pi}\right)^2 \left(\frac{4\pi e^{-\gamma_E}\mu^2}{|s|}\right)^{\epsilon} \mathcal{M}_4^{\rm tree} \times \left\{\frac{2}{\epsilon}[s\log(-s) + t\log(-t) + u\log(-u)] + F_4^{(1),\mathcal{N}=M}\right\}, \quad (4.1)$$

where $\kappa = \sqrt{32\pi G_N}$, γ_E is Euler's constant, and $F_4^{(1),\mathcal{N}=M}$ is an IR-finite contribution dependent on the degree of supersymmetry.

The infrared-singular part of Eq. (4.1), as remarked in the previous section, is universal at one-loop order. In the physical region $s > 0$; t, $u < 0$, it is given by

$$\mathcal{M}_4^{(1),\mathcal{N}=M}\Big|_{\rm IR-divergent} = \left(\frac{\kappa}{8\pi}\right)^2 \mathcal{M}_4^{\rm tree} \frac{2}{\epsilon}[s\ \log\ s - i\pi s + t\ \log(-t) + u\ \log(-u)], \quad (4.2)$$

where $\log(-s) = \log|s| - i\pi$. Setting $u = -s - t$, and expanding about the Regge limit $s \gg -t$, one obtains

$$\mathcal{M}_4^{(1),\mathcal{N}=M}|_{\rm IR-divergent} = -\frac{\kappa^2}{32\pi^2\epsilon}\left[i\pi s + t\ \log\left(\frac{s}{-t}\right) + t + \mathcal{O}\left(\frac{t^2}{s}\right)\right]\mathcal{M}_4^{\rm tree}. \quad (4.3)$$

As expected, this agrees with the result (3.8) obtained from the Wilson line calculation [up to the nonlogarithmic $\mathcal{O}(t)$ term neglected in the latter].

Next we consider the IR-finite part of the amplitude in the Regge limit. The Regge limit corresponds to $x \to 0$ with s fixed, where

[8]Note that the quantity $G(s)$ is referred to as $\alpha(s)$ in Ref. [84].

$$x \equiv \frac{-t}{s}. \tag{4.4}$$

The various remainder terms $F_4^{(1)}$ for different supergravity theories are collected in the appendix of Ref. [106]. We substitute these into Eq. (4.1), set $u = -t - s$, and keep the first two terms in the expansion about $x = 0$ to obtain (for $s > 0$, $t < 0$)

$$\begin{aligned}
\mathcal{M}_4^{(1),\mathcal{N}=8} &= \left(\frac{\kappa}{8\pi}\right)^2 \left(\frac{4\pi e^{-\gamma_E}\mu^2}{-t}\right)^{\epsilon} \mathcal{M}_4^{\text{tree}} \left\{\frac{s}{\epsilon}[-2i\pi + 2x(L+1)] + sx[-2L^2 + 2i\pi L]\right\},\\
\mathcal{M}_4^{(1),\mathcal{N}=6} &= \left(\frac{\kappa}{8\pi}\right)^2 \left(\frac{4\pi e^{-\gamma_E}\mu^2}{-t}\right)^{\epsilon} \mathcal{M}_4^{\text{tree}} \left\{\frac{s}{\epsilon}[-2i\pi + 2x(L+1)] + sx[-L^2 + 2i\pi L + \pi^2]\right\},\\
\mathcal{M}_4^{(1),\mathcal{N}=5} &= \left(\frac{\kappa}{8\pi}\right)^2 \left(\frac{4\pi e^{-\gamma_E}\mu^2}{-t}\right)^{\epsilon} \mathcal{M}_4^{\text{tree}} \left\{\frac{s}{\epsilon}[-2i\pi + 2x(L+1)] + sx\left[-\frac{L^2}{2} + 2i\pi L + \frac{3\pi^2}{2}\right]\right\},\\
\mathcal{M}_4^{(1),\mathcal{N}=4} &= \left(\frac{\kappa}{8\pi}\right)^2 \left(\frac{4\pi e^{-\gamma_E}\mu^2}{-t}\right)^{\epsilon} \mathcal{M}_4^{\text{tree}} \left\{\frac{s}{\epsilon}[-2i\pi + 2x(L+1)] + sx[2\pi i L - L + 2\pi^2 + 1]\right\},
\end{aligned} \tag{4.5}$$

where for convenience we define $L = \log(s/-t)$. These results may be compactly summarized as

$$\begin{aligned}
\mathcal{M}_4^{(1),\mathcal{N}=M} = \left(\frac{\kappa}{8\pi}\right)^2 \left(\frac{4\pi e^{-\gamma_E}\mu^2}{-t}\right)^{\epsilon} \mathcal{M}_4^{\text{tree}} \Bigg\{&\frac{s}{\epsilon}[-2i\pi + 2xL + 2x] + sx\left[\left(\frac{4-M}{2}\right)L^2 + \left(\frac{8-M}{2}\right)\pi^2 + 2i\pi L\right.\\
&\left. + \delta_{M4}(1-L)\right] + \mathcal{O}(sx^2) + \mathcal{O}(\epsilon)\Bigg\},
\end{aligned} \tag{4.6}$$

which makes clear the dependence on the degree of supersymmetry M.

The first two terms in the infrared-singular part of Eq. (4.6), as discussed at length in the previous section, correspond to the eikonal phase and Reggeization of the graviton respectively, where the latter is kinematically suppressed ($\mathcal{O}(x)$ in the present notation). However, the first term of the infrared-finite part contains the double log contribution (ignoring prefactors)

$$sx\left(\frac{4-M}{2}\right)L^2 = \left(\frac{M-4}{2}\right)t\log^2\left(\frac{s}{-t}\right) \tag{4.7}$$

as observed in Ref. [85]. This does not correspond to Reggeization of the graviton which, as we have already seen, is purely infrared singular at this order and can involve only a single log. Nevertheless, the double logarithmic contribution is of the same order (linear in x) as the Reggeization term, and in fact superleading (logarithmically in x) with respect to the Regge logs. The fact that the coefficient of the double logarithmic contribution is sensitive to the additional matter content of the theory (via the degree of supersymmetry M) tells us that one is not sensitive to this contribution in the Wilson line approach, which picks up only graviton-related contributions at one loop (this also explains why the double log is in the infrared finite part).

Another way to see that the double logs at one loop are not associated with Reggeization is to examine their origin in terms of the Feynman diagrams contributing to the amplitude. Taking the example of $\mathcal{N} = 8$ supergravity, the one-loop amplitude may be written as [102]

$$\begin{aligned}
\mathcal{M}_4^{(1),\mathcal{N}=8} = -i\left(\frac{\kappa}{2}\right)^2 stu[&\mathcal{I}_4^{(1)}(s,t) + \mathcal{I}_4^{(1)}(t,u)\\
&+ \mathcal{I}_4^{(1)}(s,u)]\mathcal{M}_4^{\text{tree}},
\end{aligned} \tag{4.8}$$

where

$$\mathcal{I}_4^{(1)}(s,t) = \mu^{4-d}\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-p_2)^2(k-p_2-p_1)^2(k+p_4)^2} \tag{4.9}$$

is the first scalar box integral shown in Fig. 4. The terms $\mathcal{I}_4^{(1)}(t,u)$ and $\mathcal{I}_4^{(1)}(s,u)$ then correspond to the second and third box diagrams in the figure. The result for the integral may be written [116]

$$\begin{aligned}
\mathcal{I}_4^{(1)}(s,t) = \frac{ie^{-\epsilon\gamma_E}(4\pi)^{\epsilon-2}}{st}\Bigg\{&\frac{4}{\epsilon^2} - \frac{2}{\epsilon}\log\left(\frac{-s}{\mu^2}\right)\\
&- \frac{2}{\epsilon}\log\left(\frac{-t}{\mu^2}\right) + 2\log\left(\frac{-s}{\mu^2}\right)\log\left(\frac{-t}{\mu^2}\right)\\
&- \frac{4\pi^2}{3} + \mathcal{O}(\epsilon)\Bigg\}
\end{aligned} \tag{4.10}$$

in $d = 4 - 2\epsilon$ dimensions in the region $s, t < 0$. From this result, we see that the double logarithmic contribution comes from $\mathcal{I}_4^{(1)}(s,u)$, corresponding to the third diagram in Fig. 4. This is neither a ladder nor a crossed ladder, and

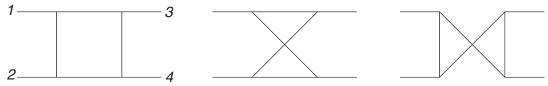

FIG. 4. Diagrams contributing to the one-loop four-graviton amplitude in $\mathcal{N}=8$ supergravity.

thus is responsible neither for the eikonal phase, nor for the Reggeization of the graviton.

The question then arises how to interpret the additional double logarithmic contributions, and whether or not they exponentiate. This has been discussed in Ref. [85], which argues for two sources of double logarithms. The first is from ladder contributions, including infrared-finite effects of the graviton Regge trajectory [59–61]. The second is that backward-scattering contributions are important at this order in $t/(-s)$, an observation corroborated by the fact that such contributions arise from the dressed u-channel diagram, the third in Fig. 4. The authors of Ref. [85] write an evolution equation for the leading partial wave contributing to the amplitude in the limit in which double logarithms are important, whose solution is argued to resum these contributions. Note, however, that such contributions become more and more kinematically suppressed at higher orders in perturbation theory, giving rise to terms

$$\left[\kappa^2 t \log^2\left(\frac{s}{-t}\right)\right]^n \sim \kappa^{2n} s^n x^n L^{2n}, \tag{4.11}$$

which are $\mathcal{O}(x^n)$.

B. Two-loop results

In the previous section, we have seen that the interpretation of Reggeization of the graviton is interrupted at one-loop order by the presence of double logarithmic contributions in the infrared finite part of the amplitude, which are the same order in t/s as the Reggeization terms. This motivates an examination of the four-graviton amplitude at two loops, with a view to seeing which structures exponentiate, and which do not.

Following Ref. [106], we write the two-loop amplitude as

$$\frac{\mathcal{M}_4^{(2),\mathcal{N}=M}(\epsilon)}{\mathcal{M}_4^{\rm tree}} = \frac{1}{2}\left[\frac{\mathcal{M}_4^{(1),\mathcal{N}=M}(\epsilon)}{\mathcal{M}_4^{\rm tree}}\right]^2 + \left(\frac{\kappa}{8\pi}\right)^4 F_4^{(2),\mathcal{N}=M} + \mathcal{O}(\epsilon), \tag{4.12}$$

where the IR-finite remainder function $F_4^{(2),\mathcal{N}=M}$ corresponds to the part of the two-loop result that is not generated by exponentiation of the one-loop result[9]. The remainder function for $\mathcal{N}=8$ supergravity was computed in Refs. [103,104] using the results of Refs. [102,117,118], and for $\mathcal{N}=M<8$ supergravity in Ref. [106]. Again defining $x=-t/s$ and expanding about the Regge limit $x\to 0$ (keeping terms up to linear in x), one finds that the behavior of each remainder function is

$$F_4^{(2),\mathcal{N}=8} = s^2 x\Big\{-2\pi^2\log^2 x - 4\pi^2 \log x + \pi^4 + 4\pi^2 + i\pi\left[\frac{4}{3}\log^3 x + 4\log^2 x - \left(8+\frac{8\pi^2}{3}\right)\log x + 16\zeta_3 + \frac{8\pi^2}{3} + 8\right]\Big\} + \cdots \tag{4.13}$$

$$F_4^{(2),\mathcal{N}=6} = s^2 x\Big\{-2\pi^2\log^2 x - 4\pi^2 \log x + \frac{59\pi^4}{90} + 4\pi^2 + i\pi\left[\frac{2}{3}\log^3 x + 4\log^2 x - \left(8+\frac{6\pi^2}{3}\right)\log x + 4\zeta_3 + \frac{16\pi^2}{3} + 8\right]\Big\} + \cdots \tag{4.14}$$

$$F_4^{(2),\mathcal{N}=5} = s^2 x\Big\{-2\pi^2\log^2 x - 4\pi^2 \log x + \frac{2\pi^4}{3} + 4\pi^2 + i\pi\left[\frac{1}{3}\log^3 x + 4\log^2 x - \left(8+\frac{5\pi^2}{3}\right)\log x + 4\zeta_3 + \frac{20\pi^2}{3} + 8\right]\Big\} + \cdots \tag{4.15}$$

$$F_4^{(2),\mathcal{N}=4} = s^2 x\Big\{-2\pi^2\log^2 x - 4\pi^2 \log x + \frac{13\pi^4}{30} + \frac{22\pi^2}{3} - 1 + i\pi\left[3\log^2 x - \left(14+\frac{4\pi^2}{3}\right)\log x - 4\zeta_3 + \frac{71\pi^2}{9} + \frac{32}{3}\right]\Big\} + \cdots. \tag{4.16}$$

Note that the IR-finite remainder functions vanish in the strict Regge limit $x\to 0$. This is because the amplitude in this limit is dominated by the eikonal phase dressing the tree-level result. The eikonal phase contribution exponentiates (at least) up to this order, and thus the remainder must vanish in the limit.

One may summarize the *logarithmic* terms of the remainder function, for general M, as

$$F_4^{(2),\mathcal{N}=M} = s^2 x\Big\{-2\pi^2\log^2 x - 4\pi^2 \log x + i\pi\left[\left(\frac{M-4}{3}\right)\log^3 x + (4-\delta_{M4})\log^2 x - \left(8+\frac{M\pi^2}{3}+6\delta_{M4}\right)\log x\right]\Big\} + \cdots \tag{4.17}$$

[9]As the notation in Eq. (4.12) suggests, in constructing the remainder one must be mindful of terms generated due to the cross talk between $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^{-1})$ terms when squaring the one-loop result.

where, as in the one-loop amplitude, $M = 4$ is a somewhat exceptional case, due presumably to the decreasing amount of supersymmetry as one counts down from $M = 8$. In both the one- and two-loop amplitudes, a number of terms are independent of M, and thus are common to all the supergravity theories considered here. Such terms presumably arise from finite contributions involving the graviton alone. At two loops such contributions would be ultraviolet divergent in pure Einstein-Hilbert gravity. Here, the results are made finite by the additional matter content of the various supergravities.

In Eqs. (4.13)–(4.16), we have not displayed terms of $\mathcal{O}(s^2x^2)$, but as noted in Ref. [85], the remainder function can contain quartic logarithms at this order

$$F_4^{(2),\mathcal{N}=M} = \cdots + c_M s^2 x^2 \log^4\left(\frac{s}{-t}\right) + \cdots,$$
$$\text{where} \quad \begin{cases} c_8 = -\frac{1}{3} \\ c_6 = 0 \\ c_5 = \frac{1}{24} \\ c_4 = 0 \end{cases} \tag{4.18}$$

showing that, for $M = 5$ and $M = 8$, the one-loop double logarithmic terms do not formally exponentiate. The authors of Ref. [85] argue that these terms can be resummed to all orders. In any case, Eq. (4.17) makes clear that there is a more dominant source of IR-finite corrections at two-loop order, namely those which are $\mathcal{O}(s^2x)$.

These $\mathcal{O}(s^2x)$ terms also threaten a simple interpretation of Reggeization at this order, as they introduce a dependence on $s/(-t)$ which is kinematically enhanced relative to the Reggeization of the graviton at this order. It is interesting to ponder whether any of the terms in the above remainders can be shown to exponentiate, or be resummable in some other form. It is known, for example, that $t/(-s)$ corrections to the eikonal phase should come into play in describing black-hole formation [75,80]. That such features are suppressed in this manner is partly due to the fact that they are not described by the eikonal approximation, which reproduces the bound states associated with only the perturbative (Coulomb-like) part of the gravitational potential. Black holes should be associated with nonperturbative dynamics, as discussed in Ref. [84].

Because the two-loop remainder functions vanish in the strict Regge limit $x \to 0$ for arbitrary degrees of supersymmetry, the four-graviton scattering amplitude is reproduced exactly in this limit by the exponentiation of the one-loop result at two-loop order. One may wonder whether this remains true at higher orders. To this end, it is interesting to note that the eikonal result itself does not satisfy this requirement at three-loop order and beyond. To see this, note that the ratio of Euler gamma functions in Eq. (3.13) can be expanded to give

$$\frac{\Gamma[1-iG]}{\Gamma[1+iG]} = e^{2i\gamma_E G}\left[1 + \frac{i}{3}\Psi^{(2)}(1)G^3 + \mathcal{O}(G^4)\right], \tag{4.19}$$

where $\Psi^{(n)}(x)$ is the n^{th} derivative of the digamma function

$$\Psi(x) = \frac{d}{dx}\log\Gamma(x). \tag{4.20}$$

Equation (4.19) does not have a purely exponential form, and shows that one-loop exactness of the Regge limit of the amplitude may be broken at three-loop level and beyond by infrared-finite contributions. This is not a firm conclusion, given that there may be infrared-finite corrections to the amplitude which are not captured by the eikonal approximation which leads to Eq. (4.19). However, the ratio of gamma functions resums contributions of known physical origin (the formation of bound states in the s-channel), and so presumably describes genuine behavior to all orders in perturbation theory.

V. THE REGGE LIMIT OF MULTIGRAVITON AMPLITUDES

In previous sections, we have considered the four-graviton scattering amplitude, consisting of $2 \to 2$ scattering dressed by virtual graviton exchanges. The Regge limit has also been widely studied for the case of general L-point scattering, with $L > 4$ (for a pedagogical review in a QCD context, see Ref. [119]). This was studied from an infrared point of view in Refs. [31,32], which confirmed the result that in the high energy limit, scattering is dominated by multiple t-channel exchanges, as shown in Fig. 5, where each strut of the ladder is dressed by a Reggeized propagator involving the relevant quadratic Casimir for the exchanged object.

Given the results of Sec. III for the $2 \to 2$ scattering in gravity, it is instructive to examine the high energy limit of

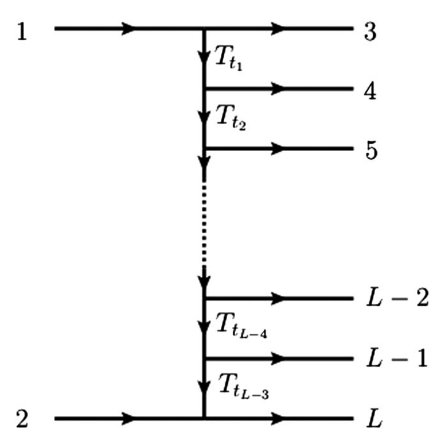

FIG. 5. A general L-parton scattering process in the MRK limit, consisting of strongly ordered rapidities in the final state. Here T_{t_i} is a quadratic Casimir operator associated with a given strut of the ladder.

multigraviton scattering using the Wilson line approach. As for the four-point amplitude, this provides an interesting comparative study with respect to non-Abelian gauge theory. Furthermore, it is useful to clarify the role of the eikonal phase in this context.

First, let us briefly review the QCD case, differing from Refs. [31,32] in that we use a Wilson line calculation, rather than the dipole formula [33–35] as a starting point. We will consider massive particles, keeping a mass m only when regulating collinear singularities [as is done in e.g. Eq. (2.26)]. For ease of comparison with Refs. [31,32] (and also for the sake of brevity in the following formulas), we will reverse the sign of the color generators associated with the incoming legs i.e. $\mathbf{T}_{1,2} \to -\mathbf{T}_{1,2}$. Treating each external leg of the amplitude as a separate Wilson line, the set of all contributing one-loop diagrams consists of gluon emissions between pairs of external lines. From Eq. (2.26), one infers that the sum of these diagrams gives (including the color adjustment mentioned above)

$$M_L^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\Lambda_{UV}^2)^\epsilon}{2\epsilon}\left\{i\pi\left(\mathbf{T}_1\cdot\mathbf{T}_2+\sum_{i=3}^{L-1}\sum_{j>i}\mathbf{T}_i\cdot\mathbf{T}_j\right)\right.$$
$$-\mathbf{T}_1\cdot\mathbf{T}_2\log\left(\frac{s}{m^2}\right)-\sum_{i=3}^{L-1}\sum_{i>j}\mathbf{T}_i\cdot\mathbf{T}_j\log\left(\frac{s_{ij}}{m^2}\right)$$
$$\left.-\sum_{i=3}^{L}\left[\mathbf{T}_1\cdot\mathbf{T}_i\log\left(-\frac{s_{1i}}{m^2}\right)+\mathbf{T}_2\cdot\mathbf{T}_i\log\left(-\frac{s_{2i}}{m^2}\right)\right]\right\} \tag{5.1}$$

where

$$s=(p_1+p_2)^2,\quad s_{ij}=(p_i+p_j)^2,$$
$$s_{1i}=(p_1-p_i)^2,\quad s_{2i}=(p_2-p_i)^2;\quad i,j>3, \tag{5.2}$$

and $\Lambda_{\rm UV}^2$ is an ultraviolet regulator, which we have chosen to be the same for all diagrams. We may now use the fact that the high energy limit of multiparton scattering corresponds to the *multi-Regge-kinematic* (MRK) regime in which the outgoing particles are widely separated in rapidity. One may then replace the various invariants appearing in Eq. (5.1) with (see e.g. Ref. [119])

$$s\simeq|k_{3\perp}||k_{L\perp}|e^{y_3-y_L},$$
$$-s_{1i}\simeq|k_{3\perp}||k_{i\perp}|e^{y_3-y_i},$$
$$-s_{2i}\simeq|k_{L\perp}||k_{i\perp}|e^{y_L-y_i},$$
$$s_{ij}\simeq|k_{i\perp}||k_{j\perp}|e^{y_i-y_j},\qquad 3\le i<j\le L, \tag{5.3}$$

where y_i and $k_{i\perp}$ are the rapidity and transverse momentum of parton i, respectively. Furthermore, given that the separation between all pairs of consecutive final state particles is asymptotically approaching infinity, this suggests that one may identify the common ultraviolet cutoff (motivated by the $2\to2$ case) with the impact parameter $\vec{z}$ corresponding to the distance of closest approach of the incoming particles. In any case, different cutoff choices will not affect the infrared behavior, only contributing additional logarithms in the infrared finite part of Eq. (5.1).

Substituting Eq. (5.3) into Eq. (5.1), one may rewrite the latter as

$$M_L^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\vec{z}^2)^\epsilon}{2\epsilon}\left\{-\sum_{i=1}^{L-1}\sum_{j>i}|y_i-y_j|\mathbf{T}_i\cdot\mathbf{T}_j\right.$$
$$+i\pi\left(\mathbf{T}_1\cdot\mathbf{T}_2+\sum_{i=3}^{L-1}\sum_{j>i}\mathbf{T}_i\cdot\mathbf{T}_j\right)$$
$$\left.+\sum_{i=1}^{L}C_i\log\left(\frac{|k_{i\perp}|}{m}\right)\right\}, \tag{5.4}$$

where $C_i=\mathbf{T}_i^2$ is the quadratic Casimir in the representation of leg i, and we made repeated use of the color conservation equation

$$\sum_{i=1}^{L}\mathbf{T}_i=0. \tag{5.5}$$

Also in Eq. (5.4), we have introduced the (unphysical) rapidities $y_1\equiv y_3$, $y_2\equiv y_L$, in order to simplify the notation. Introducing the s-channel quadratic Casimir

$$\mathbf{T}_s^2=(\mathbf{T}_1+\mathbf{T}_2)^2=\left(\sum_{i=3}^{L}\mathbf{T}_i\right)^2, \tag{5.6}$$

one may also write Eq. (5.4) as

$$M_L^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\vec{z}^2)^\epsilon}{2\epsilon}\left\{-\sum_{i=1}^{L-1}\sum_{j>i}|y_i-y_j|\mathbf{T}_i\cdot\mathbf{T}_j\right.$$
$$\left.+i\pi\mathbf{T}_s^2+\sum_{i=1}^{L}C_i\left[\log\left(\frac{|k_{i\perp}|}{m}\right)-\frac{i\pi}{2}\right]\right\}. \tag{5.7}$$

One may now use the identity, proven in Ref. [32],

$$\sum_{i=1}^{L-1}\sum_{j>i}|y_i-y_j|\mathbf{T}_i\cdot\mathbf{T}_j=-\sum_{k=3}^{L-1}\mathbf{T}_{t_{k-2}}^2\Delta y_k, \tag{5.8}$$

where $\mathbf{T}_{t_i}$ is a quadratic Casimir operator for a given strut of the t-channel ladder in Fig. 5, and $\Delta y_k=y_k-y_{k+1}$ the associated rapidity difference. Equation (5.7) then becomes

$$M_L^{(1)} = \frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\vec{z}^2)^\epsilon}{2\epsilon}\left\{\sum_{k=3}^{L-1}\mathbf{T}_{t_{k-2}}^2\Delta y_k+i\pi\mathbf{T}_s^2\right.$$
$$\left.+\sum_{i=1}^{L}C_i\left[\log\left(\frac{|k_{i\perp}|}{m}\right)-\frac{i\pi}{2}\right]\right\}, \tag{5.9}$$

which differs from the result in Refs. [31,32] owing to the use of a mass regulator for collinear singularities adopted here. As usual, one may exponentiate the one-loop soft function to obtain

$$\exp\left\{\frac{g_s^2\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\bar{z}^2)^\epsilon}{2\epsilon}\left\{\sum_{k=3}^{L-1}\mathbf{T}^2_{t_{k-2}}\Delta y_k + i\pi\mathbf{T}_s^2 + \sum_{i=1}^{L} C_i\left[\log\left(\frac{|k_{i\perp}|}{m}\right) - \frac{i\pi}{2}\right]\right\}\right\}, \tag{5.10}$$

which acts on the hard interaction consisting of L-point scattering undressed by virtual emissions. The leading high energy behavior (corresponding to leading logarithms in rapidity) is given by the first term in the exponent acting on the hard function, and produces a tower of Reggeized gluon exchanges, each dressed by the appropriate quadratic Casimir. The analysis is in fact more general than this—even if the struts of the ladder have different exchanges, each will Reggeize separately given that the t-channel operators for different struts commute with each other [32]. Note that an eikonal phase term remains present, weighted as in the $2 \to 2$ case by the quadratic Casimir operator for s-channel exchanges. This has the same physical meaning in the present context—it is associated with the formation of s-channel bound states.

Having reviewed the QCD case, let us now return to gravity. As may be confirmed by more detailed calculation, the latter case is easily obtained from the former using the replacements of Eq. (3.7), so that the exponentiated gravitational soft function is

$$\exp\left\{-\left(\frac{\kappa}{2}\right)^2\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\bar{z}^2)^\epsilon}{2\epsilon}\left[\sum_{k=3}^{L-1} t_{k-2}\Delta y_k + i\pi s\right]\right\}. \tag{5.11}$$

Here t_k is the squared momentum transfer flowing in the kth strut of the ladder, as labeled in Fig. 5. Here one sees a similar story to the QCD case, namely the presence of both a Reggeization and an eikonal phase term. The former is now itself a series of terms, each of which Reggeizes the graviton in a given strut of the ladder. However, as in the $2 \to 2$ case, the Reggeization term in gravity involves the squared momentum transfer by virtue of its being the appropriate quadratic Casimir, and thus is kinematically subleading in the strict Regge limit of $s/|t| \to \infty$. Thus, multigraviton scattering for any number of gravitons is dominated by the eikonal phase term, hinting at the production of bound states in the s-channel.

It would be interesting to consider the impact of infrared finite corrections on this result. By analogy with the four point amplitude, one would expect additional logarithms to appear in the finite part, which disrupt the interpretation of graviton Reggeization. It is worth noting here also that the Regge limit of multiparticle scattering has been widely investigated in the context of the BDS conjecture [116], an all-order ansatz for the form of planar amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory. This conjecture is known to break down for six-point amplitudes at two loops, as first shown by considering the Regge limit in an unphysical region [120]. One might expect similar structures to occur in (super)-gravity theories, using double copy [88–90] considerations.

VI. CONCLUSIONS

In this paper, we have considered the Regge limit of gravity from a Wilson line point of view, adopting an approach first used for Abelian and non-Abelian gauge theories [29,30]. Our motivation was to provide a common way of looking at Reggeization in different theories, and to clarify the role of graviton Reggeization as presented in the literature.

The Wilson line approach reveals the presence of both an eikonal phase and a Reggeization term in the soft function at one-loop, where the former is associated with the formation of s-channel bound states due to the perturbative part of the potential. In QCD, the Reggeization term dominates at leading logarithmic order, leading to automatic Reggeization of arbitrary t-channel exchanges, as discussed in Refs. [31,32], where the Regge trajectory is purely infrared singular at one-loop order, and involves the quadratic Casimir operator associated with a given t-channel exchange. Beyond this logarithmic order, cross talk occurs between the two contributions, leading to a breakdown of simple Regge pole behavior. The situation is further complicated in QCD, even for the purely infrared singular parts of the amplitude, by the presence of corrections to the exponent of the soft function.

Our gravity calculation used the Wilson line operators of Refs. [92,93], and confirmed the presence of both an eikonal phase and Reggeization term at one-loop in gravity. Here the soft function is one-loop exact, receiving no perturbative corrections in the exponent [91–94]. The eikonal phase and Regge trajectory, as expected from the QCD calculation, contain quadratic Casimir operators associated with s- and t-channel exchanges. In the gravity case, these are the Mandelstam invariants s and t themselves, and thus one finds a particularly elegant explanation for the fact that the gravitational Regge trajectory is linear in t, and thus kinematically-subleading in the Regge limit with respect to the eikonal phase $i\pi s$. We saw that cross talk between the two contributions leads to Regge cut behavior, clarifying the QCD discussion of Refs. [31,32]. We also examined Reggeization in multigraviton scattering, finding again that graviton Reggeization is subdominant with respect to the eikonal phase. The story of Reggeization in gravity is further complicated, even at one-loop order, by the presence of IR-finite $\log^2 s$ contributions. Such double-log terms arise in explicit one-loop calculations in $\mathcal{N} = 5$,

$\mathcal{N}=6$, and $\mathcal{N}=8$ supergravity [85]. Although these terms are kinematically-suppressed with respect to the eikonal phase term, they are of the same order as the Reggeization term and therefore mix with the Reggeization of the graviton.

Using known results for the two-loop amplitude in in $\mathcal{N}=4$, $\mathcal{N}=5$, $\mathcal{N}=6$, and $\mathcal{N}=8$ supergravity, we computed the Regge limit of the two-loop contribution to the logarithm of the amplitude, which measures the failure of the one-loop result to exponentiate. These correction terms are of $\mathcal{O}(st)$ and therefore kinematically-subleading with respect to the $\mathcal{O}(s^2)$ exponentiation of the eikonal phase (and hence vanish in the strict Regge limit). They are, however, kinematically-superleading with respect to the $\mathcal{O}(t^2)$ exponentiation of the one-loop Reggeization term and also with respect to the $\mathcal{O}(t^2\log^4 s)$ terms computed in Ref. [85].

Although the strict Regge limit of the two-loop amplitude was shown to be one-loop-exact for $\mathcal{N}\geq 4$ supergravity, it remains an open question whether this continues to hold at three loops and beyond, i.e. whether the strict Regge limit of the L-loop result is given by the exponential of the one-loop eikonal phase. The resolution of this question awaits the evaluation of the contributing nonplanar integrals.

It is fair to say that the higher-loop contributions to the Regge limit of gravity are still not fully understood. Investigation of these contributions in more detail may shed light on a number of unresolved issues in quantum gravity, including issues of black hole formation and unitarity (see e.g. Refs. [75,82,83] and references therein).

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APPENDIX A: WILSON LINES AND SHOCKWAVES

The authors of Ref. [86] examined the Regge limit of quantum gravity from the point of view of the double copy procedure of Refs. [88–90], which posits that gravitational scattering amplitudes can be obtained from gauge theory counterparts, by replacements of kinematic numerators by color factors (together with relevant coupling constants). That paper also pointed out that shockwave solutions—namely gauge field configurations corresponding to a single massless particle—can also be related by the double copy. In this appendix, we briefly state how shockwaves are connected to the Wilson line language.

Consider first a QED Wilson line operator

$$\exp\left[ie\int dx^\mu A_\mu(x)\right], \tag{A1}$$

with the contour chosen to be the classical straight-line trajectory of a hard emitting particle

$$x^\mu = u^\mu\tau, \tag{A2}$$

where τ is a parameter along the contour (with units of length), and $u^\mu = p^\mu/E$ the 4-velocity of a massless particle with energy E. We may rewrite Eq. (A1) in terms of a current density sourcing the gauge field, by introducing a three-dimensional delta function as follows:

$$\exp\left[ie\int dx^\mu A_\mu(x)\right] = \exp\left[ieu^\mu\int d^4x\,\delta^{(3)}(\vec{x})A_\mu(x)\right] \equiv \exp\left[-i\int d^4x\, j^\mu(x)A_\mu(x)\right], \tag{A3}$$

where

$$\delta^{(3)}(\vec{x}) = \delta(z-t)\delta(x)\delta(y), \tag{A4}$$

and without loss of generality we have taken the Wilson line to be in the $+z$ direction. We then see that the current due to the Wilson line operator is

$$j^\mu = -eu^\mu\delta(z-t)\delta(x)\delta(y). \tag{A5}$$

As pointed out in Ref. [86], this is precisely the source that gives rise to a QED shockwave, upon solving the field equations for the gauge field $A_\mu(x)$.

A similar argument may be made for gravity, and one starts by rewriting the Wilson line operator of Eq. (3.1) as[10]

$$\begin{aligned}\exp&\left[i\frac{\kappa}{2}p^\mu\int dx^\mu h_{\mu\nu}(x)\right]\\ &= \exp\left[i\frac{\kappa}{2}Eu^\mu u^\nu\int d^4x\,\delta^{(3)}(\vec{x})h_{\mu\nu}(x)\right]\\ &= \exp\left[-i\int d^4x\, j^{\mu\nu}(x)h_{\mu\nu}(x)\right].\end{aligned} \tag{A6}$$

We recognize the source current in this case as

$$j^{\mu\nu}(x) = -\frac{\kappa}{2}Eu^\mu u^\nu\delta(z-t)\delta(x)\delta(y) \equiv -\frac{\kappa}{2}T^{\mu\nu}, \tag{A7}$$

[10]Strictly speaking, in gravity there should be factors of $\sqrt{-g}$ in the volume measure, where g is the determinant of the metric tensor. However, these can be ignored in Rq. (A6) due to the fact that we are only expanding to first order in the graviton field.

where we have introduced the conventional energy-momentum tensor in the last term. This can be recognized as the energy-momentum tensor for a massless particle quoted in Ref. [86], so that solution of the field equations for $h_{\mu\nu}(x)$ gives the Aichelberg-Sexl (shock-wave) metric.

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