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Complete spectral analysis of the Jackiw-Rebbi model, including its zero mode

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In this paper we present a complete and exact spectral analysis of the $(1+1)$-dimensional model that Jackiw and Rebbi considered to show that the half-integral fermion numbers are possible due to the presence of an isolated self-charge-conjugate zero mode. The model possesses the charge and particle conjugation symmetries. These symmetries mandate the reflection symmetry of the spectrum about the line $E=0$. We obtain the bound-state energies and wave functions of the fermion in this model using two different methods, analytically and exactly, for every arbitrary choice of the parameters of the kink, i.e. its value at spatial infinity (θ_0) and its scale of variations (μ). Then, we plot the bound-state energies of the fermion as a function of θ_0. This graph enables us to consider a process of building up the kink from the trivial vacuum. We can then determine the origin and evolution of the bound-state energy levels during this process. We see that the model has a dynamical mass generation process at the first quantized level and the zero-energy fermionic mode responsible for the fractional fermion number, is always present during the construction of the kink and its origin is very peculiar, indeed. We also observe that, as expected, none of the energy levels cross one another. Moreover, we obtain analytically the continuum scattering wave functions of the fermion and then calculate the phase shifts of these wave functions. Using the information contained in the graphs of the phase shifts and the bound states, we show that our phase shifts are consistent with the weak and strong forms of the Levinson theorem. Finally, using the weak form of the Levinson theorem, we confirm that the number of the zero-energy fermionic modes is exactly one.

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I. INTRODUCTION

It is known that the modification of the spectrum of the Fermi field due to its coupling to other field configurations influences many properties of the system and causes many interesting phenomena. One of these phenomena is the appearance of a nonzero and even noninteger fermion number of the vacuum. The occurrence of the fractional fermion number was first pointed out by Jackiw and Rebbi in 1976 [1]. They considered some models possessing the charge conjugation symmetry, in which the fermion is coupled to a scalar background field in the form of a soliton. They concluded that the existence of an isolated nondegenerate zero-energy fermionic mode implies that the soliton is a degenerate doublet carrying fermion number $\pm\frac{1}{2}$. Their surprising result has motivated much of the works on this subject. This effect has been studied extensively in the literature for different physical models in many branches of physics such as particle physics [1–8], cosmology [9–13], condensed matter physics [14–17], polymer physics [18–20] and atomic physics [21–23].

In the early 1980s two systematic and elegant methods were developed as a way to evaluate the vacuum polarization of the fermions induced by the presence of prescribed static background fields. The first method called the adiabatic method, invented by Goldstone and Wilczek [3], basically consists of building up adiabatically the final configuration of the background field starting from the free vacuum. Then, the charge of the final state can be obtained by observing the fermionic current at spatial infinity, computed from the lowest-order Feynman loop diagram. This method is limited to the slowly varying background fields. For infinitely slow variations of the background, none of the bound states of the fermion crosses the line of $E=0$. Consequently, there exists only one contribution to the vacuum polarization, i.e. the adiabatic contribution. This contribution can be attributed to the change in the number of continuum states with negative energy and is responsible for the fractional part of the fermion number.

The second method was invented by MacKenzie and Wilczek [4,5], and the restriction of adiabaticity is lifted. To calculate the charge of the no-particle state in the presence of an interaction, one starts with the definition of the particle number operator in the free Dirac case and transforms it into the one which is appropriate for the basis of states in the presence of the interaction. Then, the vacuum charge turns out to be equal to the difference between the number of negative-energy states in the presence and absence of the background field. One can then return back to the first quantized level and solve for the spectrum of the Dirac field in the presence of the background field. As we couple the Dirac field to an external potential, the Dirac

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equation is altered and the spectrum of the fermion is distorted. The positive and negative continua change and bound states may appear. When the scale of spatial variation of the background field is much larger than the Compton wavelength of the fermion (λ), i.e. the adiabatic regime, the results of the two methods coincide. However, when the aforementioned scale becomes comparable to or smaller than λ, the bound-state energy levels could cross the $E = 0$ line, and the definition of the vacuum changes. This contribution to the vacuum polarization is called the nonadiabatic contribution. In this sense the second method generalizes the first.

In this paper we concentrate on the simple but important model considered by Jackiw and Rebbi. In this model a Fermi field is coupled to an external scalar field in the form of a kink in $(1 + 1)$ dimensions. This system has charge conjugation and particle conjugation symmetries. These symmetries relate each state of the fermion with positive energy E to a state with the energy $-E$. Therefore, the whole spectrum of the system is completely symmetric with respect to the line of $E = 0$. Jackiw and Rebbi stated that there is an isolated zero-energy fermionic mode in this system, which is self-charge-conjugate and showed that the vacuum polarization of the system due to the presence of this mode is $\pm\frac{1}{2}$. They have obtained the exact form of the zero mode [1,2]. Before these works, the same model had been discussed by Dashen *et al.* in 1974 [24] in a different context and with a different purpose. In that paper, they extended the semiclassical method to include fermions. They also pointed out the existence of the zero-energy fermionic mode and obtained an expression for the discrete bound levels. Later on Rajaraman gave the same expression for the bound-state energies of the Jackiw-Rebbi model while reviewing the previous works [25]. However, as far as we know, an exact solution for the whole spectrum of this model, which was introduced four decades ago has been heretofore missing. In this paper we calculate the exact spectrum of the system, i.e. the eigenstates in the continua and all of the bound eigenstates and their energies, for the whole allowed ranges of the parameters, i.e. θ_0 which denotes the value of the kink at $x \to \infty$ and μ the slope at $x = 0$. Having such solutions, we can explore what exactly happens to the spectrum of the fermion as the background field evolves from the trivial vacuum to the kink. From this evolution perspective, the origin of an isolated zero mode in a system with particle conjugation symmetry has been a mystery to us. The symmetries of the system clearly mandate the zero mode to be self-charge-conjugate, but disallow any levels crossing $E = 0$ during the evolution process, and this obfuscates the mystery even further. When this analysis illuminates the origin of this zero mode, some of its unexpected features become manifest. To this end, we obtain the bound states of the fermion for this system using two different analytical methods. In the first method we solve the equations of motion, directly. Our second method is the elegant shape invariance method [26–32]. Shape invariance indicates the presence of an integrability condition for the potential of the Schrödinger-like equations. When such a condition is satisfied for the potential, one can use the supersymmetry algebra to exactly solve the Schrödinger-like equation and obtain the stationary states and their corresponding energy eigenvalues. Our second-order equations, obtained by decoupling the two first-order equations embedded in the Dirac equation, turn out to be two Schrödinger-like equations which are surprisingly partner Hamiltonians and related to each other through a shape invariance condition. Therefore, we are able to easily obtain the solutions for our bound states using the shape invariance method. We plot the bound-state energies of the fermion as a function of θ_0. The invariance of the system under the charge and particle conjugation symmetries is obvious in this graph. We also obtain the continuum scattering states of the fermion by solving the equations of motion directly, and then calculate the phase shift of these states. One of our observations is that there is a dynamical mass-generation process operating at the first quantized level. We check the consistency of our results with both the weak and strong forms of the Levinson theorem [33,34], and use those two forms to gain further insight into the fermionic spectrum.

The outline of the paper is as follows. In Sec. II we introduce the model. In Sec. III we obtain the bound states of the fermion using two different methods. In Sec. IV we compute the continuum wave functions of the fermion and then calculate the phase shift of these states. Then, we check the consistency of our results with both the weak and strong forms of the Levinson theorem. In Sec. V we summarize and discuss the results and draw some conclusions.

II. THE PRELIMINARIES OF THE JACKIW-REBBI MODEL

In this section we review the basic definitions of the $(1 + 1)$-dimensional model studied by Jackiw and Rebbi [1]. This model includes a spinor field ψ coupled to a scalar field φ through the following Lagrangian:

$$\mathcal{L} = \bar{\psi}[i\gamma^\mu\partial_\mu - g\phi_{\rm cl}(x)]\psi, \tag{1}$$

where $g > 0$ and $\varphi_{\rm cl}(x)$ is a prescribed pseudoscalar field and chosen to be the kink of the φ^4 theory, i.e. $\phi_{\rm cl}(x) = (m/\sqrt{\lambda})\tanh(mx/\sqrt{2})$. Two important parameters which describe the kink are $\theta_0 = \phi_{\rm cl}(\infty) = \frac{m}{\sqrt{\lambda}}$ and $\mu = \frac{d\varphi_{\rm cl}}{dx}|_{x=0} = \frac{m^2}{\sqrt{2\lambda}}$. Notice that the Lagrangian has no explicit fermion mass term and the mass of the free fermion is obviously zero. However, as we shall show, the interaction term of this Lagrangian gives the mass $M_{\rm f} = |g\langle\phi_{\rm cl}\rangle| = g\frac{m}{\sqrt{\lambda}} = g\theta_0$ to the fermion, to lowest order.

We use the representation $\gamma^0 = \sigma_1$ and $\gamma^1 = i\sigma_3$ for the Dirac matrices and represent the Fermi field by

$$\psi(x,t) = \mathrm{e}^{-iEt}\begin{pmatrix}\psi^{(+)}(x)\\ \psi^{(-)}(x)\end{pmatrix}.$$

Then, the Dirac equation in the presence of the background field $\varphi_{\mathrm{cl}}(x)$ can be written as follows:

$$\begin{pmatrix}-\partial_x - g\phi_{\mathrm{cl}}(x) & E\\ E & \partial_x - g\phi_{\mathrm{cl}}(x)\end{pmatrix}\begin{pmatrix}\psi^{(+)}(x)\\ \psi^{(-)}(x)\end{pmatrix} = 0. \quad (2)$$

Our purpose is to solve this equation exactly and explore the results in detail. Before doing that, we state some important symmetries of this model. It possesses the charge conjugation symmetry. In our representation the charge conjugation operator includes σ_3 and it relates the states with positive energy to the ones with negative energy as $\psi^c_{-E} = \sigma_3\psi^*_E$. Also, if there is a zero-energy fermionic mode, it is self-conjugate, i.e. $\psi^c_0 = \sigma_3\psi^*_0 = \psi_0$. This model also possesses particle conjugation symmetry. In our representation the particle conjugation operation is given by $\psi_{-E} = \sigma_3\psi_E$. Therefore, for every state with positive energy E there exists a corresponding state with energy $-E$. The chosen model is not invariant under the parity, since the background field $\varphi_{\mathrm{cl}}(x)$ is the kink which is an odd function in space. Therefore, this model does not preserve the CP symmetry and consequently it is not invariant under the time reversal.

III. BOUND STATES OF THE FERMION IN THE PRESENCE OF THE BACKGROUND FIELD

In this section we solve Eq. (2) to find the wave functions of the bound states along with their associated discrete energies. First, we solve the equations of motion directly. Then, we solve the equations using the formalism of shape invariance, as a double-check. For an alternative derivation of the bound states see Ref. [35].

A. The direct method

Equation (2) consists of two coupled first-order differential equations. We first solve the two decoupled second-order equations and then look for a set of solutions which are consistent with the original Dirac equation (2). The decoupled equations are

$$\frac{\mathrm{d}^2\psi^{(\pm)}(x')}{\mathrm{d}x'^2} + [\epsilon_\pm - v_\pm\tanh^2(x')]\psi^{(\pm)}(x') = 0, \quad (3)$$

where we have rescaled the original parameters of the model as follows: $x' = (\mu/\theta_0)x$, $g' = (\theta_0^2/\mu)g$ and $E' = (\theta_0/\mu)E$. We have also defined two new parameters as follows: $\epsilon_\pm = (E'^\pm)^2 \pm g'$ and $v_\pm = g'^2 \pm g'$. The solutions to these two Schrödinger-like equations can be inferred from some old literature [36,37]. However, we present a very short derivation of the solutions, which we shall later combine to obtain the solution to the original Dirac equation. When $\epsilon_\pm < v_\pm$, these equations have solutions vanishing at spatial infinities and consequently for this range of parameters we can have bound states. However, for $\epsilon_\pm > v_\pm$ the continuum solutions which are oscillatory at spatial infinities are possible. To solve these equations, we substitute the ansatz $\psi^{(\pm)}(x') = \mathrm{sech}^{b_\pm}(x')F_\pm(x')$ into Eq. (3) and obtain

$$\mathrm{sech}^{b_\pm}(x')\left\{\frac{\mathrm{d}^2F_\pm(x')}{\mathrm{d}x'^2} - 2b_\pm\tanh(x')\frac{\mathrm{d}F_\pm(x')}{\mathrm{d}x'} + [\epsilon_\pm - v_\pm + b_\pm^2 + (v_\pm - b_\pm(b_\pm+1))\mathrm{sech}^2(x')]F_\pm(x')\right\} = 0. \quad (4)$$

In this equation the terms in the curly brackets should add up to zero. By choosing the arbitrary parameters $b_\pm$ such that $\epsilon_\pm - v_\pm + b_\pm^2 = 0$, and by using a change of variable $u = \frac{1}{2}[1 - \tanh(x')]$, the differential equations for $F_\pm(x')$ turn into a hypergeometric equation with the following general solution:

$$A\,{}_2F_1\left(b_\pm + \frac{1}{2} - \sqrt{v_\pm + \frac{1}{4}}, b_\pm + \frac{1}{2} + \sqrt{v_\pm + \frac{1}{4}}, 1 + b_\pm; u\right) + Bu^{-b_\pm}\,{}_2F_1\left(\frac{1}{2} - \sqrt{v_\pm + \frac{1}{4}}, \frac{1}{2} + \sqrt{v_\pm + \frac{1}{4}}, 1 - b_\pm; u\right). \quad (5)$$

For the bound states we set $b_\pm > 0$ to turn $\mathrm{sech}^{b_\pm}(x')$ into a damping factor. However, since $b_\pm > 0$, $\lim_{u\to 0}u^{-b_\pm} = \infty$ and we have to set $B = 0$. In order that the remaining solution have the proper asymptotic behavior, we have to impose the following constraint:

$$b_\pm + \frac{1}{2} - \sqrt{v_\pm + \frac{1}{4}} = -n, \quad (6)$$

where n is a semipositive integer. This constraint along with the constraint $\epsilon_\pm - v_\pm + b_\pm^2 = 0$ determine the allowed discrete energies of the system. Using these two constraints, and the definitions of $v_\pm$ and $\epsilon_\pm$, the allowed energies in terms of the original parameters θ_0 and μ are as follows:

$$E_n^+ = \pm\sqrt{2g\mu n - \frac{\mu^2}{\theta_0^2}n^2}, \quad n = 0,1,2,\ldots < \frac{g\theta_0^2}{\mu}, \quad (7)$$

$$E_n^- = \pm\sqrt{2g\mu n - \frac{\mu^2}{\theta_0^2}n^2}, \quad n = 1, 2, \ldots < \frac{g\theta_0^2}{\mu}. \tag{8}$$

We have reverted to the original parametrization since it is necessary for explaining the origin of the zero mode. The upper bounds for the integer n have been obtained using the constraint $b_\pm > 0$. The corresponding wave functions are

$$\psi_n^{(+)}(x) = N_+\left[\operatorname{sech}\left(\frac{\mu}{\theta_0}x\right)\right]^{\frac{g\theta_0^2}{\mu}-n} {}_2F_1\left(-n, 2\frac{g\theta_0^2}{\mu} - n + 1, \frac{g\theta_0^2}{\mu} - n + 1; \frac{1}{1+e^{\frac{2\mu}{\theta_0}x}}\right), \tag{9}$$

$$\psi_n^{(-)}(x) = N_-\left[\operatorname{sech}\left(\frac{\mu}{\theta_0}x\right)\right]^{\frac{g\theta_0^2}{\mu}-n} {}_2F_1\left(-n+1, 2\frac{g\theta_0^2}{\mu} - n, \frac{g\theta_0^2}{\mu} - n + 1; \frac{1}{1+e^{\frac{2\mu}{\theta_0}x}}\right). \tag{10}$$

Now, we use these solutions to construct the solutions of the original Dirac coupled first-order differential equations (2). It is important to note that the energy E of both of these solutions for any state should be the same, since E is the energy of the fermion and its wave function is the doublet

$$e^{-iEt}\begin{pmatrix}\psi^{(+)}(x)\\ \psi^{(-)}(x)\end{pmatrix}.$$

We define $E_n = E_n^+ = E_n^-$ and choose the wave function of the fermion to be in the following form:

$$\psi_n(x,t) = e^{-iE_nt}\begin{pmatrix}\psi_n^{(+)}(x)\\ \psi_n^{(-)}(x)\end{pmatrix}, \quad n = 0, 1, 2, \ldots < \frac{g\theta_0^2}{\mu}. \tag{11}$$

One can easily check that the solutions given by Eq. (11) satisfy the first-order Eq. (2), if we set $N_-/N_+ = n\mu/(\theta_0 E_n)$. Therefore, the total number of the bound states for a given value of θ_0 is $N_b = 2[\frac{g\theta_0^2}{\mu}]_F + 1$, where $[f(\theta_0)]_F$ denotes the floor function.

Now let us concentrate on the zero-energy mode since this is the most important bound state. For the $E = 0$ mode, $n = 0$ and only the upper component is nonzero. We can easily extract the explicit form of the zero mode from our exact solution given in Eqs. (7)–(11). Due to the importance of the zero mode, let us find it by a second method which we choose to be a direct calculation based on the Dirac equation (2) (see Ref. [25]). Setting $E = 0$ in the first-order Eq. (2), the two equations decouple and their solutions are as follows:

$$\psi_0^{(+)}(x) = c_+\left[\cosh\left(\frac{\mu}{\theta_0}x\right)\right]^{-\frac{g\theta_0^2}{\mu}},$$
$$\psi_0^{(-)}(x) = c_-\left[\cosh\left(\frac{\mu}{\theta_0}x\right)\right]^{\frac{g\theta_0^2}{\mu}}, \tag{12}$$

where c_+ and c_- are constant. Since $\psi_0^{(-)}(x)$ makes the fermion wave function for the zero-energy mode unnormalizable, we set $c_- = 0$. Therefore, the wave function for this mode is

$$\psi_0(x) = c_+\begin{pmatrix}\left[\cosh\left(\frac{\mu}{\theta_0}x\right)\right]^{-\frac{g\theta_0^2}{\mu}}\\ 0\end{pmatrix}. \tag{13}$$

We should mention that the asymptotic behavior of all of the bound states can be easily obtained from Eq. (3) and are as follows:

$$\lim_{|x|\to\infty}\psi_n^{(\pm)}(x) = e^{-b_n^\pm\frac{\mu}{\theta_0}|x|}. \tag{14}$$

This asymptotic behavior exactly matches the corresponding behavior of our exact solutions given in Eqs. (7)–(11).

B. The shape invariance method

In this subsection we use the shape invariance method to rederive the bound-state sector of this model. Since there are many excellent reviews on the shape invariance method (see for example Ref. [28]), we shall only use the results, with minimal introduction. Consider a sequence of Hamiltonians which are related to each other by the shape invariance condition,

$$H_k(g_1) = H_1(g_k) + \sum_{s=1}^{k-1} R(g_s), \quad k = 2, 3, \ldots, p, \tag{15}$$

where p is the number of bound states of H_1, $R(g_s)$ is a c function depending on the parameters of the Hamiltonians, and $g_s = f^{s-1}(g_1)$. The Hamiltonian H_1 and its ground state are required to have the following properties: $H_1(g) = A^\dagger(g)A(g)$, and $A(g_1)\psi_0^{(1)}(g_1) = 0$. The nth Hamiltonian H_n in this sequence has the same spectrum as H_1 except that the first $n-1$ bound states of H_1 are

absent in the spectrum of H_n. Using this relation, it is obvious that the ground state of the Hamiltonian H_k is the kth energy level of H_1. The bound-state energies of H_1 are as follows:

$$E_n^{(1)} = \sum_{s=1}^{n} R(g_s), \quad n = 1, 2, \ldots, p, \quad \text{and} \quad E_0^{(1)} = 0. \tag{16}$$

Moreover, the nth eigenstate of H_1 is as follows:

$$\psi_n^{(1)}(g_1) \propto A^\dagger(g_1)A^\dagger(g_2)\ldots A^\dagger(g_n)\psi_0^{(1)}(g_{n+1}), \quad n = 1, 2, \ldots, p. \tag{17}$$

Now, we use the shape invariance method to solve the second-order Schrödinger-like equations (3). We can write the partner Hamiltonians for our equations, in terms of the parameters μ and θ_0, as follows:

$$H_+ = A^\dagger A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g^2\theta_0^2 - g\theta_0\left(g\theta_0 + \frac{\mu}{\theta_0}\right)\mathrm{sech}^2\left(\frac{\mu}{\theta_0}x\right), \tag{18}$$

$$H_- = AA^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g^2\theta_0^2 - g\theta_0\left(g\theta_0 - \frac{\mu}{\theta_0}\right)\mathrm{sech}^2\left(\frac{\mu}{\theta_0}x\right), \tag{19}$$

where, the lowering and raising operators which build H_+ and H_- are, respectively,

$$A = -\frac{\mathrm{d}}{\mathrm{d}x} - g\theta_0 \tanh\left(\frac{\mu}{\theta_0}x\right), \tag{20}$$

$$A^\dagger = \frac{\mathrm{d}}{\mathrm{d}x} - g\theta_0 \tanh\left(\frac{\mu}{\theta_0}x\right). \tag{21}$$

It can be easily seen that the Hamiltonians (18) and (19) satisfy the following shape invariance condition:

$$H_-(g) = H_+\left(g - \frac{\mu}{\theta_0^2}\right) + \left(2g\mu - \frac{\mu^2}{\theta_0^2}\right). \tag{22}$$

Thus, in our case $f(g) = g - \frac{\mu}{\theta_0^2}$ and $R(g) = 2g\mu - \frac{\mu^2}{\theta_0^2}$. Using these two functions and Eq. (16), the energies of the bound states of H_+ are as follows:

$$E_n^+ = \pm\sqrt{2ng\mu - n^2\frac{\mu^2}{\theta_0^2}}, \quad n = 0, 1, 2, \ldots. \tag{23}$$

Substituting Eqs. (13) and (21) into Eq. (17), the bound-state wave functions of the Hamiltonian H_+ can be easily obtained. The ground state $\psi_0^{(+)}$ is already shown in Eq. (13) (upper component) and the first two excited states are as follows:

$$\psi_1^{(+)}(x) \propto \sinh\left(\frac{\mu}{\theta_0}x\right)\left[\cosh\left(\frac{\mu}{\theta_0}x\right)\right]^{-\frac{g\theta_0^2}{\mu}},$$
$$\psi_2^{(+)}(x) \propto \left[g\theta_0^2 + (\mu - g\theta_0^2)\cosh\left(\frac{2\mu}{\theta_0}x\right)\right]\left[\cosh\left(\frac{\mu}{\theta_0}x\right)\right]^{-\frac{g\theta_0^2}{\mu}}. \tag{24}$$

As it can be seen, these energies and states are exactly the ones we obtained in the previous subsection.

IV. CONTINUUM SCATTERING STATES OF THE FERMION IN THE PRESENCE OF THE BACKGROUND FIELD

Now, we focus our attention on obtaining the continuum scattering states for this model. As in the case of bound states, we choose the continuum wave functions to be in the form of $\psi_k^{(\pm)}(x) = \mathrm{sech}^{b_\pm}(\mu x/\theta_0)F_\pm(x)$. From the constraint $\epsilon_\pm - v_\pm + \frac{\mu^2}{\theta_0^2}b_\pm^2 = 0$, we have $b_\pm = \sqrt{\frac{\theta_0^2}{\mu^2}(v_\pm - \epsilon_\pm)}$. To have oscillatory behavior in spatial infinities for the continuum states, we set $b_\pm = -ik$ in which k is a real quantity. This choice corresponds to the region $\epsilon_\pm > v_\pm$. Using the definitions of the parameters $v_\pm$ and $\epsilon_\pm$ in terms of θ_0 and μ, we obtain $E^2 = \frac{\mu^2}{\theta_0^2}k^2 + g^2\theta_0^2$. This equation reveals an extremely important property of this model, namely dynamical mass generation at the tree level. This mass is given by $M_\mathrm{f} = g\theta_0$. We shall comment further on the properties of the spectrum of this model in Sec. IV A.

The continuum states of the fermion are as follows:

$$\psi_{k,\mathrm{L}}^{(\pm)}(x) = N_\pm^{k,\mathrm{L}}\cosh^{ik}\left(\frac{\mu}{\theta_0}x\right){}_2F_1\left(\frac{1}{2} - ik - \zeta_\pm, \frac{1}{2} - ik + \zeta_\pm, 1 - ik; \frac{1}{1 + \mathrm{e}^{\frac{2\mu}{\theta_0}x}}\right), \tag{25}$$

$$\psi^{(\pm)}_{k,\mathrm{R}}(x) = N^{k,\mathrm{R}}_{\pm}\cosh^{ik}\left(\frac{\mu}{\theta_0}x\right){}_2F_1\left(\frac{1}{2}-ik-\zeta_{\pm},\frac{1}{2}-ik+\zeta_{\pm},1-ik;\frac{1}{1+\mathrm{e}^{-\frac{2\mu}{\theta_0}x}}\right), \tag{26}$$

where $\zeta_{\pm}=\frac{g\theta_0^2}{\mu}\pm\frac{1}{2}$, and $N^{k,\mathrm{L}}_{\pm}$ and $N^{k,\mathrm{R}}_{\pm}$ are the normalization factors for the continuum states. We can easily check that the spinor constructed from solutions of the decoupled equations presented in Eqs. (25) or (26), with the same value of k, satisfies the original (coupled) Dirac equation (2) if we set $N^{k,\mathrm{L}}_{-}/N^{k,\mathrm{L}}_{+}=-N^{k,\mathrm{R}}_{-}/N^{k,\mathrm{R}}_{+}=(i\frac{\mu}{\theta_0}k+g\theta_0)/\sqrt{\frac{\mu^2}{\theta_0^2}k^2+g^2\theta_0^2}$. The asymptotic behavior of these wave functions at spatial infinities is as follows:

$$\psi^{(\pm)}_{k,\mathrm{L}}(x)=\begin{cases} N^{k,\mathrm{L}}_{\pm}\Gamma(1-ik)\left[\frac{\Gamma(-ik)\mathrm{e}^{ik\frac{\mu}{\theta_0}x}}{\Gamma(\frac{1}{2}-ik+\zeta_{\pm})\Gamma(\frac{1}{2}-ik-\zeta_{\pm})}+\frac{\Gamma(ik)\mathrm{e}^{-ik\frac{\mu}{\theta_0}x}}{\Gamma(\frac{1}{2}+\zeta_{\pm})\Gamma(\frac{1}{2}-\zeta_{\pm})}\right], & \text{as } x\to-\infty,\\ N^{k,\mathrm{L}}_{\pm}\mathrm{e}^{ik\frac{\mu}{\theta_0}x}, & \text{as } x\to+\infty,\end{cases} \tag{27}$$

$$\psi^{(\pm)}_{k,\mathrm{R}}(x)=\begin{cases} N^{k,\mathrm{R}}_{\pm}\mathrm{e}^{-ik\frac{\mu}{\theta_0}x}, & \text{as } x\to-\infty,\\ N^{k,\mathrm{R}}_{\pm}\Gamma(1-ik)\left[\frac{\Gamma(ik)\mathrm{e}^{ik\frac{\mu}{\theta_0}x}}{\Gamma(\frac{1}{2}+\zeta_{\pm})\Gamma(\frac{1}{2}-\zeta_{\pm})}+\frac{\Gamma(-ik)\mathrm{e}^{-ik\frac{\mu}{\theta_0}x}}{\Gamma(\frac{1}{2}-ik+\zeta_{\pm})\Gamma(\frac{1}{2}-ik-\zeta_{\pm})}\right], & \text{as } x\to+\infty.\end{cases} \tag{28}$$

This asymptotic behavior of $\psi^{(\pm)}_{k,\mathrm{L}}(x)$ [Eq. (25)], shown in Eq. (27) represents an incident wave for $x\to-\infty$ moving to the right ($\mathrm{e}^{ik\frac{\mu}{\theta_0}x}$), a reflected wave for $x\to-\infty$ moving back to the left ($\mathrm{e}^{-ik\frac{\mu}{\theta_0}x}$) and a transmitted wave for $x\to+\infty$ moving to the right ($\mathrm{e}^{ik\frac{\mu}{\theta_0}x}$). As it is apparent from Eq. (28), the solution $\psi^{(\pm)}_{k,\mathrm{R}}(x)$ given in Eq. (26) represents the opposite scattering process. By the use of these asymptotic behaviors, we can easily obtain the scattering phase shift for this model. Having obtained the phase shifts, we can use the strong form of the Levinson theorem for a double-check on our calculations and the weak form of the Levinson theorem as a counter for the bound states. In particular, we concentrate on the ever important zero mode.

A. Phase shift and the Levinson theorem

In this section we first find the phase shift for the scattering process embedded in Eqs. (25)–(28). To this end, we divide the coefficient of the transmitted wave by the coefficient of the incident wave and obtain the scattering matrix element which is related to the phase shift by the relation $S(k)=\mathrm{e}^{i\delta(k)}$. Using Eq. (27) or Eq. (28), the scattering matrix element of the fermion is as follows:

$$S_{\pm}(k)=\frac{\Gamma(\frac{1}{2}+\zeta_{\pm}-ik)\Gamma(\frac{1}{2}-\zeta_{\pm}-ik)}{\Gamma(-ik)\Gamma(1-ik)}=\mathrm{e}^{i\delta_{\pm}(k)}, \tag{29}$$

where as before $\pm$ signs refer to the upper and lower components of the continuum wave function of the fermion, respectively. As it can be seen, the S matrix is ambiguous and the upper and lower components of the fermion wave function have different phase shifts. We define the phase shift of the state to be the average of the phase shifts of the two components of the fermion wave function [34]. We shall show that this phase shift has all of the expected properties. Also, notice that the S matrix of our model and consequently the phase shifts of the fermion wave function are independent of the sign of the fermion energy, due to the symmetries of the system. Therefore, the phase shifts of the states in the Dirac sea and sky with the same value of $|E|$ are the same.

Let us plot some examples of the phase shift of the fermion and explore them in connection with the Levinson theorem. We concentrate on the case with the parameters $\mu=2.5$ and $g=2.15$ as the value of θ_0 is increased from zero. It is important to investigate the change in the values of the phase shifts at the boundaries of the continua, i.e. $\delta^{\mathrm{sky}}(0)$ and $\delta^{\mathrm{sea}}(0)$, when bound states appear. In the left panel of Fig. 1 we show the phase shifts as a function of k for $\theta_0=\{0.272,0.343242,0.414\}\pi$. We draw the phase shifts for these three values of θ_0 with solid, dashed and dot-dashed lines, respectively. In the right panel of this figure we show the fermion bound states as a function of θ_0 and indicate the same three values of θ_0 with solid, dashed and dot-dashed lines. As it can be seen, there are two threshold bound states (with $n=1$) at $\theta_0=0.343242\pi$. At $\theta_0=0.272\pi$ there is only the zero-energy bound state and at $\theta_0=0.414\pi$ there are two additional bound states.

We first utilize the weak form of the Levinson theorem for counting the exact number of the zero-energy bound states. The weak form of this theorem for the Dirac equation can be written as follows (see for example Ref. [34]):

$$\begin{aligned}\Delta\delta &\equiv [\delta^{\mathrm{sky}}(0)-\delta^{\mathrm{sky}}(\infty)]+[\delta^{\mathrm{sea}}(0)-\delta^{\mathrm{sea}}(\infty)]\\ &=\Delta\delta^{\mathrm{sky}}+\Delta\delta^{\mathrm{sea}}=\left(N+\frac{N_{\mathrm{t}}}{2}-\frac{N^0_{\mathrm{t}}}{2}\right)\pi.\end{aligned} \tag{30}$$

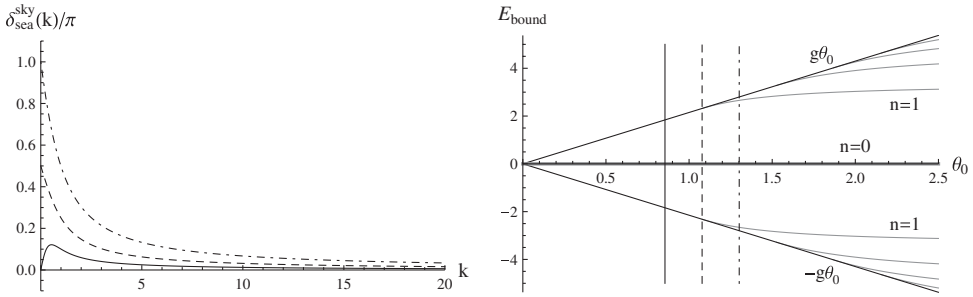

FIG. 1. Left panel: The graphical representation of $\delta^{sky}_{sea}(k)/\pi$ as a function of k. In these graphs $g = 2.15$ and $\mu = 2.5$, and the values of θ_0 for solid, dashed and dot-dashed lines are 0.272π, 0.343242π and 0.414π, respectively. Right panel: The bound states of the fermion as a function of θ_0. The three values of θ_0 are shown by the solid, dashed and dot-dashed lines.

Here N is the total number of bound states, N_t is the total number of threshold bound states at the given strength of the potential and N_t^0 is the total number of threshold bound states at the zero strength of the potential. In fact the last term takes into account the two threshold half bound states which exist for a free Dirac field in one spatial dimension. Therefore, in this model $N_t^0 = 2$. Now, we check this theorem for the sample phase shifts drawn in Fig. 1. As shown in the left graph of this figure, for the phase shift at $\theta_0 = 0.272\pi$, depicted by the solid line, we have $\Delta\delta^{sea}/\pi = \Delta\delta^{sky}/\pi = 0$. From the right graph we can see that $N_t = 0$ at $\theta_0 = 0.272\pi$, and since $N_t^0 = 2$, the Levinson theorem predicts only one bound state which is in fact the nondegenerate zero-energy bound state. By considering the phase shift at the other values of θ_0, we conclude the same result.

Now, we briefly explain the strong form of the Levinson theorem which deals with the value of the phase shift at $k = 0$ $(E = \pm M_f)$ and $k \to \infty$ $(E \to \pm\infty)$, separately [34]. For $k = 0$ this theorem can be expressed in the following form:

$$\delta(0) = (N_{exit} - N_{enter})\pi. \tag{31}$$

That is the value of the phase shift at zero momentum for each continuum is equal to the number of the bound states that exit minus the number of the bound states that enter that continuum from $E = \pm M_f$, as the strength of the potential is increased from zero to its final value. We can easily see that the sample phase shifts shown in Fig. 1 are consistent with the relation (31). For example, at $\theta_0 = 0.272\pi$ the value of the phase shifts at zero momentum $(E = \pm M_f)$ are 0. On the other hand, from the right panel of Fig. 1 we see that the total number of bound states that have exited each of the continua from the line $E = \pm M_f$ is 0. We conclude that the ever-present zero-energy bound state has been formed from the union of the two threshold (half) bound states present in the free case, i.e. at $\theta_0 = 0$. At $\theta_0 = 0.343242\pi$ two threshold bound states have formed at $E = \pm M_f$ and the corresponding phase shifts are $\delta^{sky}_{sea}(0) = \frac{\pi}{2}$. For $\theta_0 = 0.404\pi$ these states have completely separated from the continua and become full bound states. Therefore, the corresponding phase shifts attain the value π. This sort of consistency can be easily seen for other phase shifts.

The strong form of the Levinson theorem for $k \to \infty$ can be written as follows:

$$\delta(\infty) = (N_{enter} - N_{exit})\pi. \tag{32}$$

This means that the value of the phase shift for $E \to \pm\infty$ is equal to the total number of the bound states that enter minus the number of the bound states that exit that continuum from $E = \pm\infty$, as the strength of the potential is increased from zero to its final value. For the generic cases where the presence of a background field does not cause the Hamiltonian to lose its Hermiticity the spectrum remains complete, and hence $\delta^{sky}(\infty) + \delta^{sea}(\infty) = 0$. Since in this model the charge and particle conjugation symmetries imply that $\delta^{sky}(k) = \delta^{sea}(k)$ for all k, we expect $\delta^{sky}(\infty) = \delta^{sea}(\infty) = 0$, and this is precisely the result shown in Fig. 1. Now we emphasize several important properties of the spectrum of the model at this point. As is evident from the right panel of Fig. 1, there are no level crossings, the spectrum is symmetric about $E = 0$, the dynamical mass generated is $M_f = g\theta_0$, and the zero mode is always present.

V. CONCLUSION

In this paper we thoroughly investigated the $(1 + 1)$-dimensional model considered by Jackiw and Rebbi, whence they have introduced the possibility of the fractional fermion number for the ground state. In this model a Fermi field is coupled to a prescribed pseudoscalar field in the form of the kink characterized by two parameters θ_0 (the value of the kink at spatial infinity) and μ (the slope of the kink at $x = 0$). We solved the equations of motion of this system analytically and exactly, and found the bound-state wave functions and energies as well as the continuum states for arbitrary choices of the parameters θ_0 and μ. Then, we plotted the bound-state energies of the fermion as a function of the parameter θ_0. Having this complete set of solutions, we considered a process in which the

background field evolves from zero to its final form. We then observed the changes in the spectrum of the fermion during this process. We found that the interaction induces a mass for the Fermi field $M_f = g\theta_0$. That is although the free theory is massless, a band gap appears as θ_0 increases. Also, the bound states start appearing with $N_b = 2[\frac{g\theta_0^2}{\mu}]_F + 1$. The invariance of the system under the charge and particle conjugation symmetry is obvious in the graph of the bound energies, since the energy levels are totally symmetric with respect to the line of $E = 0$. We focused our attention especially on the zero-energy fermionic mode and saw that this mode is always present for every choice of the parameters. In particular, we exposed the origin of the ever-present zero mode: in the free Dirac case, i.e. when $\theta_0 = 0$, there is no mass gap and two threshold zero-energy bound states separate the continua. As θ_0 increases, a mass gap appears and the two threshold bound states merge to form the zero-energy bound state. We also found the continuum scattering states of the fermion analytically and then calculated the phase shift of these wave functions. Using the weak form of the Levinson theorem, we concluded that the self-charge-conjugate zero-energy fermionic mode is a nondegenerate mode as Jackiw and Rebbi stated.

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