

Super-renormalizable or finite completion of the Starobinsky theoryFabio Briscese,^{1,2,*} Leonardo Modesto,^{3,†} and Shinji Tsujikawa^{4,‡}¹*Istituto Nazionale di Alta Matematica Francesco Severi, Gruppo Nazionale di Fisica Matematica, Città Universitaria, P.le A. Moro 5, 00185 Rome, Italy*²*Dipartimento SBAI, Sezione di Matematica, Sapienza Università di Roma, Via Antonio Scarpa 16, 00161 Rome, Italy*³*Department of Physics & Center for Field Theory and Particle Physics, Fudan University, 200433 Shanghai, China*⁴*Department of Physics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan*

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The recent Planck data of cosmic microwave background temperature anisotropies support the Starobinsky theory in which the quadratic Ricci scalar drives cosmic inflation. We build up a multidimensional quantum consistent ultraviolet completion of the model in a phenomenological “bottom-up approach.” We present the maximal class of theories compatible with unitarity and (super-)renormalizability or finiteness which reduces to the Starobinsky theory in the low-energy limit. The outcome is a maximal extension of the Krasnikov-Tomboulis-Modesto theory including an extra scalar degree of freedom besides the graviton field. The original theory was afterwards independently discovered by Biswas-Gerwick-Koivisto-Mazumdar starting from first principles. We explicitly show power counting super-renormalizability or finiteness (in odd dimensions) and unitarity (no ghosts) of the theory. Any further extension of the theory is nonunitary, confirming the existence of at most one single extra degree of freedom, the scalaron. A mechanism to achieve the Starobinsky theory in string (field) theory is also investigated at the end of the paper.

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I. INTRODUCTION

Inflation not only resolves a number of cosmological puzzles plagued in the standard big bang cosmology [1,2], but also it can account for the origin of large-scale structure in the Universe [3]. The standard slow-roll single-field inflationary scenario gives rise to nearly scale-invariant and adiabatic primordial perturbations by stretching out quantum fluctuations over super-Hubble scales. This prediction was confirmed by the COBE [4] and WMAP [5] groups from the observations of the cosmic microwave background (CMB) temperature anisotropies.

Recently, the Planck group provided high-precision data of the CMB temperature anisotropies [6–9], by which the spectral index n_s of curvature perturbations, the tensor-to-scalar ratio r , the nonlinear parameter f_{NL} of primordial non-Gaussianities were tightly constrained relative to the bounds derived by the WMAP 9-year data [10]. Since these observables are different depending on the models of inflation [11], we can discriminate between a host of inflationary models from the Planck data. In particular the bound of the nonlinear estimator of the squeezed shape is $f_{\text{NL}}^{\text{local}} = 2.7 \pm 5.8$ (68% CL) [9], by which all of the

single-field slow-variation inflationary models are consistent with the bound of local non-Gaussianities.

The joint data analysis combined with Planck and the WMAP polarization (WP) measurement shows that the scalar spectral index is constrained to be $n_s = 0.9603 \pm 0.0073$, by which the Harrison-Zel’dovich spectrum is excluded at more than 5σ CL [8]. The tensor-to-scalar ratio is bounded to be $r < 0.12$ (95% CL), which corresponds to an upper bound for the inflationary energy scale of about $(1.9 \times 10^{16} \text{ GeV})^4$. In Ref. [12] the authors discriminated between single-field inflationary models which belong to the class of most general scalar-tensor theories with second-order equations of motion (Horndeski’s theory [13]).

There are many slow-roll single-field models which are tension with the Planck data [8,12]. For example, power-law inflation with the exponential potential $V = \Lambda^4 \exp(-\lambda\phi/M_{\text{pl}})$ [14] (where $M_{\text{pl}} = 2.435 \times 10^{18} \text{ GeV}$ is the reduced Planck mass) and chaotic inflation with the potential $V = \lambda\phi^n/n$ ($n > 2$) [15] are outside the 95% CL boundary because of the large tensor-to-scalar ratio. Even chaotic inflation with the powers $n = 1, 2/3$ [16] is under an observational pressure due to the large scalar spectral index. Hybrid inflation with the potential $V = \Lambda^4 + m^2\phi^2/2 + \dots$ [17] gives a blue-tilted scalar spectrum ($n_s > 1$) and hence it is disfavored from the data. Other models such as hilltop inflation with

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$V = \Lambda^4(1 - \phi^2/\mu^2 + \dots)$ [18] or natural inflation with $V = \Lambda^4[1 + \cos(\phi/f)]$ [19] are viable for a restricted range of parameters (e.g. $f \gtrsim 5M_{\text{pl}}$ for natural inflation).

The models favored from the Planck data are the “small-field” scenario in which the tensor-to-scalar ratio is suppressed because of the small variation of the field during inflation with the scalar spectral index $n_s \simeq 1 - 2/N$, where $N = 50 - 60$ is the number of e-foldings from the end of inflation [12]. This includes the Starobinsky model with the Lagrangian $f(R) = R + \epsilon R^2$ (R is the Ricci scalar) [1] and the nonminimally coupled model ($\xi R\phi^2/2$) with the self-coupling potential $V(\phi) = \lambda\phi^4/4$ [20] or the Higgs potential $V(\phi) = \lambda(\phi^2 - \mu^2)^2/4$ [21]. In fact, the resulting power spectra of scalar and tensor perturbations in the Starobinsky model is the same as those in the nonminimally coupled model in the limit $|\xi| \gg 1$ [22].

In the Starobinsky model the quadratic curvature term R^2 drives inflation, which is followed by the gravitational reheating with the decrease of R^2 [23–26] (see Refs. [27–29] for reviews). If we make a conformal transformation to the Einstein frame, there appears a scalar degree of freedom ϕ called “scalaron” [1] with a nearly flat potential in the regime $\phi \gg M_{\text{pl}}$. The presence of the scalaron is crucial to generate density perturbations consistent with the Planck data. Recently there have been numerous attempts to construct the Starobinsky model in the context of supergravity [30–39].

In this paper we go in search of an ultraviolet completion of Starobinsky’s $f(R)$ action, assuming a pure gravitational origin of inflation. We start restricting our attention on the most general local quadratic action for gravity [40–42],

$$\mathcal{L}_{\text{quadratic}} = R + \epsilon R^2 + \zeta C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \eta \chi_E, \quad (1)$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and χ_E is the density of the Euler number. Assuming a Friedmann-Lemaître-Robertson-Walker (homogeneous and isotropic) metric the Weyl term vanishes because the metric is conformally flat and the Lagrangian simplifies to the Starobinsky theory,

$$\mathcal{L}_{\text{Starobinsky}} = R + \epsilon R^2. \quad (2)$$

The goal of this paper is to find a (i) Lorentz invariant, (ii) (super-)renormalizable or finite, (iii) unitary theory of gravity which reduces to the Starobinsky theory in the low-energy limit. In trying to achieve this goal we are looking for a *new classical theory of gravity* which is *renormalizable at quantum level*. This point of view opposes to the emergent gravity scenery.

We can resume as follows the theoretical and observative consistency requirements for a full quantum gravity theory.

- (1) Unitarity (theoretical). A general theory is well defined if “tachyons” and “ghosts” are absent, in which case the corresponding propagator has only first poles at $k^2 - M_i^2 = 0$ with real masses (no tachyons) and with positive residues (no ghosts).

- (2) Super-renormalizability or finiteness (theoretical). This hypothesis makes consistent the theory at quantum level in analogy with all the other fundamental interactions.
- (3) Lorentz invariance (observative). This is a symmetry of nature well tested below the Planck scale.
- (4) The theory must be at least quadratic in the curvature (observative) to achieve the agreement with the recent Planck data confirming the viability of the Starobinsky theory.
- (5) The energy conditions are not violated from the matter side (observative), but can be violated because higher-derivative operators are present in the classical theory.

Possible candidate theories which satisfy the above requirements are listed below.

- (I) We have several candidate theories, but only one fulfills all the above requirements at perturbative level. We call it: “super-renormalizable or finite gravity” [43–49]. This is the theory we are going to mainly concentrate on in this paper. Another candidate perturbatively renormalizable and unitary theory has been studied in Ref. [50].
 - (II) At nonperturbative level a natural candidate theory is “asymptotic safe quantum gravity” in which the mass of the ghost diverges when the momentum scale goes to infinity and the mode decouples from the theory in the ultraviolet regime or the ghost pole is moved further by the renormalization group running [51–53].
 - (III) String theory or its field redefinition could do the job because the modification of the propagator coming from “string field theory” [54] makes it possible to get the Starobinsky model in the low-energy limit. We can easily extend supergravity in ten dimensions incorporating the modification suggested by string field theory together with diffeomorphism invariance [55–57]. We will come back to this point at the end of the paper.
- The problem lies in the spectrum of the theory which contains extra massless degrees of freedom besides the graviton field and an infinite tower of massive states. It is not clear how to select out only one scalar degree of freedom to sustain inflation and reproduce the correct perturbation spectrum, while the other massive scalars will be suppressed by a higher mass scale involving the volume of the compact space or the extra dimensions in a brane-cosmology scenery. However, a priori we cannot exclude this possibility and we strongly suggest one should investigate and engineer in this direction.

The last very important feature of the theory we are looking for is the presence of one extra scalar degree of freedom in the gravitational spectrum, without voiding unitarity and renormalizability. We call this degree of

freedom graviscalar or scalaron, which is of fundamental importance to generate primordial perturbations during inflation.

A previous work [48] presented a super-renormalizable and ghostfree theory of gravity, which, under a natural exponential ansatz of the form factor and a suitable truncation, gives rise to the Starobinsky model. However the problem with such a model is that, at quantum level, the linearized theory on a flat background has only two tensor degrees of freedom corresponding to the spin-two graviton of general relativity and no extra scalar degree of freedom. Therefore, there is no consistent way to generate primordial density perturbations in this framework due to the lack of an extra scalar playing the role of the inflaton field.

This fact could be quite obscure since the Starobinsky theory to which the super-renormalizable theory reduces, as well as any $f(R)$ gravity model [27–29], naturally encloses an extra degree of freedom. However, this apparent paradox is solved if one remembers that the super-renormalizable model introduced in Ref. [48] reduces to $R + \epsilon R^2$ gravity only after a truncation up to terms $O(1/\Lambda^4)$, where $\Lambda \sim 10^{-5} M_{\text{pl}}$ is a parameter with energy dimension. Meanwhile, the full theory where all orders in $1/\Lambda^2$ are considered, which contains infinite derivatives, does not coincide with the Starobinsky model with different degrees of freedom. In particular the full theory has no extra scalar degree of freedom. Therefore in some sense, the truncation procedure adopted in Ref. [48] is not completely consistent with the Starobinsky theory, since it does not preserve the degrees of freedom of the starting theory.

The aim of this paper is to show that the most general class of super-renormalizable theories compatible with unitarity contains at most one extra degree of freedom besides the graviton field and reduces to the Starobinsky $R + \epsilon R^2$ theory under a suitable truncation. The truncation is coherent in this case, since it preserves the degrees of freedom of the theory after truncation. Here we achieve this result and define the maximal class of such theories; we use the term “maximal” to indicate that any other extension of the theory must contain a ghost or a tachyon and therefore unitarity is violated.

This paper is organized as follows. In Sec. II we introduce the super-renormalizable action for gravity in a D -dimensional spacetime. In Sec. III we calculate the propagator for the gravitational field fluctuation and in Sec. IV we show that, requiring unitarity, the theory contains at most the spin-two graviton field plus one scalar degree of freedom that we call graviscalar or scalaron. In Sec. V we show that the theory is super-renormalizable in even dimension and finite in odd dimension. In Sec. VI the Starobinsky theory is recovered through a suitable and coherent truncation of the Lagrangian density. In Sec. VII we expand about the importance of the graviscalar degree of freedom of the theory. Finally in Sec. IX we resume the results of this paper and conclude.

Hereafter the spacetime metric tensor $g_{\mu\nu}$ has the signature $(+, -, \dots, -)$, the curvature tensor is

$R^\mu_{\nu\rho\sigma} = -\partial_\sigma \Gamma^\mu_{\nu\rho} + \dots$, the Ricci tensor is $R_{\mu\nu} = R^\sigma_{\mu\nu\sigma}$, the curvature scalar is $R = g^{\mu\nu} R_{\mu\nu}$. Moreover, we use natural units $c = 1$ and $\hbar = 1$.

II. THE MULTIDIMENSIONAL THEORY

In this section we introduce a general action for the class of super-renormalizable or finite theories under consideration in D -dimensional spacetime. Let us start with the following nonpolynomial or semipolynomial Lagrangian density

$$\begin{aligned} \mathcal{L} = & 2\kappa_D^{-2} R + \bar{\lambda} + c_1^{(1)} R^3 + \dots + c_1^{(N)} R^{N+2} \\ & + \sum_{n=0}^N [a_n R(-\square_\Lambda)^n R + b_n R_{\mu\nu}(-\square_\Lambda)^n R^{\mu\nu}] \\ & + R h_0(-\square_\Lambda) R + R_{\mu\nu} h_2(-\square_\Lambda) R^{\mu\nu}, \end{aligned} \quad (3)$$

where $\kappa_D^2 = 32\pi G$, G is Newton’s gravitational constant, $\square_\Lambda \equiv \square/\Lambda^2$ and the operator $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is constructed with covariant derivatives. The nonpolynomial operators have been introduced in the last line of (3) making use of the following two entire functions

$$\begin{aligned} h_0(z) &= -\frac{[V_0(z)^{-1} - 1]}{\kappa_D^2 \Lambda^2 z} - \sum_{n=0}^N \tilde{a}_n z^n, \\ h_2(z) &= 2\frac{[V_2(z)^{-1} - 1]}{\kappa_D^2 \Lambda^2 z} - \sum_{n=0}^N \tilde{b}_n z^n, \end{aligned} \quad (4)$$

where $V_0(z)$ and $V_2(z)$ are two entire functions that we are going to select consistently with unitarity and renormalizability. The constants \tilde{a}_n and \tilde{b}_n are just nonrunning parameters, while the running coupling constants are

$$\alpha_i \in \{\kappa_D, \bar{\lambda}, a_n, b_n, c_1^{(1)}, \dots, c_1^{(N)}\} \equiv \{\kappa_D, \bar{\lambda}, \tilde{\alpha}_n\}. \quad (5)$$

The integer N is defined as follows in order to avoid fractional powers of the D ’Alembertian operator, namely,

$$2N + 4 = D_{\text{odd}} + 1 \quad \text{in odd dimension}, \quad (6)$$

$$2N + 4 = D_{\text{even}} \quad \text{in even dimension}. \quad (7)$$

The “form factors” $V_i(z)^{-1}$ ($i = 0, 2$) will be defined later for the compatibility with unitarity and renormalizability.

The goal of this paper is to find an ultraviolet completion of the Starobinsky theory with exactly the same particle spectrum: the massless graviton and the massive graviscalar. We will see later in Sec. IV that this is the maximal particle content compatible with unitarity and super-renormalizability or finiteness at quantum level. However, a nonpolynomial minimal theory reproducing the Starobinsky action in the low-energy limit has been already introduced and studied in Refs. [45,48,49]. This action satisfies all the requirements (1–5) listed in the

previous section, but only the massless graviton propagates. The Lagrangian in Refs. [45,48,49] is the same as Eq. (3), with the identification of the two form factors as $V_0(z)^{-1} = V_2(z)^{-1} \equiv e^{\square_\Lambda}$ [49,54]. In Sec. VII we see that this prescription gives rise to the Starobinsky Lagrangian, but the problem is that it lacks the graviscalar degree of freedom required to generate scalar metric perturbations during inflation.

III. PROPAGATOR

We shall explicitly calculate the two-point propagator for the action (3) and then we impose the condition that the nonpolynomial functions $h_0(z)$ and $h_2(z)$ defined in (4) have to fulfill in order to achieve a theory which satisfies the theoretical and observative consistency requirements stated in the introduction. We stress that it is important here to obtain the expression of the two-point function since from the poles of the propagator the number of propagating degrees of freedom will be clear.

We proceed to split the spacetime metric into the flat Minkowski background plus a fluctuation $h_{\mu\nu}$ defined by

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa_D h_{\mu\nu}, \quad (8)$$

and then we expand the Lagrangian to second order in the gravitational fluctuation $h_{\mu\nu}$. Omitting total derivative operators, we end up with the following outcome [58]

$$\begin{aligned} \mathcal{L}_{\text{linear}} = & -\frac{1}{2}[h^{\mu\nu}\square h_{\mu\nu} + A_\nu^2 + (A_\nu - \phi_{,\nu})^2] \\ & + \frac{\kappa_D^2}{8}[\square h_{\mu\nu}\beta(\square)\square h^{\mu\nu} - A_{,\mu}^\mu\beta(\square)A_{,\nu}^\nu - F^{\mu\nu}\beta(\square)F_{\mu\nu} \\ & + (A_{,\alpha}^\alpha - \square\phi)(\beta(\square) + 4\alpha(\square))(A_{,\beta}^\beta - \square\phi)], \quad (9) \end{aligned}$$

where the vector and antisymmetric tensor are below defined in terms of the gravitational fluctuation,

$$\begin{aligned} A^\mu &= h^{\mu\nu}{}_{,\nu}, \\ \phi &= h^\mu{}_\mu \quad (\text{trace of } h_{\mu\nu}), \\ F_{\mu\nu} &= A_{\mu,\nu} - A_{\nu,\mu}, \quad (10) \end{aligned}$$

while the functionals of the d'Alembertian operators $\alpha(\square)$, $\beta(\square)$ are defined by

$$\begin{aligned} \alpha(\square) &:= 2 \sum_{n=0}^N a_n (-\square_\Lambda)^n + 2h_0(-\square_\Lambda), \\ \beta(\square) &:= 2 \sum_{n=0}^N b_n (-\square_\Lambda)^n + 2h_2(-\square_\Lambda). \quad (11) \end{aligned}$$

The d'Alembertian operator in $\mathcal{L}_{\text{linear}}$ and (11) should be evaluated on the flat spacetime. The linearized Lagrangian (9) is invariant under infinitesimal coordinate

transformations $x^\mu \rightarrow x^\mu + \kappa_D \xi^\mu(x)$, where $\xi^\mu(x)$ is an infinitesimal vector field of dimensions $[\xi(x)] = M^{(D-4)/2}$. Under this transformation, the graviton field turns into $h_{\mu\nu} \rightarrow h_{\mu\nu} - \xi(x)_{\mu,\nu} - \xi(x)_{\nu,\mu}$. The presence of the local gauge symmetry calls for the addition of a gauge-fixing term to the linearized Lagrangian (9). Hence, we choose the usual harmonic gauge

$$\mathcal{L}_{\text{GF}} = \xi^{-1} A_\mu \omega(-\square_\Lambda) A^\mu, \quad (12)$$

where $\omega(-\square_\Lambda)$ is a gauge weight function [40,44]. The linearized gauge-fixed Lagrangian reads

$$\mathcal{L}_{\text{linear}} + \mathcal{L}_{\text{GF}} = \frac{1}{2} h^{\mu\nu} \mathcal{O}_{\mu\nu,\rho\sigma} h^{\rho\sigma}, \quad (13)$$

where the operator \mathcal{O} has two contributions coming from the linearized Lagrangian (9) and from the gauge-fixing term (12). Inverting the operator \mathcal{O} [58], we find the following two-point function

$$\begin{aligned} \mathcal{O}^{-1} = & \frac{\xi(2P^{(1)} + \bar{P}^{(0)})}{2k^2 \omega(k^2/\Lambda^2)} + \frac{P^{(2)}}{k^2(1 + k^2 \kappa_D^2 \frac{\beta(k^2)}{4})} \\ & - \frac{P^{(0)}}{k^2(D-2 - k^2 \kappa_D^2 (D \frac{\beta(k^2)}{4} + (D-1)\alpha(k^2)))}, \quad (14) \end{aligned}$$

where the projectors in D dimensions are defined by [58,59]

$$\begin{aligned} P_{\mu\nu,\rho\sigma}^{(2)}(k) &= \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}, \\ P_{\mu\nu,\rho\sigma}^{(1)}(k) &= \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}), \\ P_{\mu\nu,\rho\sigma}^{(0)}(k) &= \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}, \quad \bar{P}_{\mu\nu,\rho\sigma}^{(0)}(k) = \omega_{\mu\nu}\omega_{\rho\sigma}, \\ \theta_{\mu\nu} &= \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \quad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}. \quad (15) \end{aligned}$$

Note that the tensorial indices for the operator \mathcal{O}^{-1} and the projectors $P^{(0)}$, $P^{(2)}$, $P^{(1)}$, $\bar{P}^{(0)}$ have been omitted.¹

¹The following identities are useful to split the terms proportional to the gravitational momentum from the remaining:

$$\begin{aligned} P_{\mu\nu,\rho\sigma}^{(2)} &= \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) - \frac{1}{D-1}\eta_{\mu\nu}\eta_{\rho\sigma} \\ &\quad - \left[P^{(1)} + \frac{D-2}{D-1}\bar{P}^{(0)} - \frac{1}{D-1}\bar{\bar{P}}^{(0)} \right]_{\mu\nu,\rho\sigma}, \\ P_{\mu\nu,\rho\sigma}^{(0)} &= \frac{1}{D-1}\eta_{\mu\nu}\eta_{\rho\sigma} - \frac{1}{D-1}[\bar{P}^{(0)} + \bar{\bar{P}}^{(0)}]_{\mu\nu,\rho\sigma}, \quad (16) \end{aligned}$$

where

$$\bar{\bar{P}}_{\mu\nu,\rho\sigma}^{(0)}(k) = \theta_{\mu\nu}\omega_{\rho\sigma} + \omega_{\mu\nu}\theta_{\rho\sigma}. \quad (17)$$

The functions $\alpha(k^2)$ and $\beta(k^2)$ are achieved by replacing $-\square \rightarrow k^2$ in the definitions (11). By looking at the last two gauge-invariant terms in Eq. (14), we deem convenient to introduce the following definitions,

$$\begin{aligned} \bar{h}_2(z) &= 1 + \frac{\kappa_D^2 \Lambda^2}{2} z \sum_{n=0}^N b_n z^n + \frac{\kappa_D^2 \Lambda^2}{2} z h_2(z), \\ \bar{h}_0(z) &= 1 - \frac{\kappa_D^2 \Lambda^2 D}{2(D-2)} z \left[\sum_{n=0}^N b_n z^n + h_2(z) \right] \\ &\quad - \frac{2\kappa_D^2 \Lambda^2 (D-1)}{D-2} z \left[\sum_{n=0}^N a_n z^n + h_0(z) \right], \end{aligned} \quad (18)$$

where again $z = -\square_\Lambda$. Through the above definitions (18), the propagator greatly simplifies to

$$\mathcal{O}^{-1} = \frac{1}{k^2} \left[\frac{P^{(2)}}{\bar{h}_2} - \frac{P^{(0)}}{(D-2)\bar{h}_0} \right] + \frac{\xi(2P^{(1)} + \bar{P}^{(0)})}{2k^2 \omega}. \quad (19)$$

In the above formula we missed the argument k^2 for the entire functions h_2 , h_0 and the weight function ω .

Once established that h_2 and h_0 are not polynomial functions, to achieve unitarity, we demand the following general properties for the transcendental entire functions $h_i(z)$ ($i = 0, 2$) and/or $\bar{h}_i(z)$ ($i = 0, 2$) [44]:

- (i) $\bar{h}_2(z)$ is real and positive on the real axis and it has no zeroes on the whole complex plane $|z| < +\infty$. $\bar{h}_0(z)$ is real with at most one zero on the real axis and then at most one zero in the whole complex plane $|z| < +\infty$. We will show in the next section that these requirements imply the maximal particle content compatible with unitarity.
- (ii) $|h_i(z)|$ has the same asymptotic behavior along the real axis at $\pm\infty$.
- (iii) There exists $\Theta > 0$ such that

$$\begin{aligned} \lim_{|z| \rightarrow +\infty} |h_i(z)| &\rightarrow |z|^{\gamma_i + N}, \\ \gamma_i &\geq D/2 \quad \text{for } D = D_{\text{even}} \quad \text{and} \\ \gamma_i &\geq (D-1)/2 \quad \text{for } D = D_{\text{odd}}, \end{aligned} \quad (20)$$

for the argument of z in the following conical regions

$$\begin{aligned} C &= \{z \mid -\Theta < \arg z < +\Theta, \pi - \Theta < \arg z < \pi + \Theta\} \\ &\text{for } 0 < \Theta < \pi/2. \end{aligned}$$

This condition is necessary to achieve the super-renormalizability of the theory that we are going to show here below. The necessary asymptotic behavior is imposed not only on the real axis, but also on the

conic regions that surround it. In a Euclidean space-time, the condition (ii) is not strictly necessary if (iii) applies.

We suppose that the theory is renormalized at some scale μ_0 . Therefore, if we set

$$\tilde{a}_n = a_n(\mu_0), \quad \tilde{b}_n = b_n(\mu_0), \quad (21)$$

we can express $\bar{h}_2(z)$ and $\bar{h}_0(z)$ in terms of the form factors $V_2(z)$ and $V_0(z)$ replacing (4) in (18), namely

$$\begin{aligned} \bar{h}_2(z) &= V_2(z)^{-1}, \\ \bar{h}_0(z) &= \frac{2}{D-2} \left[-\frac{D}{2} V_2(z)^{-1} + (D-1) V_0(z)^{-1} \right]. \end{aligned} \quad (22)$$

Let us assume for the moment that the entire functions $\bar{h}_2(z)$ and $\bar{h}_0(z)$ are each a polynomial multiplied by the exponential of an entire function, namely

$$\begin{aligned} \bar{h}_2(z) &:= e^{H_2(z)} p^{(n_2)}(z), \\ \bar{h}_0(z) &:= e^{H_0(z)} p^{(n_0)}(z), \end{aligned} \quad (23)$$

while $p^{(n_i)}(z)$ are two polynomials of degree n_i , respectively. The two polynomials will be fixed shortly in Sec. IV compatibly with unitarity. Using (22), we can invert (23) for $V_2(z)^{-1}$ and $V_0(z)^{-1}$,

$$\begin{aligned} V_2(z)^{-1} &= e^{H_2(z)} p^{(n_2)}(z), \\ V_0(z)^{-1} &= \frac{(D-2)e^{H_0(z)} p^{(n_0)}(z) + D V_2(z)^{-1}}{2(D-1)}. \end{aligned} \quad (24)$$

A class of entire functions $H_i(z)$ ($i = 2, 0$) compatible with the required properties (i)-(iii) and the definitions (4), (23) are

$$\begin{aligned} H_i(z) &= \frac{1}{2} [\gamma_E + \Gamma(0, p_{\gamma_i + N + 1}^2(z)) + \log(p_{\gamma_i + N + 1}^2(z))], \\ \text{Re}(p_{\gamma_i + N + 1}^2(z)) &> 0, \end{aligned} \quad (25)$$

where $\Gamma(a, z)$ is defined in the footnote² and the form factors can be written as

$$e^{H_i(z)} = e^{\frac{1}{2}[\Gamma(0, p_{\gamma_i + N + 1}^2(z)) + \gamma_E]} |p_{\gamma_i + N + 1}(z)|. \quad (27)$$

If we choose $p_{\gamma_i + N + 1}(z) = z^{\gamma_i + N + 1}$, $H_i(z)$ simplifies to:

² $\gamma_E = 0.577216$ is Euler's constant and

$$\Gamma(a, z) = \int_z^{+\infty} t^{a-1} e^{-t} dt \quad (26)$$

is the incomplete gamma function.

$$H_i(z) = \frac{1}{2} [\gamma_E + \Gamma(0, z^{2\gamma_i+2N+2}) + \log(z^{2\gamma_i+2N+2})],$$

$$\text{Re}(z^{2\gamma_i+2N+2}) > 0 \Rightarrow \Theta = \frac{\pi}{4(\gamma_i + N + 1)},$$

$$H_i(z) = \frac{z^{2\gamma_i+2N+2}}{2} - \frac{z^{4\gamma_i+4N+4}}{8} + \dots \text{ for } z \approx 0, \quad (28)$$

where Θ is the angle defining the cone C of the property (iii). The first correction to the form factor $e^{H_i(z)}$ goes to zero faster than any polynomial function for $z \rightarrow +\infty$, namely,

$$\lim_{z \rightarrow +\infty} e^{H_i(z)} = e^{\gamma_E/2} |z|^{\gamma_i+N+1},$$

$$\lim_{z \rightarrow +\infty} \left(\frac{e^{H_i(z)}}{e^{\gamma_E/2} |z|^{\gamma_i+N+1}} - 1 \right) z^n = 0, \quad \forall n \in \mathbb{N}. \quad (29)$$

The main result in this section is the propagator (19) together with the definitions (23) and (25).

IV. UNITARITY AND DEGREES OF FREEDOM

In this section we discuss the unitarity of the theory (3). In particular we tackle the problem of defining a theory of pure gravity with the maximal number of degrees of freedom compatible with unitarity.

If both tachyons and ghosts are absent, the stability of the theory is ensured at classical and quantum levels. In this case the corresponding propagator has only first poles at $k^2 - M_i^2 = 0$ with real masses M_i (no tachyons) and with positive residues (no ghosts).

For the evaluation of the propagator (19) we make use of the explicit definitions (23) of \bar{h}_i ($i = 2, 0$) written in terms of the entire functions $H_i(z)$ and the polynomial functions $p^{(n_i)}$ defined through (23),

$$p^{(n_2)} = \prod_{j=1}^{n_2} \left(1 - \frac{k^2}{\bar{m}_j^2} \right), \quad p^{(n_0)} = \prod_{j=1}^{n_0} \left(1 - \frac{k^2}{m_j^2} \right), \quad (30)$$

where $\bar{m}_j^2 > 0$ and $m_j^2 > 0$. If we couple the propagator (19) to the conserved stress-energy tensor $T^{\mu\nu}$ satisfying the relation $\nabla_\mu T^{\mu\nu} = 0$, the contributions coming from the terms $P^{(1)}$ and $\bar{P}^{(0)}$ vanish from the definition (15). Dropping those contributions and using Eq. (30), the propagator (19) reads

$$\mathcal{O}^{-1}(k) = \frac{P^{(2)}}{k^2 e^{H_2} p^{(n_2)}} - \frac{P^{(0)}}{(D-2)k^2 e^{H_0} p^{(n_0)}}$$

$$= \frac{P^{(2)} e^{-H_2}}{k^2 \prod_{j=1}^{n_2} \left(1 - \frac{k^2}{\bar{m}_j^2} \right)} - \frac{P^{(0)} e^{-H_0}}{(D-2)k^2 \prod_{j=1}^{n_0} \left(1 - \frac{k^2}{m_j^2} \right)}$$

$$= \frac{P^{(2)}}{e^{H_2}} \left(\frac{\bar{A}_0}{k^2} + \frac{\bar{A}_1}{k^2 - \bar{m}_1^2} + \dots + \frac{\bar{A}_{n_2}}{k^2 - \bar{m}_{n_2}^2} \right)$$

$$- \frac{P^{(0)}}{(D-2)e^{H_0}} \left(\frac{A_0}{k^2} + \frac{A_1}{k^2 - m_1^2} + \dots + \frac{A_{n_0}}{k^2 - m_{n_0}^2} \right), \quad (31)$$

where \bar{A}_j, A_j , are constants, and $\bar{m}_0^2 = 0, m_0^2 = 0$.

Let us assume that we have two real monotonic sequences of masses: $\bar{m}_1 < \bar{m}_2 < \dots < \bar{m}_{n_2}, m_1 < m_2 < \dots < m_{n_0}$. In this case the signs of the corresponding residues alternate, i.e., $\text{sign}[\text{Res}A_j] = -\text{sign}[\text{Res}A_{j+1}]$ and $\text{sign}[\text{Res}\bar{A}_j] = -\text{sign}[\text{Res}\bar{A}_{j+1}]$ [41]. From the propagator (31) we see that the residues in $k^2 = 0$ and $k^2 = m_1^2$ are positive but the residues in $k^2 = \bar{m}_1^2$ and $k^2 = m_2^2$ are negative. It follows that in order to avoid ghosts the polynomials must have, respectively, degrees

$$n_2 = 0, \quad n_0 \leq 1. \quad (32)$$

In fact, this meets the requirement (i) introduced and discussed in Sec. III.

Let us consider the explicit example for $n_2 = 1$ and $n_0 = 2$. The spin-two and spin-zero sectors of the propagator respectively read

$$\text{spin } 2: \frac{P^{(2)}}{e^{H_2}} \left(\frac{1}{k^2} - \frac{1}{k^2 - \bar{m}_1^2} \right),$$

$$\text{spin } 0: \frac{P^{(0)}}{(D-2)e^{H_0}} \left(-\frac{1}{k^2} + \frac{A_1}{k^2 - m_1^2} + \frac{A_2}{k^2 - m_2^2} \right), \quad (33)$$

where the constants A_1, A_2 are

$$A_1 = \frac{m_2^2}{m_2^2 - m_1^2} > 0, \quad A_2 = -\frac{m_1^2}{m_2^2 - m_1^2} < 0. \quad (34)$$

The quantum states have positive-definite norms and energies if the poles in the propagator have positive residues. In the example given above the residues in $k^2 = 0$ and $k^2 = m_1^2$ are positive, but the residues in $k^2 = \bar{m}_1^2$ and in $k^2 = m_2^2$ are negative.

This example confirms that the maximal theory compatible with unitarity has $n_2 = 0$ and $n_0 \leq 1$. The case with $n_0 = n_2 = 0$ corresponds to the model presented in Ref. [48], which will be discussed with more details in Sec. VII.

In the case $n_2 = 0$ and $n_0 = 1$, defining $m_1^2 \equiv m^2$, the propagator further simplifies to

$$\begin{aligned} \mathcal{O}^{-1} &= \frac{P^{(2)}}{k^2 e^{H_2}} + \frac{m^2 P^{(0)}}{k^2 e^{H_0} (k^2 - m^2) (D-2)} \\ &= \frac{P^{(2)}}{k^2 e^{H_2}} - \frac{P^{(0)}}{(D-2) k^2 e^{H_0}} + \frac{P^{(0)}}{(D-2) e^{H_0} (k^2 - m^2)}, \end{aligned} \quad (35)$$

with two single poles at $k^2 = 0$ and $k^2 = m^2$ that do have positive residues because $H_i(0) = 0$. We now expand on the tree-level unitarity coupling the propagator to external conserved stress-energy tensors $T^{\mu\nu}$, and examining the amplitude at the poles [58,59]. When we introduce a general source operator, the linearized action is replaced by

$$\mathcal{L}_{\text{linear}} + \mathcal{L}_{\text{GF}} - g h_{\mu\nu} T^{\mu\nu}, \quad (36)$$

and the transition amplitude in momentum space is

$$A = g^2 T^{\mu\nu} \mathcal{O}_{\mu\nu,\rho\sigma}^{-1} T^{\rho\sigma}, \quad (37)$$

where g is an effective coupling constant.

To make the analysis more explicit, we can expand the sources using the following set of independent vectors in the momentum space [58],

$$\begin{aligned} k^\mu &= (k^0, \vec{k}), & \tilde{k}^\mu &= (k^0, -\vec{k}), \\ \epsilon_i^\mu &= (0, \vec{\epsilon}_i), & i &= 1, \dots, D-2, \end{aligned} \quad (38)$$

where $\vec{\epsilon}_i$ are $D-2$ unit vectors orthogonal to each other and to \vec{k} ,

$$\vec{k} \cdot \vec{\epsilon}_i = 0 \rightarrow k_\mu \epsilon_i^\mu = 0, \quad \vec{\epsilon}_i \cdot \vec{\epsilon}_j = \delta_{ij}. \quad (39)$$

The most general symmetric stress-energy tensor can be expressed as

$$\begin{aligned} T^{\mu\nu} &= a k^\mu k^\nu + b \tilde{k}^\mu \tilde{k}^\nu + c^{ij} \epsilon_i^{(\mu} \epsilon_j^{\nu)} + d k^{(\mu} \tilde{k}^{\nu)} \\ &\quad + e^i k^{(\mu} \epsilon_i^{\nu)} + f^i \tilde{k}^{(\mu} \epsilon_i^{\nu)}, \end{aligned} \quad (40)$$

where we used the notation $a_{(\mu} b_{\nu)} \equiv (a_\mu b_\nu + b_\mu a_\nu)/2$. The conditions $k_\mu T^{\mu\nu} = 0$ and $k_\mu k_\nu T^{\mu\nu} = 0$ provide the following constraints and consistency conditions on the coefficients a, b, d, e^i, f^i [58]:

$$k_\mu T^{\mu\nu} = 0 \Rightarrow \left\{ \begin{array}{l} a k^2 + d(k_0^2 + \vec{k}^2)/2 = 0, \\ b(k_0^2 + \vec{k}^2) + d k^2/2 = 0, \\ e^i k^2 + f^i(k_0^2 + \vec{k}^2) = 0. \end{array} \right. \quad (41)$$

$$\Rightarrow d = 0, \quad b = 0, \quad f^i = 0 \quad \text{for } k^2 := k_0^2 - \vec{k}^2 = 0, \quad (42)$$

$$\begin{aligned} k_\mu k_\nu T^{\mu\nu} = 0 \quad (\text{consistency relation for } a, b, \text{ and } d) \\ \Rightarrow a k^4 + b(k_0^2 + \vec{k}^2)^2 + d k^2(k_0^2 + \vec{k}^2) = 0. \end{aligned} \quad (43)$$

Introducing the spin-projectors and making use of the identities (16) together with the conservation of the

stress-energy tensor $k_\mu T^{\mu\nu} = 0$, the amplitude (37) for $n_2 = 0$ and $n_1 = 1$ reads

$$\begin{aligned} A &= g^2 T^{\mu\nu} \left[\frac{P_{\mu\nu,\rho\sigma}^{(2)}}{k^2 e^{H_2} p^{(n_2)}} - \frac{P_{\mu\nu,\rho\sigma}^{(0)}}{(D-2) k^2 e^{H_0} p^{(n_0)}} \right] T^{\rho\sigma} \\ &= g^2 \left[\frac{T_{\mu\nu} T^{\mu\nu} - \frac{T^2}{D-1}}{k^2 e^{H_2}} - \frac{\frac{T^2}{D-1}}{(D-2) k^2 e^{H_0} (1 - \frac{k^2}{m^2})} \right], \end{aligned} \quad (44)$$

where $T := \eta^{\mu\nu} T_{\mu\nu}$.

We now calculate the residue of the amplitude in $k^2 = 0$ and $k^2 = m^2$. Using the properties $H_i(0) = 0$ and (42), the residue in $k^2 = 0$ reads

$$\begin{aligned} \text{Res } A|_{k^2=0} &= g^2 \left(T_{\mu\nu} T^{\mu\nu} - \frac{T^2}{D-2} \right) \Big|_{k^2=0} \\ &= g^2 \left[(c^{ij})^2 - \frac{(c^{ii})^2}{D-2} \right] \Big|_{k^2=0} > 0 \quad \text{for } D > 3. \end{aligned} \quad (45)$$

When $D = 3$, the graviton is not a dynamical degree of freedom and the amplitude is zero. The residue in $k^2 = m^2$ results

$$\text{Res } A|_{k^2=m^2} = g^2 \frac{T^2 e^{-H_0(m^2/\Lambda^2)}}{(D-1)(D-2)} > 0 \quad \text{for } D > 2, \quad (46)$$

in which case the scalar mode propagates. Thus, in the case $n_2 = 0$ and $n_0 = 1$, the spectrum consists of two particles: the graviton and the graviscalar (scalaron). We conclude that the maximal class of super-renormalizable unitary theories includes a graviscalar besides the graviton.

V. RENORMALIZABILITY AND FINITENESS

In this section we study the power counting renormalizability of the theory, showing that it is renormalizable in even spacetime dimension and finite in odd dimension.

The theory can be renormalizable if we assume the same ultraviolet behavior for the functions $\bar{h}_2(z)$ and $\bar{h}_0(z)$. For $n_2 = 0$ and $n_0 = 1$, it follows that

$$\begin{aligned} \bar{h}_2(z) &= e^{H_2} \rightarrow z^{\gamma_2 + N + 1}, \quad \text{and} \\ \bar{h}_0(z) &= e^{H_0} \left(1 - \frac{\Lambda^2 z}{m^2} \right) \rightarrow z^{\gamma_0 + N + 2}. \end{aligned} \quad (47)$$

If $\gamma_2 = \gamma_0 + 1 \equiv \gamma$, then the functions $\bar{h}_2(z)$ and $\bar{h}_0(z)$ have the same scaling property.

Replacing $\gamma_2 = \gamma$ and $\gamma_0 = \gamma - 1$ and using Eqs. (3), (4), (18), and (19), the high-energy scaling of the propagator in the momentum space and the leading interaction vertex are schematically given by

$$\mathcal{O}^{-1}(k) \sim \frac{1}{k^{2\gamma+2N+4}} \quad \text{in the ultraviolet,}$$

$$\mathcal{L}^{(n)} \sim h^n \square_\eta h h_i(-\square_\Lambda) \square_\eta h \rightarrow h^n \square_\eta h \square_\eta^{\gamma+N} \square_\eta h, \quad (48)$$

where $\square_\eta := \eta^{\mu\nu} \partial_\mu \partial_\nu$. In (48) the indices for the gravitational fluctuations $h_{\mu\nu}$ are omitted (replaced by h), and $h_i(-\square_\Lambda)$ is the entire function defined by the properties (i)-(iii). From (48), the upper bound to the superficial degree of divergence in a spacetime of “even” or “odd” dimension is

$$\begin{aligned} w(G)_{\text{even}} &= D_{\text{even}} - 2\gamma(L-1), \\ w(G)_{\text{odd}} &= D_{\text{odd}} - (2\gamma+1)(L-1), \end{aligned} \quad (49)$$

where we used the topological relation between vertices V , internal lines I , and number of loops L : $I = V + L - 1$. Thus, if $\gamma > D_{\text{even}}/2$ or $\gamma > (D_{\text{odd}} - 1)/2$, only 1-loop divergences survive in this theory. Therefore, the theory is super-renormalizable, unitary, and microcausal as pointed out also in Refs. [43,60–63]. For γ sufficiently large the divergent contributions to the β -functions (β_i) are independent from the running coupling constants (5) and then the β -functions do not depend on the energy scale μ defined through $t := \log(\mu/\mu_0)$. It follows that we can easily integrate the renormalization group equations [45], i.e.,

$$\frac{d\alpha_i}{dt} = \beta_i \Rightarrow \alpha_i(t) \sim \alpha_i(t_0) + \beta_i t. \quad (50)$$

The mass of the graviscalar is not subject to renormalization and the logarithmic quantum corrections to the propagator leave its value almost invariant because the damping factor $e^{-H_0(z)}$ suppresses any high energy shift, namely

$$\begin{aligned} \mathcal{O}^{-1} &= \frac{P^{(2)} e^{-H_2}}{k^2 [1 + e^{-H_2} k^2 (c_0 + \dots + c_N k^{2N}) \log(\frac{k^2}{\mu^2})]} \\ &+ \frac{m^2 P^{(0)} e^{-H_0}}{(D-2)k^2 [(k^2 - m^2) + e^{-H_0} k^2 (\bar{c}_0 + \dots)] \log(\frac{k^2}{\mu^2})}, \end{aligned}$$

where $c_0, \bar{c}_0, \dots, c_N$ are dimensionful constants and μ is a renormalization group invariant scale.

However, in *odd* dimension there are no local invariants (using dimensional regularization) with an odd number of derivatives which could serve as counterterms for pure gravity. This is a consequence of the rational nature of the entire functions which characterize the theory (one example of a nonrational function is $h_i(\sqrt{z})$). We conclude that all the amplitudes with an arbitrary number of loops are finite and all the beta functions are identically zero in odd dimension,

$$\begin{aligned} \beta_{a_n} = \beta_{b_n} = \beta_{c_i^{(n)}} &= 0, \\ i \in \{1, \dots, (\text{number of invariants of order } N)\}, \\ n = 1, \dots, N. \end{aligned} \quad (51)$$

It follows that we can fix to zero all the couplings $c_i^{(n)}$ and set to constants the couplings $a_n(\mu)$ and $b_n(\mu)$, namely,

$$\begin{aligned} c_i^{(n)} &= 0, \\ a_n(\mu) &= \text{constant} = \tilde{a}_n, \\ b_n(\mu) &= \text{constant} = \tilde{b}_n. \end{aligned} \quad (52)$$

Therefore, quantum gravity is finite in even dimension, as well, once the Kaluza-Klein compactification is applied [64]. The finiteness of the theory in even dimensions follows from the inclusion of an infinity tower of states which drastically affects the ultraviolet behavior.

VI. STAROBINSKY LIMIT

In the following we show how the Lagrangian (3) reduces to the Starobinsky $R + \epsilon R^2$ theory after a suitable truncation for large values of the Λ parameter. We also discuss how to fix the value of such a mass scale and the value of the graviscalar mass m .

The Lagrangian (3) can be recast as follows

$$\begin{aligned} \mathcal{L} &= \frac{2}{\kappa_D^2} \left(R - G_{\mu\nu} \frac{V_2^{-1} - 1}{\square} R^{\mu\nu} + \frac{1}{2} R \frac{V_0^{-1} - V_2^{-1}}{\square} R \right), \\ V_0^{-1} - V_2^{-1} &= \frac{D-2}{2(D-1)} \left[e^{H_0} \left(1 + \frac{\square}{m^2} \right) - e^{H_2} \right], \end{aligned} \quad (53)$$

where $G_{\mu\nu}$ is the Einstein tensor. Expanding the above Lagrangian (53) for large Λ , we find

$$\mathcal{L} = \frac{2}{\kappa_D^2} \left[R + \frac{(D-2)R^2}{4(D-1)m^2} + O\left(R \frac{\square^{2\gamma+2N-1}}{\Lambda^{4\gamma+4N}} R \right) \right]. \quad (54)$$

When

$$R \frac{\square^{2\gamma+2N-1}}{\Lambda^{4\gamma+4N}} R \ll \frac{(D-2)R^2}{4(D-1)m^2}, \quad (55)$$

the last term in (54) is negligible and in $D = 4$ dimensions the Lagrangian (54) reduces exactly to (2) with $\epsilon = 1/(6m^2)$.

It seems natural to identify Λ and the graviscalar mass m to avoid a further mass scale in the classical theory. Unlike the previous model [47], we obtain the Starobinsky $R + \epsilon R^2$ theory with exactly the same spectrum, the massless graviton and the massive graviscalar essential to generate proper primordial density perturbations.

The equation of motion up to operators $O(R\square R)$ and $O(R^3)$ reads [65–67],

$$\begin{aligned} G_{\mu\nu} + \frac{D-2}{2(D-1)m^2} R \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) \\ - \frac{D-2}{2(D-1)m^2} (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) R = \frac{\kappa_D}{4} T_{\mu\nu}, \end{aligned} \quad (56)$$

which is exactly the Starobinsky equation of motion in D dimensions³.

Let us discuss observational signatures for the model described by the Lagrangian (54) with $D = 4$ under the condition (55), i.e.,

$$\mathcal{L} = \frac{2}{\kappa_D^2} \left(R + \frac{R^2}{6m^2} \right). \quad (58)$$

The density perturbations generated in the inflationary models based on $f(R)$ gravity and scalar-tensor theory were studied in detail in Refs. [22]. The resulting power spectra of curvature perturbations \mathcal{R} and gravitational waves h_{ij} are given, respectively, by [28]

$$\mathcal{P}_{\mathcal{R}} = \frac{N^2}{24\pi^2} \left(\frac{m}{M_{\text{pl}}} \right)^2, \quad \mathcal{P}_h = \frac{1}{2\pi^2} \left(\frac{m}{M_{\text{pl}}} \right)^2, \quad (59)$$

where N is the number of e-foldings from the end of inflation to the epoch at which the perturbations relevant to the CMB anisotropies with the physical wave length a/k (a is the scale factor) crossed the Hubble radius H^{-1} . From the Planck normalization $\mathcal{P}_{\mathcal{R}} = 2.2 \times 10^{-9}$ at the pivot comoving wave number $k_0 = 0.05 \text{ Mpc}^{-1}$ [12], the scalaron mass is constrained to be

$$\frac{m}{M_{\text{pl}}} = 1.3 \times 10^{-5} \left(\frac{55}{N} \right), \quad (60)$$

which corresponds to $m \approx 3.2 \times 10^{13} \text{ GeV}$ for $N = 55$. Since the scalaron is very heavy, the fifth force is strongly suppressed in the present Universe. Hence the model is compatible with local gravity constraints in the solar system [28,68].

The scalar spectral index $n_s - 1 \equiv d \ln \mathcal{P}_{\mathcal{R}} / d \ln k|_{k=aH}$ reads [28]

$$n_s - 1 = -\frac{2}{N} = -3.6 \times 10^{-2} \left(\frac{55}{N} \right), \quad (61)$$

whereas the tensor-to-scalar ratio $r \equiv \mathcal{P}_h / \mathcal{P}_{\mathcal{R}}$ is

$$r = \frac{12}{N^2} = 4.0 \times 10^{-3} \left(\frac{55}{N} \right)^2. \quad (62)$$

³In calculating the variation of the action (53) we used the compatibility property of the metric $\nabla_\mu g_{\rho\sigma} = 0$ and the following variation of the Ricci tensor,

$$\delta R_{\mu\nu} = -\frac{1}{2} g_{\mu\alpha} g_{\nu\beta} \square \delta g^{\alpha\beta} + \frac{1}{2} [\nabla^\beta \nabla_\mu \delta g_{\beta\nu} + \nabla^\beta \nabla_\nu \delta g_{\beta\mu} - \nabla_\mu \nabla_\nu \delta g_{\alpha\alpha}], \quad (57)$$

together with $\nabla^\mu G_{\mu\nu} = 0$. The Starobinsky action is manifestly generally covariant. Therefore, its variational derivative exactly satisfies the Bianchi identity.

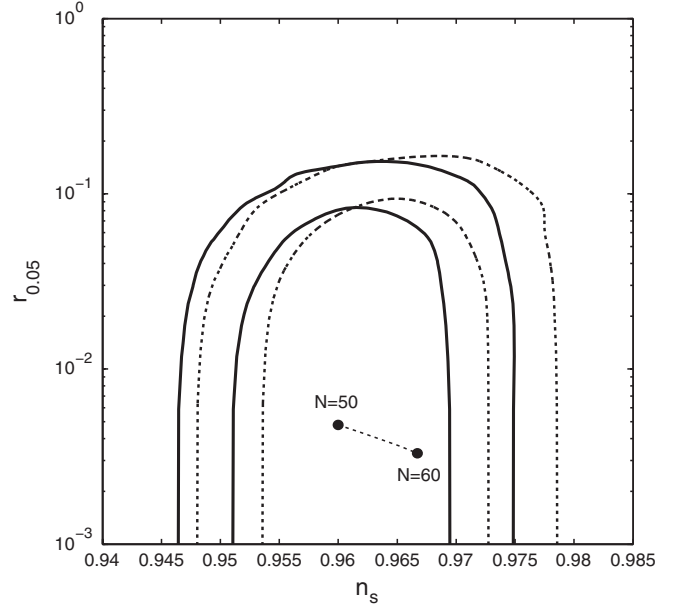


FIG. 1. 2-dimensional observational constraints on the Starobinsky's inflationary model $\mathcal{L} = (2/\kappa_D^2)[R + R^2/(6m^2)]$ in the (n_s, r) plane with the pivot wave number $k_0 = 0.05 \text{ Mpc}^{-1}$. The bold solid curves show the 68% CL (inside) and 95% CL (outside) boundaries derived by the joint data analysis of Planck+WP+BAO+high- ℓ , while the dotted curves correspond to the 68% and 95% contours constrained by Planck+WP+BAO. We plot the theoretical predictions for $N = 50, 60$ as black points.

In order to test the observational viability of the model, we run the COSMOMC code [69,70] by setting the runnings of the scalar and tensor spectral indices to be 0. The flat Λ CDM model is assumed with $N_{\text{eff}} = 3.046$ relativistic degrees of freedom and with the instant reionization. In Fig. 1 we plot the 68% CL and 95% CL boundaries (solid curves) constrained by the joint analysis of Planck [6], WP [10], baryon acoustic oscillations (BAO) [71], and high- ℓ Atacama Cosmology Telescope/South Pole Telescope temperature data [72] (solid curves), together with the boundaries constrained by Planck+WP+BAO (dotted curves). We also show the theoretical values of n_s and r for N between 50 and 60. The model is well inside the 68% CL observational boundaries.

We also note that, in the Starobinsky's model, the nonlinear parameter f_{NL} of scalar non-Gaussianities is much smaller than 1 [73]. This is consistent with the recent bounds of f_{NL} constrained by the Planck group [9].

In the next section we will argue more on the importance and the physical implications of the graviscalar degree of freedom.

VII. IMPORTANCE OF THE GRAVISCALAR DEGREE OF FREEDOM

Let us discuss the reason why it is important to consider super-renormalizable and unitary theories with one extra gravitational scalar degree of freedom.

The Starobinsky model, as well as any $f(R)$ theory [27–29], contains an extra scalar degree of freedom responsible for the generation of primordial density perturbations. This is evident after mapping the theory from the Jordan to the Einstein reference frame in which the scalaron has a nearly flat potential to drive inflation [28]. Therefore, such a scalar plays a fundamental role for the construction of a coherent cosmological model and, if it is absent, one has to resort to different mechanisms to generate primordial perturbations, e.g., Higgs inflation with nonminimal couplings [21].

In Ref. [48] a super-renormalizable model characterized by the following Lagrangian density has been proposed

$$\mathcal{L} = R - G^{\mu\nu} \left(\frac{V(\square_\Lambda)^{-1} - 1}{\square} \right) R_{\mu\nu}, \quad (63)$$

with the specific choice

$$V(\square_\Lambda) \equiv \exp(-\square_\Lambda). \quad (64)$$

This corresponds to the Lagrangian (3) with the choice $V_0(\square_\Lambda) = V_2(\square_\Lambda) = V(\square_\Lambda)$, see Eq. (53). Then, the propagator of the gravitational field on a Minkowskian background reads

$$\mathcal{O}^{-1} = \frac{V(k^2/\Lambda^2)}{k^2} \left(P^{(2)} - \frac{P^{(0)}}{D-2} \right). \quad (65)$$

Hence one has only the spin-two massless graviton and no graviscalar degree of freedom. The Lagrangian in this model is given by

$$\mathcal{L} = \frac{2}{\kappa_D^2} \left[R + \frac{R^2}{6\Lambda^2} + \mathcal{O}\left(\frac{R\square R}{\Lambda^4}\right) \right], \quad (66)$$

which reduces to

$$\mathcal{L} = 2\kappa_D^{-2} \left(R + \frac{R^2}{6\Lambda^2} \right), \quad (67)$$

for $R\square R/\Lambda^2 \ll R^2$. However this reduction is not coherent with inflation since, as we have shown above, the theory (63) contains only the spin-two graviton while the Starobinsky model includes an additional scalar degree of freedom, so the full theory starting from the Lagrangian (63) and the “reduced” one have different degrees of freedom. This means that the truncation of the $\mathcal{O}(R\square R/\Lambda^4)$ terms from the Lagrangian (66) is not consistent.

In Sec. IV we constructed super-renormalizable and unitary theories containing one scalar degree of freedom, by which gravitation is responsible for both the inflationary expansion and the generation of perturbations. These theories, after the truncation of (53), reduce in a coherent way to the Starobinsky model, since the number of degrees

of freedom is preserved in the truncation procedure. Therefore, in such a case, the graviscalar generates primordial perturbations and its mass has to fulfill the condition (60) in order to give the correct amplitude of primordial perturbations. In Sec. IV we have shown that the maximal class of unitary theories verifying this requirement contains only one extra scalar, since if any other degree of freedom is present, it must be a ghost or a tachyon. Hence the theory presented here is the maximal one.

VIII. STRING FIELD THEORY

In this section we show how the Starobinsky theory emerges from string theory when the modifications suggested by “string field theory” are taken into account.

In string field theory the propagator of the point-particle effective field theory is modified to [54]

$$\frac{1}{\square} \rightarrow \frac{e^{-\tilde{\alpha}'\square}}{\square}, \quad (68)$$

where $\tilde{\alpha}' \equiv (\alpha'/2) \ln(3\sqrt{3}/4) \approx 0.1308\alpha'$ with α' being the universal Regge slope parameter of the string. Collecting together the modification suggested by string field theory and general covariance, we propose the following effective Lagrangian for the bosonic sector of string theory,

$$\begin{aligned} \mathcal{L}_{\text{string-field}} = & 2\kappa_D^{-2} \left(R - G_{\mu\nu} \frac{e^{\tilde{\alpha}'\square} - 1}{\square} R^{\mu\nu} \right) \\ & + \frac{1}{2} \nabla_\mu \phi e^{\tilde{\alpha}'\square} \nabla^\mu \phi + \frac{1}{2n!} e^{c\phi} F_{[n]} e^{\tilde{\alpha}'\square} F_{[n]}. \end{aligned} \quad (69)$$

This is confirmed by the analysis in Ref. [74], where the authors make a field redefinition compatible with our proposal. Let us now consider the low-energy expansion of the exponential form factor in the Lagrangian (69), that is

$$e^{\tilde{\alpha}'\square} \approx 1 + \tilde{\alpha}'\square + \mathcal{O}((\tilde{\alpha}'\square)^2). \quad (70)$$

The gravity sector of the Lagrangian (69) simplifies to

$$\begin{aligned} \mathcal{L}_{\text{string-field}} & \approx 2\kappa_D^{-2} (R - \tilde{\alpha}' G_{\mu\nu} R^{\mu\nu}) \\ & = 2\kappa_D^{-2} \left(R - \tilde{\alpha}' R_{\mu\nu} R^{\mu\nu} + \frac{1}{2} \tilde{\alpha}' R^2 \right). \end{aligned} \quad (71)$$

In $D = 4$, for the Friedmann-Lemaître-Robertson-Walker metric, the following term turns out to be topological,

$$\int d^4x \sqrt{|g|} (3R_{\mu\nu} R^{\mu\nu} - R^2) = \text{topological}, \quad (72)$$

so that the truncated theory (71) reads

$$\mathcal{L}_{\text{string-field}} = 2\kappa_D^{-2} \left[R + \frac{\tilde{\alpha}'}{6} R^2 + O((\tilde{\alpha}')^2 R \square R) \right]. \quad (73)$$

When $\tilde{\alpha}' R \square R \ll R^2$, the above Lagrangian reduces to the Starobinsky model (2).

Finally, “string field theory” offers an alternative completion of the Starobinsky inflation. In order to obtain all of the results of the perturbation spectra compatible with the Planck data, we need to identify and select out an extra scalar degree of freedom from the string spectrum. However, this is not an easy task as for the theory (53) where the scalar field is a part of the gravitational sector.

In this model the graviscalar does not appear unlike the study in Sec. IV and the situation is exactly the same as the one discussed in Sec. VII. The difficult task in string theory is to pull out the extra scalar degree of freedom from the string spectrum.

IX. CONCLUSIONS

In this paper we proposed and extensively studied a class of super-renormalizable or finite theories of gravity which provide an ultraviolet completion of the Starobinsky theory. This class of theory is a generalization of the study done previously in searching for unitary and perturbatively consistent theory of quantum gravity [45,43,44]. The outcome is universal once a few theoretical and observative consistency requirements have been made.

If we require the hypothesis (1–5) listed in the introduction, together with the validity of perturbative theory, then we find only two unitary and super-renormalizable or finite theories of gravity. The minimal one contains only the graviton, but it is shown that a maximal extension is viable containing one extra scalar degree of freedom (graviscalar or scalaron). This is fundamental to generate primordial

density perturbation during inflation. The result is twofold, on the one hand in this theory the graviton and graviscalar fill up the maximal particle content compatible with unitarity and renormalizability or finiteness, on the other hand the Starobinsky model is coherently achieved in the low-energy limit.

We also mentioned other theories capable of giving a completion of the Starobinsky model. We expound about “asymptotic safe quantum gravity” where at nonperturbative level the ghost pole is moved to infinity by the renormalization group. In the string theory framework, we studied a point-particle theory incorporating the “string field theory” modified propagator together with general covariance. The resulting effective theory is in our class of super-renormalizable theories for a particular choice of the form factor and reduces to the Starobinsky model at low energy.

Finally, we believe the effort made in the search for a completion of quadratic gravity to be relevant and pertinent in the light of the recent Planck data supporting the Starobinsky inflation [9,12]. We would like to invite expert readers to invest time in this research. Our instinct is confirmed by recent papers having the same aim of this [30–39].

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