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Thermal Casimir effect in closed cosmological models with a cosmic string

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We consider the thermal corrections to the Casimir energy of a massless scalar field in the space-time with topology $S^3 \times R^1$ (Einstein and Friedmann universes) containing an idealized cosmic string. The vacuum energy of the field under consideration, in this background, can be separated in two terms: one term that is simply the known vacuum energy of the massless scalar field in the Einstein and Friedmann cosmological models and the other term that formally corresponds to the vacuum energy of the electromagnetic field, also in the Einstein and Friedmann universes, multiplied by the cosmic string parameter $\lambda = (1/\alpha) - 1$, where α is a constant related to the cosmic string tension, $G\mu$. The Casimir free energy and all the other thermodynamic expressions can also be separated in the same way. Thus, we use the expressions calculated in previous works for the massless scalar and electromagnetic fields in the closed Einstein and Friedmann models to investigate the low- and high-temperature limits of the Casimir free energy, internal energy, and entropy and show the role played by the presence of a cosmic string.

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I. INTRODUCTION

The Casimir effect [1] is a physical phenomenon with applications in different areas such as condensed matter and atomic physics, elementary particle physics, gravitation, and cosmology. In its original form, this effect arises associated to quantum fluctuations in the vacuum of the electromagnetic field due to the presence of metallic plates and, as a consequence, a finite vacuum energy appears induced by these material boundaries if compared with the Minkowski space. In fact, the effect occurs with any quantum field and is strongly depending on the geometry under consideration [2–6]. From a purely formal perspective, this phenomenon offers a large realm of studies [7–9] (see also references therein).

Measurements of the Casimir force have been performed (see [10–12] for a review), and the obtained results have been used to model phenomena in context of standard model [13], as well as to impose constraints on parameters coming from theories beyond it [14–18].

It was noticed that in gravitation and cosmology, the Casimir effect also occurs as a consequence of a nontrivial topology of space-time [19–22]. In this case the conditions due to material boundaries are substituted by some identification conditions imposed on the field and dictated by the topology of space. Along this line of research, some cosmological models with non-Euclidean topology have been investigated by many authors [23–30].

The evolution of the Universe is based on its thermal history. Taking into account that the Casimir effect is a quantum phenomenon, it is reasonable to expect that it may play an important role at the early stages of the Universe's evolution. Therefore, the studies of the thermal Casimir effect in the cosmological models scenario are of great physical importance. For this reason, some of the works on this phenomenon in cosmology were devoted to the thermal Casimir effect [31–36], and corrections of the Casimir energy [37,38]. The Casimir effect at nonzero temperature of a massive scalar field in a closed universe with three-torus topology was studied in [39]. More recently, the thermal Casimir effect in the closed Friedmann and Einstein universes was revisited using an updated approach [40,41]. In these papers, the vacuum energy and its thermal correction, together with the Casimir free energy, internal energy, pressure, and the Casimir entropy were calculated for a massless scalar [40], and electromagnetic and neutrino fields [41].

The cosmic string is a topological defect predicted in some gauge field theories and may have been formed as a result of a spontaneous symmetry breaking due to a phase transition in the very early universe [42,43], via the Kibble mechanism [44]. It can either form closed loops or extend to infinity, and is characterized by its tension, μ, which depends on the mass per unit length of the string, and is of the order of $\mu \sim \eta^2$, where η is the energy scale of symmetry breaking. The strength of the gravitational interaction of the cosmic string is characterized by the dimensionless cosmic string tension $G\mu \sim (\eta/M_p)^2$, where G is the Newton's gravitational constant and M_p is the Planck mass. For grand unification scales, observations of the cosmic microwave background (CMB) [45] provide an upper bound on the string tension of the order of $G\mu \sim 10^{-7}$. This topological defect offered the possibility to explain the origin of the primordial density perturbations leading to the formation of the observed structures in the Universe, as an alternative to inflation. This idea was abandoned with the confirmation

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that this scenario is not consistent with the anisotropies of the CMB. However, recently it was shown that cosmic strings are formed at the end of an inflationary era and should be taken into account as a subdominant partner of inflation [46].

In view of these, in recent years, there has been a renewed interest in cosmic strings due to the fact that supersymmetric grand unified theory (GUTs) and string theory also predict and seems to demand the existence of macroscopic defects such as cosmic string [47]. This new prediction establishes a connection between cosmic strings and fundamental strings. Besides, cosmic superstring (how it is called in the framework of supersymmetric GUTs) can play the role of cosmic string in the braneworld cosmology [48], and certainly, it makes sense to include cosmic string in the framework of cosmological models.

The space-time associated with a straight and infinite cosmic string has an azimuthal deficit angle given by $\Delta\varphi = 8\pi G\mu$. This means that this space-time is locally flat, but globally it is slightly curved [49–51]. The local flatness of the space-time surrounding a cosmic string also means that there is no local gravity. However, there exist some interesting gravitational effects associated with the nontrivial topology of the spacelike section around the cosmic string. Among these effects, a cosmic string can act as a gravitational lens [52], it can produce the Casimir effect [53] and many others in different contexts [54,55].

It is known that in the very early universe the temperature was extremely hot and that cosmic strings may have been formed through the process mentioned before. Thus, in this paper we study the thermal Casimir effect of a massless scalar field in the Einstein and Friedmann universes in the presence of a cosmic string. We notice that the vacuum energy in this background can be separated in two terms: one that is simply the vacuum energy of the massless scalar field in the Einstein and Friedmann universes, plus another term that can be interpreted as the vacuum energy of the electromagnetic field, also in the Einstein and Friedmann universes, multiplied by the cosmic string parameter $\lambda = (1/\alpha) - 1$. Therewith, we can simply use the results obtained previously for the massless scalar field [40] and for the electromagnetic and neutrino fields [41]. We also notice that the additional electromagneticlike term is the dominant one if one considers the constraint $G\mu \lesssim 10^{-7}$ arising from CMB observation. In spaces with nontrivial topology, the stress-energy tensor of a quantum field in its vacuum state depends strongly on the topological features of the manifolds. For instance, in the space-time of a cosmic string, the vacuum expectation value of the stress-energy tensor of the electromagnetic field is affected by the quantity that codifies the gravitational interaction of the cosmic string, namely, its tension [56].

Taking into account that the Casimir effect is a relativistic quantum phenomenon, we should expect that it may play an important role in the earlier stages of the Universe, as well as in the present problem of dark energy [57]. In other cosmological models, as in braneworlds, the Casimir effect is also a key element [58]. On the other hand, if cosmic strings exist they were formed at the end of inflationary era [59] and would be equally important in a cosmological scenario. Thus, taking into account cosmological models including string configuration, investigating the thermal correction to the Casimir effect may be important to understand the role played by the geometrical features of the Friedmann universe, as well as by the topological features of the cosmic string on this quantum physical phenomenon.

This paper is organized as follows. In Sec. II we present the Einstein and Friedmann cosmological models in the presence of a cosmic string and obtain the eigenfrequencies of the massless scalar field in these models. The vacuum energy and its thermal correction, together with other thermodynamic expressions, are discussed in Sec. III. The limits of low and high temperatures are considered in Sec. IV, and finally the conclusion is presented in Sec. V.

II. MASSLESS SCALAR FIELD IN THE EINSTEIN AND FRIEDMANN COSMOLOGICAL MODELS WITH A COSMIC STRING

In this section we present the closed Einstein cosmological model with a cosmic string and discuss how all the results obtained in this model can also be extended to the closed Friedmann cosmological model with a cosmic string. We also obtain the eigenfrequencies which will be used to calculate the vacuum energy and the corresponding thermal correction.

The line element of a space-time with an infinitely thin and straight cosmic string can be constructed by the following way: write the metric in cylindrical or spherical coordinates and insert into the metric a deficit of polar angle. Doing this procedure, the line element corresponding to the Einstein cosmological model with a topology $S^3 \times R^1$ and an angular deficit in the azimuthal angle, due to the presence of a cosmic string, can be written as [42]

$$ds^2 = c^2 d\tau^2 - a_0^2 d\sigma^2, \tag{1}$$

where

$$d\sigma^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \alpha^2 \sin^2\theta d\phi^2), \tag{2}$$

$0 \le \psi \le \pi$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ are dimensionless coordinates on a three-space of constant curvature $+1$, τ is the proper synchronous time, $a_0 = \text{const}$ is the scale factor and $\alpha = 1 - 4G\mu$ is the angular parameter of the cosmic string. This is a model with a finite spatial volume

$$\begin{aligned} V_\alpha &= \int \sqrt{-g^{(3)}} d\psi d\theta d\phi \\ &= a_0^3 \alpha \int_0^\pi \sin^2\psi d\psi \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi^2 \alpha a_0^3, \end{aligned} \tag{3}$$

with $g^{(3)}$ being the determinant of the spatial part of the line element given by Eq. (2).

The Friedmann metric with a cosmic string is obtained from Eq. (1) by doing the conformal transformation

$$d\bar{s}^2 = \Omega^{-2} ds^2, \tag{4}$$

where $\Omega(t) = a_0/a(t)$, with $a(t)$ being the scale factor in the expanding universe. One should notice that it is also necessary to rescale the time coordinate in (1), first by $d\tau = a_0 dt$ and then, after doing the transformation (4), by $dt = a^{-1}(t)d\tau$. Moreover, as pointed out in Ref. [41], the stress-energy tensor has additional contributions in the Friedmann cosmological model that are due to creation of particles and conformal anomaly [60,61]. Therefore, by doing the changing $a_0 \to a(t)$ in all expressions obtained below in the Einstein model, we can apply all the obtained results to the Friedmann model.

We are going to consider a real massless quantum scalar field in thermal equilibrium, at some temperature T, in the geometry of the closed universe described by the Einstein cosmological model, with a cosmic string embedded along the axis of rotational symmetry. In the Einstein universe, the vacuum energy and the thermodynamic quantities are not affected by the time dependence of the metric in the Friedmann model so that all the results obtained here are also applicable for the Friedmann cosmological model [26,40,41].

Now, let us determine the eigenfrequencies of a real massless quantum scalar field conformally coupled to the backgorund under consideration. This field, φ, satisfies the following Klein-Gordon equation:

$$\Box\varphi + \frac{1}{6}R\varphi = 0, \tag{5}$$

where $\Box = \partial_\nu\partial^\nu$ is the D'Alembertian operator and $R = 6a_0^{-2}$ [for the line element (1)] is the scalar curvature. The solution of Eq. (5), in the background described by Eq. (1), can be written as

$$\varphi(t,\psi,\theta,\phi) = R(\psi)P_{l_\alpha}^{m_\alpha}(\cos\theta)e^{im\phi}e^{-i\omega t}, \tag{6}$$

where $P_{l_\alpha}^{m_\alpha}$ is the associated Legendre function and

$$\begin{aligned} l_\alpha &= n + m_\alpha, \quad n = 0,1,2,\ldots, \\ m_\alpha &= \frac{m}{\alpha}, \quad -l \le m \le l. \end{aligned} \tag{7}$$

The quantum number n does not depend on the parameter α, so that we can obtain the following relation from Eq. (7):

$$l_\alpha = l + |m|\left(\frac{1}{\alpha} - 1\right), \quad l = 0,1,2,\ldots, \tag{8}$$

with $l = n + m$. Putting Eq. (6) into Eq. (5), we get

$$\frac{d}{d\psi}\left(\sin^2\psi\frac{dR}{d\psi}\right) + \left(\frac{\omega^2}{c^2} - \frac{1}{6}R\right)a_0^2\sin^2\psi R - l_\alpha(l_\alpha+1)R = 0. \tag{9}$$

The solution of Eq. (9) is given in terms of the Gegenbauer functions C_{n-l}^{l+1} as

$$R(\psi) \propto (\sin\psi)^{l_\alpha} C_{n-l_\alpha}^{l_\alpha+1}(\cos\psi). \tag{10}$$

Thus, the eigenfrequencies of the massless scalar field in the Einstein cosmological model with a cosmic string are

$$\omega_{n,\lambda} = \frac{cn}{a_0} + \frac{c|m|\lambda}{a_0}, \quad n = 1,2,3\ldots, \tag{11}$$

where $\lambda = (\frac{1}{\alpha} - 1)$. If $\lambda = 0$ (no string), we recover the well-known eigenfrequencies of the massless scalar field in the Einstein universe,

$$\omega_n = \frac{cn}{a_0}, \quad n = 1,2,3\ldots. \tag{12}$$

Therefore, the presence of a cosmic string changes the eigenfrequencies (12) by a factor proportional to λ, as it can be seen from Eq. (11).

III. THE CASIMIR ENERGY

The vacuum energy at zero temperature of the real massless quantum scalar field conformally coupled to the gravitational background of the static Einstein universe with a cosmic string, is given by

$$\begin{aligned} E_{0,\lambda} &= V\langle 0|T_0^0|0\rangle \\ &= \frac{\hbar}{2}\sum_{n=1}^{\infty}\sum_{l=0}^{n}\sum_{m=-l}^{l}\omega_{n,\lambda}^{(0)}, \\ &= \frac{\hbar}{2}\sum_{n=1}^{\infty}n^2\omega_n^{(0)} + \frac{\lambda}{6}\hbar\sum_{n=2}^{\infty}(n^2-1)\omega_n^{(1)}, \end{aligned} \tag{13}$$

where V is the three-volume of the universe, $\omega_n^{(0)}$ and $\omega_n^{(1)}$ are the eigenfrequencies of the scalar and electromagnetic fields, respectively, in the Einstein space-time with no cosmic string. It is interesting to call attention to the fact that $\omega_n^{(0)} = \omega_n^{(1)}$ and both are given by the expression for ω_n in (12).

The first term in the second line of Eq. (13) is the vacuum energy of the massless scalar field in the Einstein universe, and the summation in the second term multiplying the factor $\lambda/6$ can be interpreted as the vaccum energy of the electromagnetic field, also in the Einstein universe [41]. Thus, Eq. (13) can be rewritten as

$$E_{0,\lambda} = E_0^{(0)} + \frac{\lambda}{6}E_0^{(1)}, \tag{14}$$

where

$$E_0^{(0)} = \frac{\hbar}{2}\sum_{n=1}^{\infty} n^2\omega_n^{(0)},$$
$$E_0^{(1)} = \hbar\sum_{n=2}^{\infty}(n^2-1)\omega_n^{(1)}, \quad (15)$$

where the superscripts 0 and 1 in the vacuum energy expressions indicate the spins of the fields.

The above results show us that the effect of the presence of a cosmic string in the Einstein universe is to add to the usual vacuum energy of the massless scalar field, $E_0^{(0)}$, a term that can be seen as the usual vacuum energy of the electromagnetic field multiplied by $\lambda/6$. Thus, instead of working with a background in which the volume is given by Eq. (3), we can work in the background described by the usual Einstein universe which has volume given by $V = 2\pi^2 a_0^3$.

The renormalization of the vacuum energy $E_{0,\lambda}$ in Eq. (14) is obtained through the renormalization of the vacuum energies $E_0^{(0)}$ and $E_0^{(1)}$ for the massless scalar and electromagnetic fields, respectively. The renormalization procedure is performed by using the Abel-Plana formula [19,21,62,63], which is given by

$$\sum_{n=1}^{\infty}\Phi(n) - \int_0^{\infty}\Phi(t)dt = -\frac{1}{2}\Phi(0) + i\int_0^{\infty}\frac{\Phi(it)-\Phi(-it)}{e^{2\pi t}-1}dt, \quad (16)$$

where $\Phi(x)$ is an analytic function. The renormalization of the vacuum energy E_0^s, where $s=0$ or $s=1$, for scalar and electromagnetic fields, respectively, is performed by subtracting the vacuum energy of the tangential Minkowsky space at zero temperature. The result corresponding to the renormalized vacuum energy of the scalar was obtained some time ago and is given by [23–26].

In this way, the renormalized energies for the massless scalar and electromagnetic fields are [40,41]

$$E_{0,\mathrm{ren}}^{(0)} = \frac{\hbar c}{240 a_0}. \quad (17)$$

For the electromagnetic field, the renormalized vacuum energy was already obtained [26,33], and the result is

$$E_{0,\mathrm{ren}}^{(1)} = \frac{11\hbar c}{120 a_0}. \quad (18)$$

Thus, the renormalized energy which corresponds to Eq. (14) is written as

$$E_{0,\lambda,\mathrm{ren}} = E_{0,\mathrm{ren}}^{(0)} + \frac{\lambda}{6}E_{0,\mathrm{ren}}^{(1)}. \quad (19)$$

If one considers the CMB data that constrains the cosmic string tension as $G\mu \lesssim 10^{-7}$, the cosmic string parameter in Eq. (19) will be $\lambda \sim 10^7$. As $E_{0,\mathrm{ren}}^{(0)}$ and $E_{0,\mathrm{ren}}^{(1)}$ are approximately of the same order of magnitude (actually $E_{0,\mathrm{ren}}^{(1)}$ is one order of magnitude bigger), thus we have

$$E_{0,\lambda,\mathrm{ren}} \sim \frac{\lambda}{6}E_{0,\mathrm{ren}}^{(1)}. \quad (20)$$

In this case, the term proportional to the renormalized vacuum energy of the electromagnetic field is the dominant one since it is multiplied by $\lambda/6$. Therefore, the presence of a cosmic string not only is responsible for generate an electromagnetic vacuum energylike but it also increases the vacuum energy, by a factor which depends on the mass density of the cosmic string.

IV. THERMAL CORRECTION IN THE CASIMIR FREE ENERGY

In the last section we considered the vacuum state $|0\rangle$ to calculate the free energy. Now, let us consider a state describing the thermal equilibrium of the scalar field, φ, at some nonzero temperature T. In this case, the vacuum energy given by Eq. (13) is substituted by the free energy [32,33]

$$F_\lambda(T) = E_{0,\lambda} + \Delta F_\lambda(T), \quad (21)$$

where $E_{0,\lambda}$ is the vacuum energy at zero temperature given by Eq. (14). If one considers the renormalized vacuum energy $E_{0,\lambda,\mathrm{ren}}$ in Eqs. (19), (21) turns into the expression for the total free energy, which can be written as

$$F_{\mathrm{tot},\lambda}(T) = E_{0,\lambda,\mathrm{ren}} + \Delta F_\lambda(T). \quad (22)$$

The renormalization procedure for the thermal Casimir free energy will be the same one discussed and adopted in Refs. [40,41], in which the obtained results for this quantity, at high temperature, are in agreement with the classical limit.

According to [36], the asymptotic expression for the Casimir free energy at high temperature T, in a finite volume V contains the following terms of quantum nature:

$$\alpha_0\frac{(k_BT)^4}{(\hbar c)^3}, \qquad \alpha_1\frac{(k_BT)^3}{(\hbar c)^2}, \qquad \alpha_2\frac{(k_BT)^2}{\hbar c}, \quad (23)$$

where the coefficients α_0, α_1, α_2 depend on the spin of the field to be considered. For a massless scalar field in the Einstein universe, these coefficients are given by [40]

$$\alpha_0^{(0)} = -\frac{\pi^2}{90}V, \qquad \alpha_1^{(0)} = 0, \qquad \alpha_2^{(0)} = 0. \quad (24)$$

If we consider the electromagnetic field, these coefficients are given by [41]

$$\alpha_0^{(1)} = -\frac{\pi^2}{45}V, \qquad \alpha_1^{(1)} = 0, \qquad \alpha_2^{(1)} = \frac{\pi^2 a_0}{3}. \tag{25}$$

The finite renormalization of the thermal correction $\Delta F^{(s)}(T)$ is obtained also by using Eq. (16), which corresponds to the Abel-Plana formula and the results are given in [40], for the massless scalar field, and in [41], for the electromagnetic field. Then, after doing the renormalization of $\Delta F_\lambda(T)$ in Eq. (21), one obtains the Casimir free energy

$$F_{C,\lambda}(T) = E_{0,\lambda,\mathrm{ren}} + \Delta F_{C,\lambda}(T), \tag{26}$$

where $\Delta F_{C,\lambda}(T)$ is the renormalized thermal correction which is obtained from $\Delta F_\lambda^{(s)}(T)$ by using the Abel-Plana formula in the following way:

$$\Delta F_{C,\lambda}(T) = \Delta F_\lambda(T) - \int_0^\infty \Phi_\lambda(t)dt. \tag{27}$$

The thermal correction $\Delta F_\lambda(T)$ of the vacuum energy (14) is given by

$$\Delta F_\lambda(T) = \Delta F^{(0)}(T) + \frac{\lambda}{6}\Delta F^{(1)}(T), \tag{28}$$

where

$$\Delta F^{(0)}(T) = k_B T \sum_{n=1}^\infty n^2 \ln[1 - e^{-(\hbar\omega_n^{(0)}/k_B T)}], \tag{29}$$

$$\Delta F^{(1)}(T) = 2k_B T \sum_{n=1}^\infty (n^2 - 1) \ln[1 - e^{-(\hbar\omega_n^{(1)}/k_B T)}], \tag{30}$$

for the massless scalar and electromagnetic fields [40,41], respectively. Equation (29) was written in this way in order to recover the same structure of the vacuum energy, given by Eq. (19). The identification of the free energy for each field, scalar or electromagnetic, comes from the degeneracies n^2 $n^2 - 1$, which contain the information about the presence of the cosmic string according to Eqs. (14) and (15).

To obtain the Casimir free energy, we need to calculate the integral in the right-hand side of Eq. (27). The result of this integral is given by

$$\int_0^\infty \Phi_\lambda(t)dt = \alpha_0^{(0)}\frac{(k_BT)^4}{(\hbar c)^3} + \frac{\lambda}{6}\left(\alpha_0^{(1)}\frac{(k_BT)^4}{(\hbar c)^3} + \alpha_2^{(1)}\frac{(k_BT)^2}{\hbar c}\right). \tag{31}$$

The integral (31) provides the quantum contributions given by Eqs. (24) and (25). Thus, one can see that for the massless scalar field in the Einstein universe with a cosmic string, the integral (31) has not only the term proportional to $\alpha_0^{(0)}$ in Eq. (24) but also the terms proportional to $\alpha_0^{(1)}$ and $\alpha_2^{(1)}$, which are given in (25) and are related to the electromagnetic field. Then, the Casimir free energy can be obtained from Eq. (26) as

$$F_{C,\lambda}(T) = F_{\mathrm{tot},\lambda}(T) - \alpha_0^{(0)}\frac{(k_BT)^4}{(\hbar c)^3} - \frac{\lambda}{6}\left(\alpha_0^{(1)}\frac{(k_BT)^4}{(\hbar c)^3} + \alpha_2^{(1)}\frac{(k_BT)^2}{\hbar c}\right). \tag{32}$$

Once again, if we consider the approximated value $\lambda \sim 10^6$, Eqs. (28) and (32) turn into

$$\Delta F_\lambda(T) \cong \frac{\lambda}{6}\Delta F^{(1)}(T), \tag{33}$$

$$F_{C,\lambda}(T) \cong F_{\mathrm{tot},\lambda}(T) - \frac{\lambda}{6}\left(\alpha_0^{(1)}\frac{(k_BT)^4}{(\hbar c)^3} + \alpha_2^{(1)}\frac{(k_BT)^2}{\hbar c}\right). \tag{34}$$

V. LIMITS OF LOW AND HIGH TEMPERATURES

First, we will consider the low-temperature limit $k_B T \ll \hbar c/a_0$. In this limit, in order to obtain the asymptotic expression for the Casimir free energy, it is most convenient to write Eq. (26) in the following form:

$$F_{C,\lambda}(T) = F_C^{(0)}(T) + \frac{\lambda}{6}F_C^{(1)}(T), \tag{35}$$

where

$$F_C^{(0)}(T) = \frac{\hbar c}{240 a_0} + \frac{\pi^2}{90}V\frac{(k_BT)^4}{(\hbar c)^3} - k_B T e^{-(\hbar c/a_0 k_B T)} \tag{36}$$

is the low-temperature limit for the Casimir free energy of the massless scalar field [40] and

$$F_C^{(1)}(T) = \frac{11\hbar c}{120 a_0} + \frac{\pi^2}{45}V\frac{(k_BT)^4}{(\hbar c)^3} - \frac{\pi^2 a_0}{3}\frac{(k_BT)^2}{\hbar c} - 6k_B T e^{-(2\hbar c/a_0 k_B T)} \tag{37}$$

is the low-temperature limit for the Casimir free energy of the electromagnetic field [41]. The Casimir internal energy can also be obtained in this limit. Let us, then, use the expression for the internal energy, at a temperature T, which is defined by [64]

$$U(T) = -T^2\frac{\partial}{\partial T}\left[\frac{F(T)}{T}\right]. \tag{38}$$

Using Eqs. (38) and (26), we get the Casimir internal energy

$$U_{C,\lambda}(T) = U_C^{(0)}(T) + \frac{\lambda}{6} U_C^{(1)}(T), \tag{39}$$

where

$$U_C^{(0)}(T) = \frac{\hbar c}{240 a_0} - \frac{\pi^2}{30} V \frac{(k_B T)^4}{(\hbar c)^3} + \frac{\hbar c}{a_0} e^{-(\hbar c/a_0 k_B T)} \tag{40}$$

is the Casimir internal energy of the massless scalar field [40] and

$$U_C^{(1)}(T) = \frac{11\hbar c}{120 a_0} - \frac{\pi^2}{15} V \frac{(k_B T)^4}{(\hbar c)^3} + \frac{\pi^2 a_0}{3} \frac{(k_B T)^2}{(\hbar c)} + \frac{12\hbar c}{a_0} e^{-(2\hbar c/a_0 k_B T)} \tag{41}$$

is the Casimir internal energy of the electromagnetic field [41]. A quantity which is closely related to the free energy is the entropy. It is defined by

$$S(T) = -\frac{\partial F(T)}{\partial T}. \tag{42}$$

The Casimir entropy for the massless scalar field can be obtained from Eq. (42) by using the expression for the Casimir free energy given by Eq. (26). Thus, we get

$$S_{C,\lambda}(T) = S_C^{(0)}(T) + \frac{\lambda}{6} S_C^{(1)}(T), \tag{43}$$

where

$$S_C^{(0)}(T) = -\frac{2\pi^2}{45} k_B \left(\frac{k_B T}{\hbar c}\right)^3 V + \frac{\hbar c}{a_0 T} e^{-(\hbar c/a_0 k_B T)} \tag{44}$$

is the Casimir entropy of the massless scalar field obtained in [40] and

$$S_C^{(1)}(T) = 2\pi^2 k_B \frac{a_0 k_B T}{3\hbar c} \left[1 - \frac{4\pi^2}{15} \frac{(a_0 k_B T)^2}{(\hbar c)^2}\right] + \frac{12\hbar c}{a_0 T} e^{-(2\hbar c/a_0 k_B T)}. \tag{45}$$

is the Casimir entropy of the electromagnetic field [41]. From Eqs. (44) and (44) we conclude that the Casimir entropy goes to zero when the temperature vanishes, in accordance with the third law of thermodynamics (Nernst heat theorem).

Now, let us consider the limiting case of high temperature, which means that $k_B T \gg \hbar c/a_0$. The asymptotic expression for the Casimir free energy, in this limit, can be obtained from Eq. (35), with $F_C^{(0)}(T)$ and $F_C^{(1)}(T)$ given by

$$F_C^{(0)}(T) = \frac{k_B T}{4\pi^2} \zeta(3) + 4\pi^2 \left(\frac{a_0 k_B T}{\hbar c}\right)^3 \frac{\hbar c}{a_0} e^{-4\pi^2 (a_0 k_B T/\hbar c)}, \tag{46}$$

which is the asymptotic expression for the Casimir free energy of the massless scalar field [40] in the high-temperature limit, and

$$F_C^{(1)}(T) = -k_B T \ln \frac{a_0 k_B T}{\hbar c} - R k_B T + 8\pi^2 a_0^2 \frac{(k_B T)^3}{(\hbar c)^2} e^{-(4\pi^2 a_0 k_B T/\hbar c)}, \tag{47}$$

which is the Casimir free energy of the electromagnetic field, in the high-temperature limit, with $R = 1.77698$ being a constant [41].

In this limit, the Casimir internal energy can be calculated by using Eq. (39), with $U_C^{(0)}(T)$ and $U_C^{(1)}(T)$, in the high temperature limit, being given by

$$U_C^{(0)}(T) = 16\pi^4 \left(\frac{a_0 k_B T}{\hbar c}\right)^4 \frac{\hbar c}{a_0} e^{-4\pi^2 (a_0 k_B T/\hbar c)} \tag{48}$$

for the internal energy of the massless scalar field [40] and

$$U_C^{(1)}(T) = k_B T + \frac{32\pi^4 a_0^3 (k_B T)^4}{(\hbar c)^3} e^{-(4\pi^2 a_0 k_B T/\hbar c)} \tag{49}$$

for the internal energy of the electromagnetic field [41].

The total free energy and its limits of low and high temperatures can be obtained from Eq. (32). It is worth calling attention to the fact that as the cosmic string parameter is of order of $\lambda \sim 10^7$, then the Casimir free energy, internal energy, and Casimir entropy, which are given by Eqs. (35), (39), and (43), are dominated by the electromagneticlike terms, which means that the presence of the cosmic string increases these quantities as compared with a scenario without this topological defect.

VI. CONCLUDING REMARKS

We have considered the thermal Casimir effect for a massless scalar field in the Einstein and Friedmann universe with a cosmic string with mass per unit length μ embedded along the axis of rotational symmetry. This embedding does not locally affect the Einstein and Friedmann universes, but globally it does. In these space-times with original topology $S^3 \times R^1$, we investigated how the topological features arising from the presence of the cosmic string will affect the Casimir effect.

Using the appropriate method of renormalization adopted in [40,41], we have shown that the renormalized quantum vacuum energy is expressed as a sum of known terms, namely, one associated with the scalar field in Einstein universe without the cosmic string and another electromagneticlike term induced by the presence of the cosmic string. This second term is given by the Casimir energy of the electromagnetic field in an Einstein universe with no cosmic string multiplied by a parameter which depends on the linear mass density of this defect, contained in the parameter $\lambda = (1/\alpha) - 1$. It is worth stressing that the contribution which arises due to the presence of the cosmic string is the dominant one, due to the upper limit on λ obtained from cosmological observations.

The expression for the renormalized free energy was obtained in a form which is convenient to obtain the high and low temperature limits of the internal energy and entropy. Taking into account the expressions for the Casimir free energy and the internal energy, the high and low temperature limits of these thermodynamical quantities are found. They are in agreement with the results already known, when we take the appropriate limit, in which λ goes to zero, which means that the cosmic string is absent. The Casimir free energy at high temperature has the terms directly proportional to the temperature independent of the cosmological parameter a_0.

The expression for the renormalized free energy, and consequently for all the thermodynamical quantities such as internal energy, entropy, and pressure can also be written as a single sum whose first term is associated with the contribution arising from the massless scalar field in an Einstein universe without a cosmic string and whose second term corresponds to the contribution arising from the electromagnetic field in the Einstein universe without a cosmic string multiplied by a parameter that depends on the linear mass density of the cosmic string. Based on this, the high- and low-temperature limits of the considered thermodynamical quantities are given in terms of the known thermodynamical quantities associated with the massless scalar and electromagnetic fields in the closed Einstein and Friedmann cosmological models [40,41].

In low-temperature limit, the Casimir entropy of the system was found obeying the third law of thermodynamics; i.e., it goes to zero for vanishing temperature. Specifically, for the Friedmann cosmological model, if one considers the relation $T \propto 1/a(t)$ [65] in the radiation-dominated era, we find that the Casimir entropy tends to a constant value independent of the geometrical parameter a_0. Furthermore, the Casimir pressure in the low-temperature limit is negative and goes to zero when the temperature vanishes. In the opposed limit, this pressure is negative and grows illimitably for $T \to \infty$, which suggests a relevant role played by the vacuum energy in primordial inflationary processes.

Finally, we have calculated the internal energy of the massless scalar field and conclude that in very high temperatures it tends to $(\lambda/6)k_B T$, and in the low-temperature limit it goes to a constant value as shown in Eq. (19). One can note that without the cosmic string, the Casimir internal energy in the Einstein universe vanishes when $T \to \infty$.

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